Hypertoric category $\mathcal{O}$

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Abstract. We study the representation theory of the invariant subalgebra of the Weyl algebra under a torus action, which we call a "hypertoric enveloping algebra." We define an analogue of BGG category $\mathcal{O}$ for this algebra, and identify it with a certain category of sheaves on a hypertoric variety. We prove that a regular block of this category is highest weight and Koszul, identify its Koszul dual, compute its center, and study its cell structure. We also consider a collection of derived auto-equivalences analogous to the shuffling and twisting functors for BGG category $\mathcal{O}$.

1 Introduction

In this paper we study an algebra U analogous to the universal enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$. Just as the central quotients of $U(\mathfrak{g})$ are quantizations of the ring of functions on the cotangent bundle to the flag variety, the central quotients of our algebra are quantizations of the ring of functions on a hypertoric variety; for this reason, we call U the **hypertoric enveloping algebra**. The most important structure from our perspective is a category $\mathcal{O}$ of U-modules analogous to the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ of modules over the universal enveloping algebra. Our category $\mathcal{O}$ shares many beautiful structures and properties with the BGG category $\mathcal{O}$, including a Koszul grading, the presence of "standard objects" (analogues of Verma modules), a "cell" partition of the set of simple objects in a block, and two commuting actions (shuffling and twisting) of discrete groups by derived auto-equivalences.

In the first part of the introduction, we will review how these structures arise in Lie theory, before describing the analogous phenomena in the hypertoric setting. Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. **BGG category $\mathcal{O}$** is defined to be the category of finitely generated $U(\mathfrak{g})$-modules on which $U(\mathfrak{h})$ acts semisimply and $U(\mathfrak{b})$ acts locally finitely. The center of $U(\mathfrak{g})$ also acts locally finitely on objects of $\mathcal{O}$, which implies that $\mathcal{O}$ decomposes into **infinitesimal blocks** $\mathcal{O}_\lambda$ indexed by central characters λ of $U(\mathfrak{g})$.

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Each infinitesimal block has finitely many simple objects (at most the order of the Weyl group), and may decompose further, depending on the existence of nontrivial extensions between the simple objects. In this manner we decompose all of $\mathcal{O}$ into irreducible **blocks**. If λ is generic, then $\mathcal{O}_\lambda$ is semisimple, and breaks into one block for each simple object. At the other extreme, when λ is **integral**, the infinitesimal block $\mathcal{O}_\lambda$ is itself already a block. An infinitesimal block is called **regular** if the number of (isomorphism classes of) simple objects is equal to the order of the Weyl group. In the non-integral case, the constituent blocks of a regular infinitesimal block are called regular, as well. Each of these blocks is equivalent to a regular integral block for a Levi subalgebra of $\mathfrak{g}$, a phenomenon which is paralleled in the hypertoric setting.

One of the most powerful tools for studying BGG category $\mathcal{O}$ is the geometry of the flag variety G/B. For any central character λ of $U(\mathfrak{g})$, one can define λ-twisted D-modules on G/B. If the infinitesimal block $\mathcal{O}_\lambda$ is regular, then it is equivalent via the Beilinson-Bernstein localization theorem to the category of finitely generated λ-twisted D-modules on G/B with regular singularities and singular supports in the conormal varieties to the Schubert strata. Such D-modules may also be regarded as sheaves on the cotangent bundle $T^*(G/B)$, the "Springer resolution" of the nilpotent cone in $\mathfrak{g}$. This perspective allows one to define a Koszul grading on $\mathcal{O}_\lambda$ and to understand various algebraic properties of the category in terms of the geometry of the flag variety and the Springer resolution.

Kazhdan and Lusztig [KL79] define a partition of the simple objects of $\mathcal{O}$ called the **two-sided cell** partition, which can be further refined into **left** and **right cell** partitions. The two-sided cell partition induces a direct sum decomposition of the Grothendieck group $K(\mathcal{O}_\lambda)_\mathbb{C}$, in which the subspaces are spanned by the classes of projective covers of simples in a given two-sided cell. The set of cells has a natural partial order, and this defines for us a filtration of $K(\mathcal{O}_\lambda)_\mathbb{C}$ that we will call the **cell filtration**.

Two-sided cells are in bijection with special G-orbits in the nilpotent cone, and the partial order is given by inclusions of closures of orbits. Let $\mathbb{T} := \mathbb{C}^\times$ act on G/B in such a way so that the associated Białynicki-Birula stratification agrees with the Schubert stratification. Let $d := \dim G/B$. If we regard objects of $\mathcal{O}_\lambda$ as twisted D-modules on G/B, then the singular support map defines an isomorphism from $K(\mathcal{O}_\lambda)_\mathbb{C}$ to $H^{2d}_\mathbb{T}\big(T^*(G/B);\mathbb{C}\big)$. In the case where $\mathfrak{g} = \mathfrak{sl}_N$, this isomorphism takes the cell filtration of $K(\mathcal{O}_\lambda)_\mathbb{C}$ to the filtration of $H^{2d}_\mathbb{T}\big(T^*(G/B);\mathbb{C}\big)$ determined by the Beilinson-Bernstein-Deligne (BBD) decomposition theorem [BBD82, CG97].[4]

We now summarize some of the well known and important properties of regular integral blocks of BGG category $\mathcal{O}$, many of which we have already stated above.

Theorem 1.1. *Let $\mathcal{O}_\lambda$ be a regular integral block of BGG category $\mathcal{O}$.*

1. *$\mathcal{O}_\lambda$ is highest weight (equivalent to the representations of a quasihereditary algebra) with respect to some partial ordering of the simple objects [CPS88].*
2. *$\mathcal{O}_\lambda$ is Koszul (equivalent to the representations of an algebra with a Koszul grading) [BGS96].*

[4] A statement of this form can be made for arbitrary $\mathfrak{g}$, but it is more complicated due to the existence of non-special nilpotent orbits which show up in the BBD picture but not in the cell picture.

3. *$\mathcal{O}_\lambda$ is equivalent via the localization functor to a certain category of sheaves of modules over a quantization of the structure sheaf of $T^*(G/B)$ [BB81].*
4. *The center of the Yoneda algebra of $\mathcal{O}_\lambda$ is isomorphic to $H^*(G/B;\mathbb{C})$ [Soe90].*
5. *The complexified Grothendieck group $K(\mathcal{O}_\lambda)_\mathbb{C}$ is isomorphic to $H^{2d}_\mathbb{T}(G/B;\mathbb{C})$, and if $\mathfrak{g} = \mathfrak{sl}_N$, the cell filtration corresponds to the BBD filtration [BBD82, CG97].*
6. *There are two collections of derived auto-equivalences of $\mathcal{O}_\lambda$ (shuffling and twisting) obtained by varying the central character and the Borel subalgebra. These auto-equivalences define commuting actions of the fundamental group of the complement to the Coxeter arrangement for $\mathfrak{g}$ (the pure Artin braid group of $\mathfrak{g}$) [AS03, Irv93].*
7. *$\mathcal{O}_\lambda$ is Koszul self-dual [BGS96]. The corresponding derived auto-equivalence exchanges shuffling and twisting functors. The induced permutation of the set of simple objects sends left cells to right cells, right cells to left cells, and two-sided cells to two-sided cells. It is order-reversing for the highest weight ordering of the simple objects, and also order-reversing on the set of two-sided cells.*

Now we turn to the hypertoric enveloping algebra. This algebra, which was originally studied by Musson and Van den Bergh [MV98], is easy to define. Start with the ring $\mathbb{C}[x_1, \partial_1, \ldots, x_n, \partial_n]$ of polynomial differential operators on $\mathbb{C}^n$, which is equipped with the action of an algebraic torus $T \cong (\mathbb{C}^\times)^n$. The algebra U is defined to be the invariant ring with respect to a subtorus $K \subset T$. Inside of U is the polynomial subalgebra $\mathbf{H}$ generated by $x_1\partial_1, \ldots, x_n\partial_n$, which plays the role of the Cartan subalgebra. The role of the Borel subalgebra is played by the subalgebra $U^+ \subset U$ consisting of elements of non-negative weight for a certain action of the multiplicative group $\mathbb{T}$; this subalgebra always contains $\mathbf{H}$. We note that in our situation there is no analogue of the conjugacy of Borel subgroups; different choices of $\mathbb{T}$-action result in non-isomorphic subalgebras U^+.

We define **hypertoric category** $\mathcal{O}$ to be the category of finitely generated U-modules with the property that U^+ acts locally finitely and the center $Z(U)$ acts semisimply. Note that this definition differs in a key way from the definition of BGG category $\mathcal{O}$:

- in BGG category $\mathcal{O}$ the Cartan algebra acts semisimply, while the center acts locally finitely;
- in our definition, the center acts semisimply, but $\mathbf{H}$ only acts locally finitely.

In fact, the distinction vanishes if we look only at regular blocks or regular infinitesimal blocks. A theorem of Soergel [Soe86] says that a regular infinitesimal block $\mathcal{O}_\lambda$ of BGG category $\mathcal{O}$ is equivalent to the "reversed" category obtained by allowing the Cartan subalgebra to act locally finitely but requiring that the center act semisimply with character λ. Indeed, the proof of Part (3) of Theorem 1.1 goes through this equivalence, thus one can argue that even in the Lie-theoretic setting the reversed category is the more fundamental of the two.

The geometric perspective on BGG category $\mathcal{O}$ begins with the observation that a central quotient of $U(\mathfrak{g})$ can be realized as the ring of global twisted differential operators on G/B, or equivalently as the ring of $\mathbb{S}$-invariant global sections of an equivariant quantization of $T^*(G/B)$,

where $\mathbb{S} := \mathbb{C}^\times$ acts by scaling the fibers. In our setting the analogue of $T^*(G/B)$ is a hypertoric variety $\mathfrak{M}$. Though $\mathfrak{M}$ is not itself a cotangent bundle, it is a symplectic variety that admits a Hamiltonian $\mathbb{T}$-action analogous to the induced $\mathbb{T}$-action on $T^*(G/B)$ as well as a $\mathbb{S}$-action analogous to the scaling action on the cotangent fibers.[5] In Section 5 we construct an equivariant quantization of $\mathfrak{M}$ whose ring of $\mathbb{S}$-invariant global sections is isomorphic to a central quotient of U (there is a unique equivariant quantization for each central quotient). This quantization has already been studied by Bellamy and Kuwabara [BK], who prove an analogue of the Beilinson-Bernstein localization theorem in this context.

The data required to construct a hypertoric variety along with the necessary group actions are encoded by a linear algebraic object called a **polarized arrangement**. For any polarized arrangement X, let $\mathfrak{M}(X)$ be the associated hypertoric variety. The data required to construct a block of hypertoric category $\mathcal{O}$ (a subtorus $K \subset T$, a subalgebra $U^+ \subset U$, a central character of U, plus a little bit more data[6]) are encoded by another, slightly more complicated linear algebraic object called a **quantized polarized arrangement**. For any quantized polarized arrangement $\mathbf{X}$, let $\mathcal{O}(\mathbf{X})$ be the associated block. We define what it means for $\mathbf{X}$ to be **integral** and **regular**, and we show that the regular integral blocks are exactly those that have the largest possible number of isomorphism classes of simple objects (Remark 4.6). Quantized polarized arrangements and polarized arrangements are closely related; in Section 2.5 we make this precise by defining certain pairs X and $\mathbf{X}$ to be **linked**.

Every statement in Theorem 1.1 has a hypertoric equivalent. For example, we define left, right, and two-sided cells in the category $\mathcal{O}(\mathbf{X})$, as well as a support isomorphism from the Grothendieck group of $\mathcal{O}(\mathbf{X})$ to the degree $2d$ equivariant cohomology group of $\mathfrak{M}(X)$, where $2d$ is the dimension of $\mathfrak{M}(X)$. The statement of Theorem 1.1 is simplified by the fact that the category $\mathcal{O}_\lambda$ is Koszul dual to itself. In the hypertoric setting any regular polarized arrangement X has a **Gale dual** $X^!$, and we will see that Gale duality of arrangements corresponds to Koszul duality of categories.

Theorem 1.2. *Suppose that* $\mathbf{X}$ *and* $\mathbf{X}^!$ *are regular, integral, and linked to a dual pair* X *and* $X^!$.

1. $\mathcal{O}(\mathbf{X})$ *is highest weight with respect to a partial ordering of the simple objects (Corollary 4.10).*
2. $\mathcal{O}(\mathbf{X})$ *is Koszul (Corollary 4.10).*
3. $\mathcal{O}(\mathbf{X})$ *is equivalent to a certain category of sheaves of modules over a quantization of the structure sheaf of* $\mathfrak{M}(X)$ *(Corollary 6.5).*
4. *The center of the Yoneda algebra of* $\mathcal{O}(\mathbf{X})$ *is isomorphic to* $H^*(\mathfrak{M}(X);\mathbb{C})$ *(Theorem 5.3).*
5. *The complexified Grothendieck group* $K(\mathcal{O}(\mathbf{X}))_\mathbb{C}$ *is isomorphic to* $H^{2d}_\mathbb{T}(\mathfrak{M}(X);\mathbb{C})$*, and the cell filtration corresponds to the BBD filtration (Theorem 7.18).*
6. *There are two collections of derived auto-equivalences of* $\mathcal{O}(\mathbf{X})$ *(shuffling and twisting) obtained by varying the central character or the subalgebra* $U^+ \subset U$*. These auto-equivalences*

[5] The groups $\mathbb{T}$ and $\mathbb{S}$ are both copies of the multiplicative group, but they play very different roles in this paper. In particular, $\mathbb{T}$ acts on both $T^*(G/B)$ and $\mathfrak{M}$ preserving the symplectic forms, while $\mathbb{S}$ does not.

[6] The "little bit more data" is needed if and only if Λ_0 fails to be unimodular or the central character fails to be integral (Remark 4.2).

define commuting actions of the fundamental groups of the complements of the discriminantal arrangements for X and $X^!$ (Corollary 8.10, Theorems 8.13 and 8.19).

7. *$\mathcal{O}(\mathbf{X})$ is Koszul dual to $\mathcal{O}(\mathbf{X}^!)$. The associated derived equivalence exchanges shuffling and twisting functors. The induced bijection between sets of simple objects sends left cells to right cells, right cells to left cells, and two-sided cells to two-sided cells. It is order-reversing for the highest weight orderings of the simple objects of $\mathcal{O}(\mathbf{X})$ and $\mathcal{O}(\mathbf{X}^!)$, and also order-reversing from the set of two-sided cells of $\mathcal{O}(\mathbf{X})$ to the set of two-sided cells of $\mathcal{O}(\mathbf{X}^!)$ (Corollary 4.10, Theorems 7.15 and 8.19).*[7]

Remark 1.3. A consequence of Parts (5) and (7) of Theorem 1.2 is that the cohomology groups of $\mathfrak{M}(X)$ and $\mathfrak{M}(X^!)$ have dual BBD filtrations (Corollary 7.19). This fact has combinatorial implications that we explore in Remark 7.21.

Remark 1.4. Several parts of Theorem 1.2 are proved by means of an equivalence between $\mathcal{O}(\mathbf{X})$ and the category of modules over a finite dimensional algebra $A(X)$ that we introduced in [BLPWb] (this equivalence is stated in Theorems 4.7 and 4.8 of this paper). In [BLPWb] we proved that $A(X)$ is quasihereditary and Koszul, computed the center of its Yoneda algebra, gave a combinatorial construction of shuffling functors, and showed that $A(X)$ is Koszul dual to $A(X^!)$, so this equivalence immediately implies the corresponding statements for $\mathcal{O}(\mathbf{X})$. To establish this equivalence we rely heavily on work of Musson and Van den Bergh [MV98]. They give an detailed analysis of U and some of its representation categories, but they never consider the subalgebra $U^+ \subset U$ or our category $\mathcal{O}(\mathbf{X})$. Everything involving the geometry of $\mathfrak{M}(X)$ or the cell structure of $\mathcal{O}(\mathbf{X})$ is new to this paper.

Remark 1.5. This paper is a part of a larger program initiated by the authors, in which $\mathfrak{M}$ will be replaced by an equivariant symplectic resolution of an affine cone, and U will be replaced by an algebra whose central quotients are quantizations of the algebra of functions on $\mathfrak{M}$ [BLPWa]. When two symplectic resolutions yield categories that are Koszul dual as in Parts (2) of Theorems 1.1 and 1.2, we call those resolutions a **symplectic dual pair**. Above we see that $T^*(G/B)$ is self-dual and $\mathfrak{M}(X)$ is dual to $\mathfrak{M}(X^!)$. Other conjectural examples of dual pairs include Hilbert schemes on ALE spaces, which we expect to be dual to certain moduli spaces of instantons on $\mathbb{C}^2$, and quiver varieties of simply laced Dynkin type, which we expect to be dual to resolutions of slices to certain subvarieties of the affine Grassmannian. We expect further examples to arise from physics as Higgs branches of the moduli space of vacua for mirror dual 3-dimensional $\mathcal{N} = 4$ superconformal field theories, or as the Higgs and Coulomb branches of a single such theory. That hypertoric varieties occur in mirror dual theories was observed by Kapustin and Strassler in [KS99].

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[7] Unlike in the Lie-theoretic setting, one cannot conclude the reversal of either of the highest weight partial order on the simple objects of $\mathcal{O}(\mathbf{X})$ or the partial order on the set of two-sided cells from reversal of the other, so these are truly independent statements.

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2 Linear algebraic data

In this section we introduce the basic linear algebraic constructions that will be crucial to our analysis of hypertoric category $\mathcal{O}$. We define **polarized arrangements**, which are used to construct hypertoric varieties, and **quantized polarized arrangements**, whose combinatorics control our category in a manner similar to the way the Weyl group and associated Coxeter arrangement control BGG category $\mathcal{O}$.

2.1 The polynomial rings

Fix a positive integer n, and consider the ring

$$\mathbf{H} := \mathbb{C}[h_1^{\pm}, \dots, h_n^{\pm}] / \langle h_i^+ - h_i^- + 1 \mid i, \dots, n \rangle.$$

It is isomorphic to a polynomial ring on n generators, but it is naturally filtered by the semigroup $2\mathbb{N}$ rather than graded, where the $(2k)^{\text{th}}$ piece $F_{2k}\mathbf{H}$ of the filtration is the space of polynomials of degree $\leq k$ in the h_i^+ (or equivalently in the h_i^-).[8] Its associated graded is the polynomial ring

$$H := \operatorname{gr} \mathbf{H} = \mathbb{C}[h_1, \dots, h_n], \quad \text{where } h_i = h_i^+ + F_0\mathbf{H} = h_i^- + F_0\mathbf{H}.$$

The maximal spectrum $\mathbf{W} := \operatorname{Specm} \mathbf{H}$ is an n-dimensional complex affine space. It is naturally a torsor for the vector space $W := \operatorname{Specm} H$.

Both $\mathbf{W}$ and W have distinguished integral structures, given by

$$\mathbf{W}_{\mathbb{Z}} := \{v \in \mathbf{W} \mid h_i^{\pm}(v) \in \mathbb{Z} \text{ for all } i\} \qquad \text{and} \qquad W_{\mathbb{Z}} := \{v \in W \mid h_i(v) \in \mathbb{Z} \text{ for all } i\}.$$

These in turn induce real structures

$$\mathbf{W}_{\mathbb{R}} := \mathbf{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R} \qquad \text{and} \qquad W_{\mathbb{R}} := W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

2.2 Polarized arrangements

In this section we use only the graded ring H and the vector space W from Section 2.1; the filtered ring $\mathbf{H}$ and the affine space $\mathbf{W}$ will be used in the next section.

Definition 2.1. A **polarized arrangement** is a triple $X = (\Lambda_0, \eta, \xi)$, where $\Lambda_0 \subset W_{\mathbb{Z}}$ is a direct summand, η is a Λ_0-orbit in $W_{\mathbb{Z}}$, and $\xi \in \Lambda_0^*$. For simplicity, we will also assume that $h_i(\Lambda_0) \neq 0$ for every i.

[8] It will become clear in Section 3.1 why we have indexed our filtration by the even natural numbers.

Given a polarized arrangement, let

$$V_0 := \mathbb{C}\Lambda_0 \subset W \qquad \text{and} \qquad V_{0,\mathbb{R}} := \mathbb{R}\Lambda_0 \subset W_{\mathbb{R}}$$

be the complex and real vector spaces spanned by Λ_0, and let

$$V_{\mathbb{R}} := \eta + V_{0,\mathbb{R}} \subset W_{\mathbb{R}}.$$

For each $i \in \{1, \dots, n\}$, let

$$H_i := \{v \in V_{\mathbb{R}} \mid h_i(v) = 0\},$$

and let $\mathcal{H} := \{H_i \mid i = 1, \dots, n\}$ be the associated (multi)arrangement. The assumption that $h_i(\Lambda_0) \neq 0$ ensures that each H_i is really a hyperplane.

We will also need the corresponding central arrangement $\mathcal{H}_0$ in $V_{0,\mathbb{R}}$, but to avoid notational unpleasantness, we will refrain from giving names to its hyperplanes. If we identify $V_{\mathbb{R}}$ with $V_{0,\mathbb{R}}$ by choosing an origin (that is, by choosing a lift of η to $W_{\mathbb{Z}}$), then the arrangement $\mathcal{H}_0$ is obtained from $\mathcal{H}$ by translating all of the hyperplanes to the origin. We say that η is **regular** if the arrangement $\mathcal{H}$ is simple, which means that no point in $V_{\mathbb{R}}$ lies on more than $\dim V$ hyperplanes. This is a genericity assumption with respect to the positions (but not the slopes) of the hyperplanes in $\mathcal{H}$.

A **flat** of the arrangement $\mathcal{H}$ or $\mathcal{H}_0$ is any nonempty intersection of the hyperplanes. The parameter ξ determines a linear functional on $V_{0,\mathbb{R}}$, and an affine linear functional on $V_{\mathbb{R}}$ that is only defined up to a constant. We say that the covector ξ is **regular** if it is not constant on any one-dimensional flat of $\mathcal{H}_0$, or equivalently on any one-dimensional flat of $\mathcal{H}$. We say that $X = (\Lambda_0, \eta, \xi)$ is **regular** if both η and ξ are regular.

Remark 2.2. This definition differs slightly from the one in [BLPWb, 2.1]. In that paper η and ξ were allowed to be real rather than integral, but they were always assumed to be regular.

2.3 Quantized polarized arrangements

In this section we introduce objects that are analogous to those considered in Section 2.2, but with H and W replaced by $\mathbf{H}$ and $\mathbf{W}$.

Definition 2.3. A **quantized polarized arrangement** is a triple $\mathbf{X} = (\Lambda_0, \mathbf{\Lambda}, \xi)$, where $\Lambda_0 \subset W_{\mathbb{Z}}$ is a direct summand, $\mathbf{\Lambda}$ is a Λ_0-orbit in $\mathbf{W}$, and $\xi \in \Lambda_0^*$.

Given a quantized polarized arrangement, let

$$\mathbf{V} := \mathbf{\Lambda} + V_0 = \mathbf{\Lambda} + \mathbb{C}\Lambda_0 \subset \mathbf{W} \qquad \text{and} \qquad \mathbf{V}_{\mathbb{R}} := \mathbf{\Lambda} + V_{0,\mathbb{R}} = \mathbf{\Lambda} + \mathbb{R}\Lambda_0 \subset \mathbf{W}_{\mathbb{R}}.$$

Let $I_{\mathbf{\Lambda}}$ be the set of indices $i \in \{1, \dots, n\}$ for which $h_i^+(\mathbf{\Lambda}) \subset \mathbb{Z}$ (or equivalently $h_i^-(\mathbf{\Lambda}) \subset \mathbb{Z}$). We say that $\mathbf{X}$ and $\mathbf{\Lambda}$ are **integral** if $\mathbf{\Lambda} \subset \mathbf{W}_{\mathbb{Z}}$, or equivalently if $I_{\mathbf{\Lambda}} = \{1, \dots, n\}$. For each $i \in I_{\mathbf{\Lambda}}$, let

$$H_i^+ := \{v \in \mathbf{V}_{\mathbb{R}} \mid h_i^+(v) = 0\} \qquad \text{and} \qquad H_i^- := \{v \in \mathbf{V}_{\mathbb{R}} \mid h_i^-(v) = 0\},$$

and let $\boldsymbol{\mathcal{H}} := \{H_i^{\pm} \mid i \in I_{\boldsymbol{\Lambda}}\}$ be the associated (multi)arrangement.

Remark 2.4. The definitions of polarized and quantized polarized arrangements are clearly very close. One important difference is that the parameter η is required to be integral (it is a Λ_0-orbit in $W_{\mathbb{Z}}$), while the parameter $\boldsymbol{\Lambda}$ can sit anywhere in the complex vector space $\mathbf{W}$. This difference is unavoidable: to define a hypertoric variety it is necessary for η to be integral, but to study the hypertoric enveloping algebra it is necessary to consider arbitrary $\boldsymbol{\Lambda}$.

The definition of regularity for the parameter $\boldsymbol{\Lambda}$ is more subtle than for η, because of the more complicated geometry of the doubled hyperplanes and certain integrality issues. Ignoring the integrality issues for the moment, we make the following definitions. A hyperplane arrangement is called **essential** if it has a zero-dimensional flat. The arrangement $\mathcal{H}$ is always essential, but $\boldsymbol{\mathcal{H}}$ is not, since the index set $I_{\boldsymbol{\Lambda}}$ may be very small or even empty. For a Λ_0-orbit $\boldsymbol{\Lambda}$ in $\mathbf{W}$, let m be the maximum over all points in $\mathbf{V}_{\mathbb{R}}$ of the number of pairs of hyperplanes $H_i^{\pm}$ in between which the point lies:

$$m := \max_{v \in \mathbf{V}_{\mathbb{R}}} \left| \{ i \in I_{\boldsymbol{\Lambda}} \mid 0 < h_i^-(v) < 1 \} \right|.$$

If $\boldsymbol{\mathcal{H}}$ is essential, then we necessarily have $m \geq \dim V$.

Definition 2.5. We say that $\boldsymbol{\Lambda}$ is **quasi-regular** if $\boldsymbol{\mathcal{H}}$ is essential and $m = \dim V$. This means that any arrangement obtained by replacing each pair $H_i^{\pm}$ with a single hyperplane lying strictly between them is simple.

The definition of regularity of $\boldsymbol{\Lambda}$ and $\mathbf{X}$ will appear in the next section.

2.4 Boundedness and feasibility

Fix a polarized arrangement $X = (\Lambda_0, \eta, \xi)$ and a quantized polarized arrangement $\mathbf{X} = (\Lambda_0, \boldsymbol{\Lambda}, \xi)$ with the same underlying lattice Λ_0 and the same covector $\xi \in \Lambda_0^*$. The following definitions for X are repeated from [BLPWb]; we then adapt them for $\mathbf{X}$.

For a sign vector $\alpha \in \{+,-\}^n$, define the chamber $\Delta_\alpha \subset V_{\mathbb{R}}$ to be the polyhedron cut out by the inequalities

$$h_i \geq 0 \text{ for all } i \text{ with } \alpha(i) = + \qquad \text{and} \qquad h_i \leq 0 \text{ for all } i \text{ with } \alpha(i) = -,$$

and let $\Delta_{0,\alpha}$ be the polyhedral cone in $V_{0,\mathbb{R}}$ cut out by the same inequalities. If $\Delta_\alpha \neq \emptyset$, we say that α is **feasible** for X; let $\mathcal{F}_X$ be the set of feasible sign vectors. If α is feasible, then $\Delta_{0,\alpha}$ is the cone of unbounded directions in Δ_α. Note, however, that $\Delta_{0,\alpha}$ is always nonempty, even if α is infeasible.

We say that α is **bounded** for X if the restriction of ξ is proper and bounded above on the cone $\Delta_{0,\alpha}$. Note that if ξ is regular, we can drop the properness hypothesis. Also note that if α is feasible, then α is bounded if and only if ξ is proper and bounded above on Δ_α. Let $\mathcal{B}_X$ be the set of bounded sign vectors, and let $\mathcal{P}_X = \mathcal{F}_X \cap \mathcal{B}_X$.

Remark 2.6. When X is regular, the set $\mathcal{P}_X$ is in natural bijection with the set of vertices of the hyperplane arrangement $\mathcal{H}$, as each vertex appears as the ξ-maximal point of Δ_α for a unique bounded feasible sign vector α.

Remark 2.7. It will often be the case that we will fix a lattice Λ_0 and vary the parameters η and ξ. Since $\mathcal{F}_X$ depends only on Λ_0 and η, we will often abusively write $\mathcal{F}_\eta$ rather than $\mathcal{F}_X$. Likewise, since $\mathcal{B}_X$ depends only on Λ_0 and ξ, we will often write $\mathcal{B}_\xi$ rather than $\mathcal{B}_X$. We will then write $\mathcal{P}_{\eta,\xi} = \mathcal{F}_\eta \cap \mathcal{B}_\xi = \mathcal{P}_X$. This notation coincides with that of [BLPWb].

Remark 2.8. We have $\Delta_{0,\alpha} = \{0\}$ if and only if $\alpha \in \mathcal{B}_\xi$ for any choice of ξ. We call such a sign vector **totally bounded**. If α is feasible, then it is totally bounded if and only if the polyhedron Δ_α is compact. The set of totally bounded sign vectors depends only on Λ_0; it is independent of both η and ξ.

Turning to the quantized polarized arrangement $\mathbf{X}$, we define the chamber $\mathbf{\Delta}_\alpha$ corresponding to $\alpha \in \{+,-\}^{I_{\mathbf{\Lambda}}}$ to be the subset of the affine space $\mathbf{V}_\mathbb{R}$ cut out by the inequalities

$$h_i^+ \geq 0 \text{ for all } i \in I_{\mathbf{\Lambda}} \text{ with } \alpha(i) = + \qquad \text{and} \qquad h_i^- \leq 0 \text{ for all } i \in I_{\mathbf{\Lambda}} \text{ with } \alpha(i) = -.$$

If $\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$ is nonempty, we say that α is **feasible** for $\mathbf{X}$, and we denote the set of feasible sign vectors $\mathcal{F}_{\mathbf{X}}$. If ξ restricts to a proper, bounded function on $\mathbf{\Delta}_{0,\alpha}$, we say that α is **bounded** for $\mathbf{X}$, and we denote the set of bounded sign vectors $\mathcal{B}_{\mathbf{X}}$. Note that if α is feasible, then α is bounded if and only if ξ is proper and bounded above on $\mathbf{\Delta}_\alpha$ (or equivalently on $\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$).

Remark 2.9. Following the conventions of Remark 2.7, we will often use the notation $\mathcal{F}_{\mathbf{\Lambda}}$ in place of $\mathcal{F}_{\mathbf{X}}$, since this set does not depend on ξ. Unlike in the non-quantized case, the set $\mathcal{B}_{\mathbf{X}}$ does depend on $\mathbf{\Lambda}$ because $I_{\mathbf{\Lambda}}$ depends on $\mathbf{\Lambda}$. We will thus write $\mathcal{B}_{\mathbf{\Lambda},\xi}$ in place of $\mathcal{B}_{\mathbf{X}}$, and $\mathcal{P}_{\mathbf{\Lambda},\xi} = \mathcal{F}_{\mathbf{\Lambda}} \cap \mathcal{B}_{\mathbf{\Lambda},\xi} = \mathcal{P}_{\mathbf{X}}$.

It is possible for the polyhedron $\mathbf{\Delta}_\alpha$ to be nonempty, but so small that it does not contain an element of $\mathbf{\Lambda}$. We are primarily interested in quantized polarized arrangements for which this does not happen, thus we incorporate this condition into our definition of regularity.

Definition 2.10. We say that $\mathbf{\Lambda}$ is **regular** if it is quasi-regular and $\mathbf{\Delta}_\alpha \neq \emptyset$ implies $\alpha \in \mathcal{F}_{\mathbf{\Lambda}}$ for every $\alpha \in \{+,-\}^n$.

Remark 2.11. We say that Λ_0 is **unimodular** if its image under the projection $W_\mathbb{Z} \cong \mathbb{Z}^n \to \mathbb{Z}^I$ is a direct summand of $\mathbb{Z}^I$ for every $I \subset \{1,\ldots,n\}$. In this case regularity and quasi-regularity are equivalent, since any vertex of a chamber $\mathbf{\Delta}_\alpha$ must lie in $\mathbf{\Lambda}$.

2.5 Linked arrangements

In this section we fix a direct summand $\Lambda_0 \subset W_\mathbb{Z}$ and an arbitrary element $\xi \in \Lambda_0^*$. For any Λ_0-orbit $\mathbf{\Lambda} \subset \mathbf{W}$, let $\pi : \{+,-\}^n \to \{+,-\}^{I_{\mathbf{\Lambda}}}$ be the projection.

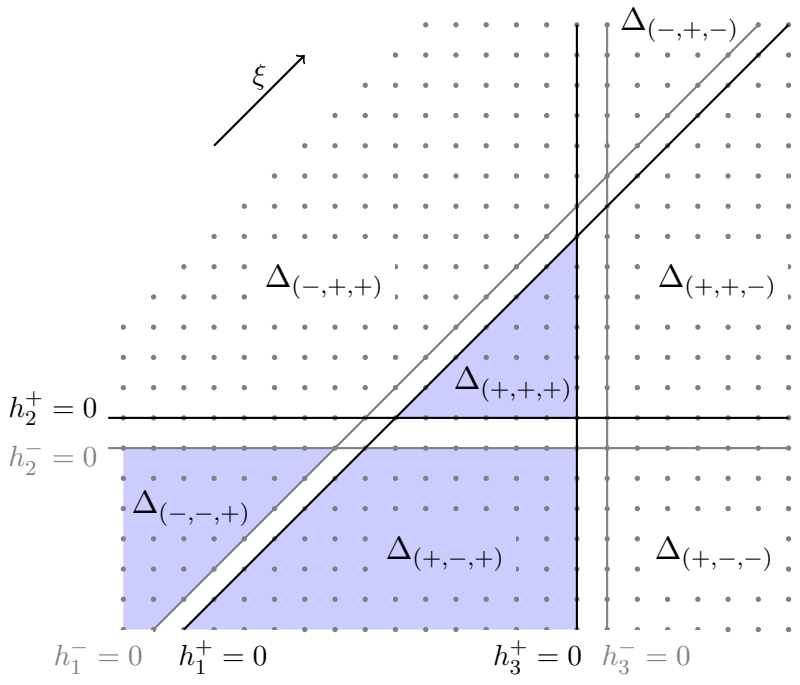


Figure 1: A regular quantized polarized arrangement. There are 7 feasible sign vectors, 3 of which are bounded. The polyhedra associated to bounded feasible sign vectors are shaded.

Definition 2.12. We say that two regular parameters η and $\mathbf{\Lambda}$ are **linked** if $\pi(\mathcal{F}_\eta) = \mathcal{F}_{\mathbf{\Lambda}}$. In this case we also say that the polarized arrangement $X = (\Lambda_0, \eta, \xi)$ is linked to the quantized polarized arrangement $\mathbf{X} = (\Lambda_0, \Lambda, \xi)$.

Remark 2.13. In general, a polarized arrangement does not determine a quantized polarized arrangement, nor the other way around. Linkage is the only property that we will consider that relates these two types of objects.

Definition 2.14. We will consider two polarized arrangements (respectively quantized polarized arrangements) to be **equivalent** if they have the same direct summand $\Lambda_0 \subset W_{\mathbb{Z}}$ and the same sets $\mathcal{F}_\eta$ and $\mathcal{B}_\xi$ (respectively $I_{\mathbf{\Lambda}}$, $\mathcal{F}_{\mathbf{\Lambda}}$, and $\mathcal{B}_{\mathbf{\Lambda},\xi}$) of feasible and bounded sign vectors.

It is clear that every regular η is linked to some regular integral $\mathbf{\Lambda}$ and *vice versa*, hence the concept of linkage provides a bijection between equivalence classes of regular polarized arrangements and equivalence classes of regular integral quantized polarized arrangements with the same lattice Λ_0 and covector ξ. The following proposition will be a key tool for us in Section 6.2.

Proposition 2.15. *Two regular parameters η and $\mathbf{\Lambda}$ are linked if and only if there exists a positive integer k such that $\mathcal{F}_{\mathbf{\Lambda}} = \mathcal{F}_{\mathbf{\Lambda}+rk\eta}$ for all positive integers r. If Λ_0 is unimodular, then k may be taken equal to 1.*

Since we need to consider the polyhedra Δ_α and $\mathbf{\Delta}_\alpha$ and arrangements $\boldsymbol{\mathcal{H}}$ for varying choices of η and $\mathbf{\Lambda}$, we add them to the notation for this section only, writing $\Delta_{\eta,\alpha}$, $\mathbf{\Delta}_{\mathbf{\Lambda},\alpha}$, and $\boldsymbol{\mathcal{H}}_{\mathbf{\Lambda}}$. Let

$N_{\mathbf{\Lambda}}$ be the number of subsets $I \subset I_{\mathbf{\Lambda}}$ such that the composition $\Lambda_{\mathbb{R}} \subset W_{\mathbb{R}} \twoheadrightarrow \mathbb{R}^I$ is surjective; such subsets are commonly known as **independent sets** of the matroid associated to $\mathcal{H}_0$.

Lemma 2.16. *For any $\mathbf{\Lambda}$, $|\mathcal{F}_{\mathbf{\Lambda}}| \leq N_{\mathbf{\Lambda}}$, and equality is attained if and only if $\mathbf{\Lambda}$ is regular.*

Proof. If $\mathbf{\Lambda}$ is regular, then it is possible to replace each pair $H_i^{\pm}$ of hyperplanes in $\boldsymbol{\mathcal{H}}_{\mathbf{\Lambda}}$ with a single hyperplane lying strictly in between them in such a way so that the new arrangement of $|I_{\mathbf{\Lambda}}|$ hyperplanes is simple, and every chamber of the new arrangement contains a unique non-empty set of the form $\boldsymbol{\Delta}_{\mathbf{\Lambda},\alpha} \cap \mathbf{\Lambda}$. (To visualize this, see Figure 1.) Thus the feasible sign vectors $\mathcal{F}_{\mathbf{\Lambda}}$ are in bijection with the chambers of the new arrangement, which is a simplification of our central arrangement $\mathcal{H}_0$.

The Orlik-Solomon algebra of any arrangement has dimension equal to the number of chambers, and if the arrangement is simple, it has a basis indexed by the independent sets of the matroid associated to the corresponding central arrangement [OT92, 3.45 & 5.95]. Since the number of chambers is equal to $|\mathcal{F}_{\mathbf{\Lambda}}|$ and the size of the basis is equal to $N_{\mathbf{\Lambda}}$, this proves the equality when $\mathbf{\Lambda}$ is regular.

If $\mathbf{\Lambda}$ is not regular, it is still possible to construct a simplification of $\mathcal{H}_0$ in the manner described above, but some of the chambers will not contain any nonempty sets of the form $\boldsymbol{\Delta}_{\mathbf{\Lambda},\alpha} \cap \mathbf{\Lambda}$. Thus the number of chambers is strictly greater than $|\mathcal{F}_{\mathbf{\Lambda}}|$, which proves the inequality. □

Proof of Proposition 2.15: Suppose that η and $\mathbf{\Lambda}$ are linked. Choose a positive integer k such that $\Delta_{k\eta,\alpha} \cap W_{\mathbb{Z}} \neq \emptyset$ for every $\alpha \in \mathcal{F}_{\eta}$. The same condition will hold when k is replaced with any positive multiple rk. When Λ_0 is unimodular, we can take $k = 1$ (see Remark 2.11).

For any positive integer r and any $\alpha \in \{+,-\}^n$, we have

$$(\Delta_{rk\eta,\alpha} \cap W_{\mathbb{Z}}) + (\boldsymbol{\Delta}_{\mathbf{\Lambda},\pi(\alpha)} \cap \mathbf{\Lambda}) \subset (\boldsymbol{\Delta}_{\mathbf{\Lambda}+rk\eta,\pi(\alpha)} \cap \mathbf{\Lambda}),$$

which tells us that $\mathcal{F}_{\mathbf{\Lambda}} \subset \mathcal{F}_{\mathbf{\Lambda}+rk\eta}$. Since

$$|\mathcal{F}_{\mathbf{\Lambda}}| = N_{\mathbf{\Lambda}} = N_{\mathbf{\Lambda}+rk\eta} \geq |\mathcal{F}_{\mathbf{\Lambda}+rk\eta}|,$$

this inclusion must be an equality.

Conversely, suppose that there exists a positive integer k such that $\mathcal{F}_{\mathbf{\Lambda}} = \mathcal{F}_{\mathbf{\Lambda}+rk\eta}$ for all positive integers r. Choose an arbitrary element of $\mathbf{W}$; this allows us to identify $\mathbf{W}$ with W. Choose an arbitrary sign vector α. As r approaches ∞, the set $\dfrac{1}{rk}\boldsymbol{\Delta}_{\mathbf{\Lambda}+rk\eta,\pi(\alpha)} \subset \mathbf{W} \cong W$ approaches a set containing $\Delta_{\eta,\alpha} \subset W$. In particular, when r is very large, one set is non-empty if and only if the other is, so we have $\mathcal{F}_{\eta} = \mathcal{F}_{\mathbf{\Lambda}+rk\eta} = \mathcal{F}_{\mathbf{\Lambda}}$. Thus η and $\mathbf{\Lambda}$ are linked. □

2.6 Gale duality

We recall the notion of Gale duality for polarized arrangements from [BLPWb, 2.3]. Fix a positive integer n, and let $X = (\Lambda_0, \eta, \xi)$ and $X^! = (\Lambda_0^!, \eta^!, \xi^!)$ be two polarized arrangements with n hyperplanes.

Definition 2.17. We say that $X^!$ is **Gale dual** to X if

- $\Lambda_0^!$ and Λ_0 are complementary with respect to the coordinate inner product on $W_\mathbb{Z} \cong \mathbb{Z}^n$
- $\eta^! = -\xi$ and $\xi^! = -\eta$ under the resulting identifications $W_\mathbb{Z}/\Lambda_0^! \cong \Lambda_0^*$ and $(\Lambda_0^!)^* \cong W_\mathbb{Z}/\Lambda_0$.

Theorem 2.18 ([BLPWb, 2.4])**.** *If X and $X^!$ are Gale dual, then*

$$\mathcal{F}_\eta = \mathcal{B}_{\xi^!} \qquad \text{and} \qquad \mathcal{F}_{\eta^!} = \mathcal{B}_\xi.$$

In particular, we have $\mathcal{P}_X = \mathcal{P}_{X^!}$. Furthermore, η is regular if and only if $\xi^!$ is regular, and ξ is regular if and only if $\eta^!$ is regular.

There is no direct way to define Gale duality for quantized polarized arrangements. However, if both $\mathbf{X}$ and $\mathbf{X}^!$ are regular and integral, then we will say that $\mathbf{X}$ and $\mathbf{X}^!$ are Gale dual if they are linked to a pair of Gale dual polarized arrangements X and $X^!$.

3 The hypertoric enveloping algebra

Recall that a quantized polarized arrangement consists of a triple $(\Lambda_0, \mathbf{\Lambda}, \xi)$. The hypertoric enveloping algebra itself is determined by the parameter Λ_0, and the affine space $\mathbf{V} = \mathbf{\Lambda} + \mathbb{C}\Lambda_0 \subset \mathbf{W}$ determines a central character of the algebra (Section 3.2). There are of course many different choices of $\mathbf{\Lambda}$ that yield the same $\mathbf{V}$; this choice of a lattice $\mathbf{\Lambda} \subset \mathbf{V}$ determines a certain subcategory of modules over the algebra with central character given by $\mathbf{V}$ (Section 3.3). The entirety of Section 3 is devoted to understanding this subcategory, which we describe in combinatorial terms in Section 3.4. In Section 3.5, we consider translation functors between these subcategories for various choices of $\mathbf{V}$ and $\mathbf{\Lambda}$.

The parameter ξ will not enter the picture until Section 4, where we use it to define hypertoric category $\mathcal{O}$. The intersection of $\mathcal{O}$ with the category studied in Section 3 will be a block of $\mathcal{O}$.

3.1 The Weyl algebra

Fix an integer n, and consider the n-dimensional torus $T := \operatorname{Specm} \mathbb{C}[W_\mathbb{Z}]$, where $\mathbb{C}[W_\mathbb{Z}]$ denotes the group ring of the lattice $W_\mathbb{Z}$ that was introduced in Section 2.1. Thus the Lie algebra $\mathfrak{t}$ of T is naturally identified with $\mathbb{C}\{h_1, \ldots, h_n\}$, $\operatorname{Sym}(\mathfrak{t})$ is naturally identified with H, and $\mathfrak{t}^*$ is naturally identified with W. The character lattice $\mathfrak{t}^*_\mathbb{Z}$ is equal to $W_\mathbb{Z}$, and the cocharacter lattice $\mathfrak{t}_\mathbb{Z}$ is the lattice spanned by $h_1, \ldots, h_n$.

Consider the coordinate vector space $\mathbb{C}^n := \operatorname{Specm} \mathbb{C}[x_1, \ldots, x_n]$. We let T act on $\mathbb{C}^n$ in a manner such that the induced action of $\mathfrak{t}$ on the ring of regular functions is given by $h_i \cdot x_j = \delta_{ij} x_j$.[9] Let $\mathbb{D}$ be the Weyl algebra of polynomial differential operators on $\mathbb{C}^n$. The ring $\mathbb{D}$ is generated over

[9]Note that if we use the basis $\{h_1, \ldots, h_n\}$ of $\mathfrak{t}$ to identify T with the coordinate torus $(\mathbb{C}^\times)^n$, then the action of T on $\mathbb{C}^n$ is the *opposite* of the standard action.

$\mathbb{C}[x_1, \ldots, x_n]$ by pairwise commuting elements $\{\partial_1, \ldots, \partial_n\}$ that satisfy the relations $[\partial_i, x_j] = \delta_{ij}$ for all $1 \leq i, j \leq n$. The action of T on $\mathbb{C}^n$ induces an action on $\mathbb{D}$, along with a grading

$$\mathbb{D} = \bigoplus_{w \in W_{\mathbb{Z}}} \mathbb{D}_z$$

of $\mathbb{D}$ by the character lattice of T. We identify the 0^{th} graded piece

$$\mathbb{D}_0 = \mathbb{D}^T = \mathbb{C}[x_1\partial_1, \ldots, x_n\partial_n] = \mathbb{C}[\partial_1 x_1, \ldots, \partial_n x_n]$$

with the ring $\mathbf{H}$ from Section 2.1 by sending $x_i\partial_i$ to h_i^+ and $\partial_i x_i$ to h_i^-. This algebra will play a role for us analogous to a Cartan subalgebra of a semisimple Lie algebra.

We filter $\mathbb{D}$ by the semigroup $\mathbb{N}$ by letting $F_k\mathbb{D}$ be the linear span of all monomials of total degree $\leq k$. In particular, we have $F_{2k}\mathbf{H} = \mathbf{H} \cap F_{2k}\mathbb{D}$, where $F_{2k}\mathbf{H}$ is the filtered piece introduced in Section 2.1. The associated graded ring $\operatorname{gr}\mathbb{D}$ is canonically identified with functions on $T^*\mathbb{C}^n$, with its T-action induced by the action on $\mathbb{C}^n$. We have the following link between the algebra structure on $\mathbb{D}$ and the T-action: let $\boldsymbol{\sigma} \in F_2\mathbf{H} \cong F_2\mathbb{D}_0$, and let σ be its image in the degree 2 part of $H = \operatorname{gr}\mathbf{H}$, which we have identified with $\mathfrak{t}$. Then for all $z \in W_{\mathbb{Z}} \cong \mathfrak{t}^*$ and $a \in \mathbb{D}_z$, we have

$$[\boldsymbol{\sigma},\, a] = z(\sigma)a \in \mathbb{D}_z. \tag{1}$$

3.2 The hypertoric enveloping algebra

In this section we fix a direct summand $\Lambda_0 \subset W_{\mathbb{Z}}$, and use it to define the hypertoric enveloping algebra. Let $V_0 := \mathbb{C}\Lambda_0 \subset W \cong \mathfrak{t}^*$, and let $\mathfrak{k} := V_0^{\perp} \subset W^* \cong \mathfrak{t}$. Let $K \subset T$ be the connected subtorus with Lie algebra $\mathfrak{k}$, so that Λ_0 may be identified with the character lattice of T/K and $W_{\mathbb{Z}}/\Lambda_0$ may be identified with the character lattice of K. Our main object of study is the ring of K-invariants

$$U := \mathbb{D}^K = \bigoplus_{z \in \Lambda_0} \mathbb{D}_z,$$

which we will call the **hypertoric enveloping algebra** associated to Λ_0.

From Equation (1) it is easy to see that the center $Z(U)$ is the subalgebra generated by all $\boldsymbol{\sigma} \in F_2\mathbf{H}$ whose image $\sigma \in \mathfrak{t}$ lies in $\mathfrak{k}$. In particular, $Z(U) \subset \mathbf{H}$, and its maximal spectrum $\operatorname{Specm} Z(U)$ is naturally the quotient of the affine space $\mathbf{W} = \operatorname{Specm}\mathbf{H}$ by the action of $V_0 \subset W$.

Let $\mathbf{V} \subset \mathbf{W}$ be a V_0-orbit, and let $\lambda\colon Z(U) \to \mathbb{C}$ be the associated central character of U. We stress that $\mathbf{V}$ and λ completely determine each other. Let $U_\lambda := U/U\langle\ker\lambda\rangle$ be the corresponding central quotient, and let $\mathbf{H}_\lambda := \mathbf{H}/\mathbf{H}\langle\ker\lambda\rangle$ be the image of $\mathbf{H}$ in U_λ.

Remark 3.1. There is a unique splitting of the surjection $F_2Z(U) \to \mathfrak{k}$ whose image is the kernel of λ. The induced map $\mu_\lambda : \operatorname{Sym}(\mathfrak{k}) \to Z(U) \to \mathbb{D}$ is known as a **quantized moment map** for the K-action, since the associated graded map

$$\operatorname{gr}\mu_\lambda : \operatorname{Sym}(\mathfrak{k}) \xrightarrow{\sim} \operatorname{gr} Z(U) \to \operatorname{gr}\mathbb{D} \cong \mathbb{C}[T^*\mathbb{C}^n]$$

is given by composition with the classical moment map $T^*\mathbb{C}^n \to \mathfrak{k}^*$. The ring U_λ is known as a **noncommutative Hamiltonian reduction** of $\mathbb{D}$ by K [CBEG07]. The associated graded algebra

$$\begin{aligned} \operatorname{gr} U_\lambda &= \operatorname{gr}\left(\mathbb{D}^K \Big/ \langle \mu_\alpha(\mathfrak{k}) \rangle\right) \\ &\cong (\operatorname{gr}\mathbb{D})^K \Big/ \langle \operatorname{gr}\mu_\alpha(\mathfrak{k}) \rangle \\ &\cong \mathbb{C}[T^*\mathbb{C}^n]^K \Big/ \langle \operatorname{gr}\mu_\alpha(\mathfrak{k}) \rangle \end{aligned}$$

is isomorphic to the coordinate ring of the symplectic quotient of $T^*\mathbb{C}^n$ by K, which is a hypertoric variety (see Proposition 5.2). This ring inherits a natural Poisson structure from the symplectic structure on the hypertoric variety, and U_λ is a quantization of this Poisson structure. Since $\operatorname{gr} U_\lambda$ is a finitely generated $\mathbb{C}$-algebra, U_λ is finitely generated as well.

Remark 3.2. The algebra U_λ can also be realized as the ring of K-equivariant endomorphisms of the right $\mathbb{D}$-module

$$Y_\lambda := \mathbb{D} \Big/ \langle \ker\lambda \rangle \mathbb{D}.$$

The isomorphism is given by sending an endomorphism ψ to $\psi(1) \in Y_\lambda^K \cong U_\lambda$.

3.3 Weight modules

Let $\mathbb{D}-\mathsf{mod}$, $U-\mathsf{mod}$, and $U_\lambda-\mathsf{mod}$ denote the category of finitely generated left $\mathbb{D}$-modules, U-modules, and U_λ-modules, respectively. (All modules over any ring in this paper will be assumed to be finitely generated.) We will mainly be interested in modules which decompose into (generalized) weight spaces for the action of $\mathbf{H}$. For $v \in \mathbf{W} = \operatorname{Specm}\mathbf{H}$, let $\mathfrak{I}_v \subset \mathbf{H}$ be the associated maximal ideal. For any module $M \in U-\mathsf{mod}$, define the ***v*-weight space** of M to be

$$M_v := \{m \in M \mid \mathfrak{I}_v^k m = 0 \text{ for } k \gg 0\}.$$

Remark 3.3. It is more conventional to call M_v a "generalized weight space", and reserve the term "weight space" for the more restrictive $k = 1$ condition. In this article, however, we will never be interested in weight spaces in the usual sense, and will always use "weight space" to mean the subspace defined above.

Equation (1) implies that $\mathfrak{I}_{z+v}a = a\mathfrak{I}_v$ for all $z \in \Lambda_0 \subset W_\mathbb{Z}$, $v \in \mathbf{W}$, and $a \in \mathbb{D}_z$, and therefore that

$$\mathbb{D}_z \cdot M_v \subset M_{z+v}.$$

For any U-module M, define its **support** by

$$\operatorname{Supp} M := \{v \in \mathbf{W} \mid M_v \neq 0\}.$$

Let $\mathbb{D}-\mathsf{mod}_{\mathrm{lf}}$, $U-\mathsf{mod}_{\mathrm{lf}}$, and $U_\lambda-\mathsf{mod}_{\mathrm{lf}}$ be the full subcategories of $\mathbb{D}-\mathsf{mod}$, $U-\mathsf{mod}$, and $U_\lambda-\mathsf{mod}$ consisting of modules for which H acts locally finitely. Objects of these categories will be called

weight modules; these are exactly the modules that are isomorphic to the direct sum of their weight spaces.

So far we have chosen an integer n, a direct summand $\Lambda_0 \subset W_{\mathbb{Z}}$, and a central character $\lambda : Z(U) \to \mathbb{C}$. Now choose a Λ_0-orbit $\mathbf{\Lambda} \subset \mathbf{V}$, and let $U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}$ be the full subcategory of $U_\lambda\text{−}\mathsf{mod}_{\mathrm{lf}}$ consisting of modules supported in $\mathbf{\Lambda}$. This is the category on which we will focus our attention for the rest of Section 3.

Remark 3.4. Since a point $v \in \mathbf{W}$ lies in $\mathbf{V}$ if and only if $\ker\lambda \subset \mathfrak{I}_v$, the category $U_\lambda\text{−}\mathsf{mod}_{\mathrm{lf}}$ can be thought of as the subcategory of $U\text{−}\mathsf{mod}_{\mathrm{lf}}$ consisting of objects that are *scheme-theoretically* supported on $\mathbf{V}$. However, an object M of $U\text{−}\mathsf{mod}_{\mathrm{lf}}$ can have $\operatorname{Supp} M \subset \mathbf{V}$ and still not lie in $U_\lambda\text{−}\mathsf{mod}_{\mathrm{lf}}$. We will consider the larger category of weight modules for U with set-theoretic support in $\mathbf{\Lambda}$ in Section 4.5 when we discuss a deformation of our category $\mathcal{O}$.

Define functors

$$(\cdot)^\lambda \colon \mathbb{D}\text{−}\mathsf{mod}_{\mathrm{lf}} \to U_\lambda\text{−}\mathsf{mod}_{\mathrm{lf}} \quad \text{and} \quad (\cdot)^{\mathbf{\Lambda}} \colon \mathbb{D}\text{−}\mathsf{mod}_{\mathrm{lf}} \to U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}$$

by

$$M^\lambda := \left\{ m \in \bigoplus_{v \in \mathbf{V}} M_v \;\middle|\; (\ker\lambda)m = 0 \right\} \qquad \text{and} \qquad M^{\mathbf{\Lambda}} := \left\{ m \in \bigoplus_{v \in \mathbf{\Lambda}} M_v \;\middle|\; (\ker\lambda)m = 0 \right\}.$$

These functors are left exact, and they are also right exact when restricted to the full subcategory of objects annihilated by $\ker\lambda$. They are right adjoint to the functor $(\mathbb{D} \otimes_U -)$ restricted to $U_\lambda\text{−}\mathsf{mod}_{\mathrm{lf}}$ or $U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}$, respectively. Furthermore, the adjunction map $M \to (\mathbb{D} \otimes_U M)^\lambda$ is an isomorphism for any $M \in U_\lambda\text{−}\mathsf{mod}_{\mathrm{lf}}$.

Fix a weight $v \in \mathbf{\Lambda}$. For any sign vector $\alpha \in \{+,-\}^{I_{\mathbf{\Lambda}}}$, consider the simple $\mathbb{D}$-module

$$L_\alpha := \mathbb{D} \Big/ \mathbb{D}\langle \partial_i \mid \alpha(i) = + \rangle + \mathbb{D}\langle x_i \mid \alpha(i) = - \rangle + \mathbb{D}\langle h_i^+ - h_i^+(v) \mid i \notin I_{\mathbf{\Lambda}} \rangle.$$

It is easy to check that the isomorphism class of L_α does not depend on the choice of v and that

$$\operatorname{Supp} L_\alpha^{\mathbf{\Lambda}} = \mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}. \tag{2}$$

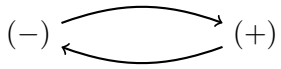
Proposition 3.5. *The modules $\{L_\alpha^{\mathbf{\Lambda}} \mid \alpha \in \mathcal{F}_{\mathbf{\Lambda}}\}$ give a complete and irredundant set of representatives for the isomorphism classes of simple objects of $U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}$.*

Proof. Since the adjoint action of $\mathbf{H}$ on U_λ is semisimple and the resulting weight spaces are all cyclic left $\mathbf{H}$-modules, the theory developed in [MV98] applies to U_λ. The modules $\{L_\alpha \mid \alpha \in \{+,-\}^{I_{\mathbf{\Lambda}}}\}$ are precisely the simple weight modules for $\mathbb{D}$ supported in the lattice $\mathbf{\Lambda} + W_{\mathbb{Z}}$, and the results [MV98, 4.2.1, 4.3.1, & 7.2.4] imply that the isomorphism classes of simples in $U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}$ are given by the set of all nonzero $L_\alpha^{\mathbf{\Lambda}}$. Since $L_\alpha^{\mathbf{\Lambda}}$ is a weight module, we have $L_\alpha^{\mathbf{\Lambda}} \neq 0$ if and only if $\operatorname{Supp} L_\alpha^{\mathbf{\Lambda}} \neq \emptyset$. By Equation (2), this is exactly the condition that $\alpha \in \mathcal{F}_{\mathbf{\Lambda}}$. □

3.4 Quiver description of weight modules

The results of Musson and Van den Bergh [MV98] give an equivalence between $U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}}$ and the category of finite dimensional modules over a certain algebra, which we now describe.

Let Q be the path algebra over $\mathbb{C}$ of the quiver



with vertices labeled $+$ and $-$ and one arrow in each direction, and let $Q_n = Q \otimes_{\mathbb{C}} Q \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} Q$ be the tensor product of n copies of Q. The algebra Q_n is the path algebra of the quiver whose vertices are labeled by $\{+,-\}^n$, the set of vertices of an n-cube, with an edge connecting α to β whenever α and β differ in exactly one position, modulo the relations that whenever α and γ differ in exactly two positions, the two paths $\alpha \to \beta \to \gamma$ and $\alpha \to \beta' \to \gamma$ are equal in Q_n. Consider the grading on Q_n for which a path of length d has degree d, and let $\widehat{Q}_n$ be the completion of Q_n with respect to the grading.

For each $\alpha \in \{+,-\}^n$ and each $1 \le i \le n$, let $\theta_{\alpha,i}$ be the element of Q_n represented by the path $\alpha \to \beta \to \alpha$, where β agrees with α except in the i^{th} place, and put $\theta_i := \sum_{\alpha \in \{+,-\}^n} \theta_{\alpha,i}$. The center $Z(Q_n)$ is a polynomial algebra on the elements $\{\theta_i \mid 1 \le i \le n\}$, and the center of $Z(\widehat{Q}_n)$ is the completion of this polynomial algebra with respect to the grading. Let $\vartheta\colon \mathfrak{t} \to Z(Q_n)$ denote the linear map which sends h_i to θ_i.

Next, let $Q_{\mathbf{\Lambda}} \subset Q_n$ and $\widehat{Q}_{\mathbf{\Lambda}} \subset \widehat{Q}_n$ be the centralizers of all length 1 paths $\alpha \to \beta$ where α and β agree in every position except for $i \notin I_{\mathbf{\Lambda}}$. If $\mathbf{\Lambda}$ is integral, we have $Q_{\mathbf{\Lambda}} = Q_n$ and $\widehat{Q}_{\mathbf{\Lambda}} = \widehat{Q}_n$. Otherwise, $Q_{\mathbf{\Lambda}}$ is isomorphic to

$$Q_{|I_{\mathbf{\Lambda}}|} \otimes_{\mathbb{C}} \mathbb{C}[\theta_i \mid i \notin I_{\mathbf{\Lambda}}],$$

and $\widehat{Q}_{\mathbf{\Lambda}}$ is its completion. The primitive idempotents of $\widehat{Q}_{\mathbf{\Lambda}}$ are indexed by $\alpha \in \{+,-\}^{I_{\mathbf{\Lambda}}}$. The idempotent e_α corresponding to α is the sum of the primitive idempotents of $\widehat{Q}_n$ for all vertices in the fiber over α of the projection $\{+,-\}^n \to \{+,-\}^{I_{\mathbf{\Lambda}}}$ forgetting the indices $i \notin I_{\mathbf{\Lambda}}$. Let $e_{\mathbf{\Lambda}} := \sum_{\alpha \in \mathcal{F}_{\mathbf{\Lambda}}} e_\alpha$.

Theorem 3.6. *There is an equivalence of categories between $U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}}$ and the category of finite dimensional modules over the ring $\left(e_{\mathbf{\Lambda}}\, \widehat{Q}_{\mathbf{\Lambda}}\, e_{\mathbf{\Lambda}}\right) \Big/ \langle \vartheta(x) e_{\mathbf{\Lambda}} \mid x \in \mathfrak{k} \rangle$.*

Proof. [MV98, 3.5.6 & 6.3] give an equivalence between the category of weight modules for $\mathbb{D}$ with weights in $\mathbf{\Lambda} + W_{\mathbb{Z}}$ and the category of finite dimensional $\widehat{Q}_{\mathbf{\Lambda}}$-modules. The result now follows from Proposition 3.5 and [MV98, 4.4.1]. □

Remark 3.7. In the proof of Theorem 3.11 below we explain in more detail how the equivalence of Theorem 3.6 is constructed.

3.5 Translation functors

Let $\lambda, \lambda' \colon Z(U) \to \mathbb{C}$ be two central characters, and let $\mathbf{V}, \mathbf{V}'$ be the corresponding V_0-orbits in $\mathbf{W}$. Since $\mathbf{W}$ is a torsor for W, the difference $\mathbf{V} - \mathbf{V}'$ is naturally a V_0-orbit in W. Let

$$\mathbb{D}^{\lambda-\lambda'} := \bigoplus_{z \in \mathbf{V}-\mathbf{V}'} \mathbb{D}_z,$$

and consider the $(U_\lambda, U_{\lambda'})$-bimodule

$$\begin{aligned} {}_\lambda T_{\lambda'} &:= \mathbb{D}^{\lambda-\lambda'} \big/ \mathbb{D}^{\lambda-\lambda'} \langle \ker \lambda' \rangle \\ &= \mathbb{D}^{\lambda-\lambda'} \big/ \langle \ker \lambda \rangle \mathbb{D}^{\lambda-\lambda'} \\ &\cong \mathrm{Hom}_K(Y_\lambda, Y_{\lambda'}). \end{aligned}$$

Note that ${}_\lambda T_{\lambda'}$ is nonzero if and only if $\lambda - \lambda'$ is integral, meaning that $\mathbf{V} - \mathbf{V}'$ contains an element of $W_\mathbb{Z}$. We have an associative collection of maps

$${}_\lambda T_{\lambda'} \otimes_{U_{\lambda'}} {}_{\lambda'} T_{\lambda''} \to {}_\lambda T_{\lambda''} \tag{3}$$

given by compositions of homomorphisms. If $\lambda = \lambda'$, then we have ${}_\lambda T_{\lambda'} \cong U_\lambda$ and Equation (3) is the obvious isomorphism. The following proposition is proved in [MV98, 4.4.4].

Proposition 3.8. *Assume $\lambda - \lambda'$ is integral. Then to any Λ_0-orbit $\mathbf{\Lambda}' \subset \mathbf{V}'$ there is a unique Λ_0-orbit $\mathbf{\Lambda} \subset \mathbf{V}$ such that $\mathbf{\Lambda}' - \mathbf{\Lambda} \subset W_\mathbb{Z}$. The bimodule map*

$${}_\lambda T_{\lambda'} \otimes_{U_{\lambda'}} {}_{\lambda'} T_\lambda \to {}_\lambda T_\lambda \cong U_\lambda$$

is an isomorphism if and only if $\mathcal{F}_{\mathbf{\Lambda}'} = \mathcal{F}_{\mathbf{\Lambda}}$ for all Λ_0-orbits $\mathbf{\Lambda}' \subset \mathbf{V}'$.

Remark 3.9. The functor of tensoring with ${}_\lambda T_{\lambda'}$ can be considered a kind of "translation functor" on weight modules, and on the category $\mathcal{O}$ which we will define in Section 4. The reader should be warned, however, that it behaves quite differently from the translation functors on the BGG category $\mathcal{O}$. In particular, tensoring with ${}_\lambda T_{\lambda'}$ is only right exact. (See Remark 4.3 below for more about the difference between our framework and classical BGG category $\mathcal{O}$.) These functors will be important in Section 6 when we use them to study the localization of U_λ-modules to a hypertoric variety, in Section 7 when we define right cells in our category $\mathcal{O}$, and in Section 8, where use them to construct certain derived equivalences between different blocks of our hypertoric category $\mathcal{O}$.

The following easy result is useful in describing the effect of tensoring with ${}_\lambda T_{\lambda'}$ on weight modules.

Lemma 3.10. *There is an equivalence of functors*

$$({}_\lambda T_{\lambda'} \otimes_{U_{\lambda'}} -) \cong (\mathbb{D} \otimes_U -)^\lambda \colon U_{\lambda'}\text{–}\mathsf{mod}_{\mathrm{lf}} \to U_\lambda\text{–}\mathsf{mod}_{\mathrm{lf}}.$$

We have the following explicit formula for tensoring with ${}_\lambda T_{\lambda'}$ in terms of the quiver algebra in Theorem 3.6. Let $\mathbf{\Lambda} \subset \mathbf{V}$ and $\mathbf{\Lambda}' \subset \mathbf{V}'$ be as in Proposition 3.8, so that $\mathbf{\Lambda}' - \mathbf{\Lambda} \subset W_\mathbb{Z}$. Let R be the ring $\widehat{Q}_{\mathbf{\Lambda}} \big/ \langle \vartheta(x) \mid x \in \mathfrak{k} \rangle$, and put $e = e_{\mathbf{\Lambda}}$, $e' = e_{\mathbf{\Lambda}'}$ so that Theorem 3.6 gives equivalences

$$F \colon U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}} \to eRe\text{–}\mathsf{mod}_{\mathrm{fin}} \qquad \text{and} \qquad F' \colon U_{\lambda'}\text{–}\mathsf{mod}_{\mathbf{\Lambda}'} \to e'Re'\text{–}\mathsf{mod}_{\mathrm{fin}}.$$

(Note that $\mathbf{\Lambda}' - \mathbf{\Lambda} \subset W_\mathbb{Z}$ implies that $I_{\mathbf{\Lambda}} = I_{\mathbf{\Lambda}'}$, so $\widehat{Q}_{\mathbf{\Lambda}} = \widehat{Q}_{\mathbf{\Lambda}'}$.)

Theorem 3.11. *The square*

$$\begin{array}{ccc} U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}} & \xrightarrow{F} & eRe\text{–}\mathsf{mod}_{\mathrm{fin}} \\ {\scriptstyle ({}_{\lambda'}T_\lambda \otimes_{U_\lambda} -)}\downarrow & & \downarrow{\scriptstyle (e'Re \otimes_{eRe} -)} \\ U_{\lambda'}\text{–}\mathsf{mod}_{\mathbf{\Lambda}'} & \xrightarrow{F'} & e'Re'\text{–}\mathsf{mod}_{\mathrm{fin}} \end{array}$$

commutes up to natural isomorphism.

Proof. First we explain more carefully how the functors F and F' are constructed using the methods of [MV98]. For each $\alpha \in \{+,-\}^{I_{\mathbf{\Lambda}}}$, choose an element $v_\alpha \in \mathbf{\Lambda} + W_\mathbb{Z}$ satisfying the inequalities

$$h_i^+(v_\alpha) \geq 0 \text{ for all } i \in I_{\mathbf{\Lambda}} \text{ with } \alpha(i) = + \qquad \text{and} \qquad h_i^-(v_\alpha) \leq 0 \text{ for all } i \in I_{\mathbf{\Lambda}} \text{ with } \alpha(i) = -.$$

Note that these are exactly the inequalities that cut $\mathbf{\Delta}_\alpha$ out of $\mathbf{V}_\mathbb{R}$. Thus when $\alpha \in \mathcal{F}_{\mathbf{\Lambda}}$, we can and will assume that $v_\alpha \in \mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$.

Recall from Section 3.3 that $\mathfrak{I}_{v_\alpha} \subset \mathbf{H}$ is the vanishing ideal of the point $v_\alpha \in \mathbf{W}$. Musson and van den Bergh [MV98, 6.3] give an isomorphism

$$\varprojlim_k \mathrm{End}_{\mathbb{D}} \left(\bigoplus_{\alpha \in \{+,-\}^{I_{\mathbf{\Lambda}}}} \mathbb{D}/\mathbb{D}\mathfrak{I}_{v_\alpha}^{k+1} \right)^{\mathrm{op}} \cong \widehat{Q}_{\mathbf{\Lambda}},$$

and we have an equivalence of categories

$$F_{\mathbb{D}} : \mathbb{D}\text{–}\mathsf{mod}_{\mathbf{\Lambda}+W_\mathbb{Z}} \to \widehat{Q}_{\mathbf{\Lambda}}\text{–}\mathsf{mod}_{\mathrm{fin}}$$

given by

$$F_{\mathbb{D}}(M) := \varprojlim_k \mathrm{Hom}_{\mathbb{D}} \left(\bigoplus_{\alpha \in \{+,-\}^{I_{\mathbf{\Lambda}}}} \mathbb{D}/\mathbb{D}\mathfrak{I}_{v_\alpha}^{k+1},\ M \right).$$

In particular, we have

$$F_{\mathbb{D}} \left(\mathbb{D}/\mathbb{D}\mathfrak{I}_{v_\alpha}^{k+1} \right) \cong \left(\widehat{Q}_{\mathbf{\Lambda}} / \langle \theta_i \rangle^{k+1} \right) e_\alpha.$$

Let M be an object of $\mathbb{D}\text{–}\mathsf{mod}_{\mathbf{\Lambda}+W_\mathbb{Z}}$. For any $m \in M$ and $v \in \mathbf{\Lambda} + W_\mathbb{Z}$, let m_v denote the projection of m onto M_v, so that $m = \sum m_v$. For each $1 \leq i \leq n$, consider the endomorphism $\Theta_i \in \mathrm{End}_{\mathbb{D}}(M)$ given by $\Theta_i\big(\sum m_v\big) := \sum \big(h_i^+ - h_i^+(v)\big) m_v$. Using [MV98, 6.3] it is easy to check

that Θ_i corresponds under the equivalence $F_{\mathbb{D}}$ to multiplication by the central element $\theta_i \in \widehat{Q}_{\mathbf{\Lambda}}$. When $M = \mathbb{D}/\mathbb{D}\mathfrak{I}_{v_\alpha}^{k+1}$, Θ_i is given by *right* multiplication by $\Theta_i(1) = h_i^+ - h_i^+(v_\alpha)$.

For each $\alpha \in \{+,-\}^{I_{\mathbf{\Lambda}}}$, let λ_α be the unique character of $Z(U)$ for which $\ker \lambda_\alpha$ is contained in $\mathfrak{I}_{v_\alpha}$ (equivalently, the character corresponding to the V_0-orbit $v_\alpha + V_0 \subset \mathbf{W}$), and put

$$P_\alpha^{(k)} := \mathbb{D}\Big/\big(\mathbb{D}\mathfrak{I}_{v_\alpha}^{k+1} + \mathbb{D}\langle \ker \lambda_\alpha\rangle\big).$$

This module is the quotient of $\mathbb{D}/\mathbb{D}\mathfrak{I}_{v_\alpha}^{k+1}$ by the sum of the images of the operators

$$\sum_{i=1}^{n} x_i\Theta_i, \;\; x \in \mathfrak{k},$$

and since $F_{\mathbb{D}}$ is an equivalence, it follows that

$$F_{\mathbb{D}}(P_\alpha^{(k)}) \cong \Big(\widehat{Q}_{\mathbf{\Lambda}}\big/\langle\theta_i\rangle^{k+1} + \langle \vartheta(x) \mid x \in \mathfrak{k}\rangle\Big)\, e_\alpha.$$

This in turn implies that

$$\varprojlim \mathrm{End}_{\mathbb{D}}\Big(\bigoplus_{\alpha\in\{+,-\}^{I_{\mathbf{\Lambda}}}} P_\alpha^{(k)}\Big) \cong R,$$

and the functor F is given by $\varprojlim \mathrm{Hom}_{U_\lambda}\Big(\bigoplus_{\alpha\in\mathcal{F}_{\mathbf{\Lambda}}}(P_\alpha^{(k)})^{\mathbf{\Lambda}}, -\Big)$.

The key property of the modules $P_\alpha^{(k)}$ that we will need is the fact that for any $\alpha \in \mathcal{F}_{\mathbf{\Lambda}}$ we have

$$\mathbb{D} \otimes_U (P_\alpha^{(k)})^{\mathbf{\Lambda}} \cong \mathbb{D} \otimes_U \Big(U/U(\mathfrak{I}_{v_\alpha}^{k+1} + \langle \ker \lambda_\alpha\rangle)\Big) \cong P_\alpha^{(k)}, \tag{4}$$

since $v_\alpha \in \mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$. Note that (4) does *not* hold for $\alpha \notin \mathcal{F}_{\mathbf{\Lambda}}$.

Now let M be an object of $U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}}$. Then if $\alpha \in \mathcal{F}_{\mathbf{\Lambda}}$, we have

$$e_\alpha F(M) = \mathrm{Hom}_U\big((P_\alpha^{(k)})^{\mathbf{\Lambda}}, M\big) \cong \mathrm{Hom}_U\big((P_\alpha^{(k)})^{\mathbf{\Lambda}}, (\mathbb{D}\otimes_U M)^{\mathbf{\Lambda}}\big) \cong \mathrm{Hom}_{\mathbb{D}}(P_\alpha^{(k)}, \mathbb{D}\otimes_U M),$$

where the second isomorphism comes from (4) and adjunction. On the other hand, if $\beta \in \mathcal{F}_{\mathbf{\Lambda}'}$, we get

$$e_\beta F'({}_{\lambda'}T_\lambda \otimes_{U_\lambda} M) \cong \mathrm{Hom}_U\big((P_\beta^{(k)})^{\mathbf{\Lambda}'}, (\mathbb{D}\otimes_U M)^{\mathbf{\Lambda}'}\big) \cong \mathrm{Hom}_{\mathbb{D}}(P_\beta^{(k)}, \mathbb{D}\otimes_U M).$$

The required natural transformation $\phi_M \colon e'Re \otimes_{eRe} F(M) \to F'({}_{\lambda'}T_\lambda \otimes_{U_\lambda} M)$ therefore comes from taking inverse limits of the composition

$$\mathrm{Hom}_{\mathbb{D}}(P_\beta^{(k)}, P_\alpha^{(k)}) \otimes \mathrm{Hom}_{\mathbb{D}}(P_\alpha^{(k)}, \mathbb{D}\otimes_U M) \to \mathrm{Hom}_{\mathbb{D}}(P_\beta^{(k)}, \mathbb{D}\otimes_U M).$$

Furthermore, it is clear from (4) that ϕ is an isomorphism on $(P_\alpha^{(k)})^{\mathbf{\Lambda}}$, $\alpha \in \mathcal{F}_{\mathbf{\Lambda}}$. This object is the projective cover of $L_\alpha^{\mathbf{\Lambda}}$ in $U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}}^{(k)}$, the full subcategory of $M \in U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}}$ for which $\mathfrak{I}_v^{k+1}M_v = 0$ for all $v \in \mathbf{\Lambda}$. This category has enough projectives, and any $M \in U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}}$ lies in $U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}}^{(k)}$ for some k, so we can find an exact sequence $P_1 \to P_0 \to M \to 0$ where ϕ_{P_0} and ϕ_{P_1} are isomorphisms.

Since both the source and target of ϕ are right exact functors, ϕ_M is an isomorphism for all M. $\square$

4 Hypertoric category $\mathcal{O}$

Fix a quantized polarized arrangement $\mathbf{X} = (\Lambda_0, \mathbf{\Lambda}, \xi)$. In Section 3 we explained how Λ_0 determines an algebra U, how $\mathbf{V} = \mathbf{\Lambda} + \mathbb{C}\Lambda_0 \subset \mathbf{W}$ determines a central character $\lambda : Z(U) \to \mathbb{C}$, and how the lattice $\mathbf{\Lambda} \subset \mathbf{V}$ determines a subcategory $U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}$ of the category of weight modules over U_λ. In this section we use the parameter $\xi \in \Lambda_0^*$ to restrict our categories even further, and thus obtain the hypertoric category $\mathcal{O}$.

4.1 Definition of the category

Recall that U has the decomposition $U = \bigoplus_{z \in \Lambda_0} U_z$, where $U_z = \mathbb{D}_z$ is the z-isotypic piece of $\mathbb{D}$. For any $k \in \mathbb{Z}$, put

$$U^k := \bigoplus_{\xi(z)=k} U_z.$$

Then put

$$U^+ := \bigoplus_{k \geq 0} U^k \qquad \text{and} \qquad U^- := \bigoplus_{k \leq 0} U^k.$$

The algebra U^+ will play a role for us similar to the role played by the enveloping algebra of a Borel in the definition BGG category $\mathcal{O}$. Note that the analogy is not exact; in particular, we have a surjection

$$U^+ \otimes_{U^0} U^- \to U,$$

but it is not an isomorphism.

These subalgebras and subspaces induce corresponding subalgebras and subspaces of the central quotient U_λ: we let $U_{\lambda,z}$, U_λ^k, U_λ^+, and U_λ^- be the images of U_z, U^k, U^+, and U^-, respectively, under the quotient map $U \to U_\lambda$.

Definition 4.1. We define **hypertoric category $\mathcal{O}$** to be the full subcategory of $U\text{−}\mathsf{mod}$ consisting of modules that are U^+-locally finite and semisimple over the center $Z(U)$. We define $\mathcal{O}_\lambda$ to be the full subcategory of $\mathcal{O}$ consisting of modules on which U acts with central character λ; equivalently, it is the full subcategory of $U_\lambda\text{−}\mathsf{mod}$ consisting of modules that are U_λ^+-locally finite. We define $\mathcal{O}(\mathbf{X})$ to be the full subcategory of $\mathcal{O}_\lambda$ consisting of modules supported in $\mathbf{\Lambda}$; equivalently, it is the full subcategory of $U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}$ consisting of modules that are U_λ^+-locally finite.

We have a direct sum decomposition

$$\mathcal{O} = \bigoplus_{\mathbf{\Lambda} \in \mathbf{W}/\Lambda_0} \mathcal{O}(\Lambda_0, \mathbf{\Lambda}, \xi).$$

It follows from Theorem 4.7 below that these summands of $\mathcal{O}$ are blocks, that is, they are the

smallest possible direct summands. In accordance with the terminology in Lie theory, we will call

$$\mathcal{O}_\lambda := \bigoplus_{\mathbf{\Lambda} \in \mathbf{V}/\Lambda_0} \mathcal{O}(\Lambda_0, \mathbf{\Lambda}, \xi)$$

an **infinitesimal block** of $\mathcal{O}$. We will call the block $\mathcal{O}(\mathbf{X})$ **regular** (respectively **integral**) if $\mathbf{X}$ is regular (respectively integral) as defined in Section 2.3; we call an infinitesimal block regular if its constituent blocks are all regular.

Remark 4.2. If Λ_0 is unimodular (Remark 2.11) and $\mathbf{X}$ is integral, then $I_{\mathbf{\Lambda}'} = \emptyset$ for all $I_{\mathbf{\Lambda}'} \subset \mathbf{V}$ different from $\mathbf{\Lambda}$, and therefore $\mathcal{O}(\mathbf{X}) = \mathcal{O}_\lambda$. If Λ_0 is not unimodular, however, then the regular infinitesimal blocks of $\mathcal{O}$ are never themselves blocks.

Remark 4.3. We warn the reader of a subtle but important difference between hypertoric category $\mathcal{O}$ and the classical BGG category $\mathcal{O}$. Let $\mathfrak{g}$ be a semisimple Lie algebra with $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ a Cartan and Borel subalgebra. BGG category $\mathcal{O}$ is the full subcategory of finitely generated $U(\mathfrak{g})$-modules for which $U(\mathfrak{b})$ acts locally finitely and $U(\mathfrak{h})$ acts semisimply, while hypertoric category $\mathcal{O}$ is the full subcategory of finitely generated U-modules for which U^+ acts locally finitely and $Z(U)$ acts semisimply. The analogy is imprecise because $U(\mathfrak{h})$ is not the center of $U(\mathfrak{g})$.

In the case of a regular infinitesimal block, we are rescued by a theorem of Soergel: a regular infinitesimal block of BGG category $\mathcal{O}$ is equivalent to a regular infinitesimal block of the category obtained by requiring the center of $U(\mathfrak{g})$, rather than the Cartan $U(\mathfrak{h})$, to act semisimply [Soe86]. Thus, it is reasonable to regard regular infinitesimal blocks (or blocks) of hypertoric category $\mathcal{O}$ as analogues of regular infinitesimal blocks (or blocks) of BGG category $\mathcal{O}$. Furthermore, Theorems 1.1 and 1.2 demonstrate that regular integral blocks of BGG category $\mathcal{O}$ and hypertoric category $\mathcal{O}$ have many properties in common.

Consider the natural projection $W_{\mathbb{Z}}^* \to \Lambda_0^*$, and choose any lift $\tilde{\xi} \in W_{\mathbb{Z}}^*$ of ξ. Recall from Section 3.1 that we have identified W with $\mathfrak{t}^*$ and $\mathfrak{t}$ with the degree 2 part of H, thus we may regard $\tilde{\xi}$ as a linear combination of $\{h_1, \ldots, h_n\}$. Lift it further to an element $\hat{\xi} \in F_2\mathbf{H}$, that is, to a linear combination of $\{h_1^\pm, \ldots, h_n^\pm\}$. Note that, by Equation (1), the subspace $U^k \subset U$ is exactly the space on which the conjugation operator $\mathrm{ad}(\hat{\xi})$ acts with eigenvalue k.

Lemma 4.4. *Let M be a finitely generated U_λ-module. Then the following are equivalent:*

1. *M is an object of $\mathcal{O}$.*
2. *M is generated by a finite-dimensional U^+-invariant subspace S.*
3. *$\hat{\xi} \in \mathbf{H} \subset U^0$ acts locally finitely on M with finite dimensional generalized eigenspaces and the eigenvalues that appear are bounded above.*

Proof. (1) $\Rightarrow$ (2): Given a finite generating set for M, we can apply U^+ to obtain a finite-dimensional U^+-invariant generating set.

(2) $\Rightarrow$ (3): Suppose that S is a finite-dimensional U^+-invariant generating set for M. Choose finitely many algebra generators $u_1, \dots, u_\ell$ for U which are $\mathbb{T}$-weight vectors, and let $k_i \in \mathbb{Z}$ be the weight of u_i. Let $U' \subset U$ be the subalgebra generated by those u_i for which $k_i < 0$. Then we have $U'U^+ = U$, since this holds after taking the associated graded. This in turn implies that $M = U'S$. Since $U' \cap U^k$ is finite dimensional for all k and is zero for $k > 0$, M has finite dimensional $\hat{\xi}$-weight spaces, all of which have no higher weight than the highest occurring in S.
(3) $\Rightarrow$ (1): If the generalized $\hat{\xi}$ eigenspaces of M are both finite dimensional and bounded above, then for all $m \in M$, U^+m is a subspace of finitely many generalized eigenspaces and thus finite dimensional. $\square$

Recall from Section 2.4 that $\mathcal{F}_{\mathbf{\Lambda}}$ is defined to be the set of all $\alpha \in \{+,-\}^{I_{\mathbf{\Lambda}}}$ such that $\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$ is nonempty, and $\mathcal{P}_{\mathbf{\Lambda},\xi} \subset \mathcal{F}_{\mathbf{\Lambda}}$ is the subset for which ξ is proper and bounded above on $\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$. Thus Lemma 4.4 has the following immediate corollary.

Corollary 4.5. *Suppose that $\alpha \in \mathcal{F}_\Lambda$, so $L_\alpha^{\mathbf{\Lambda}}$ is a simple object of $U_\lambda\text{-}\mathsf{mod}_{\mathbf{\Lambda}}$. Then $L_\alpha^{\mathbf{\Lambda}} \in \mathcal{O}(\mathbf{X})$ if and only if $\alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi}$.*

Remark 4.6. Corollary 4.5 tells us that the block $\mathcal{O}(\mathbf{X})$ has $|\mathcal{P}_{\mathbf{\Lambda},\xi}|$ isomorphism classes of simple objects. This number is always less than or equal to the number of bases for the matroid associated to $\mathcal{H}_0$, or equivalently the number of vertices of $\mathcal{H}_\eta$ for a regular value of η (Remark 2.6). Equality is achieved if and only if $\mathbf{X}$ is both regular and integral. If $\mathbf{X}$ fails to be integral, then there will be too few hyperplanes in $\boldsymbol{\mathcal{H}}$, and therefore too few chambers $\mathbf{\Delta}_\alpha$. If $\mathbf{X}$ is integral but fails to be quasi-regular, then there will again be too few chambers. If $\mathbf{X}$ is integral and quasi-regular but not regular, then we will have the right number of chambers, but some of them will not contain any points of the lattice $\mathbf{\Lambda}$.

4.2 Quiver description of $\mathcal{O}(\mathbf{X})$

Theorem 3.6 and Corollary 4.5 combine to give us the following result. Let

$$e_\xi \;:=\; \sum_{\alpha \notin \mathcal{B}_{\mathbf{\Lambda},\xi}} e_\alpha \ .$$

Theorem 4.7. *The category $\mathcal{O}(\mathbf{X})$ is equivalent to the category of finite dimensional modules over*

$$A(\mathbf{X}) := \Big(e_{\mathbf{\Lambda}}\, \widehat{Q}_{\mathbf{\Lambda}}\, e_{\mathbf{\Lambda}}\Big) \Big/ \big\langle e_\xi e_{\mathbf{\Lambda}} \big\rangle + \big\langle \vartheta(x) e_{\mathbf{\Lambda}} \mid x \in \mathfrak{k} \cap \mathfrak{t}^{I_{\mathbf{\Lambda}}} \big\rangle.$$

In [BLPWb, 3.1] we constructed an algebra $A(X)$ from a regular polarized arrangement X.

Theorem 4.8. *If $\mathbf{X}$ is integral and regular and linked to X, then $A(\mathbf{X}) \cong A(X)$.*

Proof. The linkage of of X and $\mathbf{X}$ implies that $\mathcal{P}_{\mathbf{\Lambda},\xi} = \mathcal{P}_{\eta,\xi}$, and knowing this, it is straightforward to check that the algebra defined in [BLPWb, 3.1] is equal to the "polynomial part" of $A(\mathbf{X})$,

namely the uncompleted algebra

$$(e_{\mathbf{\Lambda}}\, Q_{\mathbf{\Lambda}}\, e_{\mathbf{\Lambda}}) \Big/ \langle e_\xi e_{\mathbf{\Lambda}} \rangle + \langle \vartheta(x) e_{\mathbf{\Lambda}} \mid x \in \mathfrak{k} \rangle.$$

But [BLPWb, 4.14] implies that this algebra is finite dimensional, so it is isomorphic to its completion. □

This result implies that if $\mathbf{X}$ is a regular integral quantized polarized arrangement then, up to equivalence, the category $\mathcal{O}(\mathbf{X})$ only depends on the equivalence class of $\mathbf{X}$ (Definition 2.14). More precisely, if $\mathbf{X}$ and $\mathbf{X}'$ are regular integral polarized arrangements in the same equivalence class, then Theorem 3.11 implies that tensoring with a bimodule ${}_{\lambda'}T_\lambda$ gives an equivalence of categories from $\mathcal{O}(\mathbf{X})$ to $\mathcal{O}(\mathbf{X}')$.

When the quantized polarized arrangement $\mathbf{X}$ is not integral, there are two possibilities. First, the hyperplane arrangement $\boldsymbol{\mathcal{H}}$ associated to $\mathbf{X}$ could be inessential (Section 2.3). In this case the arrangement $\boldsymbol{\mathcal{H}}$ has no vertices; this happens, for example, when $I_{\mathbf{\Lambda}} = \emptyset$. For any $\alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi}$ the polyhedron $\mathbf{\Delta}_\alpha$ has a vertex of $\boldsymbol{\mathcal{H}}$ as a ξ-maximal point. Thus, when $\boldsymbol{\mathcal{H}}$ is inessential, $\mathcal{P}_{\mathbf{\Lambda},\xi}$ must be empty, so there are no nonzero objects in $\mathcal{O}(\mathbf{X})$.

Suppose on the other hand that $\boldsymbol{\mathcal{H}}$ is essential. Then we can define an integral quantized polarized arrangement $\mathbf{X}' = (\Lambda_0', \mathbf{\Lambda}, \xi')$ with $n' := |I_{\mathbf{\Lambda}}|$ hyperplanes along with an isomorphism $\mathbf{V}_{\mathbb{R}} \cong \mathbf{V}'_{\mathbb{R}}$ that takes $\boldsymbol{\mathcal{H}}$ to $\boldsymbol{\mathcal{H}}'$. To define $\mathbf{X}'$, let W' and $\mathbf{W}'$ be the vector space and affine space defined in Section 2.2, but with n replaced by n'. More precisely, they are the spectra of the subrings of H and $\mathbf{H}$ generated by $h_i^{\pm}$ and h_i for all $i \in I_{\mathbf{\Lambda}}$. Then let Λ_0' and $\mathbf{\Lambda}'$ be the images of Λ_0 and $\mathbf{\Lambda}$ under the natural projections $W \to W'$ and $\mathbf{W} \to \mathbf{W}'$. The condition that $\boldsymbol{\mathcal{H}}$ has a vertex implies that the projections induce isomorphisms $\Lambda_0' \cong \Lambda_0$ and $\mathbf{\Lambda}' \cong \mathbf{\Lambda}$. Letting ξ' be the composition of ξ with the isomorphism $\Lambda_0' \cong \Lambda_0$, we obtain equalities

$$\mathcal{F}_{\mathbf{\Lambda}'} = \mathcal{F}_{\mathbf{\Lambda}}, \qquad \mathcal{B}_{\mathbf{\Lambda}',\xi'} = \mathcal{B}_{\mathbf{\Lambda},\xi}, \qquad \text{and} \qquad \mathcal{P}_{\mathbf{\Lambda}',\xi'} = \mathcal{P}_{\mathbf{\Lambda},\xi}$$

of subsets of $\{+,-\}^{I_{\mathbf{\Lambda}}}$. The following result now allows us to reduce the study of arbitrary blocks of hypertoric category $\mathcal{O}$ to the integral case.

Theorem 4.9. *If $\boldsymbol{\mathcal{H}}$ is essential, then $\mathcal{O}(\mathbf{X})$ is equivalent to $\mathcal{O}(\mathbf{X}')$ and $A(\mathbf{X}) \cong A(\mathbf{X}')$.*

Proof. As noted in Section 3.4, there is an isomorphism $\widehat{Q}_{\mathbf{\Lambda}} \cong \widehat{Q}_{|I_{\mathbf{\Lambda}}|} \,\widehat{\otimes}_{\mathbb{C}}\, \mathbb{C}[[\theta_i \mid i \notin I_{\mathbf{\Lambda}}]]$, and the map ϑ gives an isomorphism from $\mathfrak{t} \cong \mathbb{C}^n$ to the degree two part of $Z(\widehat{Q}_{\mathbf{\Lambda}})$. Let $\mathfrak{t}^{I_{\mathbf{\Lambda}}} \subset \mathfrak{t}$ be the space of vectors whose i^{th} coordinates vanish for all $i \notin I_{\mathbf{\Lambda}}$, so $\vartheta(\mathfrak{t}^{I_{\mathbf{\Lambda}}}) = Z(\widehat{Q}_{|I_{\mathbf{\Lambda}}|})_2 \otimes \mathbb{C}$. Then the assumption that $\boldsymbol{\mathcal{H}}$ is essential implies that the composition

$$\mathfrak{k} \hookrightarrow \mathfrak{t} \to \mathfrak{t}/\mathfrak{t}^{I_{\mathbf{\Lambda}}}$$

is a surjection. Thus

$$\widehat{Q}_{\mathbf{\Lambda}}/\langle \vartheta(\mathfrak{k}) \rangle \cong \widehat{Q}_{|I_{\mathbf{\Lambda}}|}/\langle \vartheta'(\mathfrak{k} \cap \mathfrak{t}^{I_{\mathbf{\Lambda}}}) \rangle,$$

where $\vartheta' \colon \mathfrak{t}^{I_\Lambda} \to Z(\widehat{Q}_{|I_\Lambda|})$ is the induced map. The isomorphism $A(\mathbf{X}) \cong A(\mathbf{X}')$ follows, since $\mathfrak{k} \cap \mathfrak{t}^{I_\Lambda}$ is the annihilator of Λ_0' under the natural isomorphism $\mathfrak{t}^{I_\Lambda} \cong (W')^*$. The equivalence of $\mathcal{O}(\mathbf{X})$ and $\mathcal{O}(\mathbf{X}')$ now follows from Theorem 4.7. □

As a corollary we obtain the first two parts of Theorem 1.2, along with the beginning of the seventh part.

Corollary 4.10. *If* $\mathbf{X}$ *is regular, then the category* $\mathcal{O}(\mathbf{X})$ *is highest weight and Koszul. If* $\mathbf{X}$ *and* $\mathbf{X}^!$ *are regular, integral, and Gale dual to each other, then the rings* $A(\mathbf{X})$ *and* $A(\mathbf{X}^!)$ *are Koszul dual.*

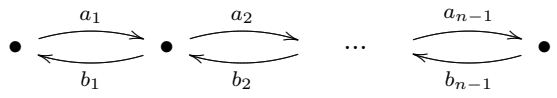
Proof. These facts follow from Theorems 4.8 and 4.9 and [BLPWb, 3.11, 5.23, & 5.24]. □

Example 4.11. We illustrate the preceding results with a simple example. Let $K \subset T$ be the 1-dimensional torus acting diagonally on $\mathbb{C}^n$, so the corresponding $(n-1)$-dimensional lattice $\Lambda_0 \subset W_\mathbb{Z}$ is the vanishing locus of $h_1 + \ldots + h_n$. For $c \in \mathbb{Z}$, let $\mathbf{\Lambda}_c := \{v \in \mathbf{W}_\mathbb{Z} \mid \sum_{i=1}^n h_i^+(v) = c\}$; it is regular if and only if $c \geq 0$ or $c \leq -n$. The quantized polarized arrangement $\boldsymbol{\mathcal{H}}$ consists of n pairs of parallel hyperplanes in a vector space of dimension $n-1$, with each pair in general position with respect to each other pair. Take any covector $\xi \in \Lambda_0^*$ which is positive on the weight of $x_i\partial_j$ if and only if $i < j$. These weights are exactly the directions of the 1-dimensional flats, so ξ is regular. Let $\mathbf{X}_c = (\Lambda_0, \mathbf{\Lambda}_c, \xi)$. An example with $n = 3$ is illustrated in Figure 1. We warn the reader that even when $\mathbf{\Lambda}_c$ is regular it is possible to obtain a picture in which three hyperplanes pass through a single point (see Definition 2.5). In fact, when $c = 0$, the n hyperplanes $\{H_1^+, \ldots, H_n^+\}$ all pass through a single point.

Let us examine the case $c = 0$ in more detail. We have

$$\mathcal{P}_{\mathbf{\Lambda}_0,\xi} = \{(+,+,\ldots,+), (-,+,\ldots,+), (-,-,+,\ldots,+), \ldots, (-,-,\ldots,-,+)\}.$$

The algebra $A(\mathbf{X}_0)$ is the quotient of the path algebra of the quiver



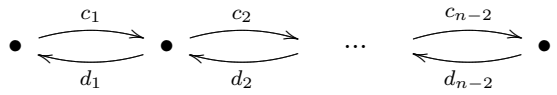
by the relations $b_1a_1 = 0$, and $a_{i+1}a_i = 0$, $b_ib_{i+1} = 0$ and $a_ib_i = b_{i+1}a_{i+1}$ for all $1 \leq i \leq n-2$. The corresponding block $\mathcal{O}(\mathbf{X}_0)$ of our hypertoric category $\mathcal{O}$ is equivalent to a parabolic block of classical BGG category $\mathcal{O}$ for $\mathfrak{g} := \mathfrak{gl}_n$ in the following way. There is a surjective homomorphism $U(\mathfrak{g}) \to U$ which takes the elementary matrix E_{ij} to $x_i\partial_j$ and sends $U(\mathfrak{b})$ to U^+. Pulling back by this homomorphism is a full and faithful functor from hypertoric category $\mathcal{O}$ into the category $\mathcal{O}'(\mathfrak{g})$ of $U(\mathfrak{g})$-modules on which $U(\mathfrak{b})$ acts locally finitely and $Z(U(\mathfrak{g}))$ acts semisimply. The image of this functor is the subcategory generated by the simple modules with highest weights $w \cdot 0 = w(\rho) - \rho$, where ρ is half the sum of the positive roots of $\mathfrak{g}$ and w runs over all smallest representatives in S_n for the left cosets $S_n/(1 \times S_{n-1})$.

The equivalence of [Soe86] gives an equivalence between this category and the subcategory of BGG category $\mathcal{O}(\mathfrak{g})$ generated by simple modules with highest weights $w^{-1} \cdot 0$, where w runs over the same set; this subcategory is a block of the parabolic category $\mathcal{O}^{\mathfrak{p}}(\mathfrak{g})$ consisting of $\mathfrak{p}$-locally finite modules in $\mathcal{O}(\mathfrak{g})$, where $\mathfrak{p} \subset \mathfrak{g}$ is the set of block matrices preserving the first basis vector of $\mathbb{C}^n$.

Example 4.12. We now give an example that shows how Corollary 4.10 can fail when $\mathbf{X}$ is not regular. Let $K \subset T$ be the $(n-1)$-dimensional subtorus that acts on $\mathbb{C}^n$ with determinant 1, so that $\Lambda_0 \subset W_{\mathbb{Z}}$ is the 1-dimensional lattice cut out by the equations $h_i = h_j$ for all $i, j \leq n$. For appropriate choices of regular $\mathbf{\Lambda}$ and ξ, the resulting quantized polarized arrangement $\mathbf{X} = (\Lambda_0, \mathbf{\Lambda}, \xi)$ will be Gale dual to the one from example 4.11. On the other hand, if either $\mathbf{\Lambda}$ or ξ is not regular, then $\mathcal{O}(\mathbf{X})$ can fail to be highest weight and Koszul.

Consider first the example where $\mathbf{\Lambda} = \{v \in \mathbf{W}_{\mathbb{Z}} \mid h_i^+(v) = h_j^+(v) \text{ for all } i, j \leq n\}$ (which is not regular) and $\xi \neq 0$. Then $\mathcal{F}_{\mathbf{\Lambda}} = \{(-,-,\ldots,-), (+,+,\ldots,+)\}$ and $\mathcal{P}_{\mathbf{\Lambda},\xi}$ contains exactly one of these sign vectors (which one depends on our choice of ξ.) Theorem 3.6 tells us that $A(\mathbf{X}) \cong \mathbb{C}[\theta]/\langle\theta^n\rangle$, which is not quasihereditary when $n > 1$ and is not Koszul when $n > 2$.

On the other hand, suppose that $\mathbf{\Lambda}$ is integral and regular, so $\mathcal{F}_{\mathbf{\Lambda}}$ contains $n+1$ sign vectors. If $\xi = 0$, then $\mathcal{P}_{\mathbf{\Lambda},\xi}$ contains exactly those $n-1$ sign vectors whose associated polyhedra are compact. The resulting algebra $A(\mathbf{X})$ is the quotient of the path algebra of the quiver



by the relations $d_1c_1 = 0$, $c_{n-2}d_{n-2} = 0$, and $c_id_i = d_{i+1}c_{i+1}$ for $1 \leq i \leq n-3$. This algebra is not quasihereditary when $n > 1$ and is not Koszul when $n > 2$.

4.3 Projective objects in $\mathcal{O}(\mathbf{X})$ and ξ-truncation

Let $\mathbf{X} = (\Lambda_0, \mathbf{\Lambda}, \xi)$ be a quantized polarized arrangement, and let λ be the corresponding character of $Z(U)$. The inclusion functor $\iota\colon \mathcal{O}(\mathbf{X}) \to U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}}$ has a left adjoint π_ξ, which we call ξ-truncation. To define it, take $M \in U_\lambda\text{–}\mathsf{mod}_{\mathrm{lf}}$ and let $\pi_\xi(M)$ be the quotient of M by the submodule generated by all weight spaces M_v with $v \notin \bigcup_{\alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi}} \mathbf{\Delta}_\alpha$. It is evident that this functor is right exact, and that $\pi_\xi \circ \iota$ is the identity functor on $\mathcal{O}(\mathbf{X})$.

Equivalently, $\pi_\xi(M)$ is the largest quotient of M which is U^+-locally finite. This leads immediately to a description of truncation in terms of the equivalent quiver categories. We use the notation of Theorems 3.11 and 4.7, so $e = e_{\mathbf{\Lambda}}$, and $A(\mathbf{X})$ is the eRe algebra $eRe/eRe_\xi eRe$.

Proposition 4.13. *The diagram*

$$\begin{array}{ccc} U_\lambda\text{–}\mathsf{mod}_{\mathbf{\Lambda}} & \longrightarrow & eRe\text{–}\mathsf{mod}_{\mathrm{fin}} \\ \Big\downarrow{\scriptstyle \pi_\xi} & & \Big\downarrow{\scriptstyle (A(\mathbf{X}) \otimes_{eRe} -)} \\ \mathcal{O}(\mathbf{X}) & \longrightarrow & A(\mathbf{X})\text{–}\mathsf{mod} \end{array}$$

commutes up to natural equivalence of functors, where the horizontal functors are the equivalences of Theorems 3.6 and 4.7.

We use the ξ-truncation functor to construct the projective objects in our category as follows. Take $\alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi}$, and recall the U_λ-module $(P_\alpha^{(k)})^{\mathbf{\Lambda}}$ from the proof of Theorem 3.11. This module, which is projective in the subcategory $U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}^{(k)} \subset U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}$, represents the functor $M \mapsto M_v$ for any $v \in \mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$. It follows that for k large enough (say $k > \dim_{\mathbb{C}} A(\mathbf{X})$), the truncated module $\pi_\xi(P_\alpha^{(k)})^{\mathbf{\Lambda}}$ is independent of k, and represents the v-weight space functor on $\mathcal{O}(\mathbf{X})$. It is thus the projective cover of $L_\alpha^{\mathbf{\Lambda}}$.

4.4 Standard modules of $\mathcal{O}(\mathbf{X})$

Let us assume that $\mathbf{X} = (\Lambda_0, \mathbf{\Lambda}, \xi)$ is regular, so that by Corollary 4.10 the category $\mathcal{O}(\mathbf{X})$ is highest weight. Being highest weight means that $\mathcal{O}(\mathbf{X})$ contains certain objects called **standard modules**, which are analogues of the Verma modules of BGG category $\mathcal{O}$. The definition of standard modules involves a partial order on the set $\mathcal{P}_{\mathbf{\Lambda},\xi}$ that indexes the simple objects of $\mathcal{O}(\mathbf{X})$; this partial order was introduced in [BLPWb, §2.6]. For $\alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi}$, the standard module $S_\alpha^{\mathbf{\Lambda}}$ is defined to be the projective cover of $L_\alpha^{\mathbf{\Lambda}}$ in the full subcategory $\mathcal{O}(\mathbf{X})_{\leq\alpha} \subset \mathcal{O}(\mathbf{X})$ consisting of modules whose composition factors $L_\beta^{\mathbf{\Lambda}}$ satisfy $\beta \leq \alpha$. In this section we construct the standard modules explicitly.

Fix an element $\alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi}$, and let v_ξ be the ξ-maximal point of the polyhedron $\mathbf{\Delta}_\alpha \subset \mathbf{V}_{\mathbb{R}}$ (regularity of ξ implies that there is a unique such point). Fix any weight $v \in \mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$, and let $J_\alpha \subset \mathbb{D}$ be the left ideal generated by the following elements:

- ∂_i for all i such that $v_\xi \in H_i^+$
- x_i for all i such that $v_\xi \in H_i^-$
- $h_i^{\alpha(i)}$ for all $i \in I_{\mathbf{\Lambda}}$
- $h_i^+ - h_i^+(v)$ for all $i \notin I_{\mathbf{\Lambda}}$.

Let $S_\alpha := \mathbb{D}/J_\alpha$. This module lies in $\mathbb{D}\text{−}\mathsf{mod}_{\mathbf{\Lambda}+W_{\mathbb{Z}}}$, and there is a natural surjection $S_\alpha \to L_\alpha$ of $\mathbb{D}$-modules. Each simple object L_β in $\mathbb{D}\text{−}\mathsf{mod}_{\mathbf{\Lambda}+W_{\mathbb{Z}}}$ appears in the composition series of S_α with multiplicity either 0 or 1, with the multiplicity being 1 if and only if $\beta(i) = \alpha(i)$ for all $i \in I_{\mathbf{\Lambda}}$ such that $v_\xi \in H_i^\pm$. Since all the weight spaces are 1-dimensional, S_α is semisimple over $Z(U)$, and so $S_\alpha^{\mathbf{\Lambda}} = \bigoplus_{v\in\mathbf{\Lambda}}(S_\alpha)_v$. It follows that that the simple object $L_\beta^{\mathbf{\Lambda}}$ in $U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}$ appears in the composition series of $S_\alpha^{\mathbf{\Lambda}}$ if and only if $\beta \in \mathcal{F}_{\mathbf{\Lambda}}$ and $\beta(i) = \alpha(i)$ for all i such that $v_\xi \in H_i^\pm$. It follows from Corollary 4.5 that $S_\alpha^{\mathbf{\Lambda}} \in \mathcal{O}(\mathbf{X})$, and from [BLPWb, 2.11] that $S_\alpha^{\mathbf{\Lambda}} \in \mathcal{O}(\mathbf{X})_{\leq\alpha}$.

For use in Section 6.3, it will be convenient if we restate these observations for integral $\mathbf{X}$ in terms of a polarized arrangement X linked to $\mathbf{X}$.

Proposition 4.14. *Suppose that $\mathbf{X}$ is integral and linked to X. Choose any $\alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi}$, and let $\Sigma_\alpha \subset V_{\mathbb{R}}$ be the unique cone that coincides with Δ_α in a neighborhood of the ξ-maximal point of*

Δ_α. *The simple module* $L_\beta^{\mathbf{\Lambda}}$ *appears in the composition series of* $S_\alpha^{\mathbf{\Lambda}}$ *if and only if* $\Delta_\beta \subset \Sigma_\alpha$, *in which case it appears exactly once.*

Proposition 4.15. *The module* $S_\alpha^{\mathbf{\Lambda}}$ *is a projective cover of* $L_\alpha^{\mathbf{\Lambda}}$ *in the category* $\mathcal{O}(\mathbf{X})_{\leq\alpha}$.

Proof. The v-weight space of S_α is 1-dimensional, and $S_\alpha^{\mathbf{\Lambda}}$ is a cyclic U_λ-module generated by any nonzero element x of this weight space. In particular, $S_\alpha^{\mathbf{\Lambda}}$ is indecomposable, and it remains only to show that it is projective in $\mathcal{O}(\mathbf{X})_{\leq\alpha}$.

To do this, we will show that for any object M of $\mathcal{O}(\mathbf{X})_{\leq\alpha}$, the map

$$\mathrm{Hom}_{U_\lambda}(S_\alpha^{\mathbf{\Lambda}}, M) \to M_v, \quad \phi \mapsto \phi(x)$$

is surjective. Since this map is obviously injective, it will follow that $S_\alpha^{\mathbf{\Lambda}}$ represents the exact functor $M \mapsto M_v$, and therefore that $S_\alpha^{\mathbf{\Lambda}}$ is projective in $\mathcal{O}(\mathbf{X})_{\leq\alpha}$.

To see surjectivity, let $\tilde{M} := \mathbb{D} \otimes_U M$, and consider the analogous map for $\mathbb{D}$-modules

$$\mathrm{Hom}_{\mathbb{D}}(S_\alpha, \tilde{M}) \to \tilde{M}_v, \quad \psi \mapsto \psi(x).$$

It suffices to show that this map is surjective since it factors

$$\mathrm{Hom}_{\mathbb{D}}(S_\alpha, \tilde{M}) \xrightarrow{(\cdot)^{\mathbf{\Lambda}}} \mathrm{Hom}_{U_\lambda}(S_\alpha^{\mathbf{\Lambda}}, M) \to M_v \cong \tilde{M}_v,$$

and $(\cdot)^{\mathbf{\Lambda}}$ is surjective.

Let v_α be the weight of the element $1 \in S_\alpha$; the associated ideal $\mathfrak{I}_{v_\alpha} \subset \mathbf{H}$ is spanned by $\{h_i^{\alpha(i)} \mid i \in I_{\mathbf{\Lambda}}\} \cup \{h_i^+ - h_i^+(v) \mid i \notin I_{\mathbf{\Lambda}}\}$. Let m be a monomial which generates $\mathbb{D}_{v-v_\alpha}$ as a left or right $\mathbf{H}$-module. For any $\mathbb{D}$-module N, multiplication by m gives an isomorphism $N_{v_\alpha} \to N_v$. Thus it is sufficient to show that the map

$$\mathrm{Hom}_{\mathbb{D}}(S_\alpha, \tilde{M}) \to \tilde{M}_{v_\alpha}$$

taking ψ to $\psi(1)$ is surjective. In other words, we must show that the defining ideal J_α of S_α annihilates $\tilde{M}_{v_\alpha}$.

The ideal J_α has four types of generators; we treat them one at a time. If $v_\xi \in H_i^+$, then ∂_i takes $\tilde{M}_{v_\alpha}$ to $\tilde{M}_{v_\beta}$, where β differs from α only in the i^{th} coordinate. But such a β cannot satisfy $\beta \leq \alpha$, thus $M \in \mathcal{O}(\mathbf{X})_{\leq\alpha}$ implies that $\tilde{M}_{v_\beta} = 0$. Hence ∂_i annihilates $\tilde{M}_{v_\alpha}$. By similar reasoning, if $v_\xi \in H_i^-$, then x_i annihilates $\tilde{M}_{v_\alpha}$. Now consider the third and fourth set of generators in the list. These lie in the ideal $\mathfrak{I}_{v_\alpha}$, and we know (by definition) that some power of $\mathfrak{I}_{v_\alpha}$ annihilates $\tilde{M}_{v_\alpha}$. We only need to show that the first power does the trick.

Let $\lambda_\alpha \colon Z(U) \to \mathbb{C}$ be the character for which $\ker\lambda_\alpha \subset \mathfrak{I}_{v_\alpha}$. Since M is semisimple over $Z(U)$, so is $\tilde{M}$, and so $(\ker\lambda_\alpha)\tilde{M}_{v_\alpha} = 0$. Furthermore, if $v_\xi \in H_i^{\alpha(i)}$, then $h_i^{\alpha(i)}$ annihilates $\tilde{M}_{v_\alpha}$ by our analysis of the action of the first two types of generators of J_α. Now we observe that $\mathfrak{I}_{v_\alpha}$ is generated by

$$\{h_i^{\alpha(i)} \mid v_\xi \in H_i^{\alpha(i)}\} \cup \ker\lambda_\alpha,$$

and we are done. □

4.5 Deformed category $\mathcal{O}$

In [BLP$^+$] we described a universal flat graded deformation of any Koszul graded algebra, and we studied its properties in the special case when the original module category and its Koszul dual are both highest weight. Let $\mathbf{X} = (\Lambda_0, \mathbf{\Lambda}, \xi)$ be a regular quantized polarized arrangement, and for simplicity assume that it is integral. We now explain how to interpret modules over the universal deformation of the Koszul algebra $A(\mathbf{X})$ as modules over the hypertoric enveloping algebra U. The results of this section will not be used later in the paper.

Let $\tilde{A}(\mathbf{X})$ be the universal deformation of the Koszul algebra $A(\mathbf{X})$, as defined in [BLP$^+$, 4.1]. It is a flat graded algebra over a certain polynomial ring S, and its specialization at $0 \in \operatorname{Specm} S$ is isomorphic to $A(\mathbf{X})$ as a graded algebra. The following result relates $\tilde{A}(\mathbf{X})$ to the other quiver algebras we have considered in this paper.

Proposition 4.16. *There are isomorphisms*

$$\tilde{A}(\mathbf{X}) \cong e_{\mathbf{\Lambda}} Q_{\mathbf{\Lambda}} e_{\mathbf{\Lambda}} / \langle e_\xi e_{\mathbf{\Lambda}} \rangle \qquad \textit{and} \qquad S \cong \operatorname{Sym}(\mathfrak{k}),$$

and under these isomorphisms the S-algebra structure on $\tilde{A}(\mathbf{X})$ is induced by $\vartheta\colon \mathfrak{k} \to Q_{\mathbf{\Lambda}}$.

Proof. Let X be a polarized arrangement linked to $\mathbf{X}$, so $A(\mathbf{X}) = A(X)$, the algebra defined in [BLPWb]. Then [BLPWb, 4.14] tells us that $A(X) \cong B(X^!)$ and $e_{\mathbf{\Lambda}} Q_{\mathbf{\Lambda}} e_{\mathbf{\Lambda}} / \langle e_\xi e_{\mathbf{\Lambda}} \rangle \cong \tilde{B}(X^!)$, where $X^!$ is Gale dual to X and the algebras $B(X^!)$ and $\tilde{B}(X^!)$ are defined in [BLPWb]. Thus the proposition is reduced to showing that the $\operatorname{Sym}(\mathfrak{k})$-algebra $\tilde{B}(X^!)$ is the universal deformation of $B(X^!)$; this is proven in [BLP$^+$, 8.7]. □

Let $\widehat{S}$ be the completion of $\operatorname{Sym}(\mathfrak{k})$ at the maximal ideal generated by $\mathfrak{k}$, and define

$$\widehat{A}(\mathbf{X}) := \widehat{S} \otimes_S \tilde{A}(\mathbf{X});$$

it is the completion of $\tilde{A}(\mathbf{X})$ with respect to the grading. Consider the categories $\widehat{A}(\mathbf{X})\text{−}\mathsf{mod}_{\mathrm{fin}}$ and $\widehat{A}(\mathbf{X})\text{−}\mathsf{mod}$ of finite dimensional and finitely-generated modules over $\widehat{A}(\mathbf{X})$. Our task will be to understand these two categories in terms of the hypertoric enveloping algebra.

Let $\widehat{\mathcal{O}}_{\mathrm{fin}}(\mathbf{X})$ be the full subcategory of $U\text{−}\mathsf{mod}_{\mathrm{lf}}$ consisting of weight modules whose support lies in $\mathbf{\Lambda}$ and which are locally finite over U^+. Modules of this category differ from modules in $\mathcal{O}(\mathbf{X})$ in that the center acts by a generalized character rather than an honest character. Let

$$\widehat{U}_\lambda := U \otimes_Z \widehat{Z}_\lambda,$$

where $\widehat{Z}_\lambda$ is the completion of the center $Z(U)$ at the maximal ideal $J_\lambda := \ker \lambda$. Set

$$\widehat{\mathbf{H}}_\lambda := \mathbf{H} \otimes_Z \widehat{Z}_\lambda \qquad \text{and} \qquad \widehat{U}^+_\lambda := U^+ \otimes_Z \widehat{Z}_\lambda,$$

and define $\widehat{\mathcal{O}}(\mathbf{X})$ to be the category of finitely generated $\widehat{U}_\lambda$-modules M so that

- $M = \bigoplus_{v\in\mathbf{\Lambda}} M_v$, that is, M is *set-theoretically* supported on $\mathbf{\Lambda} \subset \mathbf{W} \cong \operatorname{Specm} \widehat{\mathbf{H}}_\lambda$, and
- for any $m \in M$, $\widehat{U}_\lambda^+ \cdot m$ is a finitely generated $\widehat{Z}_\lambda$-module.

Theorem 4.17. *There are equivalences*

$$\widehat{\mathcal{O}}(\mathbf{X})_{\text{fin}} \cong \widehat{A}(\mathbf{X})\text{−}\mathsf{mod}_{\text{fin}} \qquad \textit{and} \qquad \widehat{\mathcal{O}}(\mathbf{X}) \cong \widehat{A}(\mathbf{X})\text{−}\mathsf{mod}.$$

For any $x \in \mathfrak{k}$, the action of $\mu_\lambda(x) \in Z(U)$ corresponds to multiplication by $\vartheta(x)e_{\mathbf{\Lambda}} \in Z(\widehat{A}(\mathbf{X}))$.

Proof. The first equivalence and the identification of the action of $\mu_\lambda(x)$ on the left category follows from [MV98, 3.5.6, 4.3.1, & 6.3]. For the second equivalence, note that the methods of [MV98] do not directly apply to $\widehat{U}_\lambda$, since $\widehat{H}_\lambda$ is not a quotient of a polynomial ring. However, any object M of $\widehat{\mathcal{O}}(\mathbf{X})$ is isomorphic to the limit $\varprojlim M/(J_\lambda)^k M$, and each $M/(J_\lambda)^k M$ lies in $\widehat{\mathcal{O}}_{\text{fin}}(\mathbf{X})$, or more precisely in the subcategory $\mathcal{O}_k$ of modules annihilated by $(J_\lambda)^k$. This gives an equivalence between $\widehat{\mathcal{O}}(\mathbf{X})$ and the category of inverse systems $\cdots \to M_k \to M_{k-1} \to \cdots \to M_1$, where each M_k lies in $\mathcal{O}_k$, and each map $M_k \to M_{k-1}$ induces an isomorphism $M_k/(J_\lambda)^{k-1}M_k \cong M_{k-1}$. The result now follows from the equivalence between $\mathcal{O}_k$ and the category of finitely-generated (hence finite dimensional) modules over $\widehat{A}(\mathbf{X})/\langle\vartheta(\mathfrak{k})e_{\mathbf{\Lambda}}\rangle^k$. □

5 Hypertoric varieties

In this section we recall the definition of a hypertoric variety associated to a regular polarized arrangement $X = (\Lambda_0, \eta, \xi)$, along with a quantization of its structure sheaf. The ring of equivariant sections of this quantization coincides with a central quotient of the hypertoric enveloping algebra determined by Λ_0.

5.1 The variety defined

Consider the moment map $\Phi : T^*\mathbb{C}^n \to \mathfrak{k}^*$ for the action of K on $T^*\mathbb{C}^n$. As noted in Remark 3.1, one way to define Φ is by taking the associated graded of the quantized moment map. More concretely, for any $(z, w) \in T^*\mathbb{C}^n$ and $k_1h_1 + \ldots + k_nh_n \in \mathfrak{k} \subset \mathfrak{t}$,

$$\Phi(z, w)(k_1h_1 + \ldots + k_nh_n) = k_1z_1w_1 + \ldots + k_nz_nw_n.$$

The parameter η lies in $W_\mathbb{Z}/\Lambda_0$, which is naturally identified with the character lattice of K (Section 3.2). Consider the subset $\mathfrak{X} \subset T^*\mathbb{C}^n$ of η-semistable points in the sense of geometric invariant theory. Then the **hypertoric variety** $\mathfrak{M} = \mathfrak{M}(X)$ is defined as the categorical quotient of $\mathfrak{X} \cap \Phi^{-1}(0)$ by K. Let $\pi : \mathfrak{X} \cap \Phi^{-1}(0) \to \mathfrak{M}$ be the quotient map. Since η is regular, the action of K on $\mathfrak{X} \cap \Phi^{-1}(0)$ is locally free, the quotient is geometric (that is, the fibers of π coincide with the orbits of K), and $\mathfrak{M}$ is a symplectic orbifold. It is smooth if and only if Λ_0 is unimodular [BD00, 3.3].

Let $\mathfrak{M}_0$ be the categorical quotient of $\Phi^{-1}(0)$ by K. The inclusion of $\mathfrak{X} \cap \Phi^{-1}(0)$ into $\Phi^{-1}(0)$ induces a projective map $\nu : \mathfrak{M} \to \mathfrak{M}_0$. Moreover, $\mathfrak{M}$ comes equipped with a line bundle

$$\mathfrak{L}_\eta := \mathfrak{X} \cap \Phi^{-1}(0) \times_K \mathbb{C}_\eta,$$

where K acts on $\mathbb{C}_\eta$ via the character η. This bundle is ample, and very ample relative to $\mathfrak{M}_0$. If $\mathfrak{M}$ is smooth, then ν is a resolution, and the Grauert-Riemenschneider theorem tells us that the higher cohomology of the holomorphic structure sheaf[10] $\mathfrak{S}_{\mathfrak{M}}$ vanishes [Kal, 2.1]. This is the first hint that $\mathfrak{M}$ is morally close to being affine.

Example 5.1. If $K \subset T$ is the one-dimensional diagonal subtorus as in Example 4.11, then $\mathfrak{M} \cong T^*\mathbb{C}P^{n-1}$ for any choice of regular η, and $\mathfrak{M}_0$ is obtained from $\mathfrak{M}$ by collapsing the zero section. If $K \subset T$ is the $(n-1)$-dimensional determinant 1 subtorus as in Example 4.12, then $\mathfrak{M}_0$ is isomorphic to the quotient of $\mathbb{C}^2$ by the symplectic action of $\mathbb{Z}/n\mathbb{Z}$, and for any choice of regular η, $\mathfrak{M}$ is a minimal resolution (all of which are isomorphic).

Let $\mathbb{S} := \mathbb{C}^\times$ act on $T^*\mathbb{C}^n$ by inverse scalar multiplication; that is, $s \cdot (z, w) := (s^{-1}z, s^{-1}w)$. This induces an action on both $\mathfrak{M}$ and $\mathfrak{M}_0$, and the map ν is $\mathbb{S}$-equivariant. This action does not preserve the symplectic form; rather we have $s \cdot \omega = s^2\omega$ for all $s \in \mathbb{S}$. Let $o \in \mathfrak{M}_0$ be the point represented by $0 \in T^*\mathbb{C}^n$, so that

$$\lim_{\mathbb{S} \ni s \to \infty} s \cdot p_0 = o$$

for all $p_0 \in \mathfrak{M}_0$. It follows that, for all $p \in \mathfrak{M}$, the limit $\lim_{\mathbb{S} \ni s \to \infty} s \cdot p$ exists and lies in $\nu^{-1}(o)$.

The hypertoric variety $\mathfrak{M}$ admits a Hamiltonian action of T/K, induced by the action of T on $\mathbb{C}^n$, which commutes with the action of $\mathbb{S}$. Let $\mathbb{T} := \mathbb{C}^\times$ act on $\mathfrak{M}$ via ξ, which is naturally a cocharacter of T/K (Section 3.2). The following proposition is straightforward.

Proposition 5.2. *The hypertoric variety $\mathfrak{M}_0$ is affine, and for any central character λ of the hypertoric enveloping algebra U there is a natural isomorphism*

$$\operatorname{gr} U_\lambda \cong \mathbb{C}[\mathfrak{M}_0] \cong \mathbb{C}[\mathfrak{M}].$$

The $\mathbb{Z}$-grading on U_λ given in Section 4.1 descends to the grading on $\mathbb{C}[\mathfrak{M}]$ induced by the action of $\mathbb{T}$. The $\mathbb{N}$-grading on $\operatorname{gr} U_\lambda$ induced by the associated graded construction coincides with the grading on $\mathbb{C}[\mathfrak{M}]$ induced by the action of $\mathbb{S}$.

For any quantized polarized arrangement $\mathbf{X} = (\Lambda_0, \mathbf{\Lambda}, \xi)$, let $L := \oplus_{a \in \mathcal{P}_{\mathbf{\Lambda},\xi}} L_\alpha^{\mathbf{\Lambda}}$ be the direct sum of the simple objects, and consider the **Yoneda algebra**

$$\bigoplus_{i \geq 0} \operatorname{Ext}^i(L, L).$$

[10] Here and elsewhere we use the unconventional notation $\mathfrak{S}$ for the structure sheaf of a variety, since the standard symbol $\mathcal{O}$ is being used for the category.

(This is the algebra whose module category is Koszul dual to $\mathcal{O}(\mathbf{X})$, as explained in Section 8.4.) The following result can be taken as a first piece of evidence that there is a strong relationship between $\mathcal{O}(\mathbf{X})$ and the geometry of hypertoric varieties.

Theorem 5.3. *If $\mathbf{X}$ is integral and regular and linked to X, then the center of the Yoneda algebra of $\mathcal{O}(\mathbf{X})$ is isomorphic as a graded algebra to the cohomology of $\mathfrak{M}(X)$.*

Proof. By Theorem 4.8, $\mathcal{O}(\mathbf{X})$ is equivalent to the module category of the ring $A(X)$ introduced in [BLPWb]. By [BLPWb, 5.24], the Yoneda algebra of $\mathcal{O}(\mathbf{X})$ is isomorphic to the ring $B(X)$ introduced in [BLPWb]. By [BLPWb, 4.7 & 4.16], the center of $B(X)$ is isomorphic to the cohomology ring of $\mathfrak{M}(X)$. □

Remark 5.4. By Theorem 4.9, every regular block $\mathcal{O}(\mathbf{X})$ of hypertoric category $\mathcal{O}$ is equivalent to a regular integral block $\mathcal{O}(\mathbf{X}')$ for some different quantized polarized arrangement $\mathbf{X}'$. Thus Theorem 5.3 gives a characterization of the center of the Yoneda algebra of an arbitrary regular block.

5.2 The relative core

Consider the reduced Lagrangian subvariety

$$\mathfrak{M}^+ := \{p \in \mathfrak{M} \mid \lim_{\mathbb{T} \ni t \to 0} t \cdot p \text{ exists}\}.$$

It contains the **core** $\nu^{-1}(o)$ (because the $\mathbb{S}$ and $\mathbb{T}$ actions commute and ν is projective) but is strictly bigger. We call $\mathfrak{M}^+$ the **relative core** of $\mathfrak{M}$.

Another Lagrangian subvariety of $\mathfrak{M}$ that has appeared in the literature is the **extended core**, which is equal to the zero set of the moment map for the T/K-action. The extended core contains the relative core; in fact, it is equal to the union of all of the (finitely many) possible relative cores, as ξ varies among generic parameters.

Proposition 5.5 ([BD00, 6.5])**.** *The extended core of $\mathfrak{M}$ is isomorphic to a union of toric varieties C_α given by the polyhedra Δ_α for $\alpha \in \mathcal{F}_\eta$ glued along toric subvarieties as prescribed by the incidences of the polyhedra. The relative core is the subvariety of the extended core corresponding to those α in $\mathcal{P}_{\eta,\xi} \subset \mathcal{F}_\eta$, and the core is the subvariety corresponding to those α that are totally bounded.*

As in Section 3.1, let $x_1, \dots, x_n$ be coordinates on $\mathbb{C}^n$, and let $y_1, \dots, y_n$ be dual coordinates, so that

$$\mathbb{C}[T^*\mathbb{C}^n] = \mathbb{C}[x_1, y_1, \dots, x_n, y_n].$$

The subvariety $C_\alpha \subset \mathfrak{M}$ of Proposition 5.5 is cut out by the equations

$$y_i = 0 \text{ if } \alpha(i) = + \qquad \text{and} \qquad x_i = 0 \text{ if } \alpha(i) = -. \tag{5}$$

If $\alpha \notin \mathcal{F}_\eta$, then these equations will have no solutions in $\mathfrak{M}$. Let $C_{0,\alpha} \subset \mathfrak{M}_0$ be defined by the same equations. If α is feasible, then $C_{0,\alpha}$ is the image of C_α along the map $\nu : \mathfrak{M} \to \mathfrak{M}_0$. However $C_{0,\alpha}$ is nonempty even if α is infeasible; in particular, it always contains the point o.

If M is a finitely-generated U_λ-module, we can put a filtration on M which is compatible with the filtration on U_λ by choosing a generating set for M. We can then consider the support $\mathrm{Supp}(\mathrm{gr}\, M) \subset \mathfrak{M}_0$, where we think of $\mathrm{gr}\, M$ as a sheaf on $\mathrm{gr}\, U_\lambda \cong \mathbb{C}[\mathfrak{M}_0]$. Standard arguments show that the support of this sheaf is independent of the choice of filtration.[11]

Lemma 5.6. *If $\mathbf{\Lambda} \subset \mathbf{W}$ is an integral Λ_0-orbit and $\alpha \in \mathcal{F}_{\mathbf{\Lambda}}$, then $C_{0,\alpha} = \mathrm{Supp}\left(\mathrm{gr}\, L_\alpha^{\mathbf{\Lambda}}\right)$.*

Proof. Let $W_\alpha \subset T^*\mathbb{C}^n$ be the Lagrangian subspace defined by the equations (5), so that $\mathbb{C}[C_{0,\alpha}] = \mathbb{C}[W_\alpha]^K$. If we put the filtration on L_α induced by the image of $1 \in \mathbb{D}$, then we get an isomorphism $\mathrm{gr}\, L_\alpha \cong \mathbb{C}[W_\alpha]$ of modules over $\mathrm{gr}\, \mathbb{D} \cong \mathbb{C}[T^*\mathbb{C}^n]$. This induces a filtration on $L_\alpha^{\mathbf{\Lambda}}$, making $\mathrm{gr}\, L_\alpha^{\mathbf{\Lambda}}$ a $\mathrm{gr}\, U_\lambda$-submodule of $\mathrm{gr}\, L_\alpha$. It follows immediately that $\mathrm{Supp}(\mathrm{gr}\, L_\alpha^{\mathbf{\Lambda}}) \subset C_{0,\alpha}$.

For the other inclusion, note that $\mathrm{gr}\, L_\alpha^{\mathbf{\Lambda}}$ will be isomorphic to some nonzero K-isotypic component $\mathbb{C}[W_\alpha]^{K,\theta}$ of $\mathbb{C}[W_\alpha]$. There is an injection $\mathbb{C}[W_\alpha]^K \hookrightarrow \mathbb{C}[W_\alpha]^{K,\theta}$ given by multiplication by any nonzero polynomial in $\mathbb{C}[W_\alpha]^{K,\theta}$, so we have $C_{0,\alpha} \subset \mathrm{Supp}(\mathrm{gr}\, L_\alpha^{\mathbf{\Lambda}})$. □

We end this section with two technical lemmas about $\mathfrak{M}^+$. Let $\mathfrak{J}$ be the ideal generated by functions on $\mathfrak{M}$ which are eigenvectors for the $\mathbb{S} \times \mathbb{T}$-action of positive $\mathbb{S}$-weight and non-negative $\mathbb{T}$-weight.

Lemma 5.7. *The relative core $\mathfrak{M}^+ \subset \mathfrak{M}$ is the vanishing locus of $\mathfrak{J}$.*

Proof. Let $f \in \mathfrak{J}$ be an eigenvector of positive $\mathbb{S}$-weight and non-negative $\mathbb{T}$-weight. Then f vanishes on $\mathbb{S}$-fixed points, and the core (being projective) contains at least one such point. Thus f vanishes on the entire core. For any $p \in \mathfrak{M}^+$,

$$f(p) = \lim_{\mathbb{T} \ni t \to 0} (t \cdot f)(t \cdot p) = 0,$$

since $t \cdot f$ is approaching either f or 0, and $t \cdot p$ is approaching an element of the core. Thus f vanishes on all of $\mathfrak{M}^+$, so $\mathfrak{M}^+$ is contained in the vanishing locus of $\mathfrak{J}$.

For the other inclusion, consider the ideal $\mathfrak{J}'$ generated by functions of *positive* $\mathbb{T}$-weight. Since all points of $\mathfrak{M}_0$ limit to o under the $\mathbb{S}$-action, all functions on $\mathfrak{M}$ have non-negative $\mathbb{S}$-weight, and the only functions with $\mathbb{S}$-weight zero are the constant functions, so we have $\mathfrak{J}' \subset \mathfrak{J}$. Suppose that $p \in \mathfrak{M}$ lies in the vanishing locus of $\mathfrak{J}$, and therefore of $\mathfrak{J}'$. Then for every function $f \in \mathbb{C}[\mathfrak{M}]$, the limit

$$f\left(\lim_{t \to 0} t \cdot p\right) = \lim_{t \to 0} (t^{-1} \cdot f)(p)$$

exists. (If f has non-positive $\mathbb{T}$-weight the right-hand side clearly exists, and if f has positive $\mathbb{T}$-weight then $t^{-1} \cdot f \in \mathfrak{J}$, so the right-hand side is equal to zero.) Since $\mathfrak{M}$ is projective over the affine variety $\mathfrak{M}_0$, this implies that $\lim_{t \to 0} t \cdot p$ exists, and therefore that $p \in \mathfrak{M}^+$. □

[11] The reader is cautioned not to confuse the support of $\mathrm{gr}\, M$, which is a subvariety of $\mathfrak{M}_0$, with the support of M as defined in Section 3.3, which is a subset of $\mathbf{V}$.

Let $f_{\mathbb{T}} : \mathfrak{M} \to \mathfrak{t}^* \cong \mathbb{C}$ be the moment map for the $\mathbb{T}$-action on $\mathfrak{M}$, where $\mathfrak{t}$ is the Lie algebra of $\mathbb{T}$. We say a coherent sheaf F on $\mathfrak{M}$ is **$\mathfrak{t}$-equivariant** if it is equipped with an endomorphism $d : F \to F$ such that, for all meromorphic sections γ and functions f, we have $d(f\gamma) = \{f_{\mathbb{T}}, f\}\gamma + fd(\gamma)$. Here $\{\,,\}$ is the Poisson bracket induced by the symplectic form on $\mathfrak{M}$, so $\{f_{\mathbb{T}}, f\} = \frac{d}{dt}(t \cdot f)|_{t=0}$.

Lemma 5.8. *For any $\mathfrak{t}$-equivariant coherent sheaf F on $\mathfrak{M}$ which is set-theoretically supported on $\mathfrak{M}^+$, the $\mathfrak{t}$-action on $\Gamma(\mathfrak{M}; F)$ is locally finite, the generalized eigenspaces are finite dimensional, and the eigenvalues that appear are bounded above.*

Proof. The conclusion holds for a sheaf if it holds for the successive quotients of a filtration of the sheaf, so we may assume that F is scheme-theoretically supported on one of the components C_α of $\mathfrak{M}^+$. Now consider the pushforward $\nu_* F$ to $\mathfrak{M}_0$. This is a coherent sheaf scheme-theoretically supported on $C_{0,\alpha} = \nu(C_\alpha)$, and so there is a surjective $\mathfrak{t}$-equivariant morphism $N \otimes \mathfrak{S}_{C_{0,\alpha}} \to \nu_* F$ for some finite dimensional $\mathfrak{t}$-representation N. Since $\mathfrak{M}_0$ is affine, the sections functor is exact, so we need only prove the result for the $\mathfrak{t}$-action on the ring of functions on $C_{0,\alpha}$. This follows from the fact that $\lim_{\mathbb{T} \ni t \to 0} t \cdot p_0 = o$ for all $p_0 \in C_{0,\alpha}$. □

5.3 Quantizing the hypertoric variety

In this section we explain how the algebra U_λ arises as sections of a quantization of the structure sheaf of $\mathfrak{M}$. Let

$$\mathcal{D} := \mathfrak{S}_{T^*\mathbb{C}^n}(\!(\hbar^{1/2})\!) \qquad \text{and} \qquad \mathcal{D}(0) := \mathfrak{S}_{T^*\mathbb{C}^n}[[\hbar^{1/2}]] \subset \mathcal{D},$$

where again $\mathfrak{S}_{T^*\mathbb{C}^n}$ denotes the holomorphic structure sheaf of $T^*\mathbb{C}^n$ and $\hbar^{1/2}$ is a formal parameter. Let χ be the Poisson bivector on $T^*\mathbb{C}^n$. The Moyal product on $\mathcal{D}$ is defined by

$$f \star g = m \circ e^{\hbar\chi/2}(f \otimes g),$$

where $m : \mathfrak{S}_{T^*\mathbb{C}^n} \otimes_{\mathbb{C}} \mathfrak{S}_{T^*\mathbb{C}^n} \to \mathfrak{S}_{T^*\mathbb{C}^n}$ is the multiplication map. Thus $\mathcal{D}$ is a sheaf of associative $K \times \mathbb{S}$-equivariant $\mathbb{C}(\!(\hbar^{1/2})\!)$-algebras, where $\mathbb{S}$ acts on $\hbar^{1/2}$ with weight 1, and therefore on $\hbar$ with weight 2. (This reflects the fact that $\mathbb{S}$ acts on the symplectic form of $T^*\mathbb{C}^n$ with weight 2.) This sheaf is flat over $\mathbb{C}[[\hbar^{1/2}]]$, and we have natural isomorphisms

$$\mathcal{D}(0)/\hbar^{1/2}\mathcal{D}(0) \cong \mathfrak{S}_{T^*\mathbb{C}^n} \qquad \text{and} \qquad \Gamma_{\mathbb{S}}(\mathcal{D}) \cong \mathbb{D},$$

where $\Gamma_{\mathbb{S}}$ is the functor that takes $\mathbb{S}$-invariant global sections [BK, 2.6]. The second isomorphism is given by sending the element $\hbar^{-1/2}x_i \in \Gamma_{\mathbb{S}}(\mathcal{D})$ to $x_i \in \mathbb{D}$ and $\hbar^{-1/2}y_i \in \Gamma_{\mathbb{S}}(\mathcal{D})$ to ∂_i.

The reduction procedure we applied to obtain U_λ from $\mathbb{D}$ can be "sheafified" as follows. Let $\lambda\colon Z(U) \to \mathbb{C}$ be any central character. Since $\ker\lambda \subset Z(U) \subset U \subset \mathbb{D}$, we can regard elements of $\ker\lambda$ as $\mathbb{S}$-invariant sections of $\mathcal{D}$. Let

$$\mathcal{Y}_\lambda := \mathcal{D}|_{\mathfrak{x}} \Big/ \ker\lambda \cdot \mathcal{D}|_{\mathfrak{x}} \qquad \text{and} \qquad \mathcal{Y}_\lambda(0) := \mathcal{D}(0)|_{\mathfrak{x}} \Big/ \Big(\mathcal{D}(0) \cap \ker\lambda \cdot \mathcal{D}|_{\mathfrak{x}}\Big).$$

Thus $\mathcal{Y}_\lambda$ is a $\mathbb{S} \times K$-equivariant sheaf of right $\mathcal{D}$-modules on $\mathfrak{X}$, and $\mathcal{Y}_\lambda(0)$ is a $\mathbb{S} \times K$-equivariant $\mathcal{D}(0)$-lattice. Both are supported on $\mathfrak{X} \cap \Phi^{-1}(0)$, thus

$$\mathcal{U}_\lambda := (\pi_* \mathcal{E}nd_{\mathcal{D}}(\mathcal{Y}_\lambda))^K \qquad \text{and} \qquad \mathcal{U}_\lambda(0) := \left(\pi_* \mathcal{E}nd_{\mathcal{D}(0)}(\mathcal{Y}_\lambda(0))\right)^K$$

are $\mathbb{S}$-equivariant sheaves of algebras on $\mathfrak{M}$. Consider also the sheaf of $(\mathcal{U}_\lambda, \mathcal{U}_{\lambda'})$-bimodules

$${}_\lambda\mathcal{T}_{\lambda'} := (\pi_* \mathcal{H}om_{\mathcal{D}}(\mathcal{Y}_\lambda, \mathcal{Y}_{\lambda'}))^K$$

and its $(\mathcal{U}_\lambda(0), \mathcal{U}_{\lambda'}(0))$-lattice

$${}_\lambda\mathcal{T}_{\lambda'}(0) := \left(\pi_* \mathcal{H}om_{\mathcal{D}(0)}(\mathcal{Y}_\lambda(0), \mathcal{Y}_{\lambda'}(0))\right)^K.$$

Just as $\mathcal{U}_\lambda(0)/\hbar^{1/2}\mathcal{U}_\lambda(0) \cong \mathfrak{S}_{\mathfrak{M}}$, we have that

$${}_\lambda\mathcal{T}_{\lambda'}(0)/\hbar^{1/2} \cdot {}_\lambda\mathcal{T}_{\lambda'}(0) \cong \mathfrak{L}_{\lambda'-\lambda} = \mathfrak{X} \cap \Phi^{-1}(0) \times_K \mathbb{C}_{\lambda-\lambda'}.$$

Proposition 5.9. *We have isomorphisms* $\Gamma_{\mathbb{S}}(\mathcal{U}_\lambda) \cong U_\lambda$ *and* $\Gamma_{\mathbb{S}}({}_\lambda\mathcal{T}_{\lambda'}) \cong {}_\lambda T_{\lambda'}$.

Proof. The isomorphism $\mathbb{D} \cong \Gamma_{\mathbb{S}}(\mathcal{D})$ induces a map $Y_\lambda \to \Gamma(\mathfrak{X}, \mathcal{Y}_\lambda)^{\mathbb{S}}$, where $Y_\lambda = \mathbb{D}/\langle \ker \lambda \rangle \mathbb{D}$ is the $\mathbb{D}$-module introduced in Remark 3.2. Since the complement of $\mathfrak{X}$ in $T^*\mathbb{C}^n$ has codimension at least 2, this map is an isomorphism. Since 1 is an $\mathbb{S}$-invariant section of $\mathcal{Y}_\lambda$, evaluating at 1 gives a map $\Gamma_{\mathbb{S}}(\mathcal{U}_\lambda) \to \mathrm{End}_{\mathbb{D}}(Y_\lambda) = U_\lambda$, which is easily seen to be an isomorphism. □

Following [KR08], we call a $\mathcal{U}_\lambda$-module **good** if it admits a $\mathcal{U}_\lambda(0)$-lattice and **coherent** if it is locally finitely generated and every locally defined, locally finitely generated submodule is locally of finite presentation. Let $\mathrm{Mod}_{\mathbb{S}}^{\mathrm{good}}(\mathcal{U}_\lambda)$ denote the category of good, coherent, $\mathbb{S}$-equivariant $\mathcal{U}_\lambda$-modules.

Remark 5.10. There are heuristic reasons to treat $\mathrm{Mod}_{\mathbb{S}}^{\mathrm{good}}(\mathcal{U}_\lambda)$ as a version of the Fukaya category of $\mathfrak{M}$. One justification comes from the physical theory of A-branes, which the Fukaya category attempts to formalize. Kapustin and Witten [KW07] suggest that on a hyperkähler manifold such as $\mathfrak{M}$ there are objects in an enlargement of the Fukaya category which correspond not just to Lagrangian submanifolds, but higher dimensional coisotropic submanifolds. In particular, there is a object in this category supported on all of $\mathfrak{M}$ called the **canonical coisotropic brane**. Following the prescription of Kapustin and Witten further shows that endomorphisms of this object are exactly the algebra U_λ, or if interpreted sheaf theoretically, $\mathcal{U}_\lambda$. This leads us to conjecture that there is a natural equivalence between $\mathrm{Mod}_{\mathbb{S}}^{\mathrm{good}}(\mathcal{U}_\lambda)$ and the Fukaya category of $\mathfrak{M}$, twisted by the B-field $H^2(\mathfrak{M}; \mathbb{C}^\times)$ determined[12] by λ. When $\mathfrak{M}$ is replaced by the cotangent bundle of an arbitrary real analytic manifold an analogous statement is proven by Nadler and Zaslow [NZ09].

[12]The group $H^2(\mathfrak{M}; \mathbb{C}^\times)$ is isomorphic to $H^2(\mathfrak{M}; \mathbb{C})/H^2(\mathfrak{M}; \mathbb{Z})$. The vector space $H^2(\mathfrak{M}; \mathbb{C})$ is isomorphic to $\mathfrak{k}^* \cong W/V_0$ via the Kirwan map. The parameter λ determines a V_0-orbit in $\mathbf{W}$, which determines a V_0-orbit in W up to translation by the lattice $W_{\mathbb{Z}}/\Lambda_0 \cong H^2(\mathfrak{M}; \mathbb{Z})$.

6 Localization

As in Section 5, let $X = (\Lambda_0, \eta, \xi)$ be a regular polarized arrangement, let $\mathfrak{M}$ be the hypertoric variety associated to X, let U be the hypertoric enveloping algebra associated to Λ_0. Fix a central character λ of U, and let $\mathbf{V} \subset \mathbf{W}$ be the corresponding V_0-orbit. In this section we show that the infinitesimal block $\mathcal{O}_\lambda$ of hypertoric category $\mathcal{O}$ (Definition 4.1) can be "localized" to a certain category of sheaves of modules over the quantized structure sheaf $\mathcal{U}_\lambda$, supported on the relative core $\mathfrak{M}^+ \subset \mathfrak{M}$. When Λ_0 is unimodular and $\mathbf{X}$ an integral quantized polarized arrangement linked to X, we use localization to obtain a topological interpretation of the Grothendieck group of $\mathcal{O}(\mathbf{X})$.

6.1 The localization functor

We define the functor

$$\mathrm{Loc} : U_\lambda\text{−}\mathsf{mod} \to \mathrm{Mod}_{\mathbb{S}}^{\mathrm{good}}(\mathcal{U}_\lambda)$$

by putting $\mathrm{Loc}(M) := \mathcal{U}_\lambda \otimes_{U_\lambda} M$ for any U_λ-module M. We note that any good filtration of M induces a $\mathcal{U}_\lambda(0)$-lattice for $\mathrm{Loc}(M)$, namely $\mathcal{U}_\lambda(0) \otimes_{R(U_\lambda)} R(M)$, where $R(U_\lambda)$ and $R(M)$ are the Rees algebra and Rees module of U_λ and M, respectively. The functor Loc is adjoint to the $\mathbb{S}$-invariant global sections functor

$$\Gamma_{\mathbb{S}} : \mathrm{Mod}_{\mathbb{S}}^{\mathrm{good}}(\mathcal{U}_\lambda) \to U_\lambda\text{−}\mathsf{mod}.$$

Just as a choice of a good filtration of M induces a particular lattice in $\mathrm{Loc}(M)$, a choice of lattice $\mathcal{M}(0) \subset \mathcal{M} \in \mathrm{Mod}_{\mathbb{S}}^{\mathrm{good}}(\mathcal{U}_\lambda)$ induces a good filtration of $\Gamma_{\mathbb{S}}(\mathcal{M})$ with $\operatorname{gr}\Gamma_{\mathbb{S}}(\mathcal{M}) \cong \Gamma\big(\mathcal{M}(0)/\hbar^{1/2}\mathcal{M}(0)\big)$.

The following theorem, based on the results of Kashiwara and Rouquier [KR08] and their adaptation to the hypertoric case by Bellamy and Kuwabara [BK], is an analogue of the localization theorem of Beilinson and Bernstein [BB81].

Theorem 6.1. *If Λ_0 is unimodular and $\mathcal{F}_{\mathbf{\Lambda}} = \mathcal{F}_{\mathbf{\Lambda}+r\eta}$ for all Λ_0-orbits $\mathbf{\Lambda} \subset \mathbf{V}$ and integers $r \geq 0$, then the functors $\Gamma_{\mathbb{S}}$ and* Loc *are quasi-inverse equivalences.*

Proof. This follows from [BK, 3.5] and Proposition 3.8. Note that a very similar theorem with slightly different hypotheses is proven in [BK, 5.8]. □

6.2 The category $\mathcal{Q}_\lambda$

Our next task is to determine which objects of $\mathrm{Mod}_{\mathbb{S}}^{\mathrm{good}}(\mathcal{U}_\lambda)$ are sent by the sections functor $\Gamma_{\mathbb{S}}$ to $\mathcal{O}_\lambda$, and conversely to determine the image of the localizaation functor restricted to $\mathcal{O}_\lambda$. Note that we do not assume that $\Gamma_{\mathbb{S}}$ and Loc are equivalences in this section.

Definition 6.2. Let $\mathcal{Q}_\lambda \subset \mathrm{Mod}_{\mathbb{S}}^{\mathrm{good}}(\mathcal{U}_\lambda)$ be the full subcategory consisting of objects $\mathcal{M}$ satisfying the following two additional conditions:

- there exists a $\mathcal{U}_\lambda(0)$-lattice $\mathcal{M}(0) \subset \mathcal{M}$ that is preserved by the action of $\hat{\xi} \in U_\lambda$[13]

[13]Note that $h\hat{\xi}$ is contained in $U_\lambda(0)$, but $\hat{\xi}$ is not, so this condition is not vacuous.

- $\mathcal{M}$ is supported on the relative core $\mathfrak{M}^+ \subset \mathfrak{M}$.

Theorem 6.3. *If $\mathcal{M} \in \mathcal{Q}_\lambda$, then $\Gamma_{\mathbb{S}}(\mathcal{M}) \in \mathcal{O}_\lambda$.*

Proof. Choose a $\mathcal{U}_\lambda(0)$-lattice $\mathcal{M}(0)$ that is preserved by the action of $\hat{\xi}$, and let $F := \mathcal{M}(0)/\hbar^{1/2}\mathcal{M}(0)$. The action of $\hat{\xi}$ on $\mathcal{M}(0)$ induces a $\mathfrak{t}$-equivariant structure on F; Lemma 5.8 tells us that $\Gamma(\mathfrak{M}, F)$ decomposes into finite dimensional generalized eigenspaces and that the eigenvalues that appear are bounded above. Since $\Gamma(\mathfrak{M}, F) \cong \operatorname{gr}\Gamma_{\mathbb{S}}(\mathcal{M})$ and the $\mathfrak{t}$-equivariant structure on F is induced by the action of $\hat{\xi}$ on $\mathcal{M}$, this implies that $\Gamma_{\mathbb{S}}(\mathcal{M})$ decomposes into finite dimensional generalized eigenspaces for the action of $\hat{\xi}$ and that the eigenvalues that appear are bounded above. Thus $\Gamma_{\mathbb{S}}(\mathcal{M})$ lies in $\mathcal{O}_\lambda$ by Lemma 4.4. □

Theorem 6.4. *If $M \in \mathcal{O}_\lambda$, then $\operatorname{Loc}(M) \in \mathcal{Q}_\lambda$.*

Proof. We need to verify that $\operatorname{Loc}(M)$ satisfies the two conditions of Definition 6.2. The first condition is easy; if we choose a filtration of M generated by weight vectors, then the associated lattice $\operatorname{Loc}(M)(0) \subset \operatorname{Loc}(M)$ will be preserved by the action of $\hat{\xi}$. Thus we only need to show that $\operatorname{Loc}(M)$ is supported on the relative core.

Let f be a global function on $\mathfrak{M}$ which is an $\mathbb{S} \times \mathbb{T}$-weight vector of non-negative $\mathbb{T}$-weight and positive $\mathbb{S}$-weight. Let $\mathfrak{M}_f$ be the subset of $\mathfrak{M}$ where f is not zero, and $i_f : \mathfrak{M}_f \to \mathfrak{M}$ be the inclusion. Then $i_f^{-1}\operatorname{Loc}(M)(0)$ is a sheaf of flat $\mathbb{C}[[\hbar^{1/2}]]$-algebras with fiber $\mathfrak{S}_{\mathfrak{M}_f} \otimes_{\mathbb{C}[\mathfrak{M}]} \operatorname{gr}(M)$ at $\hbar^{1/2} = 0$. Lemma 4.4 tells us the $\mathbb{T}$-weight spaces of $\operatorname{gr}(M)$ are finite dimensional and the weights that appear are bounded above. If f has positive $\mathbb{T}$-weight, then it acts nilpotently on $\operatorname{gr}(M)$ by the boundedness of the weights that appear. If f has $\mathbb{T}$-weight 0, then it still acts nilpotently by the finite dimensionality of the $\mathbb{T}$-weight spaces and the fact that f has positive $\mathbb{S}$-weight. This implies that $\mathfrak{S}_{\mathfrak{M}_f} \otimes_{\mathbb{C}[\mathfrak{M}]} \operatorname{gr}(M) = 0$, and therefore that $i_f^{-1}\operatorname{Loc}(M)(0) = 0$. Thus the support of $\operatorname{Loc}(M)$ is disjoint from $\mathfrak{M}_f$. The theorem then follows from Lemma 5.7. □

Corollary 6.5. *Suppose that Λ_0 is unimodular and $\mathbf{\Lambda} \subset \mathbf{V}$ is integral. If $\mathbf{X} := (\Lambda_0, \mathbf{\Lambda}, \xi)$ is linked to X, then the functors $\Gamma_{\mathbb{S}}$ and Loc induce quasi-inverse equivalences between $\mathcal{O}(\mathbf{X})$ and $\mathcal{Q}_\lambda$.*

Proof. By Proposition 2.15, we have $\mathcal{F}_{\mathbf{\Lambda}} = \mathcal{F}_{\mathbf{\Lambda}+r\eta}$ for all positive integers r. By unimodularity, we have $I_{\mathbf{\Lambda}'} = \emptyset = I_{\mathbf{\Lambda}'+r\eta}$ for all Λ_0-orbits $\mathbf{\Lambda}' \subset \mathbf{V}$ different from $\mathbf{\Lambda}$. By Remark 4.2, we have $\mathcal{O}(\mathbf{X}) = \mathcal{O}_\lambda$. The result then follows from Theorems 6.1, 6.3, and 6.4. □

Remark 6.6. We have assumed unimodularity of Λ_0 (which is equivalent to the statement that $\mathfrak{M}$ is a manifold rather than an orbifold) so that we can use the results of [KR08] and [BK]. We expect that this condition is not essential.

Example 6.7. Consider the example where $n = 2$ and $K \subset T$ is the diagonal subtorus (see Examples 4.11 and 5.1), so that $\mathfrak{M} \cong T^*\mathbb{C}P^1$. In this case it is possible to choose λ such that

$$U_\lambda = \mathbb{C}\langle x_1\partial_1, x_2\partial_1, x_1\partial_2, x_2\partial_2\rangle/(x_1\partial_1 + x_2\partial_2 = 0)$$

is isomorphic to the ring of polynomial differential operators on $\mathbb{C}P^1$ and $\mathrm{Mod}_{\mathbb{S}}^{\mathrm{good}}(\mathcal{U}_\lambda)$ is equivalent to the category of D-modules on $\mathbb{C}P^1$. The second condition of Definition 6.2 requires the singular support to lie in the relative core, which in this case consists of the zero section along with the fiber at a single point. The first condition is a regularity assumption that is needed for Theorem 6.3 to hold.

To see this, consider the $\mathcal{U}_\lambda$-module $\mathcal{U}_\lambda/\mathcal{U}_\lambda(x_1\partial_2 - 1)$. Restricted to the open subset where $x_1 \neq 0$, this is a non-singular connection on the trivial vector bundle; if $z = x_2/x_1$, then it is simply $\frac{d}{dz} - 1$. Thus this module satisfies the second condition of Definition 6.2. Its $\mathbb{S}$-equivariant sections, however, are isomorphic to $U_\lambda/U_\lambda(x_1\partial_2 - 1)$, and the fact that the image of $1 \in U_\lambda$ is fixed by $x_1\partial_2$ implies that this is not a weight module. This reflects the fact that the corresponding D-module has a non-regular singularity at $z = \infty$.

6.3 The Grothendieck group

Let $d := \dim V_0$. Consider the homomorphism

$$\mathrm{Supp} : K(\mathcal{Q}_\lambda)_{\mathbb{C}} \to H_{\mathbb{T}}^{2d}(\mathfrak{M}; \mathbb{C})$$

given by taking an object $\mathcal{M}$ to the support cycle of the coherent sheaf $\mathcal{M}(0)/\hbar^{1/2}\mathcal{M}(0)$. The fact that this does not depend on the choice of $\mathcal{U}_\lambda(0)$-lattice $\mathcal{M}(0) \subset \mathcal{M}$ is a sheafified version of [Gin86, 1.1.2]. The vector space $H_{\mathbb{T}}^{2d}(\mathfrak{M}; \mathbb{C})$ has a natural nondegenerate pairing given by integrating the product of two classes. Though $\mathfrak{M}$ is not compact, this integral can be defined by formally applying the Atiyah-Bott-Berline-Vergne localization formula; see [HP05, §1] for details. We will refer to this pairing as the **integration pairing** on $H_{\mathbb{T}}^{2d}(\mathfrak{M}; \mathbb{C})$.

Remark 6.8. It may seem more intuitive for the support homomorphism to take values in the group $H_{2d}^{BM}(\mathfrak{M}^+; \mathbb{C})$, since the support cycle of $\mathcal{M}(0)/\hbar^{1/2}\mathcal{M}(0)$ is always a sum of relative core components. In fact, these groups are canonically isomorphic via the isomorphisms

$$H_{2d}^{BM}\big(\mathfrak{M}^+; \mathbb{C}\big) \to H_{2d,\mathbb{T}}^{BM}\big(\mathfrak{M}^+; \mathbb{C}\big) \to H_{2d,\mathbb{T}}^{BM}(\mathfrak{M}; \mathbb{C}) \to H_{\mathbb{T}}^{2d}(\mathfrak{M}; \mathbb{C}).$$

Geometrically, this isomorphism takes the class $[C_\alpha]$ to $[C_\alpha]$. The main reason that we choose to work with $H_{\mathbb{T}}^{2d}(\mathfrak{M}; \mathbb{C})$ is that it is easier to understand the integration pairing on this space; it is also useful for the analysis below of the supports of localizations of standard modules.

Suppose now that the hypotheses of Corollary 6.5 are satisfied. Then it is easy to check that $\mathrm{Supp}[\mathrm{Loc}(L_\alpha^{\mathbf{\Lambda}})] = [C_\alpha]$ for all $\alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi}$. Since we know the multiplicities of each simple module in the standard module $S_\alpha^{\mathbf{\Lambda}}$, we can compute its image under Supp as follows. We will use the localization isomorphism $H_{\mathbb{T}}^{2d}(\mathfrak{M}; \mathbb{C}) \to H_{\mathbb{T}}^{2d}(\mathfrak{M}^{\mathbb{T}}; \mathbb{C})$. A proof of the following result in a more general setting will appear in [BLPWa].

Proposition 6.9. *The class* $\mathrm{Supp}[\mathrm{Loc}(S^{\mathbf{\Lambda}}_\alpha)]$ *restricts to the element of*

$$H^{2d}_{\mathbb{T}}(\mathfrak{M}^{\mathbb{T}};\mathbb{C}) \cong \bigoplus_{\alpha\in\mathcal{P}_{\mathbf{\Lambda},\xi}} \mathrm{Sym}(\mathfrak{t}^*)$$

which is supported at the fixed point p_α *and whose value at* p_α *is the product of the negative weights of the action of* $\mathbb{T}$ *on the tangent space* $T_{p_\alpha}\mathfrak{M}$*.*

Proof. By Proposition 4.14, $\mathrm{Supp}[\mathrm{Loc}(S^{\mathbf{\Lambda}}_\alpha)]$ is equal to the sum of $[C_\beta]$ for all β such that $\Delta_\beta \subset \Sigma_\alpha$. For any $\gamma \in \mathcal{P}_{\mathbf{\Lambda},\xi}$, let $w_\gamma \in V_{\mathbb{R}}$ be the vertex of $\mathcal{H}$ corresponding to γ by Remark 2.6. Then the restriction of $[C_\beta]$ to p_γ is equal to the product of the projections onto $\mathfrak{t}^*$ of the primitive vectors in the directions of the one-dimensional flats of $\mathcal{H}$ passing through w_γ; the signs of these vectors are determined by requiring that they point away from Δ_β. Alternatively, this may be described as the product of the weights of the normal bundle to $C_\beta \subset \mathfrak{M}$ at p_γ. For any $\gamma \neq \alpha$ such that $w_\gamma \in \Sigma_\alpha$, the contributions of the various $[C_\beta]$ to p_γ will cancel. Thus the restriction of $\mathrm{Supp}[\mathrm{Loc}(S^{\mathbf{\Lambda}}_\alpha)]$ to p_α coincides with the restriction of $[C_\alpha]$. Since $T_{p_\alpha}C_\alpha$ is equal to the sum of the positive weight spaces of $T_{p_\alpha}\mathfrak{M}$, the fiber of the normal bundle at p_α is isomorphic to the sum of the negative weight spaces, thus the restriction of $[C_\alpha]$ to p_α is equal to the product of the negative weights. □

Proposition 6.10. *Suppose that* Λ_0 *is unimodular and* $\mathbf{X}$ *is integral and linked to* X*, so that* $\mathcal{O}(\mathbf{X})$ *is equivalent to* $\mathcal{Q}_\lambda$*. The isomorphism* $\mathrm{Supp} : K(\mathcal{Q}_\lambda)_{\mathbb{C}} \to H^{2d}_{\mathbb{T}}(\mathfrak{M};\mathbb{C})$ *takes takes the Euler form to* $(-1)^d$ *times the integration pairing.*

Proof. The classes $\{[\mathrm{Loc}(S^{\mathbf{\Lambda}}_\alpha)] \mid \alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi}\}$ form an orthonormal basis for $K(\mathcal{Q}_\lambda)_{\mathbb{C}}$. The fact that the integration pairing of $\mathrm{Supp}[\mathrm{Loc}(S^{\mathbf{\Lambda}}_\alpha)]$ with $\mathrm{Supp}[\mathrm{Loc}(S^{\mathbf{\Lambda}}_\beta)]$ is zero for $\alpha \neq \beta$ follows from the fact that the restriction of $\mathrm{Supp}[\mathrm{Loc}(S^{\mathbf{\Lambda}}_\alpha)]$ to the fixed point set is supported at p_α. The integration pairing of $\mathrm{Supp}[\mathrm{Loc}(S^{\mathbf{\Lambda}}_\alpha)]$ with itself is equal to the square of the product of the negative weights of the action of $\mathbb{T}$ on $T_{p_\alpha}\mathfrak{M}$ divided by the product of all of the weights. Since the action is symplectic, the weights come in d pairs that each add to zero, so this quotient is equal to $(-1)^d$. □

7 Cells

In this section we define and study the hypertoric analogues of Kazhdan-Lusztig cells in BGG category $\mathcal{O}$. We fix quantized polarized arrangement $\mathbf{X} = (\Lambda_0, \mathbf{\Lambda}, \xi)$, which we assume to be both regular and integral. Recall that the assumption of integrality does not actually lose any generality, since Theorem 4.9 tells us that $\mathcal{O}(\mathbf{X})$ is either trivial (if $\boldsymbol{\mathcal{H}}$ is inessential) or equivalent to $\mathcal{O}(\mathbf{X}')$ for some integral quantized polarized arrangement $\mathbf{X}'$.

7.1 Left cells

Consider a pair of feasible sign vectors $\alpha, \beta \in \mathcal{F}_{\mathbf{\Lambda}}$ along with the corresponding simple objects of $L^{\mathbf{\Lambda}}_\alpha$ and $L^{\mathbf{\Lambda}}_\beta$ of $U_\lambda\text{−}\mathsf{mod}_{\mathbf{\Lambda}}$.

Definition 7.1. We say that $\alpha \overset{L}{\leq} \beta$ if and only if $\operatorname{Ann} L_\beta^{\mathbf{\Lambda}} \subset \operatorname{Ann} L_\alpha^{\mathbf{\Lambda}}$. We say that α and β are in the same **left cell** of $\mathcal{F}_{\mathbf{\Lambda}}$ if $\alpha \overset{L}{\leq} \beta$ and $\beta \overset{L}{\leq} \alpha$.

In order to give a characterization of the left cells of $\mathcal{F}_{\mathbf{\Lambda}}$, we introduce some basic notation and constructions for hyperplane arrangements. If $F \subset V_0$ is a flat of the hyperplane arrangement $\mathcal{H}_0$ (Section 2.2), we let

$$\mathcal{I}_F := \{i \mid F \subset H_{0,i}\} \subset \{1, \dots, n\}$$

be the set indexing the hyperplanes that contain F. We define the **localization**[14] of $\mathcal{H}_0$ at F to be the hyperplane arrangement

$$\mathcal{H}_0^F := \{H_{0,i}/F \mid i \in \mathcal{I}_F\}$$

in the vector space $V_{0,\mathbb{R}}/F$. Similarly, we define the localization of $\boldsymbol{\mathcal{H}}$ at F to be the hyperplane arrangement

$$\boldsymbol{\mathcal{H}}^F := \{H_i^{\pm}/F \mid i \in \mathcal{I}_F\}$$

in the affine space $\mathbf{V}_{\mathbb{R}}/F$.

Any sign vector $\alpha \in \{+,-\}^{\mathcal{I}_F}$ determines a polyhedron $\mathbf{\Delta}_\alpha^F \subset \mathbf{V}_{\mathbb{R}}/F$ given by the equations

$$h_i^+ \geq 0 \text{ for all } i \in \mathcal{I}_F \text{ with } \alpha(i) = + \qquad \text{and} \qquad h_i^- \leq 0 \text{ for all } i \in \mathcal{I}_F \text{ with } \alpha(i) = -.$$

Our assumption that $\mathbf{X}$ is regular implies that every non-empty polyhedron $\mathbf{\Delta}_\alpha^F$ will contain a point in the lattice $\mathbf{\Lambda}/(F \cap \Lambda_0)$. We call a sign vector $\alpha \in \{+,-\}^{\mathcal{I}_F}$ **compact** if $\mathbf{\Delta}_\alpha^F$ is non-empty and compact.[15] We say that F is **coloop-free** if there exists a compact sign vector in $\{+,-\}^{I_F}$ for the localized arrangement $\boldsymbol{\mathcal{H}}^F$. Our assumption of regularity ensures that this definition agrees with the standard notion of coloop-free flats. In [PW07, §2], the authors show that the symplectic leaves of $\mathfrak{M}_0$ are indexed by coloop-free flats; we will denote by $\mathfrak{M}_0^F$ the leaf indexed by F.

For any sign vector $\alpha \in \{+,-\}^n$, let $F_\alpha \subset V_{0,\mathbb{R}}$ be the linear span of the cone $\Delta_{0,\alpha}$ defined in Section 2.4, and let $\mathcal{I}_\alpha := \mathcal{I}_{F_\alpha}$. If α is feasible, then the restriction of α to $\mathcal{I}_\alpha$ is a compact sign vector for the localized arrangement $\boldsymbol{\mathcal{H}}^{F_\alpha}$. In particular, F_α is coloop-free, and every coloop-free flat arises in this manner.

Fix a polarized arrangement linked to $\mathbf{X}$, and let $\mathfrak{M}$ be the associated hypertoric variety. Let $E_\alpha \subset \mathfrak{M}$ be the closure of the unique component of $\nu^{-1}(\mathfrak{M}_0^{F_\alpha})$ that contains the relative core component C_α. Then E_α is cut out of $\mathfrak{M}$ by the equations

$$z_i = 0 \text{ for all } i \in \mathcal{I}_\alpha \text{ with } \alpha(i) = - \qquad \text{and} \qquad w_i = 0 \text{ for all } i \in \mathcal{I}_\alpha \text{ with } \alpha(i) = +. \tag{6}$$

For example, if $F_\alpha = V_{0,\mathbb{R}}$, then $\mathcal{I}_\alpha = \emptyset$, $\mathfrak{M}_0^{F_\alpha}$ is the dense leaf, and $E_\alpha = \mathfrak{M}$. At the other extreme, if $F_\alpha = \{0\}$, then $\mathcal{I}_\alpha = \{1, \dots, n\}$, $\mathfrak{M}_0^{F_\alpha} = \{o\}$, and $E_\alpha = C_\alpha$.

[14] There is no relationship between this notion of localization and the one that is the topic of Section 6.

[15] This definition is closely related to the notion of feasibility and total boundedness in Remark 2.8, but that had to do with ordinary arrangements rather than "doubled" arrangements, which we have here.

Proposition 7.2. *For all $\alpha, \beta \in \mathcal{F}_{\mathbf{\Lambda}}$, the following are equivalent:*

1. $\alpha \overset{L}{\leq} \beta$
2. $\mathcal{I}_\beta \subset \mathcal{I}_\alpha$ *(equivalently* $F_\alpha \subset F_\beta$*) and* α *and* β *agree on* $\mathcal{I}_\beta$
3. $\overline{\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}} \subset \overline{\mathbf{\Delta}_\beta \cap \mathbf{\Lambda}}$*, where the bar denotes Zariski closure*
4. $E_\alpha \subset E_\beta$*.*

Proof. The equivalence of (1) and (3) is proven by Musson and Van den Bergh [MV98, 7.3.1]. The equivalence of (2) and (4) is manifest from Equation (6). To see that (2) and (3) are equivalent, note that $\overline{\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}}$ is equal to the preimage of the image of $\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$ in $\mathbf{V}_\mathbb{R}/F_\alpha$, and the condition (2) is equivalent to the condition that $\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$ and $\mathbf{\Delta}_\beta \cap \mathbf{\Lambda}$ have the same image in $\mathbf{V}_\mathbb{R}/F_\beta$. □

Corollary 7.3. *There is a bijection between the set of left cells of $\mathcal{F}_{\mathbf{\Lambda}}$ and the set of compact sign vectors for various localizations of $\boldsymbol{\mathcal{H}}$.*

Proof. A flat F along with a compact sign vector $\beta \in \{+,-\}^{\mathcal{I}_F}$ indexes the left cell consisting of all α such that $F_\alpha = F$ and the restriction of α to $\mathcal{I}_F$ is equal to β. □

By a theorem of Ginzburg [Gin03, 2.1], the zero set in $\mathfrak{M}_0$ of the associated graded ideal $\operatorname{gr}\operatorname{Ann} L_\alpha^{\mathbf{\Lambda}} \subset \mathbb{C}[\mathfrak{M}_0]$ is the closure of a single symplectic leaf.

Proposition 7.4 ([MV98, 7.4.1])**.** *For any $\alpha \in \mathcal{F}_{\mathbf{\Lambda}}$, the zero set of* $\operatorname{gr}\operatorname{Ann} L_\alpha^{\mathbf{\Lambda}}$ *is equal to the closure of the leaf $\mathfrak{M}_0^{F_\alpha}$. Furthermore, $U_\lambda/\operatorname{Ann} L_\alpha^{\mathbf{\Lambda}}$ is a quantization of the ring of square matrices with coefficients in $\mathbb{C}[\mathfrak{M}_0^{F_\alpha}]$, and the size of the matrices (the Goldie rank of $L_\alpha^{\mathbf{\Lambda}}$) is equal to the number of components of $\overline{\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}}$.*

Remark 7.5. The number of components in the last part of Proposition 7.4 may also be described as the number of lattice points in the polytope $\mathbf{\Delta}_\beta^{F_\alpha}$, where $\beta \in \{+,-\}^{\mathcal{I}_\alpha}$ is the compact sign vector that indexes the left cell in which α lies; indeed, $\overline{\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}}$ is equal to the preimage in $\mathbf{V}$ of this finite set of lattice points. Thus, the Goldie rank polynomial of $L_\alpha^{\mathbf{\Lambda}}$ (which has been studied quite deeply in the Lie theoretic case; see, for example [BBM89]) is equal to the Erhart polynomial of $\mathbf{\Delta}_\beta^{F_\alpha}$.

7.2 Right cells

Suppose that $\mathbf{X} = (\Lambda_0, \mathbf{\Lambda}, \xi)$ and $\mathbf{X}' = (\Lambda_0, \mathbf{\Lambda}', \xi)$ are two regular, integral, quantized polarized arrangements that differ only in that $\mathbf{\Lambda} \neq \mathbf{\Lambda}'$. Because $\mathbf{X}$ and $\mathbf{X}'$ are both integral, we have $\mathcal{B}_{\mathbf{\Lambda},\xi} = \mathcal{B}_{\mathbf{\Lambda}',\xi}$; we will denote this set as $\mathcal{B}_\xi$ (see Remark 2.9). In this section we define right cells using a preorder on $\mathcal{B}_\xi$. Let λ, $\lambda' \colon Z(U) \to \mathbb{C}$ be the characters corresponding to $\mathbf{\Lambda}$,$\mathbf{\Lambda}'$, respectively.

Definition 7.6. We call a $(U_{\lambda'}, U_\lambda)$-bimodule B **Harish-Chandra** if it is finitely generated over both $U_{\lambda'}$ and U_λ, it has a filtration such that $\operatorname{gr} B$ is supported on the diagonal of $\mathfrak{M}_0 \times \mathfrak{M}_0$, and it is the sum of its generalized weight spaces for the adjoint action of $\hat{\xi}$.

We note that if M is a finitely-generated U-module which is the sum of its weight spaces, then its weight spaces are finite dimensional. However, the analogous statement does not necessarily hold for bimodules considered with the adjoint action. For example, the 0-weight space of U_λ, regarded as a bimodule over itself, is $\mathbf{H}_\lambda$.

Example 7.7. The bimodule ${}_{\lambda'}T_\lambda$ from Section 3.5 is Harish-Chandra.

Lemma 7.8. *If B is a Harish-Chandra $(U_{\lambda'}, U_\lambda)$-bimodule, then the functor $B \otimes_{U_\lambda} -$ sends objects of $\mathcal{O}_\lambda$ to objects of $\mathcal{O}_{\lambda'}$.*

Proof. Let $M^{>k}$ denote the finite-dimensional subspace of M consisting of $\hat{\xi}$-weight vectors of weight greater than k. Fix a k such that $M^{>k}$ generates M (such a k must exist by finite generation of M). Let $\{b_1, \dots, b_r\} \subset B$ be a finite set of adjoint weight vectors which generate B both as a left and a right module, let p_i be the weight of b_i, and let $p = \min\{p_1, \dots, p_r\}$. Let

$$S := \sum b_i \otimes M^{>k-p_i+p} \subset B \otimes_{U_\lambda} M.$$

The set S contains the tensor product of a left-generating set for B with a generating set for M, and therefore generates $B \otimes_{U_\lambda} M$. Since the weight spaces of M are finite dimensional and the weights that appear are bounded above (Lemma 4.4), S is finite dimensional. If $u \in U^+$, then for all i, there exist $\{u_1, \dots, u_r\}$ such that $u_j \in U^{\geq p_i - p_j}$ and $ub_i = \sum b_j u_j$. Thus, if $m \in M^{>k-p_i+p}$,

$$u \cdot b_i \otimes m = \sum b_j \otimes (u_j m) \in S.$$

Thus S is U^+-invariant, and $B \otimes_{U_\lambda} M \in \mathcal{O}_\lambda$ by Lemma 4.4. □

Let $\alpha, \beta \in \mathcal{B}_\xi$ be bounded sign vectors.

Definition 7.9. We write $\alpha \overset{R}{\leq} \beta$ if $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ may be chosen such that $\alpha \in \mathcal{F}_{\mathbf{\Lambda}'}$, $\beta \in \mathcal{F}_{\mathbf{\Lambda}}$, and there exists a Harish-Chandra $(U_{\lambda'}, U_\lambda)$-bimodule B such that the simple $U_{\lambda'}$-module $L_\alpha^{\mathbf{\Lambda}'}$ is a composition factor in $B \otimes_{U_\lambda} L_\beta^{\mathbf{\Lambda}}$. We say that α and β lie in the same **right cell** of $\mathcal{B}_\xi$ if $\alpha \overset{R}{\leq} \beta$ and $\beta \overset{R}{\leq} \alpha$.

In order to give a characterization of right cells in $\mathcal{B}_\xi$, we introduce and study a related class of $\mathbb{D}$-modules. Given any $\alpha \in \mathcal{F}_{\mathbf{\Lambda}}$ and any subset $A \subset I_{\mathbf{\Lambda}} = \{1, \dots, n\}$, define $S_{\alpha,A}$ to be the quotient of $\mathbb{D}$ by the left ideal generated by

- ∂_i for all $i \in A$ with $\alpha(i) = +$,
- x_i for all $i \in A$ with $\alpha(i) = -$, and
- $h_i^{\alpha(i)}$ for all $1 \leq i \leq n$.

If $A = \{1, \dots, n\}$ we have $S_{\alpha,A} = L_\alpha$. Moreover, following the notation of Section 4.4, if A is the set of i for which $v_\xi \in H_i^{\alpha(i)}$, then $S_{\alpha,A}$ is the module S_α defined in that section.

Let $\mathbf{\Delta}_{\alpha,A} \subset \mathbf{V}_{\mathbb{R}}$ be the polyhedron defined by the same inequalities that cut out $\mathbf{\Delta}_\alpha$, but only for the indices in A:

$$h_i^+ \geq 0 \text{ for all } i \in A \text{ with } \alpha(i) = + \qquad \text{and} \qquad h_i^- \leq 0 \text{ for all } i \in A \text{ with } \alpha(i) = -.$$

Then we have $\mathrm{Supp}\left((S_{\alpha,A})^{\mathbf{\Lambda}}\right) = \mathbf{\Delta}_{\alpha,A} \cap \mathbf{\Lambda}$. All of the weight spaces of $S_{\alpha,A}$ and are one-dimensional. It follows that $S_{\alpha,A}$ is semisimple over $Z(U)$, and so $(S_{\alpha,A})^{\mathbf{\Lambda}} = \bigoplus_{v \in \mathbf{\Lambda}} (S_{\alpha,A})_v$. For any $\beta \in \mathcal{F}_{\mathbf{\Lambda}}$, the simple module $L_\beta^{\mathbf{\Lambda}}$ appears with non-zero multiplicity in $(S_{\alpha,A})^{\mathbf{\Lambda}}$ if and only if $\mathbf{\Delta}_\beta \subset \mathbf{\Delta}_{\alpha,A}$, which is equivalent to the condition $\beta|_A = \alpha|_A$.

We say that an index $i \in A$ is **active** for α and A if there exists $\beta \in \mathcal{F}_{\mathbf{\Lambda}}$ with $\beta(i) \neq \alpha(i)$ and $\beta|_{A\setminus i} = \alpha|_{A\setminus i}$. In other words, i is active if and only if the i^{th} inequality is actually necessary to cut out the polyhedron $\mathbf{\Delta}_{\alpha,A}$. Let $A_\alpha \subset A$ be the set of indices that are active for α and A. Thus we have $\mathbf{\Delta}_{\alpha,A_\alpha} = \mathbf{\Delta}_{\alpha,A}$, and A_α is the smallest subset of A with this property.

Proposition 7.10. *There is an isomorphism*

$$\mathbb{D} \otimes_U (S_{\alpha,A})^{\mathbf{\Lambda}} \cong S_{\alpha,A_\alpha}.$$

In particular, L_β is a composition factor of $\mathbb{D} \otimes_U (S_{\alpha,A})^{\mathbf{\Lambda}}$ if and only if $\beta|_{A_\alpha} = \alpha|_{A_\alpha}$.

Proof. The natural surjection $S_{\alpha,A_\alpha} \to S_{\alpha,A}$ induces a surjection $(S_{\alpha,A_\alpha})^{\mathbf{\Lambda}} \to (S_{\alpha,A})^{\mathbf{\Lambda}}$. In fact, this surjection is an isomorphism, since both sides have one-dimensional weight spaces and the same support. Thus we need only show that the adjunction map

$$M := \mathbb{D} \otimes_U (S_{\alpha,A_\alpha})^{\mathbf{\Lambda}} \to S_{\alpha,A_\alpha}$$

is an isomorphism. It is certainly surjective, since S_{α,A_α} is generated by its v-weight space for any v in $\mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$, and this set is nonempty since $\alpha \in \mathcal{F}_{\mathbf{\Lambda}}$.

Take some element $\phi \otimes x \in M$ which maps to $1 \in S_{\alpha,A_\alpha}$; we can assume that x lies in a weight space $(S_{\alpha,A_\alpha})_v$ for $v \in \mathbf{\Delta}_\alpha \cap \mathbf{\Lambda}$ and $\phi \in \mathbb{D}_{v_\alpha - v}$, where v_α is the weight of $1 \in S_{\alpha,A_\alpha}$. It is easy to see that $\phi \otimes x$ generates M, so we only need to show that it is killed by the defining ideal of S_{α,A_α}.

The generators $h_i^{\alpha(i)}$ are in the ideal $\mathfrak{I}_{v_\alpha}$, so some power of these generators will kill $\phi \otimes x$. For the other two classes of generators, let $i \in A_\alpha$. We will assume that $\alpha(i) = +$ (the argument for $\alpha(i) = -$ is similar). The weight of $\partial_i(\phi \otimes v)$ is v_β, where $\beta(j) = \alpha(j)$ for all $j \neq i$. Suppose that $\partial_i(\phi \otimes v)$ is nonzero; then it follows that the simple module L_β appears in M with non-zero multiplicity. Since i is an active index, we have $\beta \in \mathcal{F}_\Lambda$, and so $L_\beta^{\mathbf{\Lambda}}$ has non-zero multiplicity in $M^{\mathbf{\Lambda}} = (S_{\alpha,A_\alpha})^{\mathbf{\Lambda}}$, contradicting the fact that $i \in A_\alpha$. □

Next we give several equivalent formulations of the right preorder on $\mathcal{B}_\xi$. As in the case of the left preorder, we have four characterizations: one from the definition, another in terms of flats, a third in terms of polyhedra, and a fourth in terms of subvarieties of a hypertoric variety. (In this section we only need to consider the affine hypertoric variety $\mathfrak{M}_0$, which is completely determined by Λ_0, thus we do not need to choose a polarized arrangement.)

Proposition 7.11. *For all $\alpha, \beta \in \mathcal{B}_\xi$, the following are equivalent:*

1. $\alpha \overset{R}{\leq} \beta$
2. $\mathcal{I}_\beta \subset \mathcal{I}_\alpha$ *(equivalently $F_\alpha \subset F_\beta$) and α and β agree on the complement of $\mathcal{I}_\alpha$*
3. $\Delta_{0,\alpha} \subset \Delta_{0,\beta}$
4. $C_{0,\alpha} \subset C_{0,\beta}$.

Proof. The fact that (4) implies (3) follows from the fact that $\Delta_{0,\alpha}$ is the image of $C_{0,\alpha}$ along the real moment map introduced in [BD00]. The fact that (1) implies (4) is immediate from Lemma 5.6 and Definition 7.6. The equivalence of (2) and (3) is clear from the definition of the flat F_α. Thus we only need to show that (3) implies (1).

Take $\alpha, \beta \in \{+,-\}^n$, and suppose that $\Delta_{0,\alpha} \subset \Delta_{0,\beta}$. Let $J = \{i \mid \alpha(i) = \beta(i)\}$. We define an integral character λ of $Z(U)$ as follows. Recall that, by choosing a parameter $\eta \in W_\mathbb{Z}/\Lambda_0$, we obtain a hyperplane arrangement $\mathcal{H}_\eta$ in the affine space $V_{\eta,\mathbb{R}}$. If we identify $V_{\eta,\mathbb{R}}$ with $V_{0,\mathbb{R}}$ by choosing an origin (this is equivalent to choosing a lift of η to $W_\mathbb{Z} \cong \mathbb{Z}^n$), then $\mathcal{H}_\eta$ is obtained from $\mathcal{H}_0$ by translating the i^{th} hyperplane away from the origin by the i^{th} coordinate of the lift of η. Removing the inequalities indexed by $i \notin J$ does not change the polyhedron $\Delta_{0,\beta}$; this means that a regular parameter η may be chosen such that $\Delta_{\eta,\beta} = \Delta_{0,\beta}$ but $\Delta_{\eta,\gamma}$ is empty for any $\gamma \neq \beta$ with $\gamma|_J = \beta|_J$. Choose a regular integral $\mathbf{\Lambda}$ which is linked to η, and let λ be the corresponding character of $Z(U)$.

Every index which is active for β and $\{1, \ldots, n\}$ lies in J. It follows from Proposition 7.10 that L_α is a composition factor of $\mathbb{D} \otimes_U L_\beta^{\mathbf{\Lambda}}$, so $L_\alpha^{\mathbf{\Lambda}'}$ is a composition factor of $(\mathbb{D} \otimes_U L_\beta^{\mathbf{\Lambda}})^{\mathbf{\Lambda}'}$ for any integral $\mathbf{\Lambda}'$ with $\alpha \in \mathcal{F}_{\mathbf{\Lambda}'}$. By Lemma 3.10 we have

$$(\mathbb{D} \otimes_U L_\beta^{\mathbf{\Lambda}})^{\mathbf{\Lambda}'} = (\mathbb{D} \otimes_U L_\beta^{\mathbf{\Lambda}})^{\lambda'} = {}_{\lambda'}T_\lambda \otimes_{U_\lambda} L_\beta^{\mathbf{\Lambda}},$$

where λ' is the central character determined by $\mathbf{\Lambda}'$. Thus $\alpha \overset{R}{\leq} \beta$, as desired. □

Corollary 7.12. *The right cells of $\mathcal{B}_\xi$ are in bijection with the ξ-bounded faces of $\mathcal{H}_0$.*

Proof. The ξ-bounded faces of $\mathcal{H}_0$ are exactly the polyhedra $\{\Delta_{\alpha,0} \mid \alpha \in \mathcal{B}_\xi\}$. □

7.3 Two-sided cells

Definition 7.13. We define a third preorder on $\mathcal{P}_{\mathbf{\Lambda},\xi} = \mathcal{F}_{\mathbf{\Lambda}} \cap \mathcal{B}_\xi$ by putting $\alpha \overset{2}{\leq} \beta$ if $\alpha \overset{L}{\leq} \beta$ or $\alpha \overset{R}{\leq} \beta$, and then taking the transitive closure. If $\alpha \overset{2}{\leq} \beta$ and $\beta \overset{2}{\leq} \alpha$, we say that α and β lie in the same **two-sided cell** of $\mathcal{P}_{\mathbf{\Lambda},\xi}$.

Proposition 7.14. *Two bounded feasible sign vectors $\alpha, \beta \in \mathcal{P}_{\mathbf{\Lambda},\xi}$ lie in the same two-sided cell if and only if $\mathcal{I}_\alpha = \mathcal{I}_\beta$ (equivalently $F_\alpha = F_\beta$).*

Proof. The only if statement is an immediate consequence of Propositions 7.2 and 7.11. To prove the if statement, we will show that the intersection of the left cell containing α and the right cell containing β is nonempty.

Let $\mathcal{I} = \mathcal{I}_\alpha = \mathcal{I}_\beta$, and let γ be the sign vector that agrees with α on $\mathcal{I}$ and with β on $\mathcal{I}^c$. Propositions 7.2 and 7.11 tell us that γ is in the same left cell as α and the same right cell as β provided that we can show that $\gamma \in \mathcal{P}_{\mathbf{\Lambda},\xi}$. That γ is ξ-bounded follows from the fact that $\Delta_{0,\gamma} = \Delta_{0,\beta}$.

Feasibility follows from Gale duality together with the same argument. Let $X = (\Lambda_0, \eta, \xi)$ be a polarized arrangement linked to $\mathbf{X}$, and let $X^! = (\Lambda_0, \eta^!, \xi^!)$ be its Gale dual, as defined in Section 2.6. We have $\alpha \in \mathcal{F}_{\mathbf{\Lambda}} = \mathcal{F}_\eta = \mathcal{B}_{\xi^!}$, where the last equality is by Theorem 2.18. Then the boundedness argument from the previous paragraph allows us to conclude that $\gamma \in \mathcal{B}_\xi$, and therefore $\gamma \in \mathcal{F}_{\mathbf{\Lambda}}$, as required. □

We conclude the section by showing that Gale duality exchanges left cells with right cells.

Theorem 7.15. *Suppose that* $\mathbf{X}$ *and* $\mathbf{X}^!$ *are Gale dual regular integral quantized polarized arrangements. Then the two preorders on* $\mathcal{F}_{\mathbf{\Lambda}} = \mathcal{B}_{\xi^!}$ *are opposite to each other, as are the preorders on* $\mathcal{F}_{\mathbf{\Lambda}^!} = \mathcal{B}_\xi$ *and on* $\mathcal{P}_{\mathbf{\Lambda},\xi} = \mathcal{P}_{\mathbf{\Lambda}^!,\xi^!}$.

Proof. This follows from Propositions 7.2, 7.11, and 7.14, along with the fact that Gale duality induces an order-reversing bijection of coloop-free flats. □

As a corollary, we find that left cells in $\mathcal{F}_{\mathbf{\Lambda}}$ coincide with right cells in $\mathcal{B}_{\xi^!}$, and the partial order on the set of left cells is the opposite of the partial order on the set of right cells, with similar statements holding for $\mathcal{F}_{\mathbf{\Lambda}^!} = \mathcal{B}_\xi$ and $\mathcal{P}_{\mathbf{\Lambda},\xi} = \mathcal{P}_{\mathbf{\Lambda}^!,\xi^!}$. We state this corollary explicitly for two-sided cells, since we will use it in the next section.

Corollary 7.16. *Let* $\mathcal{P} := \mathcal{P}_{\mathbf{\Lambda},\xi} = \mathcal{P}_{\mathbf{\Lambda}^!,\xi^!}$. *The two-sided cells of* $\mathcal{P}$ *induced by* $\mathbf{X}$ *are the same as the two-sided cells of* $\mathcal{P}$ *induced by* $\mathbf{X}^!$, *and the partial order on the set of cells induced by* $\mathbf{X}$ *is opposite to the partial order on the set of cells induced by* $\mathbf{X}^!$.

7.4 Cells and the BBD filtration

Let X be a polarized arrangement linked to $\mathbf{X}$. Assume that Λ_0 is unimodular, so that Corollary 6.5 provides an equivalence between the algebraic category $\mathcal{O}(\mathbf{X})$ and the geometric category $\mathcal{Q}_\lambda$ of sheaves on the hypertoric variety $\mathfrak{M} = \mathfrak{M}(X)$. In this section we use the two-sided cells to define a natural filtration of the Grothendieck group of $\mathcal{O}(\mathbf{X})$. This filtration will be shown to coincide with the Beilinson-Bernstein-Deligne (BBD) filtration of $H^{2d}_{\mathbb{T}}(\mathfrak{M};\mathbb{C})$ via the cycle map of Section 6.3.

Recall the projective morphism $\nu : \mathfrak{M} \longrightarrow \mathfrak{M}_0$ from $\mathfrak{M}$ to the affine hypertoric variety $\mathfrak{M}_0$. Since ν is semismall, the BBD decomposition theorem gives a canonical isomorphism

$$\nu_* \mathbb{C}_{\mathfrak{M}} \;\cong\; \bigoplus_F \mathrm{IC}^\bullet\left(\mathfrak{M}_0^F; \Pi_F\right), \tag{7}$$

where ν_* is the derived pushforward, F ranges over all coloop-free flats of $\mathcal{H}_0$, Π_F is the local system whose fiber over a point x is the top nonvanishing cohomology group of $\nu^{-1}(x)$, and $\mathrm{IC}^\bullet(\mathfrak{M}_0^F;\Pi_F)$ is the intersection cohomology sheaf of $\mathfrak{M}_0^F$ with coefficients in Π_F [CG97, 8.9.3]. (In fact, as shown in [PW07, 5.2], the local system Π_F is always trivial.) By applying the functor $H^{2d}_{\mathbb{T}}(\cdot)$ to both sides of Equation (7), we obtain the decomposition

$$H^{2d}_{\mathbb{T}}(\mathfrak{M};\mathbb{C}) \;\cong\; \bigoplus_F IH^{2d}_{\mathbb{T}}(\mathfrak{M}_0^F;\Pi_F).$$

We will be interested not in the full direct sum decomposition, but rather in one of the two associated filtrations. More precisely, let

$$D_F := \bigoplus_{F\subset F'} IH^{2d}_{\mathbb{T}}(\mathfrak{M}_0^{F'};\Pi_{F'}) \subset H^{2d}_{\mathbb{T}}(\mathfrak{M};\mathbb{C}) \qquad \text{and} \qquad E_F := \bigoplus_{F\subsetneq F'} IH^{2d}_{\mathbb{T}}(\mathfrak{M}_0^{F'};\Pi_{F'}) \subset D_F.$$

This filtration has a topological interpretation, which we prove (for arbitrary equivariant symplectic resolutions) in [BLPWa].

Theorem 7.17. *For each coloop-free flat F of $\mathcal{H}_0$, D_F is equal to the intersection*

$$\bigcap_{\{\alpha:F\not\subset F_\alpha\}} [C_\alpha]^\perp,$$

where $[C_\alpha]^\perp$ is the perpendicular space to $[C_\alpha]$ with respect to the integration pairing on $H^{2d}_{\mathbb{T}}(\mathfrak{M};\mathbb{C})$.

For all $\alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi}$, let $P_\alpha^{\mathbf{\Lambda}}$ be the projective cover of $L_\alpha^{\mathbf{\Lambda}}$ in $\mathcal{O}(\mathbf{X})$.

Theorem 7.18. *We have*

$$D_F = \mathbb{C}\left\{ \mathrm{Supp}[\mathrm{Loc}(P_\alpha^{\mathbf{\Lambda}})] \mid F\subset F_\alpha\right\} \qquad \textit{and} \qquad E_F = \mathbb{C}\left\{ \mathrm{Supp}[\mathrm{Loc}(P_\alpha^{\mathbf{\Lambda}})] \mid F\subsetneq F_\alpha\right\}.$$

In other words, the BBD filtration of $H^{2d}_{\mathbb{T}}(\mathfrak{M};\mathbb{C})$ corresponds, via the support isomorphism, to the filtration of $K(\mathcal{O}(\mathbf{X}))_{\mathbb{C}}$ defined by the basis of projective objects and the order on two-sided cells.

Proof. We have

$$\mathbb{C}\left\{ \mathrm{Supp}[\mathrm{Loc}(P_\alpha^{\mathbf{\Lambda}})] \mid F\subset F_\alpha\right\} = \mathbb{C}\left\{ \mathrm{Supp}[\mathrm{Loc}(L_\alpha^{\mathbf{\Lambda}})] \mid F\not\subset F_\alpha\right\}^\perp = \mathbb{C}\left\{ [C_\alpha] \mid F\not\subset F_\alpha\right\}^\perp$$

by Proposition 6.10 and the fact that the classes of simple and projective objects are dual with respect to the Euler form. The first statement of the theorem then follows from Theorem 7.17. The statement about E_F is proven in the same way. The interpretation in terms of two-sided cells follows from Proposition 7.14. □

For all $\alpha\in\{+,-\}^n$, let $d_\alpha := |\{i \mid \alpha(i)=-1\}|$. Let $X^!$ be the Gale dual of X, and let $\mathbf{X}$ be a regular, integral, quantized polarized arrangement linked to $X^!$. Consider the pairing

$$K(\mathcal{O}(\mathbf{X}))_{\mathbb{C}} \otimes K(\mathcal{O}(\mathbf{X}^!))_{\mathbb{C}} \to \mathbb{C}$$

given by the formula

$$\Big\langle [P^{\mathbf{\Lambda}}_\alpha], [P^{\mathbf{\Lambda}^!}_\beta] \Big\rangle = (-1)^{d_\alpha} \delta_{\alpha\beta}$$

for all $\alpha, \beta \in \mathcal{P}_{\mathbf{\Lambda},\xi} = \mathcal{P}_{\mathbf{\Lambda}^!,\xi^!}$. Using the support isomorphism from Section 6.3, we may interpret this pairing as a pairing between $H^{2d}_{\mathbb{T}}(\mathfrak{M}(X);\mathbb{C})$ and $H^{2d^!}_{\mathbb{T}}\big(\mathfrak{M}(X^!);\mathbb{C}\big)$, where $d^! := \dim V^!_0$. For each coloop-free flat F of $\mathcal{H}_0$, F^c is a coloop-free flat of $\mathcal{H}^!_0$, and we denote by

$$E^!_{F^c} \subset D^!_{F^c} \subset H^{2d^!}_{\mathbb{T}}\Big(\mathfrak{M}(X^!);\mathbb{C}\Big)$$

the corresponding pieces of the BBD filtration. Theorem 7.18 has the following immediate corollary.

Corollary 7.19. *The BBD filtrations of $H^{2d}_{\mathbb{T}}(\mathfrak{M}(X);\mathbb{C})$ and $H^{2d^!}_{\mathbb{T}}\big(\mathfrak{M}(X^!);\mathbb{C}\big)$ are dual to each other under the above pairing. More precisely, for any coloop-free flat F of $\mathcal{H}_0$, $E^!_{F^c}$ is the perpendicular space to D_F and $D^!_{F^c}$ is the perpendicular space to E_F. In particular, the pairing induces a canonical duality between*

$$IH^{2d}_{\mathbb{T}}\big(\mathfrak{M}_0(X)^F;\Pi_F\big) \cong D_F/E_F \qquad \text{and} \qquad IH^{2d^!}_{\mathbb{T}}\Big(\mathfrak{M}_0(X^!)^{F^c};\Pi^!_{F^c}\Big) \cong D^!_{F^c}/E^!_{F^c}.$$

Remark 7.20. Corollary 7.19 would still be true if we did not include the twist of $(-1)^{d_\alpha}$ in the definition of our pairing. We regard the twisted pairing as more natural than the untwisted pairing because the twisted pairing has the property that

$$\Big\langle [L^{\mathbf{\Lambda}}_\alpha], [L^{\mathbf{\Lambda}^!}_\beta] \Big\rangle = \Big\langle [S^{\mathbf{\Lambda}}_\alpha], [S^{\mathbf{\Lambda}^!}_\beta] \Big\rangle = \Big\langle [P^{\mathbf{\Lambda}}_\alpha], [P^{\mathbf{\Lambda}^!}_\beta] \Big\rangle = (-1)^{d_\alpha} \delta_{\alpha\beta}$$

for all $\alpha, \beta \in \mathcal{P}_{\mathbf{\Lambda},\xi} = \mathcal{P}_{\mathbf{\Lambda}^!,\xi^!}$. If the twist were not included, then the pairing would not be well-behaved with respect to the simple or standard bases.

Remark 7.21. For any coloop-free flat F, the dimension of $IH^{2d}_{\mathbb{T}}(\mathfrak{M}^F_0;\Pi_F)$ is equal to the dimension of $IH^*(\mathfrak{M}^F_0;\Pi_F)$. In fact, the full equivariant cohomology group $IH^*_{\mathbb{T}}(\mathfrak{M}^F_0;\Pi_F)$ is a flat family over $\operatorname{Spec} H^*_{\mathbb{T}}(pt) \cong \mathfrak{t}$ with general fiber isomorphic to $IH^{2d}_{\mathbb{T}}(\mathfrak{M}^F_0;\Pi_F)$ and special fiber isomorphic to $IH^*(\mathfrak{M}^F_0;\Pi_F)$. Thus Corollary 7.19 tells us that for every coloop-free flat F, a deformation of the vector space $IH^*\big(\mathfrak{M}_0(X)^F;\Pi_F\big)$ is canonically dual to a deformation of $IH^*\big(\mathfrak{M}_0(X^!)^{F^c};\Pi^!_{F^c}\big)$.

If we take $F = \{0\}$ to be the minimal flat, this says that a deformation of

$$IH^*\big(o;\Pi_{\{0\}}\big) = H^{2d}\big(\nu^{-1}(o);\mathbb{C}\big) = H^{2d}(\mathfrak{M};\mathbb{C})$$

is canonically dual to a deformation of $IH^*\big(\mathfrak{M}_0(X^!);\mathbb{C}\big)$. In light of the combinatorial interpretation of the ordinary and intersection Betti numbers of a hypertoric variety [PW07, 3.5 & 4.3], this is a geometric interpretation of the well-known combinatorial statement that the top h-number of a matroid is equal to the sum of the h-numbers of the broken circuit complex of the dual matroid.

8 Shuffling and twisting functors

In this section we explain how to relate the derived categories of various regular integral blocks of $\mathcal{O}$ which are obtained by fixing Λ_0 and varying $\mathbf{\Lambda}$ and ξ.

8.1 Defining the functors

Fix an integer n and a direct summand $\Lambda_0 \subset W_{\mathbb{Z}}$. For any pair of parameters $\mathbf{\Lambda}$ and ξ, let

$$\mathbf{X}_{\mathbf{\Lambda},\xi} := (\Lambda_0, \mathbf{\Lambda}, \xi)$$

be the corresponding quantized polarized arrangement. We will denote by $D(\mathbf{X}_{\mathbf{\Lambda},\xi})$ the bounded derived category of the associated block $\mathcal{O}(\mathbf{X}_{\mathbf{\Lambda},\xi})$ of hypertoric category $\mathcal{O}$. We introduce subscripts in the notation because we intend to vary the parameters $\mathbf{\Lambda}$ and ξ, restricting our attention to those parameters for which $\mathbf{X}_{\mathbf{\Lambda},\xi}$ is both regular and integral.

We begin by varying only $\mathbf{\Lambda}$. Consider a pair of regular integral parameters $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$, and let λ and λ' be the associated central characters of the hypertoric enveloping algebra. Fix a regular covector ξ, and consider the functor

$$\Phi_{\mathbf{\Lambda}}^{\mathbf{\Lambda}'} : D(\mathbf{X}_{\mathbf{\Lambda},\xi}) \to D(\mathbf{X}_{\mathbf{\Lambda}',\xi})$$

obtained as the derived functor of the translation functor ${}_{\lambda'}T_{\lambda} \otimes_{U_\lambda} -$ in Lemma 3.10.

Definition 8.1. A **shuffling functor** is an endofunctor of $D(\mathbf{X}_{\mathbf{\Lambda},\xi})$ obtained as a composition

$$\Phi_{\mathbf{\Lambda}_r}^{\mathbf{\Lambda}} \circ \Phi_{\mathbf{\Lambda}_{r-1}}^{\mathbf{\Lambda}_r} \circ \ldots \circ \Phi_{\mathbf{\Lambda}_1}^{\mathbf{\Lambda}_2} \circ \Phi_{\mathbf{\Lambda}}^{\mathbf{\Lambda}_1}$$

for any finite sequence $\mathbf{\Lambda}_1, \ldots, \mathbf{\Lambda}_r$ of regular integral parameters.

The following lemma tells us that shuffling within an equivalence class is trivial.

Lemma 8.2. *Suppose that $\mathbf{\Lambda}$, $\mathbf{\Lambda}'$, and $\mathbf{\Lambda}''$ are chosen such that the quantized polarized arrangements $\mathbf{X}_{\mathbf{\Lambda},\xi}$, $\mathbf{X}_{\mathbf{\Lambda}',\xi}$, and $\mathbf{X}_{\mathbf{\Lambda}'',\xi}$ are all equivalent in the sense of Definition 2.14. Then the natural transformation*

$$\Phi_{\mathbf{\Lambda}'}^{\mathbf{\Lambda}''} \circ \Phi_{\mathbf{\Lambda}}^{\mathbf{\Lambda}'} \to \Phi_{\mathbf{\Lambda}}^{\mathbf{\Lambda}''}$$

provided by Equation (3) *is a natural isomorphism.*

Proof. For each $\alpha \in \mathcal{P}_{\mathbf{\Lambda},\xi} = \mathcal{P}_{\mathbf{\Lambda}',\xi} = \mathcal{P}_{\mathbf{\Lambda}'',\xi}$, consider the projective cover $P_\alpha^{\mathbf{\Lambda}}$ of $L_\alpha^{\mathbf{\Lambda}}$ in $\mathcal{O}(\mathbf{X}_{\mathbf{\Lambda},\xi})$. The $\mathbb{D}$-module $\mathbb{D} \otimes_U P_\alpha^{\mathbf{\Lambda}}$ is the quotient of the module $P_\alpha^{(k)}$ from Section 4.3 by the submodule generated by the weight spaces corresponding to all weights $v \in \mathbf{\Delta}_\beta \cap \mathbf{\Lambda}$ for all $\beta \in \mathcal{F}_{\mathbf{\Lambda}} \smallsetminus \mathcal{P}_{\mathbf{\Lambda},\xi}$. This submodule is independent (up to isomorphism) of the choice of $\mathbf{\Lambda}$ in a fixed equivalence class, so it follows that

$${}_{\lambda'}T_{\lambda} \otimes_{U_\lambda} P_\alpha^{\mathbf{\Lambda}} \cong \left(\mathbb{D} \otimes_U P_\alpha^{\mathbf{\Lambda}}\right)^{\mathbf{\Lambda}'} \cong \left(\mathbb{D} \otimes_U P_\alpha^{\mathbf{\Lambda}'}\right)^{\mathbf{\Lambda}'} \cong P_\alpha^{\mathbf{\Lambda}'},$$

and similarly for $\mathbf{\Lambda}''$ and λ''.

Thus the functors given by tensor product with ${}_{\lambda''}T_{\lambda'} \otimes {}_{\lambda'}T_{\lambda}$ and ${}_{\lambda''}T_{\lambda}$ are both equivalences. To show that they are the *same* equivalence, it is sufficient to show that, for all $\alpha, \beta \in \mathcal{P}_{\mathbf{\Lambda},\xi} = \mathcal{P}_{\mathbf{\Lambda}'',\xi}$, they induce the same isomorphism between $\mathrm{Hom}_{U_\lambda}\left(P_\alpha^{\mathbf{\Lambda}}, P_\beta^{\mathbf{\Lambda}}\right)$ and $\mathrm{Hom}_{U_{\lambda''}}\left(P_\alpha^{\mathbf{\Lambda}''}, P_\beta^{\mathbf{\Lambda}''}\right)$. This follows from the fact that, in both cases, the isomorphism is compatible with the surjections from $\mathrm{Hom}_{\mathbb{D}}\left(P_\alpha^{(k)}, P_\beta^{(k)}\right)$ to the two groups. □

Remark 8.3. The term shuffling is derived from the fact that our shuffling functors are analogous to the derived functors of the shuffling functors which Irving [Irv93] defined for BGG category $\mathcal{O}$. Irving's functors give braid group actions [MS05, 5.10], and, as we will see in Theorem 8.13, our shuffling functors are analogous to the Irving shuffling functors associated to *pure* braids.

Now fix a regular integral parameter $\mathbf{\Lambda}$ and let ξ and ξ' be regular. Let $U_\lambda\text{−}\mathsf{mod}_{pro}$ be the category of topologically finitely generated U_λ-modules with profinite dimensional weight spaces, that is, the category of U_λ-modules which are inverse limits of weight modules such that all modules in the inverse limit have a consistent finite generating set. This category contains the objects representing each weight space, and thus has enough projectives. Furthermore, the functor F arising from Theorem 3.6 gives an equivalence from $U_\lambda\text{−}\mathsf{mod}_{pro,\mathbf{\Lambda}}$ to the category of finitely generated modules over $e_{\mathbf{\Lambda}} R e_{\mathbf{\Lambda}}$. As we will prove in Lemma 8.20, this category has finite global dimension.[16]

Consider the functor

$$\Psi_\xi^{\xi'} : D(\mathbf{X}_{\mathbf{\Lambda},\xi}) \to D(\mathbf{X}_{\mathbf{\Lambda},\xi'})$$

obtained as the composition of the derived functors

$$D(\mathcal{O}(\mathbf{X}_{\mathbf{\Lambda},\xi})) \xrightarrow{\iota} D(U_\lambda\text{−}\mathsf{mod}_{pro,\mathbf{\Lambda}}) \xrightarrow{L\pi_{\xi'}} D(\mathcal{O}(\mathbf{X}_{\mathbf{\Lambda},\xi'})),$$

where ι is the inclusion and $\pi_{\xi'}$ is the projection functor from Section 4.3. The functor ι is exact, but it does not send projectives to projectives. As a result, the composition of the above derived functors is not the same as the derived functor of the composition. The left derived functor $L\pi_{\xi'}$ preserves bounded derived categories since all the categories involved have finite global dimension.

Definition 8.4. A **twisting functor** is an endofunctor of $D(\mathbf{X}_{\mathbf{\Lambda},\xi})$ obtained as a composition

$$\Psi_{\xi_r}^{\xi} \circ \Psi_{\xi_{r-1}}^{\xi_r} \circ \ldots \circ \Psi_{\xi_1}^{\xi_2} \circ \Psi_{\xi}^{\xi_1}$$

for any finite sequence $\xi_1, \ldots, \xi_r$ of regular parameters.

If $\mathbf{X}_{\mathbf{\Lambda},\xi}$ and $\mathbf{X}_{\mathbf{\Lambda},\xi'}$ are equivalent in the sense of Definition 2.14, then the categories $\mathcal{O}(\mathbf{X}_{\mathbf{\Lambda},\xi})$ and $\mathcal{O}(\mathbf{X}_{\mathbf{\Lambda},\xi'})$ are in fact equal and $\Psi_\xi^{\xi'}$ is the identity functor. In particular, as was true for shuffling functors, twisting within an equivalence class is trivial.

[16] That theorem concerns an uncompleted version of $e_{\mathbf{\Lambda}} R e_{\mathbf{\Lambda}}$, which is sufficient for our purposes because the completion of a projective resolution is still a projective resolution.

Remark 8.5. We chose the term "twisting functor" because these functors are in a sense analogous to Arkhipov's twisting functors on BGG category $\mathcal{O}$ [Ark, AS03], though the connection is a bit more remote than it was with shuffling functors. Let us illustrate the precise relationship using the simplest possible example, the arrangement of two points in a line (the $n = 2$ case of Example 4.11). Then any regular integral block $\mathcal{O}(\mathbf{X})$ of hypertoric category $\mathcal{O}$ is equivalent to the block $\mathcal{O}_0(\mathfrak{sl}_2)$ of the BGG category.

There are only two equivalence classes of regular covectors in Λ_0^*, represented, say, by ξ and $-\xi$. Arkhipov's twisting functor arises from $\Psi_\xi^{-\xi}$ by dualizing and a shift: $H^{-1}\Psi_\xi^{-\xi}$ is identified with the functor which Khomenko and Mazorchuk [KM05] call "Joseph's version of Enright's completion functor". It is right adjoint to Arkhipov's twisting functor T, its derived functor is the inverse of the derived functor of T, and it is isomorphic to dTd, where d is the duality functor on $\mathcal{O}_0(\mathfrak{sl}_2)$ [AS03].

Furthermore, this identification relies on a feature special to this example, namely, that *both* the categories for ξ and for $-\xi$ can be identified with $\mathcal{O}_0(\mathfrak{sl}_2)$. This fact also appears in the original construction of Arkhipov's functor, wherein a module in $\mathcal{O}(\mathfrak{g})$ is first sent to a module of the equivalent category for a different Borel subgroup; this category is then identified with the original category by an automorphism of $\mathfrak{g}$. For general arrangements, identifications analogous to the conjugacy of Borels are not possible, and our "twisting functors" should be more properly considered as analogues of the (shifted, dualized) derived functors of those Arkhipov functors associated to elements of the pure braid group.

Remark 8.6. It is not immediately obvious that shuffling and twisting functors are equivalences. That they are will be a consequence of Proposition 8.8 for shuffling and Theorem 8.19 for twisting.

8.2 Combinatorial interpretations

In this section we give combinatorial interpretations of shuffling and twisting functors, and we use them to prove that shuffling and twisting commute. For various reasons it will be easier to work with polarized arrangements rather than quantized polarized arrangements; we lose no information with this choice, since shuffling and twisting are trivial within equivalence classes and linkage gives a bijection between equivalence classes of regular, integral, quantized polarized arrangements and regular polarized arrangements with the same Λ_0. Thus we will let $D(X_{\eta,\xi})$ denote the bounded derived category of modules over the algebra $A(\eta, \xi) := A(X_{\eta,\xi})$ from [BLPWb], and use the notation

$$\Phi_\eta^{\eta'} : D(X_{\eta,\xi}) \to D(X_{\eta',\xi}) \qquad \text{and} \qquad \Psi_\xi^{\xi'} : D(X_{\eta,\xi}) \to D(X_{\eta,\xi'}),$$

for the functors obtained from those in the previous section via Theorems 4.7 and 4.8.

Following the notation of Theorem 3.11, let $R := \widehat{Q}_n/\langle\vartheta(\mathfrak{k})\rangle$ (since everything in sight is integral we write $\widehat{Q}_n$ rather than $\widehat{Q}_{\mathbf{\Lambda}}$), and put

$$e_\eta := \sum_{\alpha\in\mathcal{F}_\eta} e_\alpha \qquad \text{and} \qquad e_\xi := \sum_{\beta\notin\mathcal{B}_\xi} e_\beta.$$

We define algebras

$$A(\eta, -) := e_\eta R e_\eta \qquad \text{and} \qquad A(-, \xi) := R/Re_\xi R,$$

so that we have

$$e_\eta A(-, \xi) e_\eta = A(\eta, \xi) = A(\eta, -)/A(\eta, -)e_\xi A(\eta, -).$$

Lemma 8.7. *Multiplication induces an isomorphism*

$$Re_\eta \otimes_{A(\eta,-)} A(\eta, \xi) \xrightarrow{\sim} A(-, \xi)e_\eta.$$

Proof. Clearly the multiplication map is surjective. To show injectivity it is enough to show that

$$Re_\xi Re_\eta \otimes_{A(\eta,-)} A(\eta, \xi) = 0.$$

This is an immediate consequence of the following fact, which says that any path in the quiver algebra $\widehat{Q}_n$ from a feasible sign vector to an unbounded one is equivalent to one which first goes through feasible sign vectors to one which is both feasible and unbounded: for any $\alpha \in \mathcal{F}_\eta$ and any $\beta \in \{+, -\}^n \setminus \mathcal{B}_\xi$, there exists $\gamma \in \mathcal{F}_\eta$ so that $\gamma \notin \mathcal{B}_\xi$ and

$$\text{for any } 1 \leq i \leq n, \text{ either } \gamma(i) = \alpha(i) \text{ or } \gamma(i) = \beta(i). \tag{*}$$

To see that this fact holds, we drop the integrality assumption on η and allow $\eta \in W_\mathbb{R}/V_{0,\mathbb{R}}$. The regularity of η implies that there is an open ball B with center η so that $\mathcal{F}_{\eta'} = \mathcal{F}_\eta$ for any $\eta' \in B$. Take η'' for which $\beta \in \mathcal{F}_{\eta''}$, and choose points $p \in \mathrm{int}(\Delta_{\eta,\alpha})$ and $p'' \in \mathrm{int}(\Delta_{\eta'',\beta})$. If a point $p' \in \Delta_{\eta',\gamma}$ lies on the line segment joining p and p'', then γ satisfies (*), since the line segment cannot cross a hyperplane more than once. We can take p'' so that $\xi(p'')$ is arbitrarily large (keeping η'' fixed), so we can find p' so that $\eta' \in B$ but $\xi(p')$ is as large as we like, giving $\gamma \in \mathcal{F}_{\eta'} = \mathcal{F}_\eta$ with $\gamma \notin \mathcal{B}_\xi$. □

Proposition 8.8. *The functor $\Phi_\eta^{\eta'} : D(X_{\eta,\xi}) \to D(X_{\eta',\xi})$ is given by*

$$\Phi_\eta^{\eta'}(M) = e_{\eta'} A(-, \xi) e_\eta \overset{L}{\otimes}_{A(\eta,\xi)} M.$$

In particular, $\Phi_\eta^{\eta'}$ is an equivalence.

Proof. Theorem 3.11 and Lemma 8.7 imply that the diagram

$$\begin{array}{ccc} \mathcal{O}(\mathbf{X}) & \xrightarrow{F} & A(\eta, \xi)\text{–mod} \\ \downarrow{\scriptstyle {}_{\lambda'}T_\lambda \otimes_{U_\lambda} -} & & \downarrow{\scriptstyle e_{\eta'} A(-, \xi) e_\eta \otimes_{A(\eta,\xi)} -} \\ \mathcal{O}(\mathbf{X}') & \xrightarrow[F]{} & A(\eta', \xi)\text{–mod} \end{array}$$

commutes up to natural isomorphism, where F and F' are the equivalences of Theorems 4.7 and 4.8. The Proposition then follows from passing to derived functors.

These functors are exactly those studied in [BLPWb, §6], which were proven to be equivalences in Theorem 6.13 of that paper. In that paper we considered the derived category of graded modules, but the corresponding functors (given by the same bimodules) on ungraded modules are still triangulated and fully faithful on gradable modules, and these generate the category. Thus these functors are equivalences on the ungraded categories. □

Proposition 8.9. *The functor $\Psi_\xi^{\xi'} : D(X_{\eta,\xi}) \to D(X_{\eta,\xi'})$ is given by*

$$\Psi_\xi^{\xi'}(M) = A(\eta,\xi') \overset{L}{\otimes}_{A(\eta,-)} M.$$

Proof. As before, we need only draw the commutative diagram

$$\begin{array}{ccc}
D(X_{\eta,\xi}) & \xrightarrow{F} & D(A(\eta,\xi)\text{-}\mathsf{mod}) \\
\Big\downarrow{\scriptstyle \iota} & & \Big\downarrow{\scriptstyle \inf_{A(\eta,\xi)}^{A(\eta,-)}} \\
D(U_\lambda\text{-}\mathsf{mod}_{pro,\mathbf{\Lambda}}) & \xrightarrow[F]{} & D(A(\eta,-)\text{-}\mathsf{mod}) \\
\Big\downarrow{\scriptstyle L\pi_{\xi'}} & & \Big\downarrow{\scriptstyle A(\eta,\xi') \overset{L}{\otimes}_{A(\eta,-)} -} \\
D(X_{\eta,\xi'}) & \xrightarrow[F]{} & D(A(\eta,\xi')\text{-}\mathsf{mod})
\end{array}$$

where $\inf_{A(\eta,\xi)}^{A(\eta,-)}$ denotes the inflation (or restriction) of $A(\eta,\xi)$ modules to modules over $A(\eta,-)$ by the obvious homomorphism. □

As a corollary of Propositions 8.8 and 8.9, we can prove that shuffling and twisting commute.

Corollary 8.10. *There is a natural isomorphism of functors*

$$\Phi_\eta^{\eta'} \circ \Psi_\xi^{\xi'} \cong \Psi_\xi^{\xi'} \circ \Phi_\eta^{\eta'} : D(X_{\eta,\xi}) \to D(X_{\eta',\xi'}).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
A(\eta,\xi) & \longleftarrow & A(\eta,-) & \longrightarrow & A(\eta,\xi') \\
\downarrow & & \downarrow & & \downarrow \\
A(-,\xi) & \longleftarrow & R & \longrightarrow & A(-,\xi') \\
\uparrow & & \uparrow & & \uparrow \\
A(\eta',\xi) & \longleftarrow & A(\eta',-) & \longrightarrow & A(\eta',\xi')
\end{array}$$

where all the maps are the obvious inclusions or projections. By Proposition 8.9, the functor $\Psi_{\xi}^{\xi'}$ is given by following across the top row or the bottom row of this diagram, first pulling back, then pushing forward. By Proposition 8.8, $\Phi_{\eta}^{\eta'}$ is given by pushing forward then pulling back along the left or right column. Note that since the map $A(\eta', \xi) \to A(-, \xi)$ does not take the unit to the unit, pulling back by this map involves first multiplying by the idempotent $e_{\eta'}$ and then taking the induced module.

It is sufficient to show that going along the top and right sides of each small square is naturally isomorphic to going along the left and bottom sides. For the upper right and lower left squares this is simply the fact that these squares commute. For the other two squares, this follows from Lemma 8.7. □

8.3 Fundamental group action via shuffling functors

This section is about shuffling functors only, so we fix Λ_0 and ξ and vary η among regular parameters. The set of regular η is the intersection of $\mathfrak{t}^*_{\mathbb{Z}} \cong W_{\mathbb{Z}}/\Lambda_0$ with the complement of a central hyperplane arrangement in $\mathfrak{t}^*_{\mathbb{R}} \cong W_{\mathbb{R}}/V_{0,\mathbb{R}}$. This arrangement, known as the **discriminantal arrangement**, has its hyperplanes indexed by the circuits of $\mathcal{H}_0$, (the minimal subsets of hyperplanes with dependent normal vectors). It was first considered for generic arrangements in [MS89] and for general arrangements in [BB97].

Let $\Upsilon \subset \mathfrak{t}^*_{\mathbb{R}}$ be the complement of the discriminantal arrangement. If η, η' are regular, then $X_{\eta,\xi}$ is equivalent to $X_{\eta',\xi}$ if and only if η and η' lie in the same connected component of Υ. Let $B \subset \Upsilon \cap \mathfrak{t}^*_{\mathbb{Z}}$ be a finite set consisting of one integral basepoint from each connected component of Υ. Let $\Upsilon_{\mathbb{C}}$ be the complexification of Υ, that is, the set of elements in $\mathfrak{t}^*$ whose real or imaginary part lies in Υ.

Definition 8.11. The **Deligne groupoid** of the discriminantal arrangement is the subgroupoid of the fundamental groupoid of $\Upsilon_{\mathbb{C}}$ consisting of paths that begin and end in B.

Since each connected component of Υ is simply connected, the Deligne groupoid is independent (up to canonical isomorphism) of our choice of B. Furthermore, it has an entirely combinatorial interpretation. Let the **Deligne quiver** be the quiver with with vertex set B and edges in both directions between two base points if and only if the components of Υ in which they lie are separated by a single hyperplane.

Theorem 8.12 ([Par93])**.** *The Deligne groupoid is isomorphic the quotient of the fundamental groupoid of the Deligne quiver by the identification of any pair of paths of minimal length between the same two points.*

Theorem 8.13. *By assigning the equivalence $\Phi_{\eta}^{\eta'}$ to the shortest oriented path in the Deligne quiver from η to η', we obtain an action of the Deligne groupoid on the categories $\{D(X_{\eta,\xi}) \mid \eta \in B\}$. In particular, for each $\eta \in B$, we obtain an action of $\pi_1(\Upsilon_{\mathbb{C}}, \eta)$ on $D(X_{\eta,\xi})$ by auto-equivalences.*

Proof. For any $\eta, \eta', \eta'' \in B$, we defined in [BLPWb, §6.3] a natural transformation from $\Phi_{\eta'}^{\eta''} \circ \Phi_{\eta}^{\eta'}$ to $\Phi_{\eta}^{\eta''}$. We need to show that if η' lies on a path of minimal length from η to η'' in the Deligne quiver, this natural transformation is an isomorphism.

In the proof of [BLPWb, 6.12], we showed that our natural transformation is a natural isomorphism if and only if the following condition is satisfied for each $S \subset \{1, \dots, n\}$:

> For every $\alpha \in \mathcal{P}_{\eta,\xi}$ and $\alpha'' \in \mathcal{P}_{\eta'',\xi}$ such that $\alpha|_S = \alpha''|_S$ is bounded[17] for the subarrangement $\{H_{i,0} \mid i \in S\} \subset \mathcal{H}_0$, there is a sign vector $\alpha' \in \mathcal{P}_{\eta',\xi}$ such that $\alpha|_S = \alpha'|_S = \alpha''|_S$.

We prove this statement by induction on the size of the complement of S. If $S = \{1, \dots, n\}$, the statement says that $\mathcal{P}_{\eta,\xi} \cap \mathcal{P}_{\eta'',\xi} \subset \mathcal{P}_{\eta',\xi}$; for this it is sufficient to show that $\mathcal{F}_\eta \cap \mathcal{F}_{\eta''} \subset \mathcal{F}_{\eta'}$. Indeed, a sign vector $\alpha \in \mathcal{F}_\eta \cap \mathcal{F}_{\eta''}$ fails to lie in $\mathcal{F}_{\eta'}$ if and only if there is a circuit C such that $\alpha|_C$ is infeasible for the sub-arrangement $\{H_{i,0} \mid i \in C\} \subset \mathcal{H}_0$. This would imply that the hyperplane in the discriminantal arrangement indexed by C separates η and η'' from η', which contradicts the fact that η' lies on a path of minimal length from η to η'' in the Deligne quiver.

Now consider the general case. Choose some $i \notin S$; by our inductive hypothesis, there exists a sign vector on $\{1, \dots, n\} \smallsetminus \{i\}$ that agrees with α and α'' on S and is bounded and feasible for the polarized arrangement obtained by deleting the i^{th} hyperplane of $\mathcal{H}_{\eta'}$. Both of the extensions of this sign vector to $\{1, \dots, n\}$ will be bounded, and at least one of them will lie in $\mathcal{F}_{\eta'}$, so we are done. □

For any $\eta \in B$, let $\zeta_\eta \in \pi_1(\Upsilon_{\mathbb{C}}, \eta)$ be the central element represented by the path $[0, 1] \to \Upsilon_{\mathbb{C}}$ taking t to $e^{2\pi i t}\eta$.

Theorem 8.14 ([BLPWb, 6.11])**.** *The element ζ_η acts as the Serre functor on $D(X_{\eta,\xi})$.*

Remark 8.15. For a regular block of $\mathcal{O}(\mathfrak{sl}_2)$, the derived functor of the square of the basic shuffling functor is the Serre functor [MS07, 1.3]. Thus Theorem 8.14 can be taken as further evidence for the analogy with Irving's shuffling functors for pure braids (Remark 8.3).

8.4 Koszul duality

Our last goal is to prove that shuffling and twisting functors are exchanged by Koszul duality. First we recall the basic features of the theory of Koszul duality that we will need. Throughout this section let $A = \bigoplus_{j \geq 0} A_j$ be a finite dimensional nonnegatively graded ring such that $A_0 \cong \bigoplus_{\alpha \in \mathcal{P}} \mathbb{C}e_\alpha$ is a commutative semisimple ring with primitive idempotents indexed by a finite set $\mathcal{P}$.

Let A−gr denote the category of finitely generated graded A-modules. The ring A is called **Koszul** if $\operatorname{Ext}^i_{A-\text{gr}}(A_0, A_0[-j]) = 0$ for all $i \neq j$. If we let

$$E(A) := \bigoplus_{i \geq 0} \operatorname{Ext}^i_A(A_0, A_0) = \bigoplus_{i,j \geq 0} \operatorname{Ext}^i_{A-\text{gr}}(A_0, A_0[-j])$$

[17] By this we mean that ξ is bounded above on the chamber of the subarrangement determined by α.

be the Yoneda algebra of A, then A is Koszul if and only if the two gradings on $E(A)$ coincide. If A is Koszul, then it is **quadratic**, that is, it is generated by over A_0 by A_1, with relations are generated in degree 2. Furthermore,

- $E(A)$ is also Koszul,
- there is a canonical isomorphism $A \cong E(E(A))$,
- $E(A)$ is isomorphic to the opposite algebra of the **quadratic dual** of A, which is the quadratic ring generated by A_1^* over A_0 with relations orthogonal to the relations of A.

Suppose that A is Koszul, and suppose in addition that $E(A)$ is left Noetherian. Then [BGS96, 2.12.6] gives an equivalence $D_{\mathrm{gr}}(A) \to D_{\mathrm{gr}}(E(A)^{\mathrm{op}})$ between the bounded derived categories of finitely generated graded modules. Since A^{op} is also Koszul [BGS96, 2.2.1] and $E(A^{\mathrm{op}})^{\mathrm{op}}) \cong E(A)$, we obtain an equivalence $D_{\mathrm{gr}}(A^{\mathrm{op}}) \to D_{\mathrm{gr}}(E(A))$. It will be more convenient for us to consider the contravariant equivalence

$$K\colon D_{\mathrm{gr}}(A) \to D_{\mathrm{gr}}(E(A))$$

given by composing this functor with the duality functor $D(A) \to D(A^{\mathrm{op}})$ induced by $M \mapsto M^*$. The functor K takes (shifted) simple objects to (shifted) projective objects and vice-versa: we have $K(A_0 e) = E(A)e$ and $K(Ae) = E(A)_0 e$ for any $e \in A_0 \cong E(A)_0$.

If A is quadratic (but not necessarily Koszul), we will denote by $A^!$ the opposite algebra of the quadratic dual of A. Thus if A is Koszul, we have $A^! \cong E(A)$ and

$$K\colon D_{\mathrm{gr}}(A) \to D_{\mathrm{gr}}(A^!).$$

Remark 8.16. We warn the reader that our notation conflicts with the notation in [BGS96]. In that paper $A^!$ is defined to be the quadratic dual of A, which is *opposite* to the Yoneda algebra $E(A)$. The reason that we make our definition is that we want an equivalence $D_{\mathrm{gr}}(A) \to D_{\mathrm{gr}}(A^!)$ that swaps simples and projectives. Our equivalence K has this property, but the basic BGS equivalence $D_{\mathrm{gr}}(A) \to D_{\mathrm{gr}}(E(A)^{\mathrm{op}})$ takes simples to projectives and *injectives* to simples.

We also note that our algebras $A(\mathbf{X})$ and $A(X)$ are isomorphic to their own opposites, which makes this conflict academic. However, it is still important to note that our equivalence differs from the BGS equivalence by an application of the duality functor.

We wish to study the interaction of Koszul duality with the inclusion $i\colon eAe \to A$ and the projection $q\colon A^! \to A^!/A^!\bar{e}A^!$, where $e \in A_0$ is an idempotent and $\bar{e} = 1 - e$ is the complementary idempotent. The following lemma follows immediately from the definition of quadratic duality.

Lemma 8.17. *If A and the subring eAe are both quadratic, then $(eAe)^! \cong A^!/A^!\bar{e}A^!$.*

The following theorem is the main result of this section; in the next section we will apply it to the ring $A(X)$ to prove the duality of shuffling and twisting.

Theorem 8.18. *Suppose that A and the subring eAe are both Koszul. Then $A^!/A^!\bar{e}A^!$ is Koszul, and in the diagram*

$$\begin{array}{ccc} D_{\mathrm{gr}}(eAe) & \xrightarrow{K_e} & D_{\mathrm{gr}}(A^!/A^!\bar{e}A^!) \\ {\scriptstyle i^*}\uparrow\downarrow{\scriptstyle i_*} & & {\scriptstyle q_*}\uparrow\downarrow{\scriptstyle q^*} \\ D_{\mathrm{gr}}(A) & \xrightarrow{K} & D_{\mathrm{gr}}(A^!) \end{array}$$

both squares commute up to natural isomorphism. Here the horizontal maps are the appropriate Koszul duality equivalences and the vertical maps are either pushing forward (tensoring) or pulling back (taking the induced module) by the homomorphisms i and q.

Proof. The Koszulity of $A^!/A^!\bar{e}A^!$ follows from Lemma 8.17. To prove the rest of the theorem, we must look closely at how the functor K is defined. On a complex (M^i, ∂) it is given by

$$(KM)^p_q := \bigoplus_{\substack{p=-i-j \\ q=l+j}} A^!_l \otimes_{A_0} (M^i_j)^*$$

with differential

$$\partial(a \otimes f) = (-1)^{i+j} \sum_c a\check{v}_c \otimes v_c f + a \otimes \partial f$$

for $a \in A^!_l$ and $f \in (M^i_j)^*$, where $\{v_c\}$ is a basis of A_1, $\{\check{v}_c\}$ is the dual basis of $A^!_1$, and $v_c f$ and ∂f are defined by dualizing the actions of v_c and ∂ on M.

Let us show that $K_e i^* \simeq q_* K$. For $M \in D_{\mathrm{gr}}(A)$, the underlying vector space of $K_e(i^*M)$ is

$$A^!/A^!\bar{e}A^! \otimes_{A_0/\bar{e}A_0} eM^*,$$

while $q_*(KM)$ is

$$A^!/A^!\bar{e}A^! \otimes_{A^!} \left(A^! \otimes_{A_0} M^*\right) = A^!/A^!\bar{e}A^! \otimes_{A_0} M^*.$$

It is easy to see that these are isomorphic and that the gradings agree. To check that the differentials are the same, use a basis $\{v_c\}$ of A_1 obtained by combining bases of $A_1 e$ and $A_1 \bar{e}$.

Next note that $i^* i_*$ is naturally equivalent to the identity functor on $D_{\mathrm{gr}}(eAe)$. From this it follows that

$$K_e \simeq K_e i^* i_* \simeq q_* K i^*.$$

It is also easy to see that the restriction of $q_* q^*$ to the full subcategory of $D_{\mathrm{gr}}(A^!)$ given by complexes $M^\bullet$ with $\bar{e}M^i = 0$ for all i is naturally equivalent to the identity. But since

$$K i_*(eAe) = K(Ae) = A^!_0 e,$$

the functor $K i_*$ lands in this subcategory, so we have

$$q^* K_e \simeq q^* q_* K i^* \simeq K i^*. \qquad \square$$

8.5 Duality of shuffling and twisting

Let $X_{\eta_1,\xi} = (\Lambda_0, \eta_1, \xi)$ and $X_{\eta_2,\xi} = (\Lambda_0, \eta_2, \xi)$ be regular polarized arrangements, and let $X^!_{\eta^!,\xi^!_1}$, $X^!_{\eta^!,\xi^!_2}$ be their Gale duals, as defined in Section 2.6. Our shuffling and twisting functors have graded versions, given by putting the natural grading on the bimodules that appear in Propositions 8.8 and 8.9. We denote the graded functors by the same symbols:

$$\Phi^{\eta_2}_{\eta_1} \colon D_{\mathrm{gr}}(X_{\eta_1,\xi}) \to D_{\mathrm{gr}}(X_{\eta_2,\xi}) \qquad \text{and} \qquad \Psi^{\xi^!_2}_{\xi^!_1} \colon D_{\mathrm{gr}}(X^!_{\eta^!,\xi^!_1}) \to D_{\mathrm{gr}}(X^!_{\eta^!,\xi^!_2}).$$

Theorem 8.19. *There is a natural equivalence*

$$\Psi^{\xi^!_2}_{\xi^!_1} \circ K_1 \simeq K_2 \circ \Phi^{\eta_2}_{\eta_1},$$

where $K_i \colon D_{\mathrm{gr}}(X_{\eta_i,\xi}) \to D_{\mathrm{gr}}(X^!_{\eta^!,\xi^!_i})$ is the contravariant Koszul equivalence defined in Section 8.4.

Proof. We denote by

$$A^!_{\mathrm{pol}}(\eta^!, -) := e_{\eta^!} \left(Q_n / \langle \vartheta(\mathfrak{k}^!) \rangle \right) e_{\eta^!},$$

the graded ring whose completion is $A^!(\eta^!, -)$. The theorem will follow from applying Theorem 8.18 to the diagram

$$\begin{array}{ccc}
D_{\mathrm{gr}}(A(\eta_1,\xi)) & \xrightarrow{K_1} & D_{\mathrm{gr}}(A^!(\eta^!,\xi^!_1)) \\
\downarrow & & \downarrow \\
D_{\mathrm{gr}}(A(-,\xi)) & \xrightarrow{K} & D_{\mathrm{gr}}(A^!_{\mathrm{pol}}(\eta^!,-)) \\
\downarrow & & \downarrow \\
D_{\mathrm{gr}}(A(\eta_2,\xi)) & \xrightarrow{K_2} & D_{\mathrm{gr}}(A^!(\eta^!,\xi^!_2))
\end{array}$$

where the vertical maps are the respective derived pushforwards and pullbacks, and the horizontal maps are the Koszul duality functors. Proposition 8.8 tells us that the composition of the left-hand vertical functors is $\Phi^{\eta_2}_{\eta_1}$, and Proposition 8.9 tells us that the composition of the right-hand vertical functors is $\Psi^{\xi^!_2}_{\xi^!_1}$, since the finite dimensionality of $A^!(\eta^!, \xi^!_i)$ implies that the images of $A^!_{\mathrm{pol}}(\eta^!, -)$ and $A^!(\eta^!, -)$ in this ring are the same.

In order to apply Theorem 8.18, we must verify that $A(-,\xi)$ and $A^!_{\mathrm{pol}}(\eta^!, -)$ are Koszul and dual to each other, $A(-,\xi)$ is finite dimensional, and $A^!_{\mathrm{pol}}(\eta^!, -)$ is left Noetherian. Let $R := Q_n/\langle \vartheta(\mathfrak{k}) \rangle$ and $R^! := Q_n/\langle \vartheta(\mathfrak{k}^!) \rangle$. (Note that unlike in Section 8.2, here R is a quotient of the quiver algebra Q_n rather than its completion $\widehat{Q}_n$.) Put $e := e_{\eta^!}$ and $\bar{e} := e_\xi = 1 - e$, so that

$$A^!_{\mathrm{pol}}(\eta^!, -) = eR^!e \qquad \text{and} \qquad A(-,\xi) = R/R\bar{e}R.$$

From this description it is clear that $A(-,\xi)$ is isomorphic to the ring $A_{\mathrm{ext}}(X)$ defined in [BLPWb, §6.1], where $X = X_{\eta_i,\xi}$ for $i = 1$ or 2 (the definition does not use the parameter η). The proposition [BLPWb, 6.1] says that this ring is isomorphic as a vector space to the direct sum of cohomology

groups

$$\bigoplus_{\alpha,\beta\in\mathcal{B}_\xi} H^*(C_\alpha \cap C_\beta;\mathbb{C}),$$

α and β range over the set $\mathcal{B}_\xi = \mathcal{F}_{\eta^!}$ and C_α is the relative core component indexed by α in the hypertoric variety $\mathfrak{M}(X^!_{\eta^!,\xi^!_i})$. Thus $A(-,\xi)$ is manifestly finite dimensional. To see that $A^!_{\mathrm{pol}}(\eta^!,-)$ is left Noetherian, simply note that for any $\alpha,\beta\in\mathcal{F}_\eta$,

$$e_\alpha A^!_{\mathrm{pol}}(\eta^!,-)e_\beta = e_\alpha R^! e_\beta$$

is a free module of rank one over $Z(R^!)$, which is isomorphic to the polynomial ring $\mathrm{Sym}(\mathbb{C}^n/\mathfrak{k}^!)$ via the map ϑ. Using [BLPWb, 3.8], we observe that our description of $A^!_{\mathrm{pol}}(\eta^!,-)$ coincides with that of [BLPWb, 3.1] for the polarized arrangement $X^!_{\eta^!,\xi^!_i}$, with the relation A1 deleted. The proof of [BLPWb, 3.2] then adapts immediately to show that $A^!_{\mathrm{pol}}(\eta^!,-)$ is quadratic.

At this point we simplify notation by putting

$$A := A^!_{\mathrm{pol}}(\eta^!,-) \qquad \text{and} \qquad B := A(-,\xi).$$

The rings R and $R^!$ are quadratic dual, where we let the natural basis of $R_1 = (Q_n)_1 = R^!_1$ given by making length one paths self-dual up to a sign that makes the commutation relations around each square in the cube quiver dual on the two sides (see [BLPWb, §3.3] for one way to produce these signs). It now follows from Lemma 8.17 that A and B are quadratic dual rings. Since both rings are isomorphic to their own opposites, we have $A(-,\xi)\cong A^!_{\mathrm{pol}}(\eta^!,-)^!$. It thus remains only to show that

Lemma 8.20. *A is Koszul and B is its Koszul dual. In particular, A has finite global dimension.*

By [BGS96, 2.6.1], A is Koszul if and only if its Koszul complex is a resolution of $A_0 = A/A_{>0}$. This complex can be defined as follows. As a vector space it is $A\otimes B^*$, where we put $\otimes = \otimes_{A_0}$ for the remainder of the proof. This complex is bigraded, with $A_i\otimes B^*_j$ in degree $(-j,i+j)$. It is a graded left A-module using the second grading, and a complex using the first grading, where the differential ∂ is the composition

$$A\otimes B^* \to A\otimes(B^*_1\otimes B^*) \to (A\otimes A_1)\otimes B^* \to A\otimes B^*$$

using the comultiplication on B^*, the identification $A_1 = B^*_1$, and the multiplication on A. The map $A\otimes B^*\to A_0$ sends $A_0\otimes B^*_0$ isomorphically to A_0 and kills all higher degree terms. As a result, what we need to show is that for any $\alpha,\beta\in\mathcal{F}_{\eta^!}=\mathcal{B}_\xi$ the complex $e_\alpha A\otimes B^* e_\beta$ is a resolution of $\mathbb{C}$ if $\alpha=\beta$ and is exact if $\alpha\neq\beta$.

First consider the case $\alpha\neq\beta$. Since $e_\alpha A\otimes B^* e_\beta$ is a complex of free graded modules over $Z := Z(R^!)$, it is exact if and only if the reduced complex $\mathbb{C}\otimes_Z e_\alpha A\otimes B^* e_\beta$ is exact. It will be more convenient to work with the dual complex, which we denote by $\Sigma^\bullet_{\alpha\beta}$. To describe $\Sigma^\bullet_{\alpha\beta}$ more

precisely, first note that for any $\gamma \in \mathcal{F}_{\eta^!}$, $\mathbb{C} \otimes_Z e_\alpha A e_\gamma$ is a copy of $\mathbb{C}$ placed in degree

$$d_{\alpha\gamma} := |\{1 \le i \le n \mid \alpha(i) \ne \gamma(i)\}|.$$

Following the notation of [BLPWb, §4.1], we have an isomorphism of vector spaces

$$\Sigma^\bullet_{\alpha\beta} \cong \bigoplus_{\gamma \in \mathcal{B}_\xi} H^*(C_\gamma \cap C_\beta; \mathbb{C})[-d_{\alpha\gamma} - d_{\gamma\beta}] = \bigoplus_{\gamma \in \mathcal{B}_\xi} R_{\gamma\beta}[-d_{\alpha\gamma} - d_{\gamma\beta}],$$

where $R_{\gamma\beta}$ is the reduced face ring of the simplicial complex associated to the polyhedron $\Delta_{\gamma\beta} := \Delta_\gamma \cap \Delta_\beta$. More precisely, if we let $\tilde{R}_{\gamma\beta}$ be the quotient of the polynomial ring $\mathrm{Sym}(W) \cong \mathbb{C}[e_1, \dots, e_n]$, with generators e_i in degree 2, by the ideal

$$\Big\langle \prod_{i \in S} e_i \;\Big|\; \Delta_{\gamma\beta} \cap \bigcap_{i \in S} H_i = \emptyset \Big\rangle,$$

then $R_{\gamma\beta}$ is the quotient of $\tilde{R}_{\gamma\beta}$ by the ideal generated by $\sum_i a_i e_i$ for all $a = (a_1, \dots, a_n) \in \mathfrak{k}^! \subset W$.

Consider the component

$$\partial_{\gamma\gamma'} \colon R_{\gamma\beta}[-d_{\alpha\gamma} - d_{\gamma\beta}] \to R_{\gamma'\beta}[-d_{\alpha\gamma'} - d_{\gamma\beta}]$$

of the boundary map. It is easy to see for degree reasons that $\partial_{\gamma\gamma'} = 0$ unless $d_{\gamma\gamma'} = 1$ and $d_{\alpha\gamma'} = d_{\alpha\gamma} - 1$, which means that γ' is one step closer to α than γ is. If these conditions hold and $d_{\beta\gamma} = d_{\beta\gamma'} + 1$, then $\Delta_{\gamma'\beta} \subset \Delta_{\gamma\beta}$ and up to a sign $\partial_{\gamma\gamma'}$ is the natural quotient $R_{\gamma\beta} \to R_{\gamma'\beta}$. If instead $d_{\beta\gamma} = d_{\beta\gamma'} - 1$, then $\Delta_{\gamma\beta} \subset \Delta_{\gamma'\beta}$, and up to a sign $\partial_{\gamma\gamma'}$ is induced by multiplication by e_i, where γ and γ' differ in the i^{th} place. The signs come from the choice of signs in the quadratic duality between R and $R^!$, and they are arranged so that the two paths around any square given by $\gamma_1, \dots, \gamma_4 \in \mathcal{F}_{\eta^!}$ which differ in exactly two places have opposite signs.

To show that $\Sigma^\bullet_{\alpha\beta}$ is exact, it will be enough to show exactness of the "equivariant" complex $\tilde{\Sigma}^\bullet_{\alpha\beta}$ which is obtained by replacing each summand $R_{\gamma\beta}$ with $\tilde{R}_{\gamma\beta}$ and using the same formula for the boundary map, since each $\tilde{R}_{\gamma\beta}$ is a free $\mathbb{C}[\mathfrak{k}^!]$-module. Since $\tilde{R}_{\gamma\beta}$ is a monomial module and the $\partial_{\gamma\gamma'}$ are monomial maps, we can check this one monomial at a time. For a function $\nu \colon \{1, \dots, n\} \to \mathbb{N}$ we let $e^\nu := \prod_i e_i^{\nu(i)}$ be the corresponding monomial of $\mathbb{C}[e_1, \dots, e_n]$, and let $\tilde{R}_{\gamma\beta}(\nu) \subset \tilde{R}_{\gamma\beta}$ be the vector space spanned by its image. Let $I = \{1 \le i \le n \mid \alpha(i) = \beta(i)\}$. Then $\tilde{\Sigma}^\bullet_{\alpha\beta}$ splits into a direct sum of complexes

$$\tilde{\Sigma}^\bullet_{\alpha\beta}(\nu) := \bigoplus_{\gamma \in \mathcal{F}_{\eta^!}} \tilde{R}_{\gamma\beta}(\nu - \mu(\gamma, \beta)), \tag{8}$$

where

$$\mu(\gamma, \beta)(i) = \begin{cases} 1 & \text{if } i \in I \text{ and } \gamma(i) \ne \beta(i), \\ 0 & \text{otherwise.} \end{cases}$$

Each summand in (8) has dimension zero or one, and if $d_{\gamma\gamma'} = 1$ and the corresponding terms are

nonzero, then the restriction of $\partial_{\gamma\gamma'}$ to $\tilde{\Sigma}^\bullet_{\alpha\beta}(\nu)$ is ± 1 with respect to our monomial basis.

The nonzero terms of (8) are indexed by

$$\left\{ \gamma \in \{+,-\}^n \;\middle|\; \gamma|_{I_\nu} = \alpha|_{I_\nu} = \beta|_{I_\nu} \text{ and } \Delta_{\gamma\beta} \cap \bigcap_{\nu(i)\neq 0} H_i \neq \emptyset \right\} \tag{9}$$

where $I_\nu = \{i \in I \mid \nu(i) = 0\}$.

First suppose that $\nu|_I \equiv 0$, so $I_\nu = I$. Define the set

$$J := \{i \notin I \mid H_i \cap \Delta_\beta \neq \emptyset\}$$

indexing the hyperplanes which separate Δ_α and Δ_β and meet Δ_β. Since $\alpha \neq \beta$, it follows that $J \neq \emptyset$, and since $\Delta_\alpha \neq \emptyset$, we must have $\Delta_\beta \cap \bigcap_{i\in J} J_i \neq \emptyset$. Take some $i \in J$; then for any γ in the set (9), the sign vector γ' obtained by flipping the i^{th} entry will also be in this set. It follows that $\tilde{\Sigma}^\bullet_{\alpha\beta}(\nu)$ has a filtration by two-step acyclic complexes and thus is itself acyclic.

Alternatively, if $\nu|_I \not\equiv 0$, choose $i \in I$ with $\nu(i) \neq 0$. We can assume that $H_i \cap \Delta_\beta \neq \emptyset$, since if there is no such i then $\tilde{\Sigma}^\bullet_{\alpha\beta}(\nu) = 0$. Then for any γ in (8), flipping the i^{th} entry gives another sign vector in the same set, so again $\tilde{\Sigma}^\bullet_{\alpha\beta}(\nu)$ is acyclic.

Finally, consider the case $\alpha = \beta$. Our argument above is not quite enough, since although we can use it to show that $\tilde{\Sigma}^\bullet_{\alpha\alpha}$ is a resolution of $\mathbb{C}$, we haven't proved a direct connection between $e_\alpha A \otimes B^* e_\alpha$ and $\tilde{\Sigma}^\bullet_{\alpha\alpha}$, but only between their reductions:

$$\Sigma^\bullet_{\alpha\alpha} = \mathbb{C} \otimes_{\mathrm{Sym}(\mathfrak{k}^!)} \tilde{\Sigma}^\bullet_{\alpha\alpha} \cong (\mathbb{C} \otimes_Z e_\alpha A \otimes B^* e_\alpha)^*. \tag{10}$$

Let p be a vertex of Δ_α, and set $\Gamma_p := \{\gamma \in \mathcal{F}_{\eta^!} \mid p \in \Delta_\gamma\}$. This set has 2^k elements, $k = \dim(\mathfrak{k}^!)$, namely all $\gamma \in \{+,-\}^n$ for which $\gamma(i) = \alpha(i)$ if $p \notin H_i$.

Consider the subspace $D = \bigoplus_{\gamma\in\Gamma_p} Z(a_{\alpha\gamma} \otimes b^*_{\gamma\alpha})$ of $e_\alpha A \otimes B^* e_\alpha$, where $a_{\alpha\gamma}$ and $b^*_{\gamma\alpha}$ are minimal degree elements of $e_\alpha A e_\gamma$ and $e_\gamma B^* e_\alpha$, respectively. It is a subcomplex of Z-submodules, and it is easy to check that it is a resolution of $\mathbb{C}$. On the other hand, let E be the kernel of multiplication by the monomial $\prod_{p\in H_i} e_i$ on the complex $\tilde{\Sigma}^\bullet_{\alpha\alpha}$. Then $\tilde{\Sigma}^\bullet_{\alpha\alpha}/E$ is isomorphic as a $S = \mathrm{Sym}(\mathfrak{k}^!)$-module to $\bigoplus_{\gamma\in\Gamma_p} S$, and one can check that it is a resolution of $\mathbb{C}$. It follows that E is acyclic, and therefore so is

$$\mathbb{C} \otimes_S E \cong \left[\mathbb{C} \otimes_Z (e_\alpha A \otimes B^* e_\alpha / D)\right]^*.$$

It follows that $e_\alpha A \otimes B^* e_\alpha$ is a resolution of $\mathbb{C}$, as desired. □

Remark 8.21. There is an ungraded version of this theorem as well; Koszul duality gives an equivalence between the ungraded derived category $D(X_{\eta,\xi})$ and the triangulated category of finitely generated DG-modules over $A^!(\xi^!, \eta^!)$. There is a version of shuffling and twisting functors in the DG setting as well, given by tensor product with the same bimodules, and the proof of Theorem 8.19 shows that Koszul duality sends DG shuffling to ungraded twisting and vice versa. By standard homological algebra, we can infer from Theorem 8.19 that all versions (graded, ungraded, and DG)

of shuffling functors are equivalences: they are full and faithful on gradable modules, and these generate the triangulated category.

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