VERTEX OPERATORS AND 2-REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

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Abstract. We construct 2-representations of quantum affine algebras from 2-representations of Heisenberg algebras. The main tools in the construction are categorical vertex operators, which are special complexes in a Heisenberg 2-representation that recover vertex operators after passing to the Grothendieck group. As an application we categorify the Frenkel-Kac-Segal homogeneous realization of the basic representation of quantum affine algebras of simply-laced type. This gives rise to categorical actions of quantum affine (and toroidal) algebras on derived categories of coherent sheaves on Hilbert schemes of points of ALE spaces.

Dedicated to our advisors Igor Frenkel and Joe Harris on the occasion of their sixtieth birthdays.

Contents

1. Introduction

Affine Lie algebras are central objects at the intersection of representation theory and mathematical physics. They have a rich representation theory generalizing that of finite dimensional semi-simple Lie algebras, and their integrable representations admit various explicit constructions. Distinguished among these is the homogeneous vertex operator construction of the basic representation due to Frenkel-Kac [FK] and Segal [S].

In this paper we categorify this construction. The key ingredients are the homogeneous Heisenberg subalgebra, its Fock space representation, and the associated vertex operators. The first two ingredients were categorified in [CLi1]. The main result in this paper is a categorification of the vertex operators.

A geometric construction of the basic representation was given by Nakajima [N1, N2] and Grojnowski [G]. In their setting, the underlying vector space of the basic representation is the cohomology of the moduli spaces of rank one torsion-free sheaves on the resolution $\widehat{\mathbb{C}^2/\Gamma}$ of a singularity of type A, D

Date: December 19, 2011.

or E. They define a Heisenberg algebra action on this cohomology and use the Frenkel-Kac-Segal construction to induce an action of the affine Lie algebra.

In [CLi1] we lift the Nakajima-Grojnowski Heisenberg action to a 2-representation of the *quantum* Heisenberg algebra. This gives a 2-representation on the derived category of $\mathbb{C}^\times$-equivariant coherent sheaves on the Hilbert schemes of points of $\widehat{\mathbb{C}^2/\Gamma}$. The current paper extends this to a 2-representation of the entire quantum affine algebra. In fact, we can extend to a 2-representation of quantum toroidal algebras. Subsequently, we recover quantum toroidal algebra actions on the $\mathbb{C}^\times$-equivariant K-theory of the moduli spaces of rank one sheaves, as conjectured by several mathematicians in the 1990s.

2-representations of quantum affine algebras have been studied in several circumstances [KL2, KL3, R, CK, CKL3]. The 2-categories considered appear in the modular representation theory of symmetric groups and of Hecke algebras at roots of unity, in the geometry of moduli spaces of sheaves on surfaces, and elsewhere. However, all these examples use the Kac-Moody presentation of quantum affine algebras. In this paper, the quantum affine algebra arises naturally in the alternative loop (or Drinfeld) presentation. The higher representation theory of loop algebras is not well understood yet, and one of our goals is to propose a definition for 2-representations of quantum affine algebras in their loop realization.

The loop realization of quantum affine algebras is important if one wants to understand relationships to conformal field theory and low dimensional topology (see section 1.1). A significant difference between this realization and the Kac-Moody realization is that we must deal with 2-categories which are triangulated rather than just additive. Nevertheless, our definition should be related (in a precise but non-trivial way) to its Kac-Moody analogue.

We expect that this categorification of vertex operators in the Frenkel-Kac-Segal construction is the beginning of a larger categorification program for vertex algebras. For example, the entire vertex operator algebra structure in the basic representation, which contains not only the vertex operators of this paper but also the Virasoro algebra and other structure, should be categorified. The problem of categorifying vertex algebras was posed over ten years ago by Igor Frenkel.

1.1. **Relations to mathematical physics and invariants in low dimensional topology.** One reason to be interested in categorifying representations of affine Lie algebra is the relationship to conformal field theory. In particular, the Wess-Zumino-Witten model in conformal field theory take as input the representation theory of affine Lie algebras. Spaces of conformal blocks are defined using the fusion product on these representations giving vector bundles over a Riemann surface. Monodromy on these bundles induces an action of the mapping class group which, in turn, gives rise to 3-manifold invariants [Ko]. The existence of categorified vertex operators suggests that these 3-manifold invariants, and perhaps much of conformal field theory itself, should be categorified.

The vertex operators used in the construction of the basic representation [FK, S] first arose in physics literature of dual resonance models. We are curious about the physical significance, if any, of the categorified vertex operators we consider here.

1.2. **Vertex operators in the homogeneous realization.** In the remainder of this introduction we sketch the Frenkel-Kac-Segal homogeneous realization of the basic representation and our categorification of it. Let $\mathfrak{g}$ be a finite dimensional Lie algebra of type A, D or E, and let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be its affinization. The enveloping algebra $U(\widehat{\mathfrak{g}})$ contains a subalgebra $\widehat{\mathfrak{h}}$, known as the homogeneous Heisenberg subalgebra. The algebra $\widehat{\mathfrak{h}}$ has essentially one irreducible representation $\mathcal{F}$, known as the Fock space.

Let V_{Λ_0} be the basic representation of $\widehat{\mathfrak{g}}$, that is, the highest weight irreducible representation of highest weight Λ_0. The basic representation is the "simplest" of the representations of $\widehat{\mathfrak{g}}$ that can be integrated to the group. The Frenkel-Kac-Segal construction of V_{Λ_0} begins with the observation that the restriction of V_{Λ_0} from $\widehat{\mathfrak{g}}$ to the homogeneous Heisenberg algebra $\widehat{\mathfrak{h}}$ decomposes as a direct sum of

copies of the Fock space

$$V_{\Lambda_0} = \bigoplus_{\lambda \in Y} \mathcal{F},$$

with the summands indexed by the root lattice Y of $\mathfrak{g}$. Thus, in order to give an explicit construction of V_{Λ_0}, one can take a direct sum of copies of the Fock space, one for each element of the root lattice, and explain how to extend the action on this space from $\widehat{\mathfrak{h}}$ to $\widehat{\mathfrak{g}}$.

Frenkel-Kac [FK] and Segal [S] show that the action of $\widehat{\mathfrak{g}}$ can be constructed from the action of $\widehat{\mathfrak{h}}$ and translation in the root lattice Y via the use of vertex operators. A basic example of a vertex operator is the formal series

$$X(i,z) = \exp\big(\sum_{n=1}^{\infty} \frac{z^n}{n} h_{i,-n}\big) \exp\big(-\sum_{n=1}^{\infty} \frac{z^{-n}}{n} h_{i,n}\big) \exp(\log z \cdot \alpha_i(0) + \alpha_i)$$

where $h_{i,n}$ are generators of $\widehat{\mathfrak{h}}$ and α_i is a simple root of $\mathfrak{g}$. The above expression contains three exponentials. The homogeneous components in z of the first two exponentials are endomorphisms of the Fock space $\mathcal{F}$, while the homogeneous components of the term $\exp(\log z \cdot \alpha_i(0) + \alpha_i)$ are "lattice translation" operators which moves the various copies of $\mathcal{F}$ along the lattice (we refer to [FK] for the precise definition of the lattice translation operator). Each homogeneous component of $X(i,z)$ is thus an endomorphism of V_{Λ_0}. These endomorphisms, together with their adjoints, generate the action of the affine Lie algebra, giving an explicit construction of the basic representation.

It was later shown in [FJ] that this construction admits a q-deformation. In this q-deformation, the Heisenberg algebra $\widehat{\mathfrak{h}}$ is replaced by the quantum Heisenberg algebra, and the vertex operators $X(i,z)$ are replaced by q-vertex operators

$$X_q(i,z) = \exp\big(\sum_{n=1}^{\infty} \frac{z^n}{[n]} h_{i,-n}\big) \exp\big(-\sum_{n=1}^{\infty} \frac{z^{-n}}{[n]} h_{i,n}\big) \exp(\log z \cdot \alpha_i(0) + \alpha_i)$$

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ is the quantum integer. The homogeneous components of the $X_q(i,z)$, together with their adjoints, then give the basic representation of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$.

In order to categorify the vertex operators $X_q(i,z)$ we use new operators $P_i^{(n)}$ and $Q_i^{(1^n)}$ defined by the formulas

$$\exp\left(\sum_{m \geq 1} \frac{h_{i,-m}}{[m]} z^m\right) = \sum_{n \geq 0} P_i^{(n)} z^n \text{ and } \exp\left(-\sum_{m \geq 1} \frac{h_{i,m}}{[m]} z^m\right) = \sum_{n \geq 0} (-1)^n Q_i^{(1^n)} z^n.$$

In terms of these operators, a single homogeneous component in z of $X_q(i,z)$ becomes an expression of the form

$$\big[\sum_{n \geq 0} (-1)^n P_i^{(n+k)} Q_i^{(1^n)}\big] t_i, \tag{1}$$

where t_i acts only in the lattice and $\sum_{n \geq 0} (-1)^n P_i^{(n+k)} Q_i^{(1^n)}$ acts only in the Fock space. Thus, the q-deformed Frenkel-Kac-Segal construction says that these alternating sums, together with their adjoints, generate an action of the quantum affine Lie algebra $U_q(\widehat{\mathfrak{g}})$.

1.3. **Categorification of the basic representation.** A categorification of the Heisenberg algebra $\widehat{\mathfrak{h}}$ and its Fock space representation was given in [CLi1] (see also [K], [LS] for related definitions). We recall this definition in section 3.2, where we also define a the notion of a 2-representations of $\widehat{\mathfrak{h}}$. Very roughly, a 2-representation of $\widehat{\mathfrak{h}}$ consists of a 2-category $\mathcal{K}$ where the objects are indexed by the natural numbers, the 1-morphisms are compositions of generating 1-morphisms P_i and Q_i, and there are 2-morphisms with specified relations (these relations are described using a graphical calculus of planar

diagrams). The relations for 2-morphisms imply that the 1-morphisms P_i and Q_i satisfy categorical analogs of the relations in the Heisenberg algebra. Thus a 2-representation of $\widehat{\mathfrak{h}}$ is a categorification of a representation of the Heisenberg algebra.

To categorify the Frenkel-Kac-Segal construction of the basic representation of $\widehat{\mathfrak{g}}$ we need to lift the operators in (1) from vector spaces to categories. In a 2-representation $\mathcal{K}$ of $\widehat{\mathfrak{h}}$, the operators $P_i^{(n)}$ and $Q_i^{(1^n)}$ are lifted to indecomposable 1-morphisms in $\mathcal{K}$. Thus, it is natural that alternating sums like (1) should lift to complexes in the homotopy category $\mathsf{Kom}(\mathcal{K})$ of $\mathcal{K}$,

$$\cdots \to \mathsf{P}_i^{(n+k)}\mathsf{Q}_i^{(1^n)}\langle l\rangle \to \mathsf{P}_i^{(n+k+1)}\mathsf{Q}_i^{(1^{n+1})}\langle l+1\rangle \to \dots$$

In Section 5.1 we define these complexes.

It is immediate from the definition that our complexes descend to the homogeneous components of vertex operators after passing to the Grothendieck group. In fact, this would be true regardless of the differentials we choose in our complex (indeed, even the zero differential would give a complex which descends to the correct operator in the Grothendieck group). The interesting content is that these complexes satisfy categorical relations of the quantum affine algebra inside $\mathsf{Kom}(\mathcal{K})$ itself, before passing to the Grothendieck group. A summary of the relations we check is in the statement of Theorem 5.2.

The (somewhat long) list of relations satisfied by the complexes of Section 5.1 can be summarized by saying that they define a 2-representation of $\widehat{\mathfrak{g}}$. More precisely, in section 4.2 we define what it means to have a 2-representation of $\widehat{\mathfrak{g}}$. The content of our theorem 5.2 is then that, given a 2-representation of $\widehat{\mathfrak{h}}$, our complexes define a 2-representation of $\widehat{\mathfrak{g}}$.

Our constructions also apply with no extra work to give a 2-representation of the quantum toroidal algebra $U_q(\widehat{\widehat{\mathfrak{g}}})$. This algebra is a loop affinization of the quantum affine algebra in its Kac-Moody presentation. In section 9 we explain how to obtain a 2-representation of the quantum toroidal algebra on the derived categories of coherent sheaves on Hilbert schemes of points on $\widehat{\mathbb{C}^2/\Gamma}$.

1.4. **Generalization to other Kac-Moody algebras.** The essential idea at the core of our construction is that 2-representations of $\widehat{\mathfrak{h}}$ (respectively $\widehat{\widehat{\mathfrak{h}}}$) give rise to 2-representation of $\widehat{\mathfrak{g}}$ (respecitively $\widehat{\widehat{\mathfrak{g}}}$). On the other hand, the only input data used to define 2-representations of $\widehat{\mathfrak{h}}$ is essentially the finite dimensional algebra $B_\Gamma := \mathbb{C}[\Gamma] \ltimes \wedge^*(\mathbb{C}^2)$, where $\Gamma \subset SL_2(\mathbb{C})$ is the finite subgroup associated to the Dynkin diagram of $\widehat{\mathfrak{g}}$ using the McKay correspondence. Thus we obtain what can be viewed as a categorical form of the McKay correspondence: starting from the finite subgroup $\Gamma \subset SL_2(\mathbb{C})$, we construct the finite-dimensional algebra B_Γ, and using this algebra we construct 2-representations of the associated quantum affine (and toroidal) algebra. Moreover, the algebra B_Γ can be described directly in terms of the underlying affine Dynkin diagram without direct reference to the finite group Γ. This description suggests how to generalize the constructions of this paper to other Kac-Moody type. More general Kac-Moody algebras are not the main subject of this paper and will therefore be discussed elsewhere, but we will briefly describe how to generalize the algebra B_Γ to other Dynkin type.

Fix a simply laced Dynkin diagram D, and choose an orientation of the edges. Let $\mathbb{C}[dD]$ denote the path algebra of the doubled quiver dD. Thus a path in dD is described as a sequence of vertices $(a_1|a_2|\dots|a_k)$ where a_i and a_{i+1} are connected by an edge in D. We define B_D to be the quotient of $\mathbb{C}[dD]$ by the two sided ideal generated by

- $(a|b|c)$ if $a \neq c$ and
- $(a|b|a) + (a|c|a)$ whenever a is connected to both b and c.

The algebra B_D can be used to define a 2-category which categorify the Heisenberg algebra $\mathfrak{h}_D$ associated to the root lattice of D. The constructions of this paper then categorify the Fock space modeled on

this lattice. In particular, the categorical vertex operators can be used to construct 2-representations of the quantum group associated to D.

The algebra B_D appears in [HK], where it is called the skew zig-zag algebra. In that paper the zigzag algebra is used to categorify the adjoint representation of $\mathfrak{g}$ for $\mathfrak{g}$ of finite type. Our categorification of the basic representation is in fact an extension of their categorification of the adjoint representation: the restriction of the basic representation V_{Λ_0} of $\widehat{\mathfrak{g}}$ to $\mathfrak{g}$ contains the adjoint representation of $\mathfrak{g}$ as a direct summand. After restricting our categorification of V_{Λ_0} from $\widehat{\mathfrak{g}}$ to the copy of the adjoint representation of $\mathfrak{g}$ sitting inside it, our complexes recover the categorification in [HK]. In this restricted construction, the complexes have length at most two, which makes checking the relations in the adjoint representation of $\mathfrak{g}$ much easier than checking the relations in the basic representation of $\widehat{\mathfrak{g}}$.

Acknowledgments: The authors benefited from discussions with Mikhail Khovanov, Aaron Lauda, Hiraku Nakajima, Raphael Rouquier, Travis Schedler and Joshua Sussan. S.C. was supported by NSF grants DMS-0964439, DMS-1101439 and the Alfred P. Sloan foundation. A.L. would like to thank the Institute for Advanced Study for support.

2. Notation and terminology

For the entirety of this paper we let $\Bbbk$ denote a field of characteristic zero. We let $\Bbbk(q)$ denote the field of rational functions of one variable, q. We denote

$$[j] := q^{-j+1} + q^{-j+3} + \cdots + q^{j-3} + q^{j-1}$$

the quantum integer. If $j \geq 0$ then V_j denotes the graded vector space

$$V_j := \Bbbk\langle j\rangle \oplus \Bbbk\langle j-2\rangle \oplus \cdots \oplus \Bbbk\langle -j+2\rangle \oplus \Bbbk\langle -j\rangle \tag{2}$$

where $\langle 1\rangle$ is a shift of 1 in the grading.

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition then $|\lambda| := \sum_i \lambda_i$ denotes its size. We say that $\lambda' \subset \lambda$ if λ' is contained in λ, meaning that $\lambda_i \geq \lambda'_i$ for all i. We denote by λ^t the transposed partition of λ (for example, $(n)^t = (1^n)$).

2.1. **Dynkin data.** From now on fix a simply-laced Dynkin diagram of affine type and denote its vertex set by $\hat{I}$. The special affine vertex in $\hat{I}$ is labeled 0 and we let $I := \hat{I} \setminus \{0\}$. The subdiagram whose vertex set is I is a Dynkin diagram of finite type A, D, E. We denote the Lie algebras associated to these Dynkin diagrams by $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$.

We denote the weight lattice of $\mathfrak{g}$ by X and the root lattice by Y. The standard pairing on the weight lattice is denoted with brackets $\langle\cdot,\cdot\rangle$, which should not be confused with the grading shift on categories. For $i \in I$, $\alpha_i \in Y$ and $\Lambda_i \in X$ will denote the simple roots and fundamental weights, respectively. These satisfy the relation $C_{i,j} = \langle \alpha_i, \alpha_j\rangle$ where $C_{i,j}$ is the Cartan matrix of $\mathfrak{g}$. In terms of the Dynkin diagram, we have

$$\langle \alpha_i, \alpha_j\rangle = \begin{cases} 2 \text{ if } i = j \\ -1 \text{ if } i \neq j \text{ are joined by an edge} \\ 0 \text{ if } i \neq j \text{ are not joined by an edge.} \end{cases}$$

Moreover, $\langle \Lambda_i, \alpha_j\rangle = \delta_{i,j}$ for all $i, j \in I$. Sometimes, if we will write $\langle i, j\rangle$ instead of $\langle \alpha_i, \alpha_j\rangle$. The dominant weights in X are denoted $X^+ \subset X$. Similarly, we denote by $\widehat{X}$ the weight lattice of $\widehat{\mathfrak{g}}$, $\widehat{Y}$ the affine root lattice, etc. We use the same notation $\langle\cdot,\cdot\rangle$ for the pairing on the affine weight lattice as for the finite weight lattice. The imaginary root, which is denoted δ, satisfies $\langle \delta, \alpha_i\rangle = 0$ for all i. The associated Weyl groups of X and $\widehat{X}$ are denoted W and $\widehat{W}$.

Fix an orientation ϵ of the Dynkin diagram of $\mathfrak{g}$. If $\langle i,j\rangle = -1$ then we set $\epsilon_{ij} = 1$ if the edge is oriented $i \to j$ by ϵ and $\epsilon_{ij} = -1$ if it oriented $j \to i$. If $\langle i,j\rangle = 0$ then we set $\epsilon_{ij} = 0$. Notice that in both cases we have $\epsilon_{ij} = -\epsilon_{ji}$.

2.2. **Graded 2-categories.** By a graded category we will mean a category equipped with an auto-equivalence $\langle 1\rangle$. We denote by $\langle l\rangle$ the auto-equivalence obtained by applying $\langle 1\rangle$ a total of l times. The Grothendieck group $K_0(\mathcal{C})$ of an additive category $\mathcal{C}$ is the abelian group generated the set $\{[A] : A \in Ob(\mathcal{C})\}$ modulo the relation $[A] + [A'] = [A'']$ if $A'' \cong A \oplus A'$. This group is a $\mathbb{Z}[q,q^{-1}]$-module where q acts by the shift $\langle 1\rangle$. We usually tensor this with the field $\Bbbk(q)$ to obtain a $\Bbbk(q)$-module.

A graded additive $\Bbbk$-linear 2-category $\mathcal{K}$ is a category enriched over graded additive $\Bbbk$-linear categories. This means that for any two objects $A, B \in \mathcal{K}$ the Hom category $\mathrm{Hom}_{\mathcal{K}}(A,B)$ is a graded additive $\Bbbk$-linear category. Moreover, the composition map $\mathrm{Hom}_{\mathcal{K}}(A,B)\times\mathrm{Hom}_{\mathcal{K}}(B,C) \to \mathrm{Hom}_{\mathcal{K}}(A,C)$ is a graded additive $\Bbbk$-linear functor.

A graded additive $\Bbbk$-linear 2-functor $F : \mathcal{K} \to \mathcal{K}'$ is a (weak) 2-functor that maps the Hom categories $\mathrm{Hom}_{\mathcal{K}}(A,B)$ to $\mathrm{Hom}_{\mathcal{K}'}(FA,FB)$ by additive functors that commute with the auto-equivalence $\langle 1\rangle$.

If $\mathcal{K}$ is an additive 2-category, the Grothendieck group $K_0(\mathcal{K})$ is a $\Bbbk(q)$-linear category whose objects are the same as those of $\mathcal{K}$ and whose morphism spaces are

$$\mathrm{Hom}_{K_0(\mathcal{K})}(A,A') = K_0(\mathrm{Hom}_{\mathcal{K}}(A,A')).$$

Example 1: Suppose B_n is a sequence of graded $\Bbbk$-algebras indexed by $n \in \mathbb{N}$. Then one can define a 2-category $\mathcal{K}$ whose objects (0-morphisms) are indexed by $\mathbb{N}$, the 1-morphisms are graded (B_m, B_n)-bimodules and the 2-morphisms are maps of graded (B_m, B_n)-bimodules.

2.2.1. *Idempotent completeness.* An additive category $\mathcal{C}$ is said to be idempotent complete when every idempotent 1-morphism splits in $\mathcal{C}$. The idempotent completion (or Karoubi envelope) $Kar(\mathcal{C})$, of an additive category $\mathcal{C}$ is defined as the category whose objects exactly the objects of $\mathcal{C}$, and whose 1-morphisms are pairs (A,e) where $e : A \to A$ is an idempotent endomorphism, $e^2 = e$. 2-morphisms $(A,e) \to (A',e')$ in $Kar(\mathcal{C})$ are morphisms $f : A \to A'$ in $\mathcal{C}$ such that the following diagram is commutative.

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle e}\downarrow & \searrow^{f} & \downarrow{\scriptstyle e'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

The idempotent completion of $\mathcal{C}$ can be viewed as a minimal enlargement of the category $\mathcal{C}$ so that idempotents split.

Similarly, we say that the additive 2-category $\mathcal{K}$ is idempotent complete when the Hom categories $\mathrm{Hom}_{\mathcal{K}}(A,B)$ are idempotent complete for any pair of objects $A, B \in \mathcal{K}$, (so that all idempotent 2-morphisms split). The idempotent completion $Kar(\mathcal{K})$ of a 2-category $\mathcal{K}$ is the 2-category with the same objects as $\mathcal{K}$, but with Hom categories given by $Kar(\mathrm{Hom}_{\mathcal{K}}(A,B))$.

2.2.2. *Triangulated 2-categories.* A graded triangulated category is a graded category equipped with a triangulated structure where the autoequivalence $\langle 1\rangle$ takes exact triangles to exact triangles. We denote the homological shift by $[\cdot]$ where $[1]$ denotes a downward shift by one. The Grothendieck group $K_0(\mathcal{C})$ of a graded triangulated category $\mathcal{C}$ is the abelian group generated the set $\{[A] : A \in Ob(\mathcal{C})\}$ modulo the relation $[A] + [A'] = [A'']$ if there exists a distinguished triangle $A \to A'' \to A'$. As before, this is a $\mathbb{Z}[q,q^{-1}]$-module where q acts by $\langle 1\rangle$.

A graded triangulated $\Bbbk$-linear 2-category $\mathcal{K}'$ is a category enriched over graded triangulated $\Bbbk$-linear categories. This means that for any two objects $A, B \in \mathcal{K}'$ the Hom category $\mathrm{Hom}_{\mathcal{K}'}(A,B)$ is a graded additive $\Bbbk$-linear triangulated category. Here are two examples to keep in mind.

Example 2: the homotopy category $\mathcal{K}' := \mathsf{Kom}(\mathcal{K})$ of a graded additive k-linear 2-category $\mathcal{K}$. The objects of $\mathcal{K}'$ are the same as the objects of $\mathcal{K}$. The 1-morphisms of $\mathcal{K}'$ are unbounded complexes of 1-morphisms in $\mathcal{K}$, and 2-morphisms are maps of complexes. Two complexes of 1-morphisms are then deemed isomorphic if they are homotopy equivalent. This makes $\mathrm{Hom}_{\mathcal{K}'}(A, B)$ into a graded triangulated category.

Example 3: the 2-category of Fourier-Mukai (FM) transforms. The objects are a set of varieties $\{Y_i\}$, where each variety defined over $\Bbbk$. The 1-morphisms are objects in the derived category of coherent sheaves $DCoh(Y_i \times Y_j)$ (these 1-morphisms are known as FM kernels). The 2-morphisms are morphisms between these FM kernels. If all varieties also carry a $\Bbbk^\times$ action then considering derived categories of $\Bbbk^\times$-equivariant coherent sheaves allows for an extra grading and yields a graded triangulated $\Bbbk$-linear 2-category.

3. Heisenberg algebras and 2-representations of $\widehat{\mathfrak{h}}$

3.1. **The quantum Heisenberg algebra.** The quantum Heisenberg algebra, which we denote by $\widehat{\mathfrak{h}}$, plays a central role in all of the constructions to come. We begin by describing this algebra and its Fock space representation.

The traditional presentation for the quantum Heisenberg algebra is as a unital algebra generated by $h_{i,n}$, where $i \in I$ and $n \in \mathbb{Z} \setminus \{0\}$. The relations are

$$[h_{i,m}, h_{j,n}] = \delta_{m,-n}[n\langle i, j\rangle]\frac{[n]}{n}. \tag{3}$$

When $q = 1$, this presentation specializes to the standard presentation of the non-quantum Heisenberg algebra. Sometimes, relation (3) appears in the literature with a minus sign on the right hand side, though this does not change the isomorphism class of the algebra itself (just replace $h_{i,m}$ with $-h_{i,m}$ if $m > 0$).

Our preferred presentation of $\widehat{\mathfrak{h}}$ is slightly less common. We use generators for $P_i^{(n)}$ and $Q_i^{(n)}$ which are obtained from the standard generators $h_{i,m}$ via the generating functions

$$\exp\left(\sum_{m\geq 1}\frac{h_{i,-m}}{[m]}z^m\right) = \sum_{n\geq 0} P_i^{(n)} z^n \text{ and } \exp\left(\sum_{m\geq 1}\frac{h_{i,m}}{[m]}z^m\right) = \sum_{n\geq 0} Q_i^{(n)} z^n. \tag{4}$$

Lemma 3.1. *The elements $\{P_i^{(n)}, Q_i^{(n)}\}_{i\in I, n\geq 0}$ also generate $\widehat{\mathfrak{h}}$. They satisfy the following relations:*

$$\begin{aligned} P_i^{(n)}P_j^{(m)} &= P_j^{(m)}P_i^{(n)} \text{ and } Q_i^{(n)}Q_j^{(m)} = Q_j^{(m)}Q_i^{(n)} \text{ for all } i, j \in I, \\ Q_i^{(n)}P_j^{(m)} &= \begin{cases} \sum_{k\geq 0}[k+1]P_i^{(m-k)}Q_i^{(n-k)} & \text{if } i = j, \\ P_j^{(m)}Q_i^{(n)} + P_j^{(m-1)}Q_i^{(n-1)} & \text{if } i \neq j \in I \text{ with } \langle i, j\rangle = -1 \\ P_j^{(m)}Q_i^{(n)} & \text{if } i \neq j \in I \text{ with } \langle i, j\rangle = 0. \end{cases} \end{aligned}$$

Proof. This is proved in Lemma 1 of [CLi1]. Note $P_i^{(0)} = Q_j^{(0)} = 1$ and $P_i^{(k)} = Q_i^{(k)} = 0$ when $k < 0$ so the summations in the relations above are all finite. □

There is an alternative generating set of $\widehat{\mathfrak{h}}$ given by elements $P_i^{(1^n)}$ and $Q_i^{(1^n)}$. These are defined using generating functions similar to (4) as follows

$$\exp\left(-\sum_{m\geq 1}\frac{h_{i,-m}}{[m]}z^m\right) = \sum_{n\geq 0}(-1)^n P_i^{(1^n)} z^n \text{ and } \exp\left(-\sum_{m\geq 1}\frac{h_{i,m}}{[m]}z^m\right) = \sum_{n\geq 0}(-1)^n Q_i^{(1^n)} z^n. \tag{5}$$

The Heisenberg algebra admits an involution $\psi : \widehat{\mathfrak{h}} \longrightarrow \widehat{\mathfrak{h}}$ defined by

$$P_i^{(n)} \mapsto P_i^{(1^n)}, \quad Q_i^{(n)} \mapsto Q_i^{(1^n)}, \quad P_i^{(1^n)} \mapsto P_i^{(n)}, \quad Q_i^{(1^n)} \mapsto Q_i^{(n)}.$$

In particular, the commutation relations among the $P_i^{(1^n)}$ and $Q_i^{(1^n)}$ are the same as those between the $P_i^{(n)}$ and $Q_i^{(n)}$ (just replace (n) by (1^n) everywhere).

The involution ψ is essentially the standard involution on symmetric functions. More precisely, fix $i \in I$ and let $\widehat{\mathfrak{h}}_i^- \subset \widehat{\mathfrak{h}}$ denote the subalgebra generated by the $\{P_i^{(n)}\}$. After setting $q = 1$, there is an isomorphism of algebras

$$\widehat{\mathfrak{h}}_i^- \cong \mathrm{Sym} = \mathbb{Z}[h_1, h_2, \ldots, h_n, \ldots]$$

which takes $P_i^{(n)}$ to the homogeneous symmetric function h_n. This isomorphism intertwines ψ with the standard involution on symmetric functions which exchanges homogeneous and elementary symmetric functions.

In addition to the relations from Lemma 3.1 we have the "mixed" relations

$$\begin{aligned} P_i^{(m)} P_j^{(1^n)} &= P_j^{(1^n)} P_i^{(m)} \text{ and } Q_i^{(m)} Q_j^{(1^n)} = Q_j^{(1^n)} Q_i^{(m)} \\ Q_i^{(1^m)} P_j^{(n)} &= \begin{cases} P_i^{(n)} Q_i^{(1^m)} + [2] P_i^{(n-1)} Q_i^{(1^{m-1})} + P_i^{(n-2)} Q_i^{(1^{m-2})} \text{ if } i = j \\ \sum_{k \geq 0} P_j^{(n-k)} Q_i^{(1^{m-k})} \text{ if } \langle i, j \rangle = -1 \\ P_j^{(n)} Q_i^{(1^m)} \text{ if } \langle i, j \rangle = 0. \end{cases} \end{aligned}$$

These relations can be checked in the same way as those in Lemma 3.1.

The unital algebra $\widehat{\mathfrak{h}}$ also admits an idempotent modification where the unit is replaced by a collection of idempotents 1_n, $n \in \mathbb{Z}$. The relations between these idempotents and the generators $P_i^{(n)}$ and $Q_j^{(n)}$ is

$$1_{n+k} P_i^{(n)} = 1_{n+k} P_i^{(n)} 1_k = P_i^{(n)} 1_k \text{ and } 1_k Q_i^{(n)} = 1_k Q_i^{(n)} 1_{n+k} = Q_i^{(n)} 1_{n+k}$$

We also have that $1_m P_i^{(n)} 1_k$ and $1_k Q_i^{(n)} 1_m$ is zero unless $m = n + k$. Hence the idempotent Heisenberg algebra can be thought of as a category, where the objects are the integers and the morphisms from n to m are given by $1_m \widehat{\mathfrak{h}} 1_n$. From this point of view we should write $P_i 1_n$ or $1_n P_i$ to make clear the domain and codomain. However, the domain or codomain will often be obvious or irrelevant in which case we just write P_i.

A representation of $\widehat{\mathfrak{h}}$ is said to be *integrable* if the object 1_n is zero for $n \ll 0$. It is *weakly integrable* if it is the direct sum of (possibly infinitely many) integrable representations.

3.1.1. *The Fock space.* The Heisenbeg algebra $\widehat{\mathfrak{h}}$ has a natural integrable representation $\mathcal{F}$, known as the Fock space. Let $\widehat{\mathfrak{h}}^+ \subset \widehat{\mathfrak{h}}$ denotes the subalgebra generated by the $Q_i^{(n)}$ for all $i \in I$ and $n \geq 0$. Let triv_0 denote the trivial representation of $\widehat{\mathfrak{h}}^+$, where all $Q_i^{(n)}$ $(n > 0)$ act by zero. Then

$$\mathcal{F} := \mathrm{Ind}_{\widehat{\mathfrak{h}}^+}^{\widehat{\mathfrak{h}}}(\mathrm{triv}_0)$$

is called the Fock space representation of $\widehat{\mathfrak{h}}$. It inherits a $\mathbb{Z}$ grading $\mathcal{F} = \oplus_{m \in \mathbb{N}} \mathcal{F}(m)$ by declaring triv_0 to have degree zero, $P_i^{(n)}$ degree n and $Q_i^{(n)}$ degree $-n$.

3.2. **2-representations of $\widehat{\mathfrak{h}}$.** We now explain what it means to have a 2-representation of $\widehat{\mathfrak{h}}$. This concept is closely related to the categorification of $\widehat{\mathfrak{h}}$ defined in [CLi1]. In that paper we defined a graded additive $\Bbbk$-linear 2-category $\mathcal{H}^\Gamma$ together with an algebra isomorphism from $K_0(\mathcal{H}^\Gamma)$ to $\widehat{\mathfrak{h}}$ (here $\Gamma \subset SL_2(\mathbb{C})$ is the finite group associated to our affine Dynkin diagram by McKay correspondence). In essence, a 2-representation of $\widehat{\mathfrak{h}}$ is a representation of the 2-category $\mathcal{H}^\Gamma$ on a graded, $\Bbbk$-linear 2-category.

A 2-representation of $\widehat{\mathfrak{h}}$ consists of a graded, idempotent complete $\Bbbk$-linear category $\mathcal{K}$ where

- 0-morphisms (objects) are denoted $\mathcal{D}(n)$ and are indexed by $n \in \mathbb{Z}$.
- 1-morphisms include the identity 1-morphisms $\mathbf{1}_n$ of $n \in \mathbb{Z}$ (these are mutually orthogonal idempotents) as well as

$$\mathsf{P}_i\mathbf{1}_n : \mathcal{D}(n) \to \mathcal{D}(n+1), \quad \mathsf{Q}_i\mathbf{1}_n : \mathcal{D}(n) \to \mathcal{D}(n-1).$$

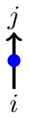
  Other 1-morphisms are obtained from these by taking compositions, direct sums and grading shifts.
- 2-morphisms include the identity 2-morphisms, cups, caps and dots (see the pictures below). Other 2-morphisms are obtained from these by composition.

3.2.1. *2-morphisms in $\mathcal{K}$.* We require that the space of 2-morphisms between any two 1-morphisms be finite dimensional and that $\mathrm{Hom}(\mathbf{1}_n, \mathbf{1}_n\langle\ell\rangle)$ is zero if $\ell < 0$ and one-dimensinal if $\ell = 0$. Moreover, the 2-morphisms must satisfy the defining relations in the Heisenberg 2-category defined in [CLi1]. We now summarize these relations.

The 2-morphisms are encoded by a graphical calculus similar to ones used in the categorifications of quatum groups and other Heisenberg algebras, for example [L, KL1, KL2, KL3, K, LS].

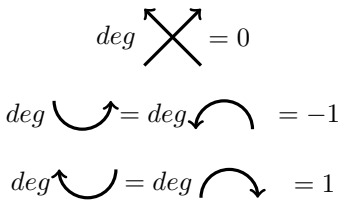
Strands will be used to denote 1-morphisms. More precisely, an upward pointing strand labeled by i denotes P_i while a downward pointing strand labeled i denotes Q_i. Composition of 1-morphisms is obtained by sideways concatenation of diagrams. The space of 2-morphisms between compositions of P_is and Q_js is a $\Bbbk$-algebra described by certain string diagrams with relations. By convention, composition of 2-morphisms is done vertically from the bottom and going up.

We have the following generating 2-morphisms. For any $i, j \in I$ with $\langle i, j\rangle = -1$ there is a 2-morphism $\mathsf{P}_i \to \mathsf{P}_j\langle 1\rangle$ which is diagrammatically denoted by a solid dot:



Note that such an $i - j$ dot is defined to have degree one. For each $i \in I$ there is also a 2-morphism $P_i \to P_i\langle 2\rangle$ of degree two.

The other generators are given by caps, cups and crossings. These, together with their gradings, are depicted below:

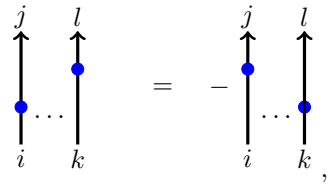


The diagrammatic relations include any planar isotopy which preserves the relative height of dots. These planar isotopy relations imply that the caps and cups give canonical adjunctions, making P_i and Q_i biadjoint up to a shift. Explicitly, using the gradings above we see that the right and left adjoints of P_i are

$$(\mathsf{P}_i)_R \cong \mathsf{Q}_i\langle -1\rangle \text{ and } (\mathsf{P}_i)_L \cong \mathsf{Q}_i\langle 1\rangle.$$

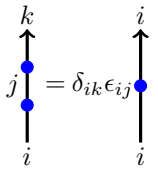
In addition to the isotopy relations we have the following extra relations. First, dots are allowed to move freely through crossings. Next, degree one dots on different strands supercommute when they pass each other, meaning that they pick up the sign $(-1)^{ab}$ where $a, b \in \{1, 2\}$ denotes their degree.

For example, if $i \neq j$ and $k \neq l$ then we have



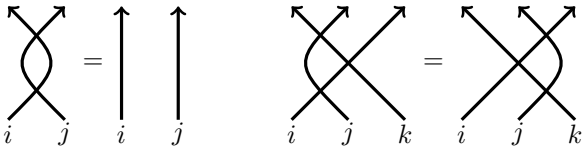
since each of these dots has degree one. The above relation is technically not a local relation, since there may be any number of vertical strands in between the dotted strands. The remaining relations listed below are all local.

The relation which governs the composition of dots on the same strand is



(6)

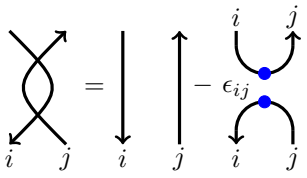
Next, for any $i, j, k \in I$ we have



(7)

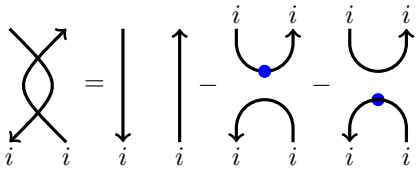
(8)

Finally, if $i \neq j$ then



(9)

while



(10)

Notice all the graphical relations are compatible with the grading assigned to generators. Relation 7 above implies that there is a natural map $\Bbbk[S_n] \longrightarrow \mathrm{End}(\mathsf{P}_i^n)$, which will be important in the next section.

3.3. **Idempotent completeness.** Since the underlying 2-category in a 2-representation of $\widehat{\mathfrak{h}}$ is assumed to be idempotent complete, any idempotent 2-morphism e of a 1-morphism A gives rise to a direct sum decomposition $\mathsf{A} \cong \mathsf{A}e \oplus \mathsf{A}(1-e)$.

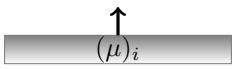
For example, since $\Bbbk[S_n] \to \mathrm{End}(\mathsf{P}_i^n)$, each idempotent in $\Bbbk[S_n]$ gives rise to a direct summand of P_i^n. We let P_i^μ be the 1-morphism of $\mathcal{K}$ corresponding to a minimal idempotent of $\Bbbk[S_n]$ associated to the partition μ of n. More explicitly, fix a labeling T of the boxes in a Young diagram of μ with the numbers $1, \dots, n$. Corresponding to T there are the subgroups $S_{row(T)}$ and $S_{col(T)}$ of S_n preserving the rows and columns respectively. These subgroups have associated Young symmetrizers

$$a_T = \sum_{g \in S_{row(T)}} g, \qquad \sum_{g \in S_{col(T)}} (-1)^{l(g)} g,$$

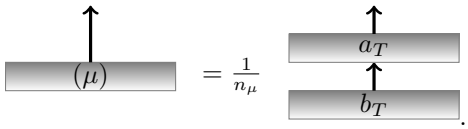
where $l(g)$ is the length of the permutation g.

We set $c_T = \frac{1}{n_\mu} a_T b_T$, an idempotent in $\Bbbk[S_n]$. Here the scalar n_μ is defined as the cardinality of the set $\{(s_1, t_1, s_2, t_2) : s_i \in row(T),\ t_i \in col(T),\ \ (-1)^{l(t_1)} = (-1)^{l(t_2)},\ \ s_1 t_1 s_2 t_2 = 1\}$. The idempotent c_T is used to construct the irreducible S_n representation $\Bbbk[S_n]c_T$ associated to the partition μ. Note that $\Bbbk[S_n]c_T \cong \Bbbk[S_n]c_{T'}$ if T and T' are different fillings of the same partition. Exchanging the roles of row and column, one can also use instead the idempotent $\tilde{c}_T = \frac{1}{n_\mu} b_T a_T$ to construct the same irreducible representation (see Chapter 7 of [F] for a discussion of the constructions of irreducible symmetric group representations from tableaux).

In our case, c_T and $\tilde{c}_T$ also define 1-morphisms $\mathsf{P}_i^n c_T, \mathsf{P}_i \tilde{c}_T$ in $\mathcal{K}$. Just as for representations of the symmetric group, these 1-morphisms do not depend on the choice of labeling T but only on the partition μ in the sense that $\mathsf{P}_i^n c_T \cong \mathsf{P}_i^n c_{T'}$ if T and T' are different labelings of μ; moreover, both of these one-morphisms are isomorphic to $\mathsf{P}_i \tilde{c}_T$. We will abuse notation slightly and write P_i^μ for $\mathsf{P}_i^n c_T$ for an arbitrary choice of T. The 1-morphism P_i^μ will generally be drawn as



When the strand label i is understood, which is often the case in this paper, it will not be drawn. We define $\mathsf{Q}_i^\mu := \mathsf{Q}_i^n c_T$ similarly and draw this 1-morphism with downward pointing arrows. In a few cases we will need to emphasize the choice of the idempotent c_T, in which case we draw



Of particular importance are the elements $\mathsf{P}_i^{(n)}, \mathsf{P}_i^{(1^n)}, \mathsf{Q}_i^{(n)}$ and $\mathsf{Q}_i^{(1^n)}$ corresponding to the partitions $\mu = (n)$ and $\mu = (1^n) = (1, 1, \dots, 1)$ (i.e. corresponding to the trivial and sign representations of S_n). To emphasize the difference between $\mathsf{P}_i^{(n)}$ and $\mathsf{P}_i^{(1^n)}$ in future calculations we will shade the box for $\mathsf{P}_i^{(1^n)}$ while leaving the box for $\mathsf{P}_i^{(n)}$ unshaded (this shading of the boxes is for visual convenience only).

The Ps and the Qs are biadjoint to each other in $\mathcal{K}$. Precisely, if μ is an arbitrary partition then the left and right adjoints are given by

$$(\mathsf{P}_i^\mu)_R \cong \mathsf{Q}_i^\mu \langle -|\mu| \rangle \text{ and } (\mathsf{P}_i^\mu)_L \cong \mathsf{Q}_i^\mu \langle |\mu| \rangle$$

where $|\mu|$ denotes the size of the partition.

3.3.1. *Integrability.* A 2-representation of $\widehat{\mathfrak{h}}$ is said to be *integrable* if the object $\mathbf{1}_n$ is zero for $n \ll 0$. It is *weakly integrable* if it is the direct sum of (possibly infinitely many) integrable 2-representations.

In this paper, all 2-representations are assumed to be weakly integrable. An example of such a 2-representation of $\widehat{\mathfrak{h}}$ was constructed in [CLi1] on categories of coherent sheaves on Hilbert schemes of points on the surface $\widehat{\mathbb{A}^2/\Gamma}$.

3.3.2. *A symmetry of 2-representations.* A 2-representation of $\widehat{\mathfrak{h}}$ admits a covariant involution $\Psi : \mathcal{K} \to \mathcal{K}$. It is defined as the identity on objects and 1-morphisms and also the identity on cups, caps, and dots while acting as multiplication by -1 on any crossing. This means that Ψ takes the idempotent 2-morphism $c_{(1^n)}$ to $c_{(n)}$ and vice versa. Subsequently

$$\Psi(\mathsf{P}_i^{(n)}) = \mathsf{P}_i^{(1^n)}, \quad \Psi(\mathsf{P}_i^{(1^n)}) = \mathsf{P}_i^{(n)}, \quad \Psi(\mathsf{Q}_i^{(n)}) = \mathsf{Q}_i^{(1^n)}, \quad \Psi(\mathsf{Q}_i^{(1^n)}) = \mathsf{Q}_i^{(n)}$$

while more generally $\Psi(\mathsf{P}_i^{(\lambda)}) = \mathsf{P}_i^{(\lambda^t)}$ and $\Psi(\mathsf{Q}_i^{(\lambda)}) = \mathsf{Q}_i^{(\lambda^t)}$. Thus Ψ categorifies the involution ψ from section 3.1.

3.3.3. *Induced relations among 1-morphisms in $\mathcal{K}$.* The relations among 2-morphisms in a 2-representation of $\widehat{\mathfrak{h}}$ imply certain isomorphisms between 1-morphisms. We recall some of these relations below.

Proposition 3.2. [CLi1] *We have the following direct sum decompositions of 1-morphisms in $\mathcal{H}$:*

(1) $\mathsf{P}_i^{(m)}\mathsf{P}_j^{(n)} \cong \begin{cases} \mathsf{P}_i^{(n)}\mathsf{P}_i^{(m)} \cong \oplus_{k=0}^{min(n,m)} \mathsf{P}_i^{(n+m-k,k)} & \text{if } i = j \\ \mathsf{P}_i^{(n)}\mathsf{P}_j^{(m)} & \text{if } i \neq j \end{cases}$

(2) $\mathsf{P}_i^{(m)}\mathsf{P}_j^{(1^n)} \cong \begin{cases} \mathsf{P}_i^{(m,1^n)} \oplus \mathsf{P}_i^{(m+1,1^{n-1})} & \text{if } i = j \\ \mathsf{P}_j^{(1^n)}\mathsf{P}_i^{(m)} & \text{if } i \neq j \end{cases}$

(3) $\mathsf{Q}_j^{(n)}\mathsf{P}_i^{(m)} \cong \begin{cases} \oplus_{k\geq 0}\mathsf{P}_i^{(m-k)}\mathsf{Q}_i^{(n-k)} \otimes_{\Bbbk} V_k & \text{if } i = j \\ \mathsf{P}_i^{(m)}\mathsf{Q}_j^{(n)} \oplus \mathsf{P}_i^{(m-1)}\mathsf{Q}_j^{(n-1)} & \text{if } \langle i, j\rangle = -1 \\ \mathsf{P}_i^{(m)}\mathsf{Q}_j^{(n)} & \text{if } \langle i, j\rangle = 0 \end{cases}$

(4) $\mathsf{Q}_j^{(1^n)}\mathsf{P}_i^{(m)} \cong \begin{cases} \mathsf{P}_i^{(m)}\mathsf{Q}_i^{(1^n)} \oplus \mathsf{P}_i^{(m-1)}\mathsf{Q}_i^{(1^{n-1})} \otimes_{\Bbbk} V_1 \oplus \mathsf{P}_i^{(m-2)}\mathsf{Q}_i^{(1^{n-2})} & \text{if } i = j \\ \oplus_{k\geq 0}\mathsf{P}_i^{(m-k)}\mathsf{Q}_j^{(1^{n-k})} & \text{if } \langle i, j\rangle = -1 \\ \mathsf{P}_i^{(m)}\mathsf{Q}_j^{(1^n)} & \text{if } \langle i, j\rangle = 0 \end{cases}$

In each case the direct summands on the right hand side are indecomposable 1-morphisms in $\mathcal{K}$.

3.4. **Technical lemmas.** We now discuss a few technical lemmas dealing with 2-representations of $\widehat{\mathfrak{h}}$. These will be used later but can be skipped on a first reading of the paper.

Lemma 3.3. *For an arbitrary partition λ we have*

$$\mathsf{Q}_i^{(\lambda)}\mathsf{P}_i \cong \mathsf{P}_i\mathsf{Q}_i^{(\lambda)} \bigoplus_{\lambda' \subset \lambda} \mathsf{Q}_i^{(\lambda')} \otimes_{\Bbbk} V_1 \text{ and } \mathsf{Q}_i\mathsf{P}_i^{(\lambda)} \cong \mathsf{P}_i^{(\lambda)}\mathsf{Q}_i \bigoplus_{\lambda' \subset \lambda} \mathsf{P}_i^{(\lambda')} \otimes_{\Bbbk} V_1$$

where the sums are over all $\lambda' \subset \lambda$ with $|\lambda'| = |\lambda| - 1$.

Proof. We prove only the first relation (the second one is proved similarly). First, using the fact that $\mathsf{P}_i\mathsf{Q}_i^{(\lambda)}$ is indecomposable and that $\mathsf{Q}_i\mathsf{P}_i\mathbf{1}_n \cong \mathsf{P}_i\mathsf{Q}_i\mathbf{1}_n \oplus \mathbf{1}_n \otimes_{\Bbbk} V_1$, it follows by induction that

$$\mathsf{Q}_i^{(\lambda)}\mathsf{P}_i \cong \mathsf{P}_i\mathsf{Q}_i^{(\lambda)} \bigoplus_{\mu \subset \lambda} \mathsf{Q}_i^{(\mu)} \otimes_{\Bbbk} V_{\mu,\lambda}$$

for some graded vector spaces $V_{\mu,\lambda}$. We will prove by induction on $|\lambda|$ that $V_{\mu,\lambda} = V_1$ if $\mu \subset \lambda$ and that $V_{\mu,\lambda} = 0$ otherwise.

Note that if $\mathsf{Q}_i^{(\mu)}\langle l\rangle$ is a summand of $\mathsf{Q}_i^{(\lambda)}\mathsf{P}_i$ then we must have

$$\mathrm{Hom}(\mathsf{Q}_i^{(\lambda)}\mathsf{P}_i, \mathsf{Q}_i^{(\mu)}\langle l\rangle) \neq 0 \text{ and } \mathrm{Hom}(\mathsf{Q}_i^{(\mu)}\langle l\rangle, \mathsf{Q}_i^{(\lambda)}\mathsf{P}_i) \neq 0.$$

Now, by adjunction

$$\begin{aligned}\mathrm{Hom}(\mathsf{Q}_i^{(\lambda)}\mathsf{P}_i, \mathsf{Q}_i^{(\mu)}\langle l\rangle) &\cong \mathrm{Hom}(\mathsf{Q}_i^{(\lambda)}, \mathsf{Q}_i^{(\mu)}\mathsf{Q}_i\langle l+1\rangle)\\ &\cong \mathrm{Hom}(\mathsf{Q}_i^{(\lambda)}, \oplus_{\mu\subset\mu'}\mathsf{Q}_i^{(\mu')}\langle l+1\rangle)\end{aligned}$$

and similarly

$$\mathrm{Hom}(\mathsf{Q}_i^{(\mu)}\langle l\rangle, \mathsf{Q}_i^{(\lambda)}\mathsf{P}_i) \cong \mathrm{Hom}(\oplus_{\mu\subset\mu'}\mathsf{Q}_i^{(\mu')}, \mathsf{Q}_i^{(\lambda)}\langle -l+1\rangle).$$

One of these two morphism spaces is zero unless $l=-1$ or $l=1$ and $\mu\subset\lambda$ in which case one of these is one dimensional. Thus $V_{\mu,\lambda}=0$ unless $\mu\subset\lambda$ in which case $V_{\mu,\lambda}\subset V_1$.

It remains to show that $V_{\mu,\lambda}$ actually equals V_1. We do this by counting dimensions. Take $\nu\subset\lambda$ with $|\lambda|=|\nu|+1$. Then by induction

$$\mathsf{Q}_i^{(\nu)}\mathsf{Q}_i\mathsf{P}_i \cong \mathsf{P}_i\mathsf{Q}_i^{(\nu)}\mathsf{Q}_i \oplus \mathsf{Q}_i^{(\nu)} \otimes_{\Bbbk} V_1 \oplus_{\nu'\subset\nu} \mathsf{Q}_i^{(\nu')}\mathsf{Q}_i \otimes_{\Bbbk} V_1.$$

On the other hand, this equals

$$\oplus_{\nu\subset\lambda'}\mathsf{Q}_i^{(\lambda')}\mathsf{P}_i \cong \oplus_{\nu\subset\lambda'}\mathsf{P}_i\mathsf{Q}_i^{(\lambda')} \oplus_{\nu,\lambda''\subset\lambda'} \mathsf{Q}_i^{(\lambda'')} \otimes_{\Bbbk} V_{\lambda'',\lambda'}.$$

Comparing summands involving only Q_i's one can check that indeed $\dim(V_{\mu,\lambda'})=2$ for any $\mu\subset\lambda'$. The result follows. □

Lemma 3.4. *Suppose λ,λ',μ and μ' are partitions such that $|\lambda|>|\lambda'|$ and $|\mu|>|\mu'|$. Then* $\dim\mathrm{Hom}(\mathsf{P}_i^{(\lambda)}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu')}\langle 1\rangle)\leq 1$ *with equality if and only if $\lambda'\subset\lambda$ and $\mu'\subset\mu$ with $|\lambda|=|\lambda'|+1$ and $|\mu|=|\mu'|+1$. In this case this space is spanned by the diagram given by a single cap.*

Likewise, if $|\lambda|<|\lambda'|$ and $|\mu|<|\mu'|$ then $\dim\mathrm{Hom}(\mathsf{P}_i^{(\lambda)}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu')}\langle 1\rangle)\leq 1$ *with equality if and only if $\lambda'\supset\lambda$ and $\mu'\supset\mu$ with $|\lambda|=|\lambda'|-1$ and $|\mu|=|\mu'|-1$. In this case this space is spanned by the diagram given by a single cup.*

Proof. We prove only the first assertion (as the second one follows similarly). If $|\lambda|-|\lambda'|\geq 2$ then the space of degree one maps is zero since any map requires at least two caps and thus has degree at least two. So from now on we assume that $|\lambda|=|\lambda'|+1$ and hence $|\mu|=|\mu'|+1$.

First we show by induction on $|\mu|$ that if $\mu'\not\subset\mu$ then the space of maps is zero. Choose any $\lambda''\subset\lambda'$ with $|\lambda'|=|\lambda''|+1$. Since $\mathsf{Q}_i\mathsf{Q}_i^{(\lambda'')}$ contains $\mathsf{Q}_i^{(\lambda')}$ as a direct summand it suffices to show that

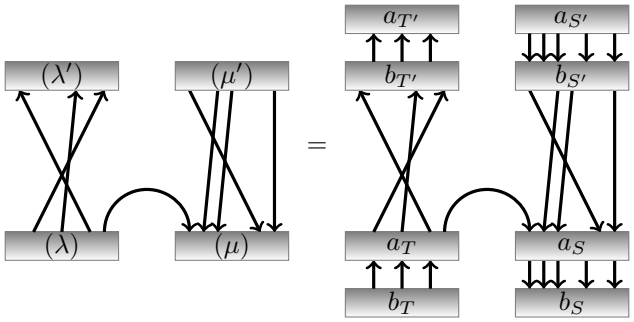
$$\mathrm{Hom}(\mathsf{P}_i^{(\lambda)}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i\mathsf{P}_i^{(\lambda'')}\mathsf{Q}_i^{(\mu')}\langle 1\rangle) = 0.$$

By adjunction and lemma 3.3 we have

$$\begin{aligned}&\mathrm{Hom}(\mathsf{P}_i^{(\lambda)}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i\mathsf{P}_i^{(\lambda'')}\mathsf{Q}_i^{(\mu')}\langle 1\rangle)\\ \cong\ &\mathrm{Hom}(\mathsf{Q}_i\mathsf{P}_i^{(\lambda)}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda'')}\mathsf{Q}_i^{(\mu')})\\ \cong\ &\mathrm{Hom}(\mathsf{P}_i^{(\lambda)}\mathsf{Q}_i\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda'')}\mathsf{Q}_i^{(\mu')}) \bigoplus_{\lambda'''\subset\lambda'} \mathrm{Hom}(\mathsf{P}_i^{(\lambda''')}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda'')}\mathsf{Q}_i^{(\mu')} \otimes_{\Bbbk} V_1)\\ \cong\ &\bigoplus_{\mu''\subset\mu} \mathrm{Hom}(\mathsf{P}_i^{(\lambda)}\mathsf{Q}_i^{(\mu'')}, \mathsf{P}_i^{(\lambda'')}\mathsf{Q}_i^{(\mu')}) \bigoplus_{\lambda'''\subset\lambda'} \mathrm{Hom}(\mathsf{P}_i^{(\lambda''')}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda'')}\mathsf{Q}_i^{(\mu')} \otimes_{\Bbbk} V_1).\end{aligned}$$

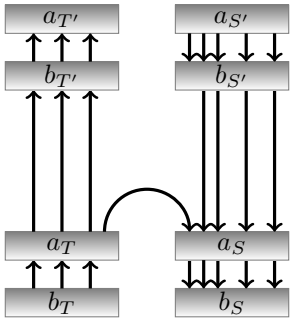
The terms in the first sum vanish since μ'' and μ' are never equal, while those in the second sum vanish by induction. Likewise, one can show that the space of maps is also zero if $\lambda'\not\subset\lambda$.

Now suppose $\lambda' \subset \lambda$ and $\mu' \subset \mu$. It follows from degree consideration that any map $\mathsf{P}_i^{(\lambda)}\mathsf{Q}_i^{(\mu)} \to \mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu')}\langle 1\rangle$ consists of a single cap composed with a diagram without local minima or local maxima. Thus we have a diagram like the following



After possibly replacing the filling T' by the filling gT' for some permutation g, we may assume that the upward pointing strands do not intersect. Similarly, replacing S' by hS' for some permutation h we can assume the downward strands do not intersect. Thus we must show that the diagram

(11)



which spans the space of maps.

It remains to show that $\dim \mathrm{Hom}(\mathsf{P}_i^{(\lambda)}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu')}\langle 1\rangle) = 1$. Now

$$\begin{aligned} & \mathrm{Hom}(\mathsf{P}_i\mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu')}\langle 1\rangle) \\ \cong\ & \mathrm{Hom}(\mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu)}, \mathsf{Q}_i\mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu')}) \\ \cong\ & \mathrm{Hom}(\mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda')}\mathsf{Q}_i\mathsf{Q}_i^{(\mu')}) \bigoplus_{\lambda'' \subset \lambda'} \mathrm{Hom}(\mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda'')}\mathsf{Q}_i^{(\mu')} \otimes_{\Bbbk} V_1) \end{aligned}$$

Now the left hand term is isomorphic to $\Bbbk$ while, by induction, the right hand term is isomorphic to $\Bbbk^\ell$ where ℓ is the number of $\lambda'' \subset \lambda'$. Thus

$$\dim \mathrm{Hom}(\mathsf{P}_i\mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu')}\langle 1\rangle) = \ell + 1.$$

On the other hand, $\mathsf{P}_i\mathsf{P}_i^{(\lambda')} \cong \oplus_{\lambda' \subset \nu}\mathsf{P}_i^{(\nu)}$. Since the number of such partitions ν is $\ell + 1$ it follows that

$$\dim \mathrm{Hom}(\mathsf{P}_i^{(\nu)}\mathsf{Q}_i^{(\mu)}, \mathsf{P}_i^{(\lambda')}\mathsf{Q}_i^{(\mu')}\langle 1\rangle) = 1$$

for any ν containing λ'. The result follows since we can take $\nu = \lambda$. □

Lemma 3.5. *If $i \neq j$ then* $\dim \mathrm{Hom}(\mathsf{P}_i^{(a)}\mathsf{P}_j^{(b)}\mathsf{Q}_i^{(1^c)}\mathsf{Q}_j^{(1^d)}, \mathsf{P}_i^{(a')}\mathsf{P}_j^{(b')}\mathsf{Q}_i^{(1^{c'})}\mathsf{Q}_j^{(1^{d'})}\langle 1\rangle) \leq 1$.

Proof. Let D be a diagram depicting a 2-morphism in

$$\mathrm{Hom}(\mathsf{P}_i^{(a)}\mathsf{P}_j^{(b)}\mathsf{Q}_i^{(1^c)}\mathsf{Q}_j^{(1^d)}, \mathsf{P}_i^{(a')}\mathsf{P}_j^{(b')}\mathsf{Q}_i^{(1^{c'})}\mathsf{Q}_j^{(1^{d'})}\langle 1\rangle).$$

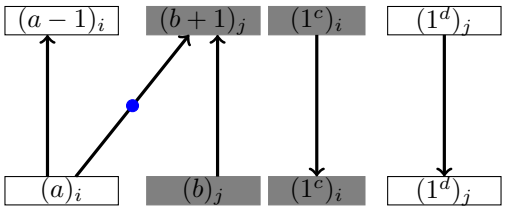
Because all P's occur to the left of all Q's one can simplify D so that it has no right-pointing cups or left-pointing caps. One can also get rid of all degree zero crossings. The remaining map is made up of dots, cups and caps all of which have positive degree. Of these only the the ij dot, the ji dot, the right cap and the left cup have degree one. It follows that D is made up precisely of one such map and hence the Hom space is zero except in the following cases:

(1) $(a', b', c', d') = (a \pm 1, b \mp 1, c, d)$,
(2) $(a', b', c', d') = (a, b, c \pm 1, d \mp 1)$
(3) $(a', b', c', d') = (a \pm 1, b, c \pm 1, d)$
(4) $(a', b', c', d') = (a, b \pm 1, c, d \pm 1)$

□

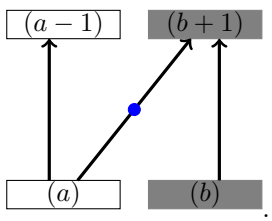
The proof above shows, for example, that $\mathrm{Hom}(\mathsf{P}_i^{(a)}\mathsf{P}_j^{(b)}\mathsf{Q}_i^{(1^c)}\mathsf{Q}_j^{(1^d)}, \mathsf{P}_i^{(a-1)}\mathsf{P}_j^{(b+1)}\mathsf{Q}_i^{(1^c)}\mathsf{Q}_j^{(1^d)}\langle 1\rangle)$ is spanned by the diagram

(12)



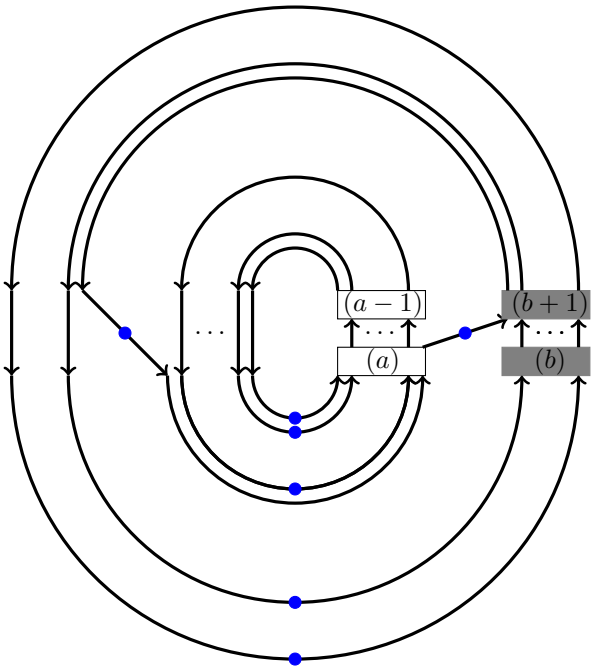
where the dot is an $i - j$ dot. A basic question is how to check that the diagram above is a nonzero 2-morphism. There are two ways to do this. One way is to check directly in some 2-representation of $\mathcal{H}$ that this diagram is represents a nonzero 2-morphism. A representation of $\mathcal{H}$ which is faithful on 2-morphisms was defined in [CLi1], so in principle one can check that the above diagram becomes a nonzero map in that representation.

A second proof that the above diagram is nonzero proceeds by closing off the diagram, simplifying using the graphical relations, and then showing that the resulting diagram is nonzero. Since we use this technique several times in future sections we now explain in this example how this procedure works. First note that it suffices to show that the diagram



is nonzero. To do this we close it off as shown in figure 13 below to get an endomorphism of the identity. Note that that there are two $i - j$ dots in the diagram (the ones on the two diagonal line segments)

while the remaining ones are either $i-i$ or $j-j$ dots.



(13)

This procedure can be thought of as defining a linear map

$$\mathrm{Hom}(\mathsf{P}_i^{(a)}\mathsf{P}_j^{(b)}\mathbf{1}_n, \mathsf{P}_i^{(a-1)}\mathsf{P}_j^{(b+1)}\mathbf{1}_n\langle 1\rangle) \longrightarrow \mathrm{Hom}(\mathbf{1}_n, \mathbf{1}_n).$$

We now explain why this defines a nonzero multiple of the identity endomorphism of $\mathbf{1}_n$. First, the idempotent labeled $(a-1)$ can be absorbed into the idempotent (a) and the idempotent (b) can be absorbed into the idempotent $(b+1)$. Now, expanding the remaining idempotents (a) and $(b+1)$ explicitly as a sum of permutations we see that almost all of the summands contain either two degree 2 dots on the same strand (giving zero) or a left-twist curl (which is also zero). Some of the remaining terms may have double crossings between strands but these double crossings can all be removed using the graphical relations.

The only remaining diagrams are a collection of disjoint counterclockwise circles with a degree 2 dot on them. Each such circle is equal to the identity and hence can be erased. This leaves us with the empty diagram, which is the identity endomorphism of $\mathbf{1}_n$ and is therefore nonzero.

Lemma 3.6. *Suppose ∂_1 and ∂_2 are nonzero 2-morphisms*

$$\partial_1 : \mathsf{P}_i^{(a)}\mathsf{P}_j^{(b)}\mathsf{Q}_i^{(1^c)}\mathsf{Q}_j^{(1^d)} \to \mathsf{P}_i^{(a')}\mathsf{P}_j^{(b')}\mathsf{Q}_i^{(1^{c'})}\mathsf{Q}_j^{(1^{d'})}\langle 1\rangle$$

$$\partial_2 : \mathsf{P}_i^{(a)}\mathsf{P}_j^{(b)}\mathsf{Q}_i^{(1^c)}\mathsf{Q}_j^{(1^d)} \to \mathsf{P}_i^{(a'')}\mathsf{P}_j^{(b'')}\mathsf{Q}_i^{(1^{c''})}\mathsf{Q}_j^{(1^{d''})}\langle 1\rangle.$$

Then there exist maps ∂_1', ∂_2' that form a commutative square

$$\begin{array}{ccc} \mathsf{P}_i^{(a)}\mathsf{P}_j^{(b)}\mathsf{Q}_i^{(1^c)}\mathsf{Q}_j^{(1^d)} & \xrightarrow{\partial_1} & \mathsf{P}_i^{(a')}\mathsf{P}_j^{(b')}\mathsf{Q}_i^{(1^{c'})}\mathsf{Q}_j^{(1^{d'})}\langle 1\rangle \\ \downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle \partial_2'} \\ \mathsf{P}_i^{(a'')}\mathsf{P}_j^{(b'')}\mathsf{Q}_i^{(1^{c''})}\mathsf{Q}_j^{(1^{d''})}\langle 1\rangle & \xrightarrow{\partial_1'} & \mathsf{P}_i^{(a''')}\mathsf{P}_j^{(b''')}\mathsf{Q}_i^{(1^{c'''})}\mathsf{Q}_j^{(1^{d'''})}\langle 2\rangle \end{array}$$

so that the compositions $\partial_2' \circ \partial_1 = \partial_1' \circ \partial_2$ are nonzero.

Proof. This follows from Lemma 3.5. For example, suppose ∂_1 is the diagram consisting of an $i-j$ dot and ∂_2 is the diagram consisting of a right cap labeled i. Then ∂_2' is defined as a right cap while ∂_1' is now an $i-j$ dot.

In the composition $\partial_2' \circ \partial_1$ the dot lies below the right cap while in $\partial_1' \circ \partial_2$ the right cap lies below the dot. These diagrams are equivalent just because of the isotopy relation which allows one to slide the portion of the diagram containing the dot past the portion containg the right cup. Similar equivalences can be checked for any other pair of nonzero 2-morphisms ∂_1, ∂_2.

Finally, one can check that these compositions are nonzero just like we proved that the map (12) is nonzero. □

3.5. **Complexes in a 2-representation of $\widehat{\mathfrak{h}}$.** Since the underlying 2-category $\mathcal{K}$ of a 2-representation of $\widehat{\mathfrak{h}}$ is a graded, additive, $\Bbbk$-linear 2-category, its homotopy category $\mathsf{Kom}(\mathcal{K})$ is a graded, additive, $\Bbbk$-linear triangulated 2-category. The objects of $\mathsf{Kom}(\mathcal{K})$ are the objects of $\mathcal{K}$, the 1-morphisms are (possibly unbounded) complexes of 1-morphisms of $\mathcal{K}$ and 2-morphisms are chain maps up to homotopy.

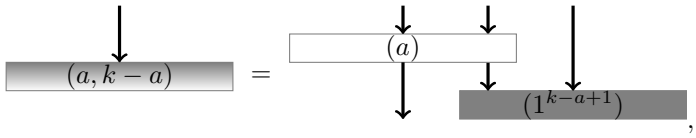
Many such complexes arise naturally in categorification of quantum affine algebras. We give an example of such a complex below. This complex will by useful later.

Fix $i \in I$. We define complexes

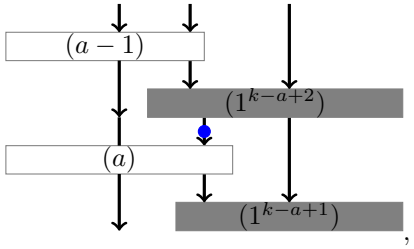
$$\mathsf{P}_i^{[1^k]} := \left[\mathsf{P}_i^{(k)}\langle -2(k-1)\rangle \to \cdots \to \mathsf{P}_i^{(3,1^{k-3})}\langle -4\rangle \to \mathsf{P}_i^{(2,1^{k-2})}\langle -2\rangle \to \mathsf{P}_i^{(1^k)}\right][1]\langle -1\rangle \tag{14}$$

$$\mathsf{Q}_i^{[1^k]} := \left[\mathsf{Q}_i^{(1^k)} \to \mathsf{Q}_i^{(2,1^{k-2})}\langle 2\rangle \to \mathsf{Q}_i^{(3,1^{k-3})}\langle 4\rangle \to \cdots \to \mathsf{Q}_i^{(k)}\langle 2(k-1)\rangle\right][-1]\langle 1\rangle \tag{15}$$

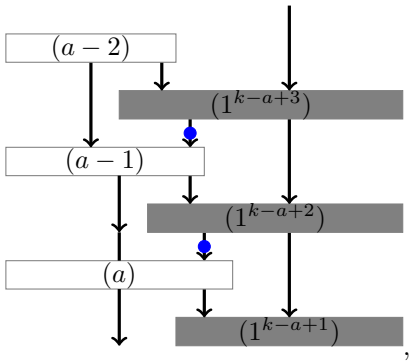
where the right hand term in (14) and the left hand term in (15) are in cohomological degree zero. The differential in the above complexes is (essentially) defined by a single dot. More precisely, consider a filling of the hook partition $(a, 1^{k-a})$ with numbers $1, \ldots, a$ in the row of the hook and $a, a+1, \ldots, k$ in the column (so that the unique box in the first row and first column is filled with a). Then the idempotent defining $\mathsf{Q}_i^{(a,k-a)}$ is the product of the trivial idempotent of $\mathbb{C}[S_a] \subset \mathbb{C}[S_k]$ (embedded as permutations which fix $a+1, \ldots, k$) and the sign idempotent of $\mathbb{C}[S_{k-a+1}]$ (embedded as permutations which fix $1, \ldots, a-1$). Diagrammatically we have



where all strands are labeled by i. The differential $\mathsf{Q}_i^{(a,1^{k-a})} \to \mathsf{Q}_i^{(a+1,1^{k-a-1})}$ in $\mathsf{Q}_i^{[1^k]}$ is defined by the diagram



Notice that there is one degree two dot on the middle downward strand. To see that this defines a differential (i.e. $d^2 = 0$) note that the picture for d^2 is



Expanding the middle idempotents (1^{k-a+2}) and $(a-1)$ explicitly as a sum of permutations we write the above as a sum of diagrams where there are three (at least) three strands connecting the top (1^{k-a+3}) box and the bottom (a) box. Each of these three strands contains $0, 1$ or 2 dots but it is easy to check that for any such configuration one gets zero because of the idempotents (1^{k-a+2}) and $(a-1)$. Therefore all the expansion terms are zero and it follows that $d^2 = 0$.

Finally, we would like to check that $d \neq 0$. To see this consider the closure of the map with a dot on each strand except for the strand already containing a dot (meaning that if we erased the idempotents in the differential we would have a collection of counterclockwise circles each of which has a dot). Expanding all the idempotents and using that a dot squares to zero one sees that all diagrams vanish except for the ones containing k counterclockwise circles, each with one dot. Since each such circle is the identity, it follows this map is nonzero and hence that $d \neq 0$.

4. Quantum affine algebras and 2-representations of $\widehat{\mathfrak{g}}$

4.1. **The idempotent vertex presentation.** The quantum Heisenberg algebra $\widehat{\mathfrak{h}}$ plays an important role in infinite dimensional representation theory in part because it occurs as a subalgebra of an associated quantum affine algebra. In this section we will define an idempotent version of this quantum affine Lie algebra, which we denote by $\dot{U}_q(\widehat{\mathfrak{g}})$. In general this definition involves a parameter c known as the level. Since we are primarily concerned with the basic representation, which is level one, we have set $c = 1$ in the definitions below. For the general definition involving an arbitrary level, see [CLi2].

We define the level one quantum affine algebra $\dot{U}_q(\widehat{\mathfrak{g}})$ to be the $\Bbbk(q)$ algebra generated by

$$E_{i,r}1_\lambda, F_{i,r}1_\lambda, Q_i^{(n)}1_\lambda, P_i^{(n)}1_\lambda, Q_i^{(1^n)}1_\lambda, P_i^{(1^n)}1_\lambda, \text{ where } i \in I, r \in \mathbb{Z} \text{ and } n \in \mathbb{N}.$$

By definition taking $n = 0$ gives the identity morphism.

We also define the following additional elements

$$Q_i^{[1^n]} := \sum_{m=0}^{n} (-q)^m [m] Q_i^{(1^{n-m})} Q_i^{(m)} \quad \text{and} \quad Q_i^{[n]} := \sum_{m=0}^{n} (-q)^m [m] Q_i^{(n-m)} Q_i^{(1^m)}$$

$$P_i^{[1^n]} := \sum_{m=0}^{n} (-q)^{-m} [m] P_i^{(1^{n-m})} P_i^{(m)} \quad \text{and} \quad P_i^{[n]} := \sum_{m=0}^{n} (-q)^{-m} [m] P_i^{(n-m)} P_i^{(1^m)}$$

Note that $P_i^{[1]} = -q^{-1}P_i$ and $Q_i^{[1]} = -q^{-1}Q_i$. The relations in $\dot{U}_q(\widehat{\mathfrak{g}})$ are as follows.

(1) $\{1_\lambda : \lambda \in \widehat{X}\}$ are mutually orthogonal idempotents, moreover

$$E_{i,r}1_\lambda = 1_\mu E_{i,r}1_\lambda = 1_\mu E_{i,r}$$
$$F_{i,r}1_\mu = 1_\lambda E_{i,r}1_\mu = 1_\lambda E_{i,r}$$

where $\mu = \lambda + \alpha_i + rc\delta$

(2) We have

$$P_i^{(n)}1_\lambda = 1_\mu P_i^{(n)}1_\lambda = 1_\mu P_i^{(n)} \quad \text{and} \quad P_i^{(1^n)}1_\lambda = 1_\mu P_i^{(1^n)}1_\lambda = 1_\mu P_i^{(1^n)}$$
$$Q_i^{(n)}1_\mu = 1_\lambda Q_i^{(n)}1_\mu = 1_\lambda Q_i^{(n)} \quad \text{and} \quad Q_i^{(1^n)}1_\mu = 1_\lambda Q_i^{(1^n)}1_\mu = 1_\lambda Q_i^{(1^n)}$$

where $\mu = \lambda + nc\delta$.

(3) The subalgebra generated by Ps and Qs is isomorphic to the quantum Heisenberg algebra $\widehat{\mathfrak{h}}$.

(4) We have

$$[Q_i^{[1^{a+1}]}, E_{i,b}]1_\lambda = \begin{cases} q^2 Q_i^{[1^a]} E_{i,b+1}1_\lambda - q^{-2} E_{i,b+1} Q_i^{[1^a]}1_\lambda \text{ if } a > 0 \\ [2]E_{i,b+1}1_\lambda \text{ if } a = 0. \end{cases}$$

$$q^{-1}[Q_i^{[1^{a+1}]}, F_{i,b}]1_\lambda = \begin{cases} q^{-2} Q_i^{[1^a]} F_{i,b+1}1_\lambda - q^2 F_{i,b+1} Q_i^{[1^a]}1_\lambda \text{ if } a > 0 \\ -[2]F_{i,b+1}1_\lambda \text{ if } a = 0. \end{cases}$$

$$q[P_i^{[1^{a+1}]}, E_{i,b+1}]1_\lambda = \begin{cases} q^2 E_{i,b} P_i^{[1^a]}1_\lambda - q^{-2} P_i^{[1^a]} E_{i,b}1_\lambda \text{ if } a > 0 \\ [2]E_{i,b}1_\lambda \text{ if } a = 0 \end{cases}$$

$$[P_i^{[1^{a+1}]}, F_{i,b+1}]1_\lambda = \begin{cases} q^{-2} F_{i,b} P_i^{[1^a]}1_\lambda - q^2 P_i^{[1^a]} F_{i,b}1_\lambda \text{ if } a > 0 \\ -[2]F_{i,b}1_\lambda \text{ if } a = 0. \end{cases}$$

while if $\langle i, j\rangle = -1$ we have

$$[Q_j^{[1^{a+1}]}, E_{i,b}]1_\lambda = \begin{cases} q E_{i,b+1} Q_j^{[1^a]}1_\lambda - q^{-1} Q_j^{[1^a]} E_{i,b+1}1_\lambda \text{ if } a > 0 \\ E_{i,b+1}1_\lambda \text{ if } a = 0. \end{cases}$$

$$q^{-1}[Q_j^{[1^{a+1}]}, F_{i,b}]1_\lambda = \begin{cases} q^{-1} F_{i,b+1} Q_j^{[1^a]}1_\lambda - q Q_j^{[1^a]} F_{i,b+1}1_\lambda \text{ if } a > 0 \\ -F_{i,b+1}1_\lambda \text{ if } a = 0 \end{cases}$$

$$q[P_j^{[1^{a+1}]}, E_{i,b+1}]1_\lambda = \begin{cases} q^{-1} E_{i,b} P_j^{[1^a]}1_\lambda - q P_j^{[1^a]} E_{i,b}1_\lambda \text{ if } a > 0 \\ E_{i,b}1_\lambda \text{ if } a = 0 \end{cases}$$

$$[P_j^{[1^{a+1}]}, F_{i,b+1}]1_\lambda = \begin{cases} q F_{i,b} P_j^{[1^a]}1_\lambda - q^{-1} P_j^{[1^a]} F_{i,b}1_\lambda \text{ if } a > 0 \\ -F_{i,b}1_\lambda \text{ if } a = 0. \end{cases}$$

If $\langle i, j\rangle = 0$ then $P_j^{[1^a]}1_\lambda$ and $Q_j^{[1^a]}1_\lambda$ commute with both $E_{i,b}1_\lambda$ and $F_{i,b}1_\lambda$.

(5) We have

$$[E_{i,a}, F_{i,b}]1_\lambda = \begin{cases} q^{-b} q^{\langle \lambda, i\rangle} Q_i^{[1^{a+b}]}1_\lambda \text{ if } a + b > 0 \\ q^{-a} q^{-\langle \lambda, i\rangle} P_i^{[1^{-a-b}]}1_\lambda \text{ if } a + b < 0 \\ [\langle \lambda, i\rangle + a]1_\lambda \text{ if } a + b = 0. \end{cases}$$

while if $i \neq j$ then $[E_{i,a}, F_{j,b}]1_\lambda = 0$.

(6) For any $m, n \in \mathbb{Z}$ we have

$$E_{i,m}E_{i,n-1}1_\lambda + E_{i,n}E_{i,m-1}1_\lambda = q^2 \left(E_{i,m-1}E_{i,n}1_\lambda + E_{i,n-1}E_{i,m}1_\lambda\right)$$
$$F_{i,n-1}F_{i,m}1_\lambda + F_{i,m-1}F_{i,n}1_\lambda = q^2 \left(F_{i,n}F_{i,m-1}1_\lambda + F_{i,m}F_{i,n-1}1_\lambda\right).$$

(7) For any $m, n \in \mathbb{Z}$, if $\langle i, j \rangle = -1$ we have

$$\begin{aligned} E_{i,m}E_{j,n+1}1_\lambda - qE_{j,n+1}E_{i,m}1_\lambda &= E_{j,n}E_{i,m+1}1_\lambda - qE_{i,m+1}E_{j,n}1_\lambda \\ F_{i,m+1}F_{j,n}1_\lambda - qF_{j,n}F_{i,m+1}1_\lambda &= F_{j,n+1}F_{i,m}1_\lambda - qF_{i,m}F_{j,n+1}1_\lambda \end{aligned}$$

while if $\langle i, j \rangle = 0$ then

$$E_{i,m}E_{j,n}1_\lambda = E_{j,n}E_{i,m}1_\lambda \text{ and } F_{i,m}F_{j,n}1_\lambda = F_{j,n}F_{i,m}1_\lambda.$$

(8) If $\langle i, j \rangle = -1$ then

$$\sum_{\sigma \in S_2} \left(E_{j,n}E_{i,m_{\sigma(1)}}E_{i,m_{\sigma(2)}}1_\lambda + E_{i,m_{\sigma(1)}}E_{i,m_{\sigma(2)}}E_{j,n}1_\lambda \right) = \sum_{\sigma \in S_2} [2]E_{i,m_{\sigma(1)}}E_{j,n}E_{i,m_{\sigma(2)}}1_\lambda$$

and similarly if we replace all Es by Fs.

This definition is an idempotent modified version of Drinfeld's new realization of the quantum affine algebra. The modification is not entirely trivial. For example, we no longer have the generators q^h, $q^{\pm d}$ or $q^{\pm c/2}$ which are standard in the Drinfeld presentation. Our presentation also differs from Drinfeld's by a renormalization which includes certain sign changes. This presentation is described in the last section of [CLi2] where we show that it is equivalent to Drinfeld's new realization.

The fact that $\dot{U}_q(\widehat{\mathfrak{g}})$ is idempotent means that we can think of it as a 1-category. The objects are labeled by the weights of $\widehat{\mathfrak{g}}$ while elements like $E_i 1_\lambda$ are 1-morphisms between λ and $\lambda + \alpha_i$.

4.1.1. *The basic representation of $U_q(\widehat{\mathfrak{g}})$.* The basic representation V_{Λ_0} of the affine Lie algebra $\widehat{\mathfrak{g}}$ is characterized by the fact that it is irreducible and that there exists a vector $v \in V_{\Lambda_0}$ such that

$$(\mathfrak{g} \otimes \mathbb{C}[t]) \cdot v = 0 \text{ and } c \cdot v = v.$$

It follows from this that the central element c acts by the identity on all of V_{Λ_0}, so that V_{Λ_0} is a level one irreducible representation. This representation deforms to give the basic representation of the quantum affine algebra $\dot{U}_q(\widehat{\mathfrak{g}})$.

An explicit construction of V_{Λ_0} was given by Frenkel-Kac [FK] and Segal [S] in the case of the affine Lie algebra ($q = 1$). The construction, which is known as the homogeneous realization of V_{Λ_0}, was extended to the quantum affine algebra in [FJ]. It begins by restricting V_{Λ_0} to the homogeneous Heisenberg subalgebra $\widehat{\mathfrak{h}} \subset \mathfrak{g}$. Each vector $u \in V_{\Lambda_0}$ such that $\widehat{\mathfrak{h}}^+ \cdot u = 0$ generates a copy of the Fock space $\mathcal{F}$. The space of such vectors has a basis given by $\widehat{W} \cdot v$, namely the affine Weyl group orbit of the highest weight vector $v \in V_{\Lambda_0}(\Lambda_0)$. This orbit can be identified with the root lattice Y of $\mathfrak{g}$. Thus, as an $\widehat{\mathfrak{h}}$ module, V_{Λ_0} decomposes as

$$V_{\Lambda_0}|_{\widehat{\mathfrak{h}}} \cong \mathcal{F} \otimes_{\Bbbk(q)} \Bbbk(q)[Y].$$

For $\alpha = w \cdot \Lambda_0 \in X$, the summand $\mathcal{F}(n) \otimes \alpha$ is the weight space in V_{Λ_0} of weight $w \cdot \Lambda_0 - n\delta$. This bijection between weight spaces of the basic representation and pairs $(\alpha, n) \in Y \times \mathbb{N}$ will be used in section 5.1. The Frenkel-Kac-Segal construction, as described in [FJ], then uses vertex operators to extend the action of $\widehat{\mathfrak{h}}$ on V_{Λ_0} to the entire quantum affine algebra.

4.2. **2-representations of $\widehat{\mathfrak{g}}$.** Since the affine Lie algebra $\widehat{\mathfrak{g}}$ also has a description as a Kac-Moody Lie algebra, there already exist several notions of a 2-representation of $\widehat{\mathfrak{g}}$ [KL3, R, CK]. Although the definitions in these papers are slightly different, they are all closely related, and are in some cases equivalent. For example, the main result of [CKL2] shows that when $\mathfrak{g} = \mathfrak{sl}_2$, a 2-representation in the sense of [CK] induces a 2-representation in the sense of [R]. Similarly, in [CLa] it is shown that a 2-representation in the sense of [R], together with a mild additional hypothesis, induces a 2-representation in the sense of [KL3].

However, in this paper $\widehat{\mathfrak{g}}$ and its q-deformation $\dot{U}_q(\widehat{\mathfrak{g}})$ appear not in their Kac-Moody presentation, but rather in their loop presentation. In particular, the Heisenberg algebra $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$, which does not appear in the theory of Kac-Moody categorification, plays a prominent role. In this section we give a definition of a 2-representation of $\widehat{\mathfrak{g}}$ which is natural from the loop point of view. This definition is independent from, but parallel to, the Kac-Moody theory.

Since the basic representation has level one, we consider here only the notion of a level one 2-representation of $\widehat{\mathfrak{g}}$. Higher level 2-representations are also natural and will be discussed in future work.

A level one 2-representation of $\widehat{\mathfrak{g}}$ consists of a graded, triangulated, $\Bbbk$-linear idempotent complete 2-category $\mathcal{K}$ where:

- 0-morphisms (objects) are denoted $\mathcal{D}(\lambda)$ and are indexed by weights $\lambda \in \widehat{X}$,
- 1-morphisms include
  (1) the idempotent $\mathbf{1}_\lambda$ which is the identity on $\mathcal{D}(\lambda)$ and zero on $\mathcal{D}(\mu)$ if $\mu \neq \lambda$
  (2) $\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda : \mathcal{D}(\lambda) \to \mathcal{D}(\lambda + r\alpha_i - mr\delta)$ and $\mathbf{1}_\lambda \mathsf{F}_{i,m}^{(r)} : \mathcal{D}(\lambda + r\alpha_i + mr\delta) \to \mathcal{D}(\lambda)$
  (3) $\mathsf{P}_i^{(n)}\mathbf{1}_\lambda, \mathsf{P}_i^{(1^n)} : \mathcal{D}(\lambda) \to \mathcal{D}(\lambda + n\delta)$ and $\mathbf{1}_\lambda \mathsf{Q}_i^{(n)}, \mathsf{Q}_i^{(1^n)} : \mathcal{D}(\lambda + n\delta) \to \mathcal{D}(\lambda)$
  where $i \in I$, $m \in \mathbb{Z}$ and $r, n \in \mathbb{N}$.
- 2-morphisms include a map $\theta_i : \mathbf{1}_\lambda \to \mathbf{1}_\lambda\langle 2\rangle$ for each $i \in I$.

Note that if $\lambda, \mu \in \widehat{X}$ then $\mathrm{Hom}(\mathcal{D}(\lambda), \mathcal{D}(\mu))$ is a graded, triangulated, $\Bbbk$-linear category where the objects are the 1-morphisms from $\mathcal{D}(\lambda)$ to $\mathcal{D}(\mu)$ and the morphisms are 2-morphisms between them. By convention $\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda$ and $\mathbf{1}_\lambda\mathsf{F}_{i,m}^{(r)}$ are zero if $r < 0$ and equal $\mathbf{1}_\lambda$ if $r = 0$. Usually we will write $\mathsf{E}_i^{(r)}$ and $\mathsf{F}_i^{(r)}$ for $\mathsf{E}_{i,0}^{(r)}$ and $\mathsf{F}_{i,0}^{(r)}$.

We say $\mathcal{K}$ is *integrable* if for any root $\alpha \in \widehat{X}$ the objects $\mathcal{D}(\lambda \pm n\alpha)$ are zero for $n \gg 0$. Note that if $\mathcal{D}(\lambda)$ is zero then, by definition, $\mathrm{Hom}_{\mathcal{K}}(\mathbf{1}_\lambda, \mathbf{1}_\lambda\langle l\rangle) = 0$ for all l. We say that $\mathcal{K}$ is *weakly integrable* if it is the direct sum of (possibly infinitely many) integrable 2-representations.

On this data we impose the following conditions.

(1) $\mathcal{K}$ is weakly integrable.
(2) If $\mathcal{D}(\lambda) \neq 0$ then $\mathrm{Hom}_{\mathcal{K}}(\mathbf{1}_\lambda, \mathbf{1}_\lambda\langle l\rangle)$ is zero if $l < 0$ and one-dimensional if $l = 0$. Moreover, the space of 2-morphisms between any two 1-morphisms is finite dimensional.
(3) $\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda$ and $\mathbf{1}_\lambda\mathsf{F}_{i,-m}^{(r)}$ are left and right adjoints of each other up to shift. More precisely
  (a) $(\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda)_R \cong \mathbf{1}_\lambda\mathsf{F}_{i,-m}^{(r)}\langle r(\langle\lambda, \alpha_i\rangle + r + m)\rangle$, and
  (b) $(\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda)_L \cong \mathbf{1}_\lambda\mathsf{F}_{i,-m}^{(r)}\langle -r(\langle\lambda, \alpha_i\rangle + r + m)\rangle$.
(4) The P_i's and Q_i's define a (weakly integrable) 2-representation of $\widehat{\mathfrak{h}}$.
(5) We have

$$\begin{aligned}
\mathsf{F}_{i,-m}\mathsf{E}_{i,m}\mathbf{1}_\lambda &\cong \mathsf{E}_{i,m}\mathsf{F}_{i,-m}\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda \otimes_{\Bbbk} V_{-\langle\lambda,\alpha_i\rangle - m - 1} \text{ if } \langle\lambda, \alpha_i\rangle + m \leq 0,\\
\mathsf{E}_{i,m}\mathsf{F}_{i,-m}\mathbf{1}_\lambda &\cong \mathsf{F}_{i,-m}\mathsf{E}_{i,m}\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda \otimes_{\Bbbk} V_{\langle\lambda,\alpha_i\rangle + m - 1} \text{ if } \langle\lambda, \alpha_i\rangle + m \geq 0.
\end{aligned}$$

(6) For $m + n \neq 0$ then there exist distinguished triangles

$$\begin{aligned}
\mathsf{Q}_i^{[1^{m+n}]}\mathbf{1}_\lambda\langle\langle\lambda, \alpha_i\rangle - n\rangle \to \mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda \to \mathsf{F}_{i,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda \quad &\text{if} \quad m + n > 0,\\
\mathsf{F}_{i,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda \to \mathsf{P}_i^{[1^{-m-n}]}\mathbf{1}_\lambda\langle -\langle\lambda, \alpha_i\rangle - m\rangle \quad &\text{if} \quad m + n < 0.
\end{aligned}$$

(7) For $i \in I$ and $m \in \mathbb{Z}$ the maps

$$\begin{aligned}
I\theta_i I : \mathsf{F}_{i,-m}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{F}_{i,-m}\mathsf{E}_{i,m}\langle 2\rangle\mathbf{1}_\lambda \quad &\text{if} \quad \langle\lambda, \alpha_i\rangle + m \leq 0 \text{ and}\\
I\theta_i I : \mathsf{E}_{i,m}\mathsf{F}_{i,-m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{F}_{i,-m}\langle 2\rangle\mathbf{1}_\lambda \quad &\text{if} \quad \langle\lambda, \alpha_i\rangle + m \geq 0
\end{aligned}$$

induce an isomorphism between all summands $\mathbf{1}_\lambda\langle\cdot\rangle$ of the same degree on either side.

(8) If $i \neq j$ then $\mathsf{E}_{i,m}$ and $\mathsf{F}_{j,n}$ commute.
(9) For $i \in I$ and $m \in \mathbb{Z}$ we have
   (a) $\mathsf{E}_{i,m}\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda \cong \mathsf{E}_{i,m}^{(r+1)}\mathbf{1}_\lambda \otimes_{\Bbbk} V_r \cong \mathsf{E}_{i,m}^{(r)}\mathsf{E}_{i,m}\mathbf{1}_\lambda$ and
   (b) $\mathsf{F}_{i,m}\mathsf{F}_{i,m}^{(r)}\mathbf{1}_\lambda \cong \mathsf{F}_{i,m}^{(r+1)}\mathbf{1}_\lambda \otimes_{\Bbbk} V_r \cong \mathsf{F}_{i,m}^{(r)}\mathsf{F}_{i,m}\mathbf{1}_\lambda$.
(10) For $i \in I$ and $m \in \mathbb{Z}$ we have the distinguished triangles

$$\mathsf{E}_{i,m-1} \otimes_{\Bbbk} V_1\langle -1\rangle \mathbf{1}_\lambda \to \mathsf{P}_i^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{P}_i^{[1]}\mathbf{1}_\lambda$$
$$\mathsf{E}_{i,m}\mathsf{Q}_i^{[1]}\mathbf{1}_\lambda \to \mathsf{Q}_i^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m+1} \otimes_{\Bbbk} V_1\mathbf{1}_\lambda$$

and their adjoints.
(11) If $\langle i,j\rangle = -1$ and $m \in \mathbb{Z}$ then there exist distinguished triangles

$$\mathsf{E}_{i,m-1}[1]\langle -1\rangle \mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{P}_j^{[1]}\mathbf{1}_\lambda \to \mathsf{P}_j^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda$$
$$\mathsf{E}_{i,m+1}\mathbf{1}_\lambda \to \mathsf{Q}_j^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{Q}_j^{[1]}\mathbf{1}_\lambda$$

and their adjoints. If $\langle i,j\rangle = 0$ then $\mathsf{E}_{i,m}\mathbf{1}_\lambda$ and $\mathsf{F}_{i,n}\mathbf{1}_\lambda$ commute with $\mathsf{P}_j^{[1]}\mathbf{1}_\lambda$ and $\mathsf{Q}_j^{[1]}\mathbf{1}_\lambda$.
(12) For $i \in I$ and $m \in \mathbb{Z}$ we have

$$\mathsf{E}_{i,m-1}\mathsf{E}_{i,m}\mathbf{1}_\lambda \cong \mathsf{E}_{i,m}\mathsf{E}_{i,m-1}\langle -2\rangle \mathbf{1}_\lambda$$
$$\mathsf{F}_{i,m-1}\mathsf{F}_{i,m}\mathbf{1}_\lambda \cong \mathsf{F}_{i,m}\mathsf{F}_{i,m-1}\langle 2\rangle \mathbf{1}_\lambda.$$

(13) In fact it follows from the relations above that if $\langle i,j\rangle = -1$ and $m \in \mathbb{Z}$ then

$$\mathrm{Hom}(\mathsf{E}_{i,m}\mathsf{E}_{j,m+1}\mathbf{1}_\lambda, \mathsf{E}_{j,m+1}\mathsf{E}_{i,m}\langle 1\rangle \mathbf{1}_\lambda) \text{ and } \mathrm{Hom}(\mathsf{E}_{j,m}\mathsf{E}_{i,m+1}\mathbf{1}_\lambda, \mathsf{E}_{i,m+1}\mathsf{E}_{j,m}\langle 1\rangle \mathbf{1}_\lambda)$$

are both one-dimensional. If α and β are 2-morphisms which span these spaces, then we require the relation $\mathrm{Cone}(\alpha) \cong \mathrm{Cone}(\beta)$.
(14) If $\langle i,j\rangle = -1$ and $m,n \in \mathbb{Z}$ then

$$\mathsf{E}_{i,m}\mathsf{E}_{j,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda \cong \mathsf{E}_{i,m}^{(2)}\mathsf{E}_{j,n}\mathbf{1}_\lambda \oplus \mathsf{E}_{j,m}\mathsf{E}_{i,m}^{(2)}\mathbf{1}_\lambda$$

and similarly if we replace E by F. Meanwhile, $\mathsf{E}_{i,m}$ commutes with $\mathsf{E}_{j,n}$ and $\mathsf{F}_{i,m}$ commutes with $\mathsf{F}_{j,n}$ if $\langle i,j\rangle = 0$.

The 1-morphisms $\mathsf{P}_i^{[1^n]}$ and $\mathsf{Q}_i^{[1^n]}$ above denote the complexes defined in (14) and (15).

One difference between the definition above and that of Kac-Moody 2-representations is the appearance of triangulated categories and distinguished triangles. It would be interesting to rephrase the definition definition above to more closely resemble the defintions from [KL3, R], where all generating 2-morphisms are written explicitly, though it is not yet clear to us how to do this. Furthermore, the presence of cones and exact triangles in the relations makes it unclear philosophically how to conveniently encode the relations for 2-morphisms in a graphical calculus.

4.3. **Decategorification.** We now explain why a (level one) 2-representation of $\widehat{\mathfrak{g}}$ decategorifies to give a (level one) representation of $\dot{U}_q(\widehat{\mathfrak{g}})$. In other words, why there exists a map $\dot{U}_q(\widehat{\mathfrak{g}}) \to K_0(\mathcal{K})$. We will denote the class of a 1-morphism by square brackets. For example, the class of P_i is $[\mathsf{P}_i]$.

Lemma 4.1. *We have* $[\mathsf{P}_i^{[1^n]}\mathbf{1}_\lambda] = P_i^{[1_n]}1_\lambda$ *and* $[\mathsf{Q}_i^{[1^n]}\mathbf{1}_\lambda] = Q_i^{[1^n]}1_\lambda$ *and likewise if we replace* $[1^n]$ *by* $[n]$.

Proof. We prove that $[\mathsf{P}_i^{[1^n]}\mathbf{1}_\lambda] = P_i^{[1_n]}1_\lambda$ (the other equalities follow similarly). First recall that $P_i^{(1^{n-m})}P_i^{(m)} = P_i^{(m,1^{n-m})} + P_i^{(m+1,1^{n-m-1})}$. So we get

$$\begin{aligned} P_i^{[1^n]} &= \sum_{m=1}^{n}(-q)^{-m}[m]\left(P_i^{(m,1^{n-m})} + P_i^{(m+1,1^{n-m-1})}\right) \\ &= \sum_{m=1}^{n} P_i^{(m,1^{n-m})}\left((-q)^{-m+1}[m-1] + (-q)^{-m}[m]\right) \\ &= \sum_{m=1}^{n}(-1)^m q^{-2m+1}P_i^{(m,1^{n-m})} \\ &= -q^{-1}\sum_{m=1}^{n}(-1)^{m-1}q^{-2(m-1)}P_i^{(m,1^{n-m})} \\ &= [\mathsf{P}_i^{[1^n]}] \end{aligned}$$

where the last equality follows from the definition of $\mathsf{P}_i^{[1^n]}$. □

The first five relations in $\dot{U}_q(\widehat{\mathfrak{g}})$ hold as immediate consequences of the corresponding relations in $\mathcal{K}$. For example, relations (10) and (11) in $\mathcal{K}$ decategorify to give relation (4) in $\dot{U}_q(\widehat{\mathfrak{g}})$. In [CLi2] we showed that relations (6) and (7) in $\dot{U}_q(\widehat{\mathfrak{g}})$ are implied by the for all m and n once they are known for the case $m = n$. The case $m = n$, in turn, is an immediate consequence of relations (12) and (13) in $\mathcal{K}$.

Finally, in [CLi2] we showed that relation (8) in $\dot{U}_q(\widehat{\mathfrak{g}})$ follows from other relations when working in an integrable representation. Together with lemma 4.1, this completes the check that a level one 2-representation of $\widehat{\mathfrak{g}}$ descends in the Grothendieck group to a representation of the quantum affine algebra.

4.4. **Toroidal modifications.** All definitions above can be extended from the affine case to the toroidal case as follows. First recall that the Heisenberg 2-category $\mathcal{H}^\Gamma$ from [CLi1] has 1-morphisms P_i and Q_i for $i \in \hat{I}$ a node in the affine Dynkin diagram (not just the finite one); thus 2-morphisms in the 2category $\mathcal{H}^\Gamma$ also incldue generators P_0 and Q_0, where $0 \in \hat{I}$ is the affine node. The 2-category $\mathcal{H}^\Gamma$ categorifies the quantum toroidal Heisenberg algebra, which is a subalgebra of the idempotent quantum toroidal algebra $\dot{U}_q(\widehat{\widehat{\mathfrak{g}}})$.

The definition of a 2-representation of $\widehat{\widehat{\mathfrak{h}}}$ and $\widehat{\widehat{\mathfrak{g}}}$ is then essentially the same as for $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{g}}$, taking into account the existence of new 1-morphisms P_0, Q_0, $\mathsf{E}_{0,b}$, $\mathsf{F}_{0,b}$, etc. The categorified vertex operators of this paper then produce the basic 2-representation of $\widehat{\widehat{\mathfrak{g}}}$ from the Fock space 2-representations of $\widehat{\widehat{\mathfrak{h}}}$.

In particular, the structure of the basic representation of $\dot{U}_q(\widehat{\widehat{\mathfrak{g}}})$ is very similar to that of the basic representation of $\dot{U}_q(\widehat{\mathfrak{g}})$. For example, the underlying vector space for the basic toroidal representation is

$$V_{\Lambda_0} := \widehat{\mathcal{F}} \otimes_{\Bbbk(q)} \Bbbk(q)[\widehat{Y}]$$

where $\widehat{\mathcal{F}}$ is the Fock space representation of the quantum toroidal Heisenberg algebra and $\widehat{Y}$ is the affine root lattice, [FJW]. In section 9 we will give algebraic and geometric categorifications of the basic representation of $\dot{U}_q(\widehat{\widehat{\mathfrak{g}}})$.

All the proofs to follow extend without extra work from the affine case to the toroidal case, with one exception. The definition of a 2-representation of $\widehat{\widehat{\mathfrak{sl}_2}}$ should be changed slightly, essentially because the Dynkin diagram of $\widehat{\mathfrak{sl}_2}$ is not simply-laced. The appropriate definition of Heisenberg category for

this case is given in the appendix of [CLi1], and after using that definition, all the theorems of the current paper then carry over.

5. Definitions of categorified vertex operators and main results

5.1. **Categorified vertex operators.** Let $\mathcal{K}$ denote the underlying 2-category of a 2-representation of $\widehat{\mathfrak{h}}$. For any $i \in I$ we define the following complexes in the homotopy category $\mathsf{Kom}(\mathcal{K})$. We let

$$\mathsf{C}_i^-(k) := \left[\cdots \to \mathsf{P}_i^{(-k+l)}\mathsf{Q}_i^{(1^l)}\langle -l\rangle \to \cdots \to \mathsf{P}_i^{(-k+1)}\mathsf{Q}_i\langle -1\rangle \to \mathsf{P}_i^{(-k)}\right]\langle k\rangle[-k] \tag{16}$$

$$\mathsf{C}_i^-(k) := \left[\cdots \to \mathsf{P}_i^{(l)}\mathsf{Q}_i^{(1^{k+l})}\langle -l\rangle \to \cdots \to \mathsf{P}_i\mathsf{Q}_i^{(1^{k+1})}\langle -1\rangle \to \mathsf{Q}_i^{(1^k)}\right] \tag{17}$$

depending on whether $k \leq 0$ or $k \geq 0$ respectively. Here the right most term is in cohomological degree zero and the minus signs indicate that the complex is unbounded below. Likewise, we let

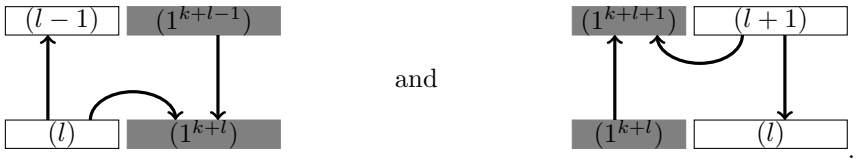
$$\mathsf{C}_i^+(k) := \left[\mathsf{Q}_i^{(-k)} \to \mathsf{P}_i\mathsf{Q}_i^{(-k+1)}\langle 1\rangle \to \cdots \to \mathsf{P}_i^{(1^l)}\mathsf{Q}_i^{(-k+l)}\langle l\rangle \to \dots\right]\langle -k\rangle[k] \tag{18}$$

$$\mathsf{C}_i^+(k) := \left[\mathsf{P}_i^{(1^k)} \to \mathsf{P}_i^{(1^{k+1})}\mathsf{Q}_i\langle 1\rangle \to \cdots \to \mathsf{P}_i^{(1^{k+l})}\mathsf{Q}_i^{(l)}\langle l\rangle \to \dots\right] \tag{19}$$

depending on whether $k \leq 0$ or $k \geq 0$. Here the left most term is in cohomological degree zero and the plus signs indicate that the complex is unbounded above. The differentials in these complexes are either given by a cap or a cup. For example, the differentials

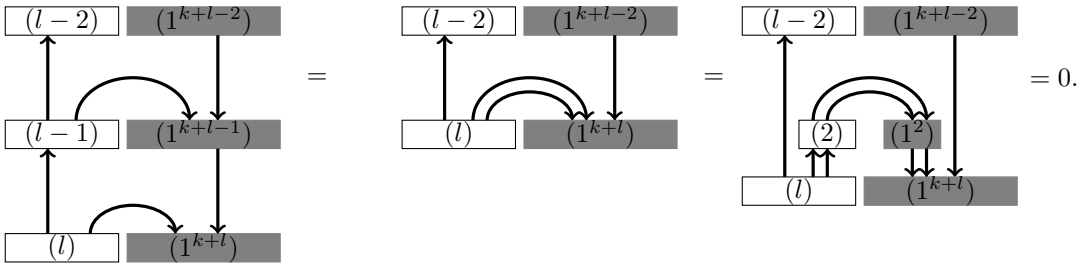
$$\begin{aligned} \mathsf{P}_i^{(l)}\mathsf{Q}_i^{(1^{k+l})}\langle -l\rangle &\longrightarrow \mathsf{P}_i^{(l-1)}\mathsf{Q}_i^{(1^{k+l-1})}\langle -l+1\rangle \text{ and} \\ \mathsf{P}_i^{(1^{k+l})}\mathsf{Q}_i^{(l)}\langle l\rangle &\longrightarrow \mathsf{P}_i^{(1^{k+l+1})}\mathsf{Q}_i^{(l+1)}\langle l+1\rangle \end{aligned}$$

in the definition of $\mathsf{C}_i^-(k)$ and $\mathsf{C}_i^+(k)$ when $k \geq 0$ are given by



Lemma 5.1. *The maps defined above define a differential (i.e. they square to zero).*

Proof. We consider C_i^-. Applying the differential twice gives the following



The first and second equalities above follows from the fact for $s \geq 0$, if we denote by $c_{(s)}$ and $c_{(1^s)}$ the idempotents in $\Bbbk[S_s]$ corresponding to the trivial and sign representation, then after embedding $\Bbbk[S_r]$ into $\Bbbk[S_{r+s}]$ in the natural way we have

$$c_{(r)}c_{(r+s)} = c_{(r+s)} = c_{(r+s)}c_{(r)} \text{ and } c_{(1^r)}c_{(1^{r+s})} = c_{(1^{r+s})} = c_{(1^{r+s})}c_{(1^r)}.$$

Now last equality in the proof of the Lemma follows from the fact that $c_{(2)}c_{(1^2)} = 0 \in \Bbbk[S_2]$. Thus C_i^- is a chain complex. The proof that C_i^+ is a chain complex is the same. □

5.2. **Defining a 2-representation of $\widehat{\mathfrak{g}}$.** Suppose $\mathcal{K}$ is an integrable 2-representation of $\widehat{\mathfrak{h}}$. Since the object labeled n is zero for $n \ll 0$ we can relabel the objects of $\mathcal{K}$ so that $\mathbf{1}_n = 0$ if $n < 0$. We will assume this from now on.

We now define a 2-representation of $\widehat{\mathfrak{g}}$ by describing the objects, 1-morphisms and 2-morphisms. If $\lambda \in \widehat{X}$ does not occur in the basic representation V_{Λ_0} of $\widehat{\mathfrak{g}}$ then we define $\mathcal{D}(\lambda) := 0$. On the other hand, any nonzero weight space of V_{Λ_0} corresponds to a weight $\lambda \in \widehat{X}$ of the form $\lambda = w \cdot \Lambda_0 - n\delta$ where $w \in W_{\widehat{\mathfrak{g}}}$ and $n \in \mathbb{N}$ (this w and n are uniquely determined by λ). In this case we define $\mathcal{D}(\lambda) := \mathcal{D}(n)$. Since the $W_{\widehat{\mathfrak{g}}}$ orbit of Λ_0 is in bijection with the root lattice Y of $\mathfrak{g}$, it follows that nonzero objects of are in bijection with pairs (α, n) where $\alpha \in Y$ and $n \in \mathbb{N}$.

Next we define 1-morphisms $\mathsf{E}_{i,m}\mathbf{1}_\lambda$ and $\mathbf{1}_\lambda\mathsf{F}_{i,m}$ as 1-morphisms in $\mathsf{Kom}(\mathcal{K})$. We set

$$\mathsf{E}_{i,m}\mathbf{1}_\lambda \mapsto \mathsf{C}_i^-(\langle\lambda,\alpha_i\rangle + 1 + m)\mathbf{1}_n \quad \text{and} \quad \mathbf{1}_\lambda\mathsf{F}_{i,m} \mapsto \mathbf{1}_n\mathsf{C}_i^+(\langle\lambda,\alpha_i\rangle + 1 - m). \tag{20}$$

The 1-morphisms $\mathsf{P}_i^{[1^n]}\mathbf{1}_\lambda$ and $\mathsf{Q}_i^{[1^n]}\mathbf{1}_\lambda$ are defined as in (14) and (15).

5.2.1. *Divided powers.* One can also define divided powers $\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda$ and $\mathbf{1}_\lambda\mathsf{F}_{i,-m}^{(r)}$ in $\mathsf{Kom}(\mathcal{K})$, though these will not play a role in what follows. If $k := -(\langle\lambda,\alpha_i\rangle + r + m) \geq 0$ we define

$$\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda := \left[\cdots \to \bigoplus_{w(\mu)\leq r, |\mu|=l} \mathsf{P}_i^{(k^r,\mu^t)}\mathsf{Q}_i^{(\mu)}\langle -l\rangle \to \cdots \to \mathsf{P}_i^{(k^r,1)}\mathsf{Q}_i\langle -1\rangle \to \mathsf{P}_i^{(k^r)}\right]\langle -kr - \binom{r}{2}\rangle[kr + \binom{r}{2}]$$

where the sum is over all partitions μ of size $|\mu|$ which fit in a box of width r ($w(\mu)$ denotes the width of μ).

Similarly, if $k := \langle\lambda,\alpha_i\rangle + r + m \geq 0$ we define

$$\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda := \left[\cdots \to \bigoplus_{w(\mu)\leq r, |\mu|=l} \mathsf{P}_i^{(\mu^t)}\mathsf{Q}_i^{(r^k,\mu)}\langle -l\rangle \to \cdots \to \mathsf{P}_i\mathsf{Q}_i^{(r^k,1)}\langle -1\rangle \to \mathsf{Q}_i^{(r^k)}\right]\langle -\binom{r}{2}\rangle[\binom{r}{2}].$$

The differentials are defined using caps as above. For example, in the first complex above there is a map

$$\mathsf{P}_i^{(k^r,\mu^t)}\mathsf{Q}_i^{(\mu)}\langle -l\rangle \to \mathsf{P}_i^{(k^r,\nu^t)}\mathsf{Q}_i^{(\nu)}\langle -l+1\rangle$$

if and only if μ is obtained from ν by adding a box. Moreover, in this case, this map is unique up to scalar and given by a cap. The scalar multiples need to be chosen so as to satisfy $\partial^2 = 0$, but it is not hard to see (using the same arguments as in lemmas 7.6 or 7.4) that any two such choices of scalars yield homotopic complexes.

The divided powers $\mathbf{1}_\lambda\mathsf{F}_{i,-m}^{(r)}$ are defined as the appropriately shifted right (or equivalently left) adjoints of the complexes above.

5.2.2. *Definition of θ.* For each $i \in I$ we have the following 2-morphism in $\mathcal{K}$

$$i \circlearrowright : \mathbf{1}_n \to \mathbf{1}_n\langle 2\rangle$$

Replacing $\mathbf{1}_n$ by $\mathbf{1}_\lambda$ defines the map $\theta_i\mathbf{1}_\lambda : \mathbf{1}_\lambda \to \mathbf{1}_\lambda\langle 2\rangle$.

5.3. **The main results.** The main work in this paper is proving that the functors defined above satisfy the relations required for a 2-representation of $\widehat{\mathfrak{g}}$. Below we summarize all the relations that we check directly.

Theorem 5.2. *In the homotopy 2-category $\mathsf{Kom}(\mathcal{K})$ we have the following relations.*

- *[Proposition 6.1] For $i \in I$ and $m \in \mathbb{Z}$, $r \in \mathbb{N}$ we have*
  (1) $(\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda)_R \cong \mathbf{1}_\lambda\mathsf{F}_{i,-m}^{(r)}\langle r(\langle\lambda,\alpha_i\rangle + r + m)\rangle$

(2) $(\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda)_L \cong \mathbf{1}_\lambda \mathsf{F}_{i,-m}^{(r)}\langle -r(\langle\lambda,\alpha_i\rangle + r + m)\rangle$
(3) $(\mathsf{P}_i^{[1^n]}\mathbf{1}_\lambda)_R \cong \mathsf{Q}_i^{[1^n]}\langle -n\rangle$
(4) $(\mathsf{P}_i^{[1^n]}\mathbf{1}_\lambda)_L \cong \mathsf{Q}_i^{[1^n]}\langle n\rangle$.

- *[Proposition 6.3] For $i \in I$ we have*

$$\begin{aligned}\mathsf{F}_{i,-m}\mathsf{E}_{i,m}\mathbf{1}_\lambda &\cong \mathsf{E}_{i,m}\mathsf{F}_{i,-m}\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda \otimes_{\Bbbk} V_{-\langle\lambda,\alpha_i\rangle - m - 1} \text{ if } \langle\lambda,\alpha_i\rangle + m \leq 0\\ \mathsf{E}_{i,m}\mathsf{F}_{i,-m}\mathbf{1}_\lambda &\cong \mathsf{F}_{i,-m}\mathsf{E}_{i,m}\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda \otimes_{\Bbbk} V_{\langle\lambda,\alpha_i\rangle + m - 1} \text{ if } \langle\lambda,\alpha_i\rangle + m \geq 0.\end{aligned}$$

- *[Proposition 6.12] Under the two isomorphisms above, for $i \in I$ and $m \in \mathbb{Z}$ the maps*

$$\begin{aligned}I\theta_i I : \mathsf{F}_{i,-m}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{F}_{i,-m}\mathsf{E}_{i,m}\langle 2\rangle\mathbf{1}_\lambda \quad &\text{if} \quad \langle\lambda,\alpha_i\rangle + m \leq 0 \text{ and}\\ I\theta_i I : \mathsf{E}_{i,m}\mathsf{F}_{i,-m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{F}_{i,-m}\langle 2\rangle\mathbf{1}_\lambda \quad &\text{if} \quad \langle\lambda,\alpha_i\rangle + m \geq 0\end{aligned}$$

  induce an isomorphism between all summands $\mathbf{1}_\lambda\langle\cdot\rangle$ of the same degree on either side.
- *[Proposition 6.8] If $m + n \neq 0$ then for any $i \in I$ there exist distinguished triangles*

$$\begin{aligned}\mathsf{Q}_i^{[1^{m+n}]}\mathbf{1}_\lambda\langle\langle\lambda,\alpha_i\rangle - n\rangle \to \mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda \to \mathsf{F}_{i,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda \quad &\text{if} \quad m + n > 0\\ \mathsf{F}_{i,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda \to \mathsf{P}_i^{[1^{-m-n}]}\mathbf{1}_\lambda\langle -\langle\lambda,\alpha_i\rangle - m\rangle \quad &\text{if} \quad m + n < 0.\end{aligned}$$

- *[Proposition 8.1] For $i \neq j \in I$ and $m, n \in \mathbb{Z}$ we have $\mathsf{F}_{j,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda \cong \mathsf{E}_{i,m}\mathsf{F}_{j,n}\mathbf{1}_\lambda$.*
- *[Proposition 7.3] For $i \in I$ and $m \in \mathbb{Z}$ we have $\mathsf{E}_{i,m}\mathsf{E}_{i,m}\mathbf{1}_\lambda \cong \mathsf{E}_{i,m}^{(2)}\langle -1\rangle\mathbf{1}_\lambda \oplus \mathsf{E}_{i,m}^{(2)}\langle 1\rangle\mathbf{1}_\lambda$.*
- *[Proposition 7.1] For $i \in I$ and $m \in \mathbb{Z}$ we have the distinguished triangles*

$$\begin{aligned}&\mathsf{E}_{i,m-1} \otimes_{\Bbbk} V_1\langle -1\rangle\mathbf{1}_\lambda \to \mathsf{P}_i^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{P}_i^{[1]}\mathbf{1}_\lambda\\ &\mathsf{E}_{i,m}\mathsf{Q}_i^{[1]}\mathbf{1}_\lambda \to \mathsf{Q}_i^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m+1} \otimes_{\Bbbk} V_1\mathbf{1}_\lambda\\ &\mathsf{Q}_i^{[1]}\mathsf{F}_{i,m}\mathbf{1}_\lambda \to \mathsf{F}_{i,m}\mathsf{Q}_i^{[1]}\mathbf{1}_\lambda \to \mathsf{F}_{i,m+1} \otimes_{\Bbbk} V_1\langle 1\rangle\mathbf{1}_\lambda\\ &\mathsf{F}_{i,m-1} \otimes_{\Bbbk} V_1\mathbf{1}_\lambda \to \mathsf{F}_{i,m}\mathsf{P}_i^{[1]}\mathbf{1}_\lambda \to \mathsf{P}_i^{[1]}\mathsf{F}_{i,m}\mathbf{1}_\lambda.\end{aligned}$$

- *[Proposition 8.2] If $\langle i,j\rangle = -1$ then there exist distinguished triangles*

$$\begin{aligned}&\mathsf{E}_{i,m-1}[1]\langle -1\rangle\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{P}_j^{[1]}\mathbf{1}_\lambda \to \mathsf{P}_j^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda\\ &\mathsf{E}_{i,m+1}\mathbf{1}_\lambda \to \mathsf{Q}_j^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{Q}_j^{[1]}\mathbf{1}_\lambda\\ &\mathsf{F}_{i,m}\mathsf{Q}_j^{[1]}\mathbf{1}_\lambda \to \mathsf{Q}_j^{[1]}\mathsf{F}_{i,m}\mathbf{1}_\lambda \to \mathsf{F}_{i,m+1}[-1]\langle 1\rangle\mathbf{1}_\lambda\\ &\mathsf{P}_j^{[1]}\mathsf{F}_{i,m}\mathbf{1}_\lambda \to \mathsf{F}_{i,m}\mathsf{P}_j^{[1]}\mathbf{1}_\lambda \to \mathsf{F}_{i,m-1}\mathbf{1}_\lambda\end{aligned}$$

  while if $\langle i,j\rangle = 0$ then $\mathsf{E}_{i,m}$ and $\mathsf{F}_{i,m}$ commute with $\mathsf{P}_j^{[1]}$ and $\mathsf{Q}_j^{[1]}$.
- *[Proposition 7.5] For $i \in I$ and any $m \in \mathbb{Z}$ we have*

$$\mathsf{E}_{i,m-1}\mathsf{E}_{i,m}\mathbf{1}_\lambda \cong \mathsf{E}_{i,m}\mathsf{E}_{i,m-1}\langle -2\rangle\mathbf{1}_\lambda.$$

- *[Proposition 8.3] If $\langle i,j\rangle = -1$ then there exist unique (up to a multiple) nonzero maps*

$$\alpha \in \mathrm{Hom}(\mathsf{E}_{i,m}\mathsf{E}_{j,m+1}\mathbf{1}_\lambda, \mathsf{E}_{j,m+1}\mathsf{E}_{i,m}\langle 1\rangle\mathbf{1}_\lambda) \text{ and } \beta \in \mathrm{Hom}(\mathsf{E}_{j,m}\mathsf{E}_{i,m+1}\mathbf{1}_\lambda, \mathsf{E}_{i,m+1}\mathsf{E}_{j,m}\langle 1\rangle\mathbf{1}_\lambda)$$

  for any $m \in \mathbb{Z}$. The cones of these maps are isomorphic: $Cone(\alpha) \cong Cone(\beta)$. Meanwhile, if $\langle i,j\rangle = 0$ then $\mathsf{E}_{i,m}$ and $\mathsf{E}_{j,n}$ commute for any $m, n \in \mathbb{Z}$.

Corollary 5.3. *Given a 2-representation $\mathcal{K}$ of $\widehat{\mathfrak{h}}$, the definitions of E's and F's from section 5.2 induce a 2-representation of $\widehat{\mathfrak{g}}$ on $\oplus_{\lambda\in Y, n\in\mathbb{Z}}\mathcal{D}(n)$.*

Proof. The relations we need to check are a consequence of the relations in theorem 5.2, with two exceptions.

First, the definition of a 2-representation of $\widehat{\mathfrak{g}}$ from section 4.2 includes the condition $\mathsf{E}_{i,m}\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda \cong \mathsf{E}_{i,m}^{(r+1)} \otimes_{\Bbbk} V_r$ although theorem 5.2 does not claim to prove this condition. The reason for this is that the existence of these divided power as well as the relations they satisfy follow formally from the other relations. This will be proved in [Cau]. For this reason we do not need to check this condition directly in this paper (except when $r = 1$). Indeed, the somewhat complicated definition of $\mathsf{E}_{i,m}^{(r)}$ indicates that such a check would be quite tedious.

Since we do not check this condition it would have been more honest to remove the condition on divided powers from our definition of a 2-representation of $\widehat{\mathfrak{g}}$. We have elected to leave it in, however, since these divided powers are natural categorically and we prefer to emphasize their appearance at the expense of some details.

Second, in theorem 5.2 we never check the Serre relation, namely that

$$\mathsf{E}_{i,m}\mathsf{E}_{j,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda \cong \mathsf{E}_{i,m}^{(2)}\mathsf{E}_{j,n}\mathbf{1}_\lambda \oplus \mathsf{E}_{j,m}\mathsf{E}_{i,m}^{(2)}\mathbf{1}_\lambda$$

if $\langle i, j\rangle = -1$. The reason is that this follows formally from the other relations. This should not come as a complete surprise since it is known that (at the decategorified level) the Serre relation in an integrable representation is redundant. The categorical analog of this fact is proved in [Cau]. □

6. The $\mathfrak{sl}_2$ relations

In this section we prove the $\mathfrak{sl}_2$ relations which appear in Theorem 5.2. Since we are only dealing with $\mathfrak{sl}_2$ we abbreviate P_i by P, Q_i by Q while λ is now just the integer $\langle \lambda, \alpha_i\rangle$.

Proposition 6.1. *The left and right adjoints of* $\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda$ *and* $\mathsf{P}_i^{[1^n]}\mathbf{1}_\lambda$ *are given by*

$$(\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda)_R \cong \mathbf{1}_\lambda\mathsf{F}_{i,-m}^{(r)}\langle r(\lambda + r + m)\rangle \quad \text{and} \quad (\mathsf{E}_{i,m}^{(r)}\mathbf{1}_\lambda)_L \cong \mathbf{1}_\lambda\mathsf{F}_{i,-m}^{(r)}\langle -r(\lambda + r + m)\rangle$$
$$(\mathsf{P}_i^{[1^n]}\mathbf{1}_\lambda)_R \cong \mathbf{1}_\lambda\mathsf{Q}_i^{[1^n]}\langle -n\rangle \quad \text{and} \quad (\mathsf{P}_i^{[1^n]}\mathbf{1}_\lambda)_L \cong \mathbf{1}_\lambda\mathsf{Q}_i^{[1^n]}\langle n\rangle.$$

Proof. This is a consequence of the fact that $\mathsf{P}_R \cong \mathsf{Q}\langle -1\rangle$ and $\mathsf{P}_L \cong \mathsf{Q}\langle 1\rangle$. □

6.1. **Gaussian elimination.** The following result is a slight generalization of a lemma which Bar-Natan [BN] calls "Gaussian elimination". We will use it to simplify complexes in the homotopy category of $\mathcal{H}$.

Lemma 6.2. *Let* X, Y, Z, W, U, V *be four objects in an additive category and consider a complex*

$$\cdots \to U \xrightarrow{u} X \oplus Y \xrightarrow{f} Z \oplus W \xrightarrow{v} V \to \dots \tag{21}$$

where $f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *and* u, v *are arbitrary morphisms. If* $D : Y \to W$ *is an isomorphism, then (21) is homotopic to a complex*

$$\cdots \to U \xrightarrow{u} X \xrightarrow{A - BD^{-1}C} Z \xrightarrow{v|_Z} V \to \dots \tag{22}$$

Proof. The key is the following commutative diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & U & \xrightarrow{u} & X \oplus Y & \xrightarrow{f} & Z \oplus W & \xrightarrow{v} & V & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\cdots & \longrightarrow & U & \xrightarrow{\alpha u} & X \oplus Y & \xrightarrow{g} & Z \oplus W & \xrightarrow{v\beta^{-1}} & V & \longrightarrow & \cdots
\end{array}$$

where

$$g = \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \quad \alpha = \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & -BD^{-1} \\ 0 & 1 \end{pmatrix}.$$

The vertical map of complexes is a homotopy equivalence and it is straightforward to check that the bottom row is homotopic to (22). □

6.2. **The basic $\mathfrak{sl}_2$ commutator relation.**

Proposition 6.3. *We have*

$$\mathsf{F}_{i,-m}\mathsf{E}_{i,m}\mathbf{1}_\lambda \cong \mathsf{E}_{i,m}\mathsf{F}_{i,-m}\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda \otimes_{\Bbbk} V_{-\lambda-m-1} \text{ if } \lambda + m \leq 0 \tag{23}$$

$$\mathsf{E}_{i,m}\mathsf{F}_{i,-m}\mathbf{1}_\lambda \cong \mathsf{F}_{i,-m}\mathsf{E}_{i,m}\mathbf{1}_\lambda \oplus \mathbf{1}_\lambda \otimes_{\Bbbk} V_{\lambda+m-1} \text{ if } \lambda + m \geq 0. \tag{24}$$

We will only prove (24) as (23) follows similarly. Because of the way $\mathsf{E}_{i,m}$ is defined it suffices to prove the case $m = 0$. To do this we identify in the following two propositions explicit expressions for $\mathsf{F}_i\mathsf{E}_i\mathbf{1}_\lambda$ and $\mathsf{E}_i\mathsf{F}_i\mathbf{1}_\lambda$ which we then compare. For simplicity we will write E and F for E_i and F_i.

Proposition 6.4. *The composition $\mathsf{FE}\mathbf{1}_\lambda$ is homotopic to the complex*

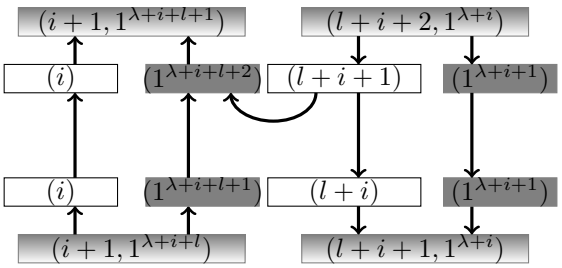
$$\cdots \to \bigoplus_{i\geq 0,-l} \mathsf{P}^{(i+1,1^{\lambda+i+l})}\mathsf{Q}^{(l+i+1,1^{\lambda+i})}\langle l\rangle \to \bigoplus_{i\geq 0,-l-1} \mathsf{P}^{(i+1,1^{\lambda+i+l+1})}\mathsf{Q}^{(l+i+2,1^{\lambda+i})}\langle l+1\rangle \to \dots \tag{25}$$

where the nonzero part of the differential maps the summand $\mathsf{P}^{(i+1,1^{\lambda+i+l})}\mathsf{Q}^{(l+i+1,1^{\lambda+i})}\langle l\rangle$ to the two terms

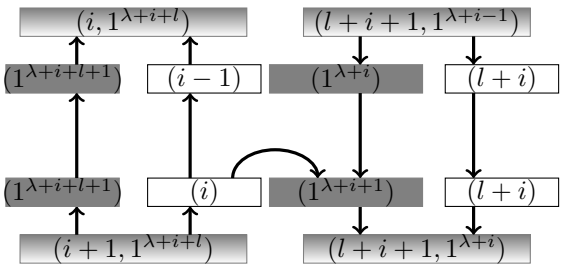
$$\mathsf{P}^{(i+1,1^{\lambda+i+l+1})}\mathsf{Q}^{(l+i+2,1^{\lambda+i})}\langle l+1\rangle \text{ and } \mathsf{P}^{(i,1^{\lambda+i+l})}\mathsf{Q}^{(l+i+1,1^{\lambda+i-1})}\langle l+1\rangle$$

using a cup and cap as in figures (26) and (27) below.

(26)



(27)



Proof. The composition $\mathsf{FE}\mathbf{1}_\lambda$ is equal to the complex

$$\cdots \to \bigoplus_{i\geq 0,-l} \mathsf{P}^{(1^{\lambda+1+i+l})}\mathsf{Q}^{(l+i)}\mathsf{P}^{(i)}\mathsf{Q}^{(1^{\lambda+1+i})}\langle l\rangle \to \dots \tag{28}$$

where the terms occurs in cohomological degree l. One can decompose the $\mathsf{Q}^{(l+i)}\mathsf{P}^{(i)}$ part of the expression to obtain

$$\cdots \to \bigoplus_{i\geq 0, -l\geq j-i} \mathsf{P}^{(1^{\lambda+1+i+l})}\mathsf{P}^{(i-j)}\mathsf{Q}^{(l+i-j)}\mathsf{Q}^{(1^{\lambda+1+i})} \otimes_{\Bbbk} V_j\langle l\rangle \to \dots \tag{29}$$

where V_j is the graded vector space in (2). Now, by Proposition 3.2, each term above breaks up into four terms. Let us consider the term $\mathsf{P}^{(i_0-j_0,1^{\lambda+i_0+l_0+1})}\mathsf{Q}^{(l_0+i_0-j_0,1^{\lambda+1+i_0})}$ for some fixed (i_0, j_0, l_0). This indecomposable 1-morphism occurs four times, namely, when

$$(i,j,l) \text{ equals } (i_0,j_0,l_0),(i_0+1,j_0+2,l_0),(i_0+1,j_0+1,l_0-1) \text{ or } (i_0,j_0+1,l_0+1). \tag{30}$$

When $-1 \leq j_0 < \min(i_0, i_0+l)$, these terms taken together form a subcomplex

$$\mathsf{P}^{(i_0-j_0,1^{\lambda+1+i_0+l_0})}\mathsf{Q}^{(l_0+i_0-j_0,1^{\lambda+1+i_0})}\langle l_0\rangle \otimes_{\Bbbk} (V_{j_0+1}\langle -1\rangle \to V_{j_0} \oplus V_{j_0+2} \to V_{j_0+1}\langle 1\rangle) \tag{31}$$

We empose the restriction $-1 \leq j_0 < \min(i_0, i_0+l_0)$ because otherwise the term in (29) disappears for at least one of the choices of (i,j,l) in (30). In lemma 6.5 below, we show that the complex in equation (31) exact. Thus we can cancel out such terms using the cancellation lemma 6.2.

If $j_0 > \min(i_0, i_0+l_0)$ then the term $\mathsf{P}^{(i_0-j_0,1^{\lambda+i_0+l_0+1})}\mathsf{Q}^{(\lambda+i_0-j_0,1^{\lambda+1+i_0})}$ vanishes because (29) disappears for all choices of (i,j,l) in (30). So it remains to study the case $j_0 = \min(i_0, i_0+l_0)$. For convenience let us assume $l_0 \geq 0$ so that $j_0 = i_0$ (the case $l_0 \leq 0$ is similar).

Case $j_0 = i_0$, $|l_0| \geq 2$. Here we are interested in terms of the form $\mathsf{P}^{(1^{\lambda+1+i_0+l_0})}\mathsf{Q}^{(l_0,1^{\lambda+1+i_0})}$ when $l_0 \geq 2$. Looking again at (29), such terms occur when

$$(i,j,l) \text{ equals } (i_0,i_0,l_0),(i_0,i_0-1,l_0-1),(i_0+1,i_0+1,l_0-1) \text{ or } (i_0+1,i_0,l_0-2).$$

So the resulting complex looks like

$$\mathsf{P}^{(1^{\lambda+1+i_0+l_0})}\mathsf{Q}^{(l_0,1^{\lambda+1+i_0})}\langle l_0-1\rangle \otimes_{\Bbbk} (V_{i_0}\langle -1\rangle \to V_{i_0+1} \oplus V_{i_0-1} \to V_{i_0}\langle 1\rangle)\,.$$

The same type of argument used in lemma 6.5 works to show that it is exact.

Case $j_0 = i_0$, $l_0 = 0, 1$. Here we are looking at terms of the form $\mathsf{P}^{(1^a)}\mathsf{Q}^{(1^a)}$. Such terms occur in (29) when

$$(i,j,l) \text{ equals } (a-\lambda, a-\lambda-1, -1),(a-\lambda, a-\lambda, 0),(a-\lambda-1, a-\lambda-2, 0) \text{ or } (a-\lambda-1, a-\lambda-1, 1)$$

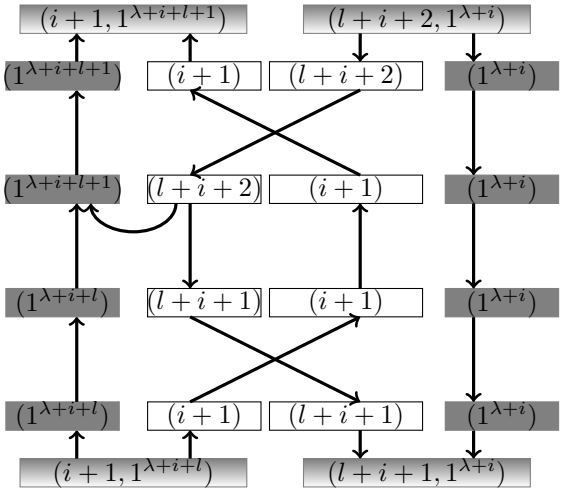
and again we get an exact complex.

Case $j_0 = -2$. So it seems like every complex is exact, but we missed one case, namely terms of the form $\mathsf{P}^{(i+1,1^{\lambda+i+l})}\mathsf{Q}^{(l+i+1,1^{\lambda+i})}$ which occur when $j_0 = -2$. Then three of the four terms in the complex in (31) are zero and we are left with just one term $\mathsf{P}^{(i+1,1^{\lambda+i+l})}\mathsf{Q}^{(l+i+1,1^{\lambda+i})}\langle l\rangle$ in degree l. Such indecomposable terms only occur once and hence cannot cancel out with anything else. Putting all these terms together we obtain a complex like in (25).

Finally, we need to determine the differential

$$\mathsf{P}^{(i+1,1^{\lambda+i+l})}\mathsf{Q}^{(l+i+1,1^{\lambda+i})}\langle l\rangle \to \mathsf{P}^{(i+1,1^{\lambda+i+l+1})}\mathsf{Q}^{(l+i+2,1^{\lambda+i})}\langle l+1\rangle.$$

For degree reasons it is not hard to see that the cancellation above does not alter the original differential between such terms. This differential is given by the following composition



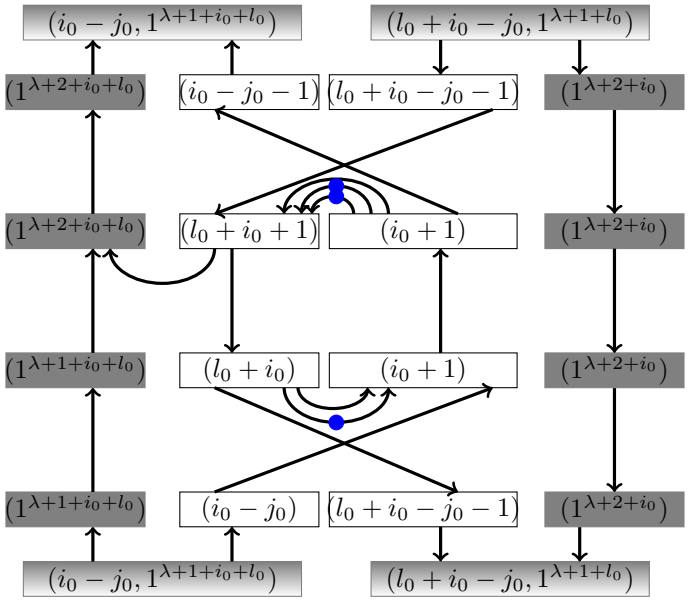
where the cup in the middle of the diagram comes from the differential in the complex for F. It is not hard to check that this diagram simplifies to give the map in (26). Likewise, one has such a diagram which simplifies to give the map in (27). Finally, all the other possible differential have to be zero by lemma 3.4. Thus, in the end, we get the complex in (25). □

Lemma 6.5. *The complex in (31) is exact.*

Proof. We need to show that the complex

$$\mathsf{P}^{(i_0-j_0,1^{\lambda+1+i_0+l_0})}\mathsf{Q}^{(l+i_0-j_0,1^{\lambda+1+i_0})}\langle l_0\rangle \otimes_{\Bbbk} (V_{j_0+1}\langle -1\rangle \to V_{j_0} \oplus V_{j_0+2} \to V_{j_0+1}\langle 1\rangle) \tag{32}$$

is exact. To do this we first show that the map $\alpha : V_{j_0+1}\langle -1\rangle \to V_{j_0+2}$ at the bottom left of the complex is injective. This map is given as a sum of the following compositions:



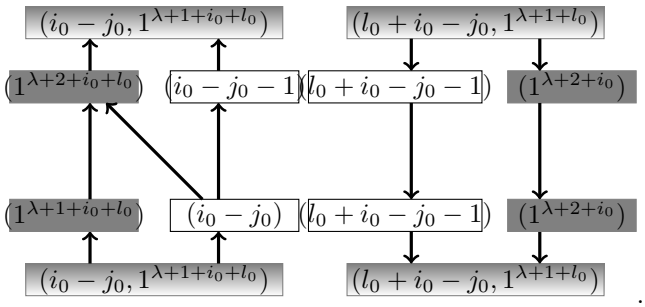
Here there are $j_0 + 1$ right pointing cups in the bottom part of the above picture, one left pointing cup in the middle, and $j_0 + 2$ left pointing caps in the top part. The sum is taken over a basis of the space of solid dots on the right pointing cups (this space is V_{j_0+1} given by $0, 1, \ldots, j_0 + 1$ dots) and a basis for the space of solid dots on the left cups (this space is V_{j_0+2} given by $0, 1, \ldots, j_0 + 2$ dots).

If there are k dots on the right pointing cups then the diagram evaluates to zero unless the left pointing caps have either $j_0 + 1 - k$ or $j_0 + 2 - k$ dots. This is because of the following three facts:

- two dots on the same strand is zero
- a counter-clockwise circle with no dots is also zero
- a counter-clockwise loop on an upward strand is zero.

Now let us look at the two cases when the diagram is nonzero.

Case 1. In the first case, when the total number of dots is $j_0 + 1$ the diagram simplifies to



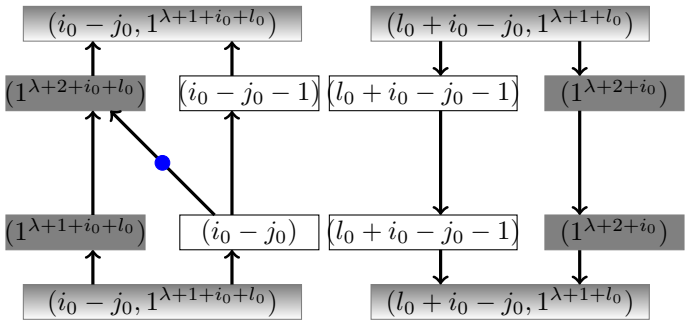
.

In this simplification we use the fact that a counter-clockwise circle with a degree two dot is equal to the identity and hence can be erased. Let us call the composition in the above diagram f_1. Now, the left part of the diagram, the part which involves only Ps, is the composition

$$\mathsf{P}^{(a,1^b)} \to \mathsf{P}^{(a)}\mathsf{P}^{(1^b)} \to \mathsf{P}^{(a-1)}\mathsf{P}\mathsf{P}^{(1^b)} \to \mathsf{P}^{(a-1)}\mathsf{P}^{(1^{b+1})} \to \mathsf{P}^{(a,1^b)}$$

where $a = i_0 - j_0$ and $b = \lambda + 1 + i_0 + l_0$. It is an exercise in the representation theory of the symmetric group that this composition is a nonzero multiple of the identity. Thus f_1 is (a nonzero multiple of) the identity.

Case 2. In the second case, when the total number of dots is $j_0 + 2$ the diagram simplifies to almost the same thing, namely:



Let us denote this map by f_2.

In terms of $f_1 = \mathrm{id}$ and f_2 the matrix for $\alpha : V_{j_0+1} \to V_{j_0+2}$ is

$$\alpha = \begin{bmatrix} 0 & \dots & 0 & f_2 \\ 0 & \dots & f_2 & \mathrm{id} \\ \dots & \dots & \dots & \dots \\ f_2 & \mathrm{id} & 0 & \dots \\ \mathrm{id} & 0 & \dots & 0. \end{bmatrix}$$

Note that there are $j_0 + 2$ rows and $j_0 + 1$ columns in the above matrix. Since this matrix has rank $j_0 + 1$, it follows that $\alpha : V_{j_0+1}\langle -1\rangle \to V_{j_0+2}$ is injective.

An almost identical analysis shows that the second map $\beta : V_{j_0+2} \to V_{j_0+1}\langle 1\rangle$ in (32) is surjective. Thus the complex in (32) is exact, since the first map is injective, the second is surjective and the dimension of the middle term is the sum of the dimensions of the right and left terms. □

Proposition 6.6. *The composition* $\mathsf{EF}\mathbf{1}_\lambda$ *is homotopic to the complex in (25) direct sum* $\mathbf{1}_\lambda \otimes_{\Bbbk} V_{\lambda-1}$.

Proof. The composition $\mathsf{EF}\mathbf{1}_\lambda$ is given by

$$\cdots \to \bigoplus_{i \geq 0} \mathsf{P}^{(i)}\mathsf{Q}^{(1^{\lambda-1+i})}\mathsf{P}^{(1^{\lambda-1+i+l})}\mathsf{Q}^{(i+l)}\langle l\rangle \to \dots \tag{33}$$

which we can rewrite as

$$\cdots \to \bigoplus_{i,j \geq 0} \mathsf{P}^{(i)}\mathsf{P}^{(1^{\lambda-1+i+l-j})}\mathsf{Q}^{(1^{\lambda-1+i-j})}\mathsf{Q}^{(i+l)} \otimes_{\Bbbk} V_j\langle l\rangle \to \dots \tag{34}$$

Now consider the term $\mathsf{P}^{(i_0,1^{\lambda-1+i_0+l_0-j_0})}\mathsf{Q}^{(i_0+l_0,1^{\lambda-1+i_0-j_0})}$. Such a term occurs when

$$(i,j,k) \text{ equals } (i_0, j_0, l_0), (i_0 - 1, j_0 - 2, l_0), (i_0, j_0 - 1, l_0 - 1) \text{ or } (i_0 - 1, j_0 - 1, l_0 + 1).$$

Thus we end up with a complex

$$\mathsf{P}^{(i_0,1^{\lambda-1+i_0+l_0-j_0})}\mathsf{Q}^{(i_0+l_0,1^{\lambda-1+i_0-j_0})}\langle l_0\rangle \otimes_k \left(V_{j_0-1}\langle -1\rangle \to V_{j_0} \oplus V_{j_0-2} \to V_{j_0-1}\langle 1\rangle\right). \tag{35}$$

The same argument as in lemma 6.5 shows that this complex is exact with the following possible exceptions: $i_0 = 0$ or $j_0 = 0$.

Case $i_0 = 0$, $j_0 > 0$, $l_0 \geq 2$. Here we have terms of the form $\mathsf{P}^{(1^{\lambda-1+l_0-j_0})}\mathsf{Q}^{(l_0,1^{\lambda-1-j_0})}$. If $l_0 \geq 2$ then one can check again by looking at (34) that there are four cases when such a term occurs, namely when

$$(i,j,l) \text{ equals } (0,j_0,l_0),(0,j_0-1,l_0-1),(0,j_0+1,l_0-1) \text{ or } (0,j_0,l_0-2).$$

Thus we end up with a complex

$$\mathsf{P}^{(1^{\lambda-1+l_0-j_0})}\mathsf{Q}^{(l_0,1^{\lambda-1-j_0})}\langle l_0-1\rangle \otimes_{\Bbbk} (V_{j_0}\langle -1\rangle \to V_{j_0-1}\oplus V_{j_0+1} \to V_{j_0}\langle 1\rangle)$$

and one can prove like before that if $j_0 > 0$ then it is exact. Likewise, one obtains an exact sequence if $l_0 \leq -2$.

Case $i_0 = 0$, $j_0 > 0$, $l_0 = -1, 0, 1$. This time we end up with terms of the form $\mathsf{P}^{(1^a)}\mathsf{Q}^{(1^a)}$ for some a. Such terms occur when

$$(i,j,l) \text{ equals } (0,\lambda-1-a,0),(1,\lambda+1-a,0),(0,\lambda-a,1) \text{ or } (1,\lambda-a,-1).$$

So again, if $a > 0$ we obtain a complex

$$\mathsf{P}^{(1^a)}\mathsf{Q}^{(1^a)} \otimes_{\Bbbk} (V_{\lambda-a}\langle -1\rangle \to V_{\lambda-1-a}\oplus V_{\lambda+1-a} \to V_{\lambda-a}\langle 1\rangle)$$

which, by the same type of argument as in lemma 6.5, is exact. The reason $a = 0$ is special is that we require that $j_0 \leq \min(\lambda-1+i, \lambda-1+i+l)$. So if $a = 0$ this condition is violated for three of the terms in the complex above and we end up with only $\mathbf{1}_\lambda \otimes_{\Bbbk} V_{\lambda-1}$.

Case $j_0 = 0$. Finally, if $j_0 = 0$ then three of the four terms in the complex (35) become zero and we end up with

$$\mathsf{P}^{(i_0,1^{\lambda-1+i_0+l_0})}\mathsf{Q}^{(i_0+l_0,1^{\lambda-1+i_0})}\langle l_0\rangle \otimes_k (0 \to V_0 \oplus 0 \to 0) \cong \mathsf{P}^{(i_0,1^{\lambda-1+i_0+l_0})}\mathsf{Q}^{(i_0+l_0,1^{\lambda-1+i_0})}\langle l_0\rangle.$$

Putting these terms together leaves us with a complex

$$\cdots \to \bigoplus_{i\geq 1,-l+1} \mathsf{P}^{(i,1^{\lambda-1+i+l})}\mathsf{Q}^{(i+l,1^{\lambda-1+i})}\langle l\rangle \to \dots$$

Notice that after replacing i by $i+1$ the terms are the same terms as those in the complex (25). Tracing through the differentials as before it is not hard to see that they are the same as those in (26) and (27), at least up to a nonzero multiple. Although these nonzero multiples may differ, one can show that the particular choice of multiples does not matter since any two such complexes must be homotopic.

Finally, one can check that the term $\mathbf{1}_\lambda \otimes_{\Bbbk} V_{\lambda-1}$ is a direct summand (i.e. there are no differentials into it or out of it). In fact, by adjunction one can check that

$$\mathrm{Hom}(\mathsf{P}^{(i+1,1^{\lambda-1+i})}\mathsf{Q}^{(i,1^{\lambda+i})}\langle -1\rangle, \mathbf{1}_\lambda \otimes_{\Bbbk} V_{\lambda-1}) = 0 = \mathrm{Hom}(\mathbf{1}_\lambda \otimes_{\Bbbk} V_{\lambda-1}, \mathsf{P}^{(i,1^{\lambda+i})}\mathsf{Q}^{(i+1,1^{\lambda-1+i})}\langle 1\rangle)$$

if $i \geq 1$. For example, the left hand space is equal to

$$\begin{aligned}\mathrm{Hom}(\mathsf{Q}^{(i,1^{\lambda+i})}\langle -1\rangle, (\mathsf{P}^{(i+1,1^{\lambda-1+i})})_R \otimes_{\Bbbk} V_{\lambda-1}) &= \mathrm{Hom}(\mathsf{Q}^{(i,1^{\lambda+i})}, \mathsf{Q}^{(i+1,1^{\lambda-1+i})} \otimes_{\Bbbk} V_{\lambda-1}\langle -\lambda-2i+1\rangle) \\ &= \oplus_{j=0}^{\lambda-1}\mathrm{Hom}(\mathsf{Q}^{(i,1^{\lambda+i})}, \mathsf{Q}^{(i+1,1^{\lambda-1+i})}\langle -2i-2j\rangle)\end{aligned}$$

which is zero since $2i+2j > 0$ if $j \geq 0$ and $i \geq 1$.

It follows that $\mathsf{EF}\mathbf{1}_\lambda$ is a direct sum of the complex in (25) and $\mathbf{1}_\lambda \otimes_{\Bbbk} V_{\lambda-1}$ which is what we needed to prove. □

Remark 6.7. If $\lambda \leq 0$ the argument is the same. In that case one shows that $\mathsf{EF}\mathbf{1}_\lambda$ is the complex

$$\cdots \to \bigoplus_{i\geq 0,-l} \mathsf{P}^{(-\lambda+i+2,1^{i+l-1})}\mathsf{Q}^{(-\lambda+i+l+2,1^{i-1})}\langle l\rangle \to \bigoplus_{i\geq 0,-l-1} \mathsf{P}^{(-\lambda+i+2,1^{i+l})}\mathsf{Q}^{(-\lambda+i+l+3,1^{i-1})}\langle l+1\rangle \to \dots$$

where the differentials are given by cups and caps like the ones in (26) and (27).

6.3. **The loop $\mathfrak{sl}_2$ commutator relation.**

Proposition 6.8. *In* $\mathsf{Kom}(\mathcal{K})$ *there exist distinguished triangles*

$$\mathsf{Q}_i^{[1^{m+n}]}\langle \lambda - n\rangle \mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda \to \mathsf{F}_{i,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda \quad \text{if} \quad m+n>0$$
$$\mathsf{F}_{i,m}\mathsf{E}_{i,n}\mathbf{1}_\lambda \to \mathsf{E}_{i,n}\mathsf{F}_{i,m}\mathbf{1}_\lambda \to \mathsf{P}_i^{[1^{-m-n}]}\langle -\lambda - m\rangle \mathbf{1}_\lambda \quad \text{if} \quad m+n<0$$

where $\lambda = \langle \lambda, \alpha_i\rangle$.

Remark 6.9. It is possible to see in simple examples that the triangles above do not split. For example, in order to split the first triangle we need (at least) a non-zero map $\mathsf{F}_{i,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda$. Now, if we take $m=1, n=0$ then by adjunction and using that $\mathsf{E}_i\mathsf{E}_{i,1} \cong \mathsf{E}_{i,1}\mathsf{E}_i\langle -2\rangle$ it is not difficult to check that

$$\mathrm{Hom}(\mathsf{F}_i\mathsf{E}_{i,1}\mathbf{1}_\lambda, \mathsf{E}_{i,1}\mathsf{F}_i\mathbf{1}_\lambda) \cong \mathrm{Hom}(\mathsf{E}_i\mathsf{E}_{i,1}\mathbf{1}_\mu, \mathsf{E}_i\mathsf{E}_{i,1}\langle -4\rangle \mathbf{1}_\mu)$$

where $\mu = \lambda - \alpha_i$. Now, you can choose μ so that $\mathbf{1}_\mu = \mathbf{1}_0$ and the complexes for $\mathsf{E}_{i,1}$ and E_i start with P_i and id respectively. Then, because $\mathsf{Q}_i\mathbf{1}_\mu = 0$, one can check that $\mathsf{E}_i\mathsf{E}_{i,1}\mathbf{1}_\mu \cong \mathsf{P}_i\mathbf{1}_0[1]$ and hence $\mathrm{Hom}(\mathsf{E}_i\mathsf{E}_{i,1}\mathbf{1}_\mu, \mathsf{E}_i\mathsf{E}_{i,1}\langle -4\rangle \mathbf{1}_\mu) \cong \mathrm{Hom}(\mathsf{P}_i, \mathsf{P}_i\langle -4\rangle) = 0$. So, in this instance, the first triangle in Proposition 6.8 does not split.

Proof. We prove only the first exact triangle (the second triangle follows by taking adjoints). We assume that $\lambda \geq n+1$ (a similar proof works if $\lambda \leq n+1$). As usual we write P and Q for P_i and Q_i.

First we identify $\mathsf{F}_{i,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda$ explicitly just like in the proof of proposition 6.3. The composition $\mathsf{F}_{i,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda$ is equal to

$$\cdots \to \bigoplus_{i\geq 0} \mathsf{P}^{(1^{\lambda+1+i+l-n})}\mathsf{Q}^{(l+i)}\mathsf{P}^{(i)}\mathsf{Q}^{(1^{\lambda+1+m+i})}\langle l\rangle \to \dots$$

just like the complex in equation (28). Then the same proof as that of proposition 6.4 shows that this is homotopic to the complex

(36)
$$\cdots \to \bigoplus_{i\geq 0,-l} \mathsf{P}^{(i+1,1^{\lambda+i+l-n})}\mathsf{Q}^{(l+i+1,1^{\lambda+m+i})}\langle l\rangle \to \bigoplus_{i\geq 0,-l-1} \mathsf{P}^{(i+1,1^{\lambda+i+l+1-n})}\mathsf{Q}^{(l+i+2,1^{\lambda+m+i})}\langle l+1\rangle \to \dots$$

which is just like the complex in (25).

On the other hand, the composition $\mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda$ is given by

(37)
$$\cdots \to \bigoplus_{i\geq 0} \mathsf{P}^{(i)}\mathsf{Q}^{(1^{\lambda-1+i+m})}\mathsf{P}^{(1^{\lambda-1+i+l-n})}\mathsf{Q}^{(i+l)}\langle l\rangle \to \dots$$

just like in equation (33). Cancelling out terms as in proposition 6.6 they all cancel with the exception of terms just like those appearing in (36) and terms involving only Q's (this is in contrast with extra copies of the identity in the computation of $\mathsf{EF}\mathbf{1}_\lambda$). Terms involving only Q's can occur in (37) only when $i=0$ and $l=0,\dots,m$. In other words, we get the complex

$$\mathsf{Q}^{(1^{\lambda-1+m})}\left[\mathsf{P}^{(1^{\lambda-1-n})} \to \mathsf{Q}\mathsf{P}^{(1^{\lambda-n})}\langle 1\rangle \to \cdots \to \mathsf{Q}^{(m)}\mathsf{P}^{(1^{\lambda-1+m-n})}\langle m\rangle\right]$$

where the left-most term is in degree zero. By lemma 6.10 the terms involving only Q's in this complex cancel out to give $\mathsf{Q}^{[1^{m+n}]}\langle \lambda - n\rangle$.

The left over terms in $\mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda$ cancel out, just like in the proof of the $[\mathsf{E}_i, \mathsf{F}_i]$ commutator relation, to leave us with the complex for $\mathsf{F}_{i,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda$ from (36). For the same degree reasons as in the proof of lemma 6.10, this cancellation does not change the differentials in $\mathsf{Q}^{[1^{m+n}]}\langle \lambda - n\rangle$.

Finally we show that, although $\mathsf{Q}^{[1^{m+n}]}\langle \lambda - n\rangle$ is not a direct summand of $\mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda$, there is a natural inclusion map

$$\mathsf{Q}^{[1^{m+n}]}\langle \lambda - n\rangle \to \mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda.$$

To see this one needs to check that there are no maps from the term $\mathsf{Q}^{(l,1^{m+n-l})}\langle\lambda-1+2l\rangle$ inside $\mathsf{Q}^{[1^{m+n}]}\langle\lambda\rangle$ (which lies in cohomological degree $l\geq 1$) to a term in cohomological degree $(l+1)$ in (36). By adjunction we have:

$$\begin{aligned}
&\mathrm{Hom}(\mathsf{Q}^{(l,1^{m+n-l})}\langle\lambda-1+2l\rangle, \mathsf{P}^{(i+1,1^{\lambda+i+l+1})}\mathsf{Q}^{(l+i+2,1^{\lambda+m+n+i})}\langle l+1\rangle)\\
\cong\ &\mathrm{Hom}(\mathsf{P}_L^{(i+1,1^{\lambda+i+l+1})}\mathsf{Q}^{(l,1^{m+n-l})}, \mathsf{Q}^{(l+i+2,1^{\lambda+m+n+i})}\langle-\lambda-l+2\rangle)\\
\cong\ &\mathrm{Hom}(\mathsf{Q}^{(i+1,1^{\lambda+i+l+1})}\mathsf{Q}^{(l,1^{m+n-l})}\langle\lambda+2i+l+2\rangle, \mathsf{Q}^{(l+i+2,1^{\lambda+m+n+i})}\langle-\lambda-l+2\rangle)\\
\cong\ &\mathrm{Hom}(\mathsf{Q}^{(i+1,1^{\lambda+i+l+1})}\mathsf{Q}^{(l,1^{m+n-l})}, \mathsf{Q}^{(l+i+2,1^{\lambda+m+n+i})}\langle-2(\lambda+i+l)\rangle)
\end{aligned}$$

which is zero since $\lambda, i\geq 0$ and $l\geq 1$. Thus we end up with an exact triangle

$$\mathsf{Q}^{[1^{m+n}]}\langle\lambda-n\rangle\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{F}_{i,n}\mathbf{1}_\lambda \to \mathsf{F}_{i,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda$$

as in the statement of the proposition. □

Lemma 6.10. *Modulo terms of the form* $\mathsf{P}^{(1^m)}\mathsf{Q}^{(1^n)}\mathsf{Q}^{(n')}$ *where* $m>0$, *the complex*

$$\mathsf{Q}^{(1^{a+l})}\left[\mathsf{P}^{(1^{a-b})}\to\cdots\to\mathsf{P}^{(1^a)}\mathsf{Q}^{(b)}\langle b\rangle\to\mathsf{P}^{(1^{a+1})}\mathsf{Q}^{(b+1)}\langle b+1\rangle\to\cdots\to\mathsf{P}^{(1^{a+l})}\mathsf{Q}^{(b+l)}\langle b+l\rangle\right]$$

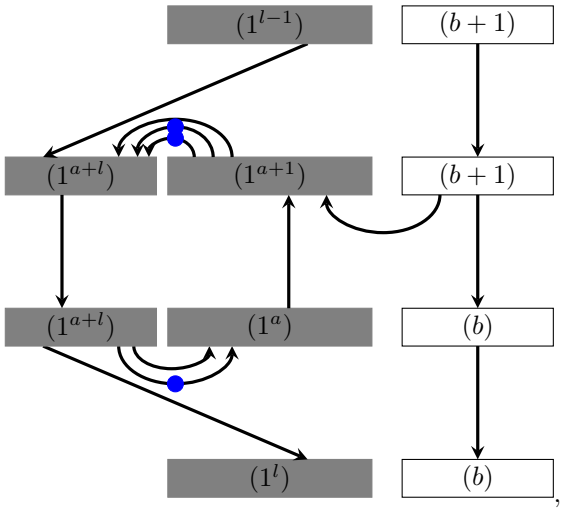
is homotopic to $\mathsf{Q}^{[1^{b+l}]}\langle a-b+1\rangle$.

Proof. Modulo terms involving P's the complex is equal to

(38)

$$\mathsf{Q}^{(1^{l+b})}\otimes_{\Bbbk}V_{a-b}\to\cdots\to\mathsf{Q}^{(1^l)}\mathsf{Q}^{(b)}\otimes_{\Bbbk}V_a\langle b\rangle\xrightarrow{\gamma_a}\mathsf{Q}^{(1^{l-1})}\mathsf{Q}^{(b+1)}\otimes_{\Bbbk}V_{a+1}\langle b+1\rangle\to\cdots\to\mathsf{Q}^{(b+l)}\otimes_{\Bbbk}V_{a+l}\langle b+l\rangle.$$

The map γ_a is given by a sum of diagrams of the form



with the summation being over the possible dots on the cups and caps. We denote the map above by γ_a^{jk} where there are $0\leq j\leq a$ and $0\leq k\leq a+1$ dots on the cups and caps respectively. Now

$$\mathsf{Q}^{(1^l)}\mathsf{Q}^{(b)}\cong\mathsf{Q}^{(b,1^l)}\oplus\mathsf{Q}^{(b+1,1^{l-1})}\text{ and }\mathsf{Q}^{(1^{l-1})}\mathsf{Q}^{(b+1)}\cong\mathsf{Q}^{(b+1,1^{l-1})}\oplus\mathsf{Q}^{(b+2,1^{l-2})}.$$

The composition

$$\mathsf{Q}^{(b+1,1^{l-1})}\hookrightarrow\mathsf{Q}^{(1^{l-1})}\mathsf{Q}^{(b+1)}\xrightarrow{\gamma_a^{jk}}\mathsf{Q}^{(1^{l-1})}\mathsf{Q}^{(b+1)}\langle 2(a+1-j-k)\rangle\to\mathsf{Q}^{(b+1,1^{l-1})}\langle 2(a+1-j-k)\rangle$$

can be shown, much like in the proof of lemma 6.5, to be an isomorphism if $j+k=a+1$. Thus γ_a induces an isomorphism between $a+1$ summands of the form $\mathsf{Q}^{(b+1,1^{l-1})}\langle\cdot\rangle$. The cancellation lemma

then leaves us with only one such summand, namely $\mathsf{Q}^{(b+1,1^{l-1})}\langle a+b+2\rangle$ on the right hand side. Similarly, one has such cancellations in every degree which leaves us with a complex

$$0 \to \mathsf{Q}^{(1^{l+b})}\langle a-b+2\rangle \to \mathsf{Q}^{(2,1^{l+b-2})}\langle a-b+4\rangle \to \cdots \to \mathsf{Q}^{(b+l-1,1)}\langle a+b+2l-2\rangle \to \mathsf{Q}^{(b+l)}\langle a+b+2l\rangle.$$

The terms here are the same as those in $\mathsf{Q}^{[1^{b+l}]}\langle a-b+1\rangle$. So we just need to check that the connecting boundary maps are also the same. To see this we first note that the restriction of the original boundary map to

$$\mathsf{Q}^{(b+1,1^{l-1})}\langle a+b+2\rangle \longrightarrow \mathsf{Q}^{(b+2,1^{l-2})}\langle a+b+4\rangle$$

simplifies to give the map in the definition of $\mathsf{Q}^{[1^{b+l}]}$ given by a dot. The computation needed to prove this is almost identical to the analysis of cases 1 and 2 in the proof of lemma 6.5 (so we omit it here).

The second step is to show that this map remains unchanged after applying the cancellation lemma 6.2 repeatedly. To do this we use that all the maps involved have even degree. Then, using the notation from the cancellation lemma 6.2, we take $X = \mathsf{Q}^{(b,1^l)}$ and $Z = \mathsf{Q}^{(b+1,1^{l-1})}\langle 2\rangle$. It follows that either B or C are degree zero. Suppose B is degree zero (the case of C having degree zero is the same). Then $B = 0$ or B is an isomorphism. The latter is impossible since each indecomposable occurs at most once in each degree in (38) and D is already an isomorphism. Thus $B = 0$ and the cancellation lemma does not change the map $A : X \to Z$ given by a dot. □

6.4. **The map θ.** The map $\theta_i : \mathbf{1}_{n+m} \to \mathbf{1}_{n+m}\langle 2\rangle$ gives a map

$$I\theta_i I : \mathsf{Q}_i^{(n)}\mathsf{P}_i^{(n)}\mathbf{1}_m \cong \mathsf{Q}_i^{(n)}\mathbf{1}_{m+n}\mathsf{P}_i^{(n)} \longrightarrow \mathsf{Q}_i^{(n)}\mathbf{1}_{m+n}\mathsf{P}_i^{(n)}\langle 2\rangle \cong \mathsf{Q}_i^{(n)}\mathsf{P}_i^{(n)}\mathbf{1}_m\langle 2\rangle. \tag{39}$$

Now recall that $\mathsf{Q}_i^{(n)}\mathsf{P}_i^{(n)}\mathbf{1}_m \cong \oplus_{k=0}^n \mathsf{P}_i^{(n-k)}\mathsf{Q}_i^{(n-k)} \otimes_{\Bbbk} V_k\mathbf{1}_m$.

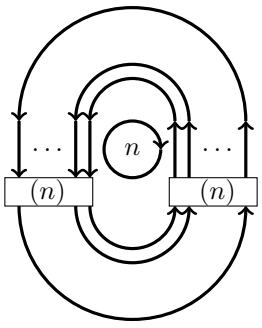
Lemma 6.11. *The map $I\theta_i I : \mathsf{Q}_i^{(n)}\mathsf{P}_i^{(n)}\mathbf{1}_m \to \mathsf{Q}_i^{(n)}\mathsf{P}_i^{(n)}\mathbf{1}_m\langle 2\rangle$ induces an isomorphism between n summands of the form $\mathbf{1}_m\langle\cdot\rangle$ on either side (in other words between all summands $\mathbf{1}_m\langle\cdot\rangle$ of the same degree on either side).*

Proof. Note that we have a canonical inclusion and projection

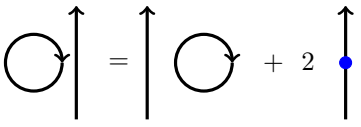
$$\iota : \mathbf{1}_m\langle n\rangle \to \mathsf{Q}_i^{(n)}\mathsf{P}_i^{(n)}\mathbf{1}_m, \quad \pi : \mathsf{Q}_i^{(n)}\mathsf{P}_i^{(n)}\mathbf{1}_m \to \mathbf{1}_m\langle n\rangle$$

given by a cup and cap. The result follows if we can show that the composition $\pi(I\theta_i I)^n\iota : \mathbf{1}_m \to \mathbf{1}_m$ is (some nonzero multiple of) the identity.

Diagrammatically, $\pi(I\theta_i I)^n\iota$ is given by the picture



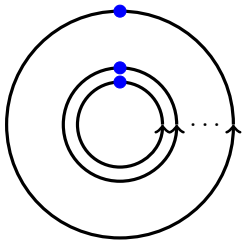
where the n in the middle of the center circle indicates that there are n concentric circles corresponding to $(I\theta_i I)^n$. Now, slide each of these circles from the inside towards the outside using that



This fact is an easy consequence of the relations among 2-morphisms in a Heisenberg 2-representation. After moving all the circles to the outside we end up with a bunch of circles and solid dots. Using that

- a dot squares to zero
- a counter-clockwise circle with no dots is zero

this simplifies to give some nonzero multiple of



where there are n circles. The result follows since each of these circles evaluates to the identity. □

Proposition 6.12. *The maps*

$$I\theta_i I : \mathsf{F}_{i,-m}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{F}_{i,-m}\mathsf{E}_{i,m}\langle 2\rangle\mathbf{1}_\lambda \quad \text{if} \quad \lambda + m \leq 0$$
$$I\theta_i I : \mathsf{E}_{i,m}\mathsf{F}_{i,-m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{F}_{i,-m}\langle 2\rangle\mathbf{1}_\lambda \quad \text{if} \quad \lambda + m \geq 0$$

induce an isomorphism between all summands $\mathbf{1}_\lambda\langle\cdot\rangle$ of the same degree on either side (here $\lambda = \langle\lambda, \alpha_i\rangle$).

Proof. We consider the case $\lambda + m \geq 0$ (the other case is very similar). Examining the proof of proposition 6.6 shows that the direct sum $\mathbf{1}_\lambda \otimes_{\Bbbk} V_{\lambda-1+m}$ inside $\mathsf{E}_{i,m}\mathsf{F}_{i,-m}\mathbf{1}_\lambda$ comes from the term $\mathsf{Q}^{(\lambda-1+m)}\mathsf{P}^{(\lambda-1+m)} \cong \mathbf{1}_\lambda \otimes_{\Bbbk} V_{\lambda-1} \bigoplus \mathrm{A}$ where the precise form of A is not important. The map $I\theta_i I$ then restricts to the endomorphism $I\theta_i I : \mathsf{Q}^{(\lambda-1+m)}\mathbf{1}_\mu\mathsf{P}^{(\lambda-1+m)} \to \mathsf{Q}^{(\lambda-1+m)}\mathbf{1}_\mu\mathsf{P}^{(\lambda-1+m)}\langle 2\rangle$ where μ is some weight. The result now follows by applying lemma 6.11. □

7. The loop relations

7.1. **The $[\mathsf{P},\mathsf{E}]$-type commutator.** In this section we examine how the functors E_i and F_i commute with P_i and Q_i. More precisely, we prove the following.

Proposition 7.1. *In $\mathsf{Kom}(\mathcal{K})$ there exist exact triangles*

$$\mathsf{E}_{i,m-1} \otimes_{\Bbbk} V_1\langle -1\rangle\mathbf{1}_\lambda \to \mathsf{P}_i^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{P}_i^{[1]}\mathbf{1}_\lambda$$
$$\mathsf{E}_{i,m}\mathsf{Q}_i^{[1]}\mathbf{1}_\lambda \to \mathsf{Q}_i^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m+1} \otimes_{\Bbbk} V_1\mathbf{1}_\lambda$$

as well as the triangles obtained by applying the (left or right) adjoint.

Proof. We prove the first relation (the second relation is similar). For simplicity we write P and Q for P_i and Q_i and assume that $m = 0$.

Case $\lambda \geq 0$. First note that

$$\mathsf{E}_i\mathbf{1}_\lambda \cong \cdots \to \mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle \to \cdots \to \mathsf{P}^{(2)}\mathsf{Q}^{(1^{\lambda+3})}\langle -2\rangle \to \mathsf{P}\mathsf{Q}^{(1^{\lambda+2})}\langle -1\rangle \to \mathsf{Q}^{(1^{\lambda+1})}$$

which means that $\mathsf{P}\mathsf{E}_i\mathbf{1}_\lambda$ is isomorphic to the complex

$$\cdots \to \mathsf{P}^{(l,1)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle \oplus \mathsf{P}^{(l+1)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle \to \cdots \to \mathsf{P}\mathsf{Q}^{(1^{\lambda+1})}. \tag{40}$$

It is easy to check that the diagonal maps in the differential, namely

$$\mathsf{P}^{(l,1)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle \to \mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l+1\rangle \text{ and } \mathsf{P}^{(l+1)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle \to \mathsf{P}^{(l-1,1)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l+1\rangle$$

are both zero.

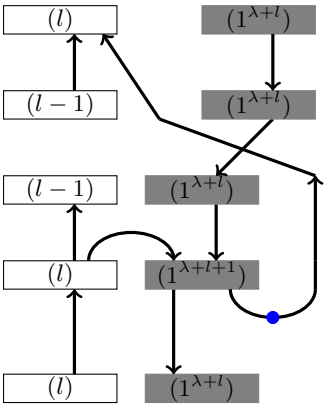
On the other hand, the general term of $\mathsf{E}_i\mathbf{1}_{\lambda+\delta}\mathsf{P}$ is

$$\mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l+1})}\mathsf{P}\langle -l\rangle \cong \mathsf{P}^{(l,1)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle \oplus \mathsf{P}^{(l+1)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle \oplus \mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l})} \otimes_{\Bbbk} V_1\langle -l\rangle.$$

If we consider terms of the form $\mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l})}$ which occur in this complex we get

$$\mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l})} \otimes_{\Bbbk} V_1\langle -l\rangle \to \mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l+1\rangle.$$

The two differential maps here are given by the following composition where the number of dots is either zero or one.



If there is no dot then the diagram simplifies to give the identity map on the terms $\mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l+1\rangle$ on either side. Applying the cancellation lemma we find that $\mathsf{E}_i\mathbf{1}_\lambda\mathsf{P}$ is homotopy equivalent to the complex

$$\cdots \to \begin{matrix}\mathsf{P}^{(l,1)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle \\ \mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l-1\rangle\end{matrix} \xrightarrow{\begin{pmatrix} A & B \\ C & D\end{pmatrix}} \begin{matrix}\mathsf{P}^{(l-1,1)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l+1\rangle \\ \mathsf{P}^{(l-1)}\mathsf{Q}^{(1^{\lambda+l-1})}\langle -l\rangle\end{matrix} \to \cdots \to \mathsf{Q}^{(1^\lambda)} \otimes_{\Bbbk} V_1. \tag{41}$$

where it is not difficult to check that A and D are given by the usual capping map as in (27), $C = 0$ and B is some map whose precise form does not matter.

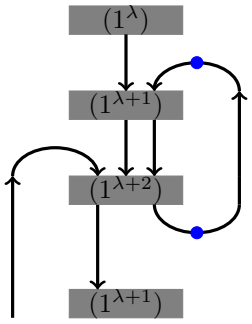
Now consider the following map of complexes from (40) to (41):

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \begin{matrix}\mathsf{P}^{(l,1)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle \\ \mathsf{P}^{(l+1)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle\end{matrix} & \longrightarrow & \begin{matrix}\mathsf{P}^{(l-1,1)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l+1\rangle \\ \mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l+1\rangle\end{matrix} & \longrightarrow \cdots \longrightarrow & \begin{matrix}\mathsf{P}^{(1,1)}\mathsf{Q}^{(1^{\lambda+2})}\langle -1\rangle \\ \mathsf{P}^{(2)}\mathsf{Q}^{(1^{\lambda+2})}\langle -1\rangle\end{matrix} & \longrightarrow & \mathsf{PQ}^{(1^{\lambda+1})} \\
& \downarrow \begin{pmatrix}\mathrm{id} & 0 \\ 0 & 0\end{pmatrix} & & \downarrow \begin{pmatrix}\mathrm{id} & 0 \\ 0 & 0\end{pmatrix} & & \downarrow \begin{pmatrix}\mathrm{id} & 0 \\ 0 & 0\end{pmatrix} & & \downarrow \begin{pmatrix}\partial \\ 0\end{pmatrix} \\
\cdots \longrightarrow & \begin{matrix}\mathsf{P}^{(l,1)}\mathsf{Q}^{(1^{\lambda+l+1})}\langle -l\rangle \\ \mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l-1\rangle\end{matrix} & \longrightarrow & \begin{matrix}\mathsf{P}^{(l-1,1)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l+1\rangle \\ \mathsf{P}^{(l-1)}\mathsf{Q}^{(1^{\lambda+l-1})}\langle -l\rangle\end{matrix} & \longrightarrow \cdots \longrightarrow & \begin{matrix}\mathsf{P}^{(1,1)}\mathsf{Q}^{(1^{\lambda+2})}\langle -1\rangle \\ \mathsf{PQ}^{(1^{\lambda+1})}\langle -2\rangle\end{matrix} & \longrightarrow & \begin{matrix}\mathsf{Q}^{(1^\lambda)}\langle 1\rangle \\ \mathsf{Q}^{(1^\lambda)}\langle -1\rangle\end{matrix}
\end{array}$$

It is not difficult to check that this is a map of complexes. Now the map

$$\mathsf{PQ}^{(1^{\lambda+1})}\langle -2\rangle \to \mathsf{Q}^{(1^\lambda)}\langle 1\rangle$$

induced by the bottom right arrow is given by



which simplifies to give zero. Thus we can apply lemma 7.2 below to find that the cone of this map of complexes is homotopic to the direct sum of two complexes

$$\mathsf{P}^{(l)}\mathsf{Q}^{(1^{\lambda+l})}\langle -l\rangle \to \mathsf{P}^{(l-1)}\mathsf{Q}^{(1^{\lambda+l-1})}\langle -l+1\rangle \to \cdots \to \mathsf{P}\mathsf{Q}^{(1^{\lambda+1})}\langle -1\rangle \to \mathsf{Q}^{(1^{\lambda})}$$

(one shifted by $\langle -1\rangle$ and the other by $\langle 1\rangle$) where the differential is induced by capping. Thus we get a distinguished triangle

$$\mathsf{P}\mathsf{E}_i\mathbf{1}_\lambda \to \mathsf{E}_i\mathsf{P}\mathbf{1}_\lambda \to \mathsf{E}_{i,-1}\otimes_{\Bbbk} V_1\mathbf{1}_\lambda.$$

The result follows since $\mathsf{P}_i^{[1]} = \mathsf{P}[1]\langle -1\rangle$.

Case $\lambda \leq 0$. We now only sketch the proof since it is similar to the one above. This time $\mathsf{E}_i\mathbf{1}_\lambda$ is congruent to

$$\left[\cdots \to \mathsf{P}^{(-\lambda-1+l)}\mathsf{Q}^{(1^l)}\langle -l\rangle \to \cdots \to \mathsf{P}^{(-\lambda+1)}\mathsf{Q}^{(1^2)}\langle -2\rangle \to \mathsf{P}^{(-\lambda)}\mathsf{Q}\langle -1\rangle \to \mathsf{P}^{(-\lambda-1)}\right]\langle \lambda+1\rangle[-\lambda-1]$$

which means that

$$\mathsf{P}\mathsf{E}_i\mathbf{1}_\lambda \cong \left[\cdots \to \begin{matrix}\mathsf{P}^{(-\lambda+2)}\mathsf{Q}^{(1^2)}\langle -2\rangle\oplus\\ \mathsf{P}^{(-\lambda+1,1)}\mathsf{Q}^{(1^2)}\langle -2\rangle\end{matrix} \to \begin{matrix}\mathsf{P}^{(-\lambda+1)}\mathsf{Q}\langle -1\rangle\oplus\\ \mathsf{P}^{(-\lambda,1)}\mathsf{Q}\langle -1\rangle\end{matrix} \to \begin{matrix}\mathsf{P}^{(-\lambda)}\oplus\\ \mathsf{P}^{(-\lambda-1,1)}\end{matrix}\right]\langle \lambda+1\rangle[-\lambda-1] \tag{42}$$

while $\mathsf{E}_i\mathsf{P}\mathbf{1}_\lambda$ is isomorphic to the complex

$$\begin{aligned}&\left[\cdots \to \mathsf{P}^{(-\lambda+1)}\mathsf{Q}^{(1^2)}\mathsf{P}\langle -2\rangle \to \mathsf{P}^{(-\lambda)}\mathsf{Q}\mathsf{P}\langle -1\rangle \to \mathsf{P}^{(-\lambda-1)}\mathsf{P}\right]\langle \lambda+1\rangle[-\lambda-1]\\ \cong\ &\left[\cdots \to \begin{matrix}\mathsf{P}^{(-\lambda+1)}\mathsf{P}\mathsf{Q}^{(1^2)}\langle -2\rangle\oplus\\ \mathsf{P}^{(-\lambda+1)}\mathsf{Q}\otimes_{\Bbbk} V_1\langle -2\rangle\end{matrix} \to \begin{matrix}\mathsf{P}^{(-\lambda)}\mathsf{P}\mathsf{Q}\langle -1\rangle\oplus\\ \mathsf{P}^{(-\lambda)}\otimes_{\Bbbk} V_1\langle -1\rangle\end{matrix} \to \mathsf{P}^{(-\lambda-1)}\mathsf{P}\right]\langle \lambda+1\rangle[-\lambda-1].\end{aligned}$$

Now cancelling terms as before this is homotopic to the complex

$$\mathsf{E}_i\mathsf{P}\mathbf{1}_\lambda \cong \left[\cdots \to \begin{matrix}\mathsf{P}^{(-\lambda+1,1)}\mathsf{Q}^{(1^2)}\langle -2\rangle\oplus\\ \mathsf{P}^{(-\lambda+1)}\mathsf{Q}\langle -3\rangle\end{matrix} \to \begin{matrix}\mathsf{P}^{(-\lambda,1)}\mathsf{Q}\langle -1\rangle\oplus\\ \mathsf{P}^{(-\lambda)}\langle -2\rangle\end{matrix} \to \mathsf{P}^{(-\lambda-1,1)}\right]\langle \lambda+1\rangle[-\lambda-1]. \tag{43}$$

Now, as before, one can write a natural map of complexes from (42) to (43) whose cone, after cancelling the terms of the form $\mathsf{P}^{(-\lambda-1+l,1)}\mathsf{Q}^{(1^l)}\langle -l\rangle$, is isomorphic to

$$\left[\cdots \to \begin{matrix}\mathsf{P}^{(-\lambda+2)}\mathsf{Q}^{(1^2)}\langle -2\rangle\oplus\\ \mathsf{P}^{(-\lambda+2)}\mathsf{Q}^{(1^2)}\langle -4\rangle\end{matrix} \to \begin{matrix}\mathsf{P}^{(-\lambda+1)}\mathsf{Q}\langle -1\rangle\oplus\\ \mathsf{P}^{(-\lambda+1)}\mathsf{Q}\langle -3\rangle\end{matrix} \to \begin{matrix}\mathsf{P}^{(-\lambda)}\oplus\\ \mathsf{P}^{(-\lambda)}\langle -2\rangle\end{matrix} \to 0\right]\langle \lambda+1\rangle[-\lambda-1].$$

This is then isomorphic to $\mathsf{E}_{i,-1}\otimes_{\Bbbk} V_1\mathbf{1}_\lambda$ and the result follows. Notice that it is important to have the shifts $\langle \lambda+1\rangle[-\lambda-1]$ in the definition of E or this calculation would be off by a shift. □

Lemma 7.2. *Consider the following map of complexes of objects in an additive category*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \begin{matrix} A_l \\ B_{l+1} \end{matrix} & \longrightarrow & \begin{matrix} A_{l-1} \\ B_l \end{matrix} & \longrightarrow \cdots \longrightarrow & \begin{matrix} A_1 \\ B_2 \end{matrix} & \longrightarrow & B_1 & \\
& \downarrow \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} \partial \\ 0 \end{pmatrix} & \\
\cdots \longrightarrow & \begin{matrix} A_l \\ C_l \end{matrix} & \longrightarrow & \begin{matrix} A_{l-1} \\ C_{l-1} \end{matrix} & \longrightarrow \cdots \longrightarrow & \begin{matrix} A_1 \\ C_1 \end{matrix} & \longrightarrow & B_0 & \\
& & & & & & & C_0 &
\end{array}$$

such that the maps satisfy the following conditions:

- *in the top line all "diagonal" maps $A_l \to B_l$ and $B_{l+1} \to A_{l-1}$ are zero,*
- $\mathrm{Hom}(B_{l+1}, C_l) = 0$ *and* $\mathrm{Hom}(A_l, C_{l-1}) = 0$ *for all l and*
- *the map $C_1 \to B_0$ is zero.*

Then the restriction of the differentials to $B_\bullet$ and $C_\bullet$ give complexes and the cone of the vertical map is homotopic to $B_\bullet \oplus C_\bullet$.

Proof. Given all the zero maps above it is easy to verify that the restriction of the horizontal maps to $B_\bullet$ and $C_\bullet$ yield complexes. Now, the cone of the vertical map is a complex

$$\cdots \longrightarrow \begin{matrix} A_l \oplus A_{l-1} \\ C_l \oplus B_l \end{matrix} \longrightarrow \begin{matrix} A_{l-1} \oplus A_{l-2} \\ C_{l-1} \oplus B_{l-1} \end{matrix} \longrightarrow \cdots \longrightarrow \begin{matrix} A_1 \oplus B_1 \\ C_1 \end{matrix} \longrightarrow \begin{matrix} B_0 \\ C_0 \end{matrix}$$

where the differentials induce isomorphisms $A_{l-1} \xrightarrow{\sim} A_{l-1}$ for all l. Also, the conditions above imply that the following maps are zero

$$B_l \to C_{l-1}, \quad A_{l-1} \to B_{l-1}, \quad C_l \to B_{l-1} \quad \text{for all } l.$$

This means that using the cancellation lemma to cancel all the A's does not change the differentials in $B_\bullet$ and $C_\bullet$ leaving us with the direct sum $B_\bullet \oplus C_\bullet$. □

7.2. **Divided powers.**

Proposition 7.3. *We have $\mathsf{E}_{i,m}\mathsf{E}_{i,m}\mathbf{1}_\lambda \cong \mathsf{E}^{(2)}_{i,m}\langle -1\rangle \mathbf{1}_\lambda \oplus \mathsf{E}^{(2)}_{i,m}\langle 1\rangle \mathbf{1}_\lambda$ and similarly for Fs.*

Proof. The proof is actually very similar to that of Proposition (7.5). We assume $m = 0$ and that $\lambda + 1 \geq 0$ (the other cases are similar). Recall that $\mathsf{E}^{(2)}_i \mathbf{1}_\lambda$ is given by the complex

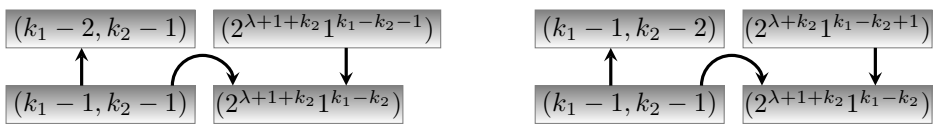
$$\left[\cdots \to \begin{matrix} \mathsf{P}^{(3)}\mathsf{Q}^{(2^{\lambda+2},1^3)}\langle -3\rangle \\ \oplus \mathsf{P}^{(2,1)}\mathsf{Q}^{(2^{\lambda+3},1)}\langle -3\rangle \end{matrix} \to \begin{matrix} \mathsf{P}^{(2)}\mathsf{Q}^{(2^{\lambda+2},1^2)}\langle -2\rangle \\ \oplus \mathsf{P}^{(1^2)}\mathsf{Q}^{(2^{\lambda+3})}\langle -2\rangle \end{matrix} \to \mathsf{P}\mathsf{Q}^{(2^{\lambda+2},1)}\langle -1\rangle \to \mathsf{Q}^{(2^{\lambda+2})} \right] [1]\langle -1\rangle \tag{44}$$

where the term in cohomological degree $-(l-1)$ is

$$\bigoplus_{\substack{k_1+k_2=l \\ k_1 \geq k_2}} \mathsf{P}^{(k_1-1,k_2-1)}\mathsf{Q}^{(2^{\lambda+1+k_2},1^{k_1-k_2})}\langle -l+1\rangle$$

The differential out of $\mathsf{P}^{(k_1-1,k_2-1)}\mathsf{Q}^{(2^{\lambda+1+k_2},1^{k_1-k_2})}$ is made up of two caps as depicted below.

(45)



Note that by lemma 7.4 it suffices to specify these two maps only up to a multiple. Now $\mathsf{E}_i\mathsf{E}_i\mathbf{1}_\lambda$ is given by the composition of complexes

$$\left[\cdots \to \mathsf{PQ}^{(1^{\lambda+4})}\langle -1\rangle \to \mathsf{Q}^{(1^{\lambda+3})}\right]\left[\cdots \to \mathsf{PQ}^{(1^{\lambda+2})}\langle -1\rangle \to \mathsf{Q}^{(1^{\lambda+1})}\right]$$

which means that the term in cohomological degree $-l$ is

$$\bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\left(\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{\lambda+3+k_2})} \oplus \mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{\lambda+2+k_2})} \otimes_{\Bbbk} V_1 \oplus \mathsf{P}^{(k_1-2)}\mathsf{Q}^{(1^{\lambda+1+k_2})}\right)\mathsf{Q}^{(1^{\lambda+1+k_1})}\langle -l\rangle.$$

If we collect terms with a shift of $\langle -l\rangle$ they must occur in cohomological degrees $(-l-1), -l$ and $(-l+1)$. These are

$$\bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-2)}\mathsf{Q}^{(1^{\lambda+1+k_2})}\mathsf{Q}^{(1^{\lambda+1+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{\lambda+2+k_2})}\mathsf{Q}^{(1^{\lambda+1+k_1})} \tag{46}$$

in cohomological degree $-l$ and $(-l+1)$ and

$$\bigoplus_{k_1+k_2=l+1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{\lambda+2+k_2})}\mathsf{Q}^{(1^{\lambda+1+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{\lambda+3+k_2})}\mathsf{Q}^{(1^{\lambda+1+k_1})} \tag{47}$$

occuring in cohomological degree $(-l-1)$ and $-l$.

Calculation of (46). As we did in the proof of Proposition 7.5 where we showed that the map in (52) was injective one can show (in a very similar manner) that the map in (46) is also injective. We do not do this in detail again but instead just keep track of what terms we have left on the right hand side of (46). First we rewrite (46) as

$$\bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{\lambda+1+k_2})}\mathsf{Q}^{(1^{\lambda+2+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{\lambda+2+k_2})}\mathsf{Q}^{(1^{\lambda+1+k_1})}$$

and then we cancel out to get

$$\bigoplus_{\substack{k_1+k_2=l-1\\ k_2\geq k_1+1}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(2^{\lambda+2+k_1},1^{k_2-k_1-1})} \longrightarrow \bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2+1}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(2^{\lambda+2+k_2},1^{k_1-k_2-1})}.$$

Now we rewrite this again as

$$\bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2+1}} \mathsf{P}^{(k_1)}\mathsf{P}^{(k_2-1)}\mathsf{Q}^{(2^{\lambda+2+k_2},1^{k_1-k_2-1})} \longrightarrow \bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2+1}} \mathsf{P}^{(k_1-1)}\mathsf{P}^{(k_2)}\mathsf{Q}^{(2^{\lambda+2+k_2},1^{k_1-k_2-1})}$$

which then cancels out to give

$$\bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2+1}} \mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{\lambda+2+k_2},1^{k_1-k_2-1})}\langle -l\rangle \cong \bigoplus_{\substack{k_1+k_2=l\\ k_1\geq k_2}} \mathsf{P}^{(k_1-1,k_2-1)}\mathsf{Q}^{(2^{\lambda+1+k_2},1^{k_1-k_2})}\langle -l\rangle \tag{48}$$

in cohomological degree $-(l-1)$ (we added back in the shift $\langle -l\rangle$). These are precisely the terms in (44) shifted by $\langle -1\rangle$.

Calculation of (47). The map in (47) can be shown to be surjective. For simplicity we will just keep track of the what terms survive after cancellation. First rewrite (47) as

$$\bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{\lambda+2+k_2})}\mathsf{Q}^{(1^{\lambda+2+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{\lambda+3+k_2})}\mathsf{Q}^{(1^{\lambda+1+k_1})}$$

and then cancelling terms we get

$$\bigoplus_{\substack{k_1+k_2=l\\ k_2\geq k_1}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(2^{\lambda+2+k_1},1^{k_2-k_1})} \longrightarrow \bigoplus_{\substack{k_1+k_2=l\\ k_1\geq k_2+2}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(2^{\lambda+1+k_2},1^{k_1-k_2-2})}$$

which we rewrite to obtain

$$\bigoplus_{\substack{k_1+k_2=l\\ k_1\geq k_2}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(2^{\lambda+2+k_2},1^{k_1-k_2})} \longrightarrow \bigoplus_{\substack{k_1+k_2=l\\ k_1\geq k_2}} \mathsf{P}^{(k_2-1)}\mathsf{P}^{(k_1+1)}\mathsf{Q}^{(2^{\lambda+2+k_2},1^{k_1-k_2})}.$$

This map is surjective and we are left with

$$\bigoplus_{\substack{k_1+k_2=l\\ k_1\geq k_2}} \mathsf{P}^{(k_1,k_2)}\mathsf{Q}^{(2^{\lambda+2+k_2},1^{k_1-k_2})}\langle -l\rangle$$

in cohomological degree $-(l+1)$. Replacing k_1, k_2 and l by k_1-1, k_2-1 and $l-2$ this is the same as the terms in (44) shifted by $\langle 1\rangle$.

To conclude that $\mathsf{E}_i\mathsf{E}_i\mathbf{1}_\lambda$ is homotopic to a direct sum of $\mathsf{E}_i^{(2)}\langle -1\rangle\mathbf{1}_\lambda$ and $\mathsf{E}_i^{(2)}\langle 1\rangle\mathbf{1}_\lambda$ one needs to check that there are no cross differentials such as

$$\mathsf{P}^{(k_1-1,k_2-1)}\mathsf{Q}^{(2^{\lambda+1+k_2},1^{k_2-k_1})} \to \mathsf{P}^{(k_1-2,k_2-1)}\mathsf{Q}^{(2^{\lambda+k_2},1^{k_2-k_1+1})}\langle 3\rangle$$
$$\text{or} \quad \mathsf{P}^{(k_1-1,k_2-1)}\mathsf{Q}^{(2^{\lambda+1+k_2},1^{k_2-k_1})} \to \mathsf{P}^{(k_1-1,k_2-2)}\mathsf{Q}^{(2^{\lambda+1+k_2},1^{k_2-k_1})}\langle 3\rangle.$$

Fortunately, such maps have degree three and cannot show up (even after various applications of the cancellation lemma) because in the original complex for $\mathsf{E}_i\mathsf{E}_i\mathbf{1}_\lambda$, all maps between irreducibles are either isomorphisms or degree one maps given by a cap. □

Lemma 7.4. *Let C and C' be two complexes whose terms are as in (44) and whose differentials are given by some nonzero multiples of the maps in (45). Then C and C' are homotopic.*

Proof. The proof is similar (but easier) than that of lemma 7.6. Each term in (44) has either one or two maps coming out (namely those in (45)). If there are two maps coming out (call them α_1 and α_2) then these can always be completed to a complex

$$A \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} B_1 \overset{\beta_1}{\underset{\beta_2}{\rightrightarrows}} C$$

like that in (51) where $\mathrm{Hom}(A,B_1) \cong \mathrm{Hom}(A,B_2) \cong \mathrm{Hom}(B_1,C) \cong \mathrm{Hom}(B_2,C) \cong \mathrm{Hom}(A,C) \cong \Bbbk$ are spanned by $\alpha_1, \alpha_2, \beta_1, \beta_2$ and $\beta_1\circ\alpha_1 = -\beta_2\circ\alpha_2$ respectively.

Thus, starting from the right end of a complex like that in (44), one can can homotope it into any other such complex using maps which act by multiples of the identity on each summand in the complex. □

7.3. **Commutation of Es or Fs.** In general $\mathsf{E}_{i,m}$ and $\mathsf{E}_{i,n}$ do not commute. But in the simplest case they do commute up to a shift.

Proposition 7.5. *For any $n\in\mathbb{Z}$ we have*

$$\mathsf{E}_{i,n-1}\mathsf{E}_{i,n}\mathbf{1}_\lambda \cong \mathsf{E}_{i,n}\mathsf{E}_{i,n-1}\langle -2\rangle\mathbf{1}_\lambda \text{ and } \mathsf{F}_{i,n-1}\mathsf{F}_{i,n}\mathbf{1}_\lambda \cong \mathsf{F}_{i,n}\mathsf{F}_{i,n-1}\langle 2\rangle\mathbf{1}_\lambda.$$

We use the rest of this section to prove the first relation above (the second relation is obtained by taking the adjoint of the first relation). The idea of the proof is to show that both sides are homotopic to a complex

$$\left[\begin{array}{c} \mathsf{P}^{(2)}\mathsf{Q}^{(2^{a+1},1^3)}\langle -2\rangle \\ \cdots \to \oplus \mathsf{P}^{(1^2)}\mathsf{Q}^{(2^{a+2},1)}\langle -2\rangle \\ \oplus \mathsf{P}^{(2)}\mathsf{Q}^{(2^{a+2},1)}\langle -2\rangle \end{array} \begin{array}{c} \to \mathsf{P}\mathsf{Q}^{(2^{a+1},1^2)}\langle -1\rangle \\ \oplus \mathsf{P}\mathsf{Q}^{(2^{a+2})}\langle -1\rangle \end{array} \to \mathsf{Q}^{(2^{a+1},1)}\right][1]\langle -2\rangle \tag{49}$$

where $a := \lambda + n \geq 0$. In the equation above, the term in cohomological degree $(-l+1)$ is

$$\bigoplus_{\substack{k_1+k_2=l-1 \\ k_1 \geq k_2}} \mathsf{P}^{(k_1,k_2-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})}\langle -l\rangle \quad \bigoplus_{\substack{k_1+k_2=l-1 \\ k_1 \geq k_2+1}} \mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})}\langle -l\rangle.$$

Using lemma 3.4 one checks that there are three possible nonzero maps out of each summand above, all of which are given by a cap:

$$\begin{aligned}
f_1 &: \mathsf{P}^{(k_1,k_2-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})} \longrightarrow \mathsf{P}^{(k_1-1,k_2-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2-1})} \\
f_2 &: \mathsf{P}^{(k_1,k_2-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})} \longrightarrow \mathsf{P}^{(k_1-1,k_2-1)}\mathsf{Q}^{(2^{a+k_2},1^{k_1-k_2+1})} \\
f_3 &: \mathsf{P}^{(k_1,k_2-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})} \longrightarrow \mathsf{P}^{(k_1,k_2-2)}\mathsf{Q}^{(2^{a+k_2},1^{k_1-k_2+1})} \\
g_1 &: \mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})} \longrightarrow \mathsf{P}^{(k_1-1,k_2-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2-1})} \\
g_2 &: \mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})} \longrightarrow \mathsf{P}^{(k_1-1,k_2-1)}\mathsf{Q}^{(2^{a+k_2},1^{k_1-k_2+1})} \\
g_3 &: \mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})} \longrightarrow \mathsf{P}^{(k_1-2,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2-1})}.
\end{aligned}$$

The differential in (49) is of the form

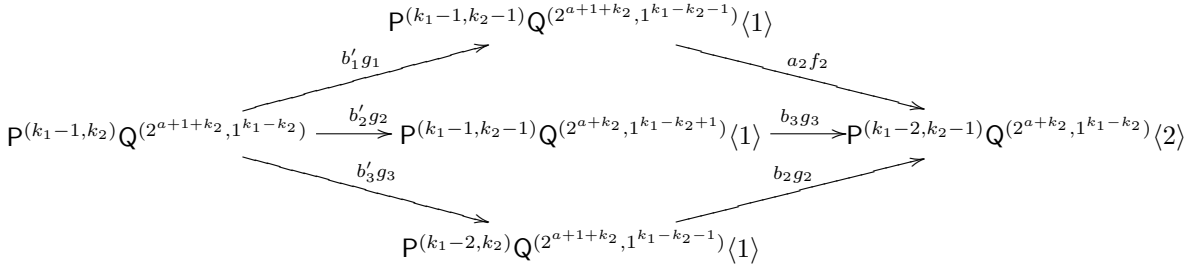
$$\partial = a_1f_1 + a_2f_2 + a_3f_3 + b_1g_1 + b_2g_2 + b_3g_3 \quad \text{for some} \quad a_1, a_2, a_3, b_1, b_2, b_3 \in \Bbbk. \tag{50}$$

Using the following lemma we will check that $a_2 = 0 = b_2$ and $a_1, a_3, b_1, b_3 \in \Bbbk^\times$ which determines the complex uniquely up to homotopy.

Lemma 7.6. *Consider a complex as in (49) with differential ∂ as in (50). If $b_1, b_3 \in \Bbbk^\times$ then $a_1, a_3 \in \Bbbk^\times$ while $a_2 = 0 = b_2$. Moreover, any two such complexes with this property are homotopy equivalent.*

Proof. Suppose we have a complex as in (49) where $b_1, b_3 \in \Bbbk^\times$.

The first step is to show that $a_2 = 0 = b_2$. To see this consider the following three compositions

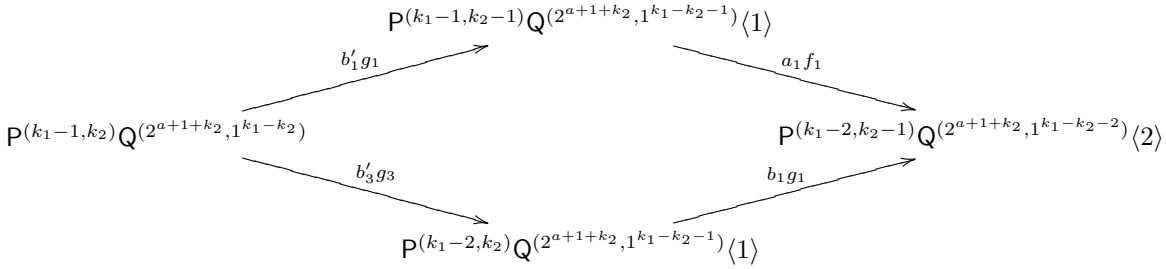


By induction we can assume that $b_2 = 0$. Looking at the top two compositions, this means that $a_2f_2 \circ b_1'g_1 + b_3g_3 \circ b_2'g_2 = 0$. But one can check that the compositions $f_2 \circ g_1$ and $g_3 \circ g_2$ are linearly independent since they span the two dimensional vector space

$$\mathrm{Hom}(\mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})}, \mathsf{P}^{(k_1-2,k_2-1)}\mathsf{Q}^{(2^{a+k_2},1^{k_1-k_2})}\langle 2\rangle) \cong \Bbbk^2.$$

This space is 2-dimensional since the two boxes we add to go from partition (k_1-2,k_2-1) to (k_1-1,k_2) and from $(2^{a+k_2},1^{k_1-k_2})$ to $(2^{a+1+k_2},1^{k_1-k_2})$ occur in different columns and rows. This implies that $a_2=0$ and $b_2'=0$. Thus, by induction, we always have $a_2=0=b_2$.

Next we show that $a_1,a_3\in\Bbbk^\times$. First consider the following two compositions

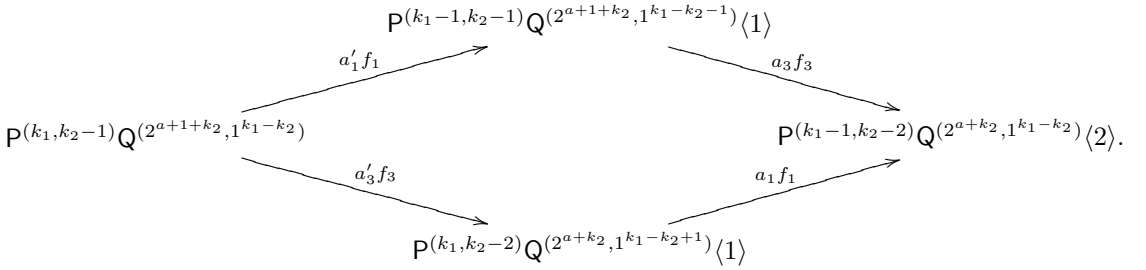


These two compositions both span the one dimensional vector space

$$\mathrm{Hom}(\mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})},\mathsf{P}^{(k_1-2,k_2-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2-2})}\langle 2\rangle)\cong\Bbbk.$$

Since b_1,b_3' are both nonzero this means $a_1\neq 0$.

Similarly, we can consider the two compositions



These two compositions both span the one dimensional vector space

$$\mathrm{Hom}(\mathsf{P}^{(k_1,k_2-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})},\mathsf{P}^{(k_1-1,k_2-2)}\mathsf{Q}^{(2^{a+k_2},1^{k_1-k_2})}\langle 2\rangle)\cong\Bbbk.$$

We know that a_1,a_1' are nonzero and by induction we can assume $a_3\neq 0$. This implies that $a_3'\neq 0$ and hence, by induction, all a_3 are nonzero.

Finally we show that any two such complexes are homotopy equivalent to each other. The idea is very simple. Suppose you have a complex

$$A\underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}}B_1\underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}}C \tag{51}$$

where $\mathrm{Hom}(A,B_1)\cong\mathrm{Hom}(A,B_2)\cong\mathrm{Hom}(B_1,C)\cong\mathrm{Hom}(B_2,C)\cong\mathrm{Hom}(A,C)\cong\Bbbk$ are spanned by $\alpha_1,\alpha_2,\beta_1,\beta_2$ and $\beta_1\circ\alpha_1=-\beta_2\circ\alpha_2$ respectively. Then any other complex where the four maps above are nonzero is homotopic to it via a map which acts by certain multiples of the identity on A,B_1,B_2 and C. This is a simple exercise which we leave to the reader.

If we now look at (49) and recall that each differential is made up of maps f_1,f_3 or g_1,g_3 it follows that (49) is made up of little complexes like (51). Thus starting from the far right, we can repeatedly apply the homotopy above to show that any two such complexes are homotopy equivalent. □

Computation of $\mathsf{E}_{i,n-1}\mathsf{E}_{i,n}\mathbf{1}_\lambda$. This composition is isomorphic to

$$\Big[\cdots\to\mathsf{P}^{(2)}\mathsf{Q}^{(1^{a+4})}\langle -2\rangle\to\mathsf{PQ}^{(1^{a+3})}\langle -1\rangle\to\mathsf{Q}^{(1^{a+2})}\Big]\Big[\cdots\to\mathsf{P}^{(2)}\mathsf{Q}^{(1^{a+3})}\langle -2\rangle\to\mathsf{PQ}^{(1^{a+2})}\langle -1\rangle\to\mathsf{Q}^{(1^{a+1})}\Big]$$

which means that the term in cohomological degree $-l$ is

$$\begin{aligned}&\bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{Q}^{(1^{a+2+k_2})}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+1+k_1})}\langle -l\rangle\\ \cong &\bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\left(\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+2+k_2})}\oplus \mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+1+k_2})}\otimes_{\Bbbk} V_1\oplus \mathsf{P}^{(k_1-2)}\mathsf{Q}^{(1^{a+k_2})}\right)\mathsf{Q}^{(1^{a+1+k_1})}\langle -l\rangle.\end{aligned}$$

If we collect terms with a shift of $\langle -l\rangle$ they must occur in cohomological degrees $(-l-1), -l$ and $(-l+1)$. These are

$$\bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-2)}\mathsf{Q}^{(1^{a+k_2})}\mathsf{Q}^{(1^{a+1+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+1+k_2})}\mathsf{Q}^{(1^{a+1+k_1})} \tag{52}$$

in cohomological degree $-l$ and $(-l+1)$ and

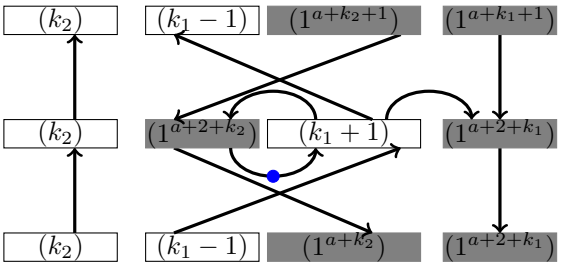
$$\bigoplus_{k_1+k_2=l+1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+1+k_2})}\mathsf{Q}^{(1^{a+1+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+2+k_2})}\mathsf{Q}^{(1^{a+1+k_1})} \tag{53}$$

occuring in cohomological degree $(-l-1)$ and $-l$.

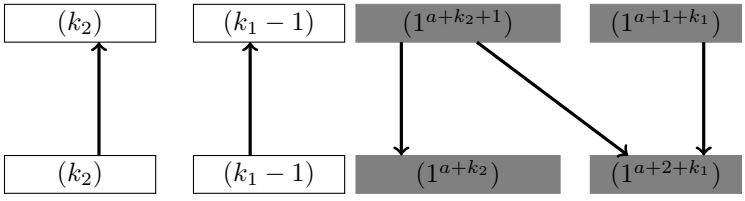
There are two parts to the differentials in (52). To describe the first we rewrite (52) as

$$\bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+k_2})}\mathsf{Q}^{(1^{a+2+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+1+k_2})}\mathsf{Q}^{(1^{a+1+k_1})}. \tag{54}$$

Then the first part of the differential is given by the composition



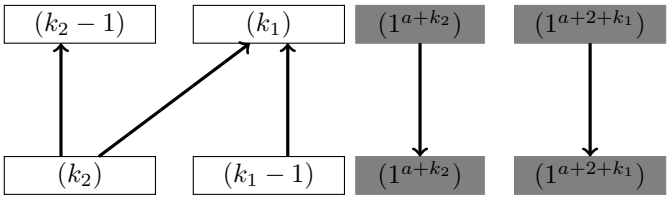
which is equal to a scalar multiple of



To see the other part of the differential we can also rewrite (52) as

$$\bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+k_2})}\mathsf{Q}^{(1^{a+2+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2-1)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+k_2})}\mathsf{Q}^{(1^{a+2+k_1})} \tag{55}$$

and then there is a similar map given by



Claim. The map in (52) is injective.

Let us consider a general indecomposable term on the left side of (54) or (55). Using proposition 3.2 such a term is of the form $\mathsf{P}^{(m,n)}\mathsf{Q}^{(2^{a+m'},1^{a+n'})}$ where $m \leq n$ and $m+n = l-1 = 2m'+n'$. There is one such summand for each (k_1,k_2) where

$$m \leq \min(k_1-1, k_2) \text{ and } m' \leq \min(k_1+2, k_2).$$

On the right hand side of (54) there is one such summand corresponding to each pair (k_1,k_2) where

$$m \leq \min(k_1-1, k_2) \text{ and } m' \leq \min(k_1+1, k_2+1)$$

and the map in figure 7.3 induces an isomorphism between any two such summands corresponding to the same pair (k_1,k_2). Likewise, on the right side of (55) there is one such summand corresponding to each (k_1,k_2) where

$$m \leq \min(k_1, k_2-1) \text{ and } m' \leq \min(k_1+2, k_2)$$

and the map in 7.3 also induces an isomorphism between summands corresponding to the same pair (k_1,k_2).

Using the inequalities above and looking at (54), we see that the map in (7.3) is injective on summands $\mathsf{P}^{(m,n)}\mathsf{Q}^{(2^{a+m'},1^{a+n'})}$ unless $k_1+2 \leq k_2$ in which case there is precisely one term, namely the one corresponding to $k_1+2 = m'$ on the left, which maps to zero. Notice that for such a term to exist on the left side of (54) we must also have $m \leq \min(m'-3, l+1-m')$ (and in particular $m \leq m'-3$).

On the other hand, looking at (55), we see that the map in (7.3) is an isomorphism between all summands $\mathsf{P}^{(m,n)}\mathsf{Q}^{(2^{a+m'},1^{a+n'})}$ unless $k_2 \leq k_1-1$ in which case there is precisely one term mapped to zero, namely the one corresponding to $k_2 = m$. This time such a term exists on the left hand side of (55) only if $m' \leq \min(l+1-m, m)$ (and in particular $m' \leq m$).

Since we cannot have both $m \leq m'-3$ and $m' \leq m$ either (7.3) or (7.3) is injective on all summands of the form $\mathsf{P}^{(m,n)}\mathsf{Q}^{(2^{a+m'},1^{a+n'})}$. The map in (52) is upper triangular and hence also injective.

Now we need to figure out what terms remain on the right hand side of (52) after cancelling terms. We can replace (54) by

$$\bigoplus_{\substack{k_1+k_2=l-1\\ k_2\geq k_1+2}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(2^{a+2+k_1},1^{k_2-k_1-2})} \longrightarrow \bigoplus_{\substack{k_1+k_2=l-1\\ k_2\leq k_1}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})}. \tag{56}$$

since, using proposition refprop:rels1, we have

$$\begin{aligned}
\mathsf{Q}^{(1^{a+k_2})}\mathsf{Q}^{(1^{a+2+k_1})} &\cong \mathsf{Q}^{(1^{a+1+k_2})}\mathsf{Q}^{(1^{a+1+k_1})} \oplus \mathsf{Q}^{(2^{a+2+k_1},1^{k_2-k_1-2})} \text{ if } k_2 \geq k_1+2\\
\mathsf{Q}^{(1^{a+1+k_2})}\mathsf{Q}^{(1^{a+1+k_1})} &\cong \mathsf{Q}^{(1^{a+k_2})}\mathsf{Q}^{(1^{a+2+k_1})} \oplus \mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})} \text{ if } k_2 \leq k_1\\
\mathsf{Q}^{(1^{a+k_2})}\mathsf{Q}^{(1^{a+2+k_1})} &\cong \mathsf{Q}^{(1^{a+1+k_2})}\mathsf{Q}^{(1^{a+1+k_1})} \text{ if } k_2 = k_1+1.
\end{aligned}$$

Now, switching the roles of k_1 and k_2 on the left hand side and replacing the new k_1 by k_1+1 and the new k_2 by k_2-1 we get that (56) is equivalent to

$$\bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2}} \mathsf{P}^{(k_1+1)}\mathsf{P}^{(k_2-2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})} \longrightarrow \bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})}. \tag{57}$$

Again, we can cancel terms using that

$$\begin{aligned}\mathsf{P}^{(k_1-1)}\mathsf{P}^{(k_2)} &\cong \mathsf{P}^{(k_1+1)}\mathsf{P}^{(k_2-2)} \oplus \mathsf{P}^{(k_1,k_2-1)} \oplus \mathsf{P}^{(k_1-1,k_2)} \text{ if } k_1 \geq k_2+1\\ \mathsf{P}^{(k_1-1)}\mathsf{P}^{(k_2)} &\cong \mathsf{P}^{(k_1+1)}\mathsf{P}^{(k_2-2)} \oplus \mathsf{P}^{(k_1,k_2-1)} \text{ if } k_1 = k_2\end{aligned}$$

to obtain

$$\bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2}} \mathsf{P}^{(k_1,k_2-1)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})}\langle -l\rangle \quad \bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2+1}} \mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})}\langle -l\rangle$$

in cohomological degree $(-l+1)$ (where we have added back the $\langle -l\rangle$ shift). Notice that these are the same as the terms in the complex (49).

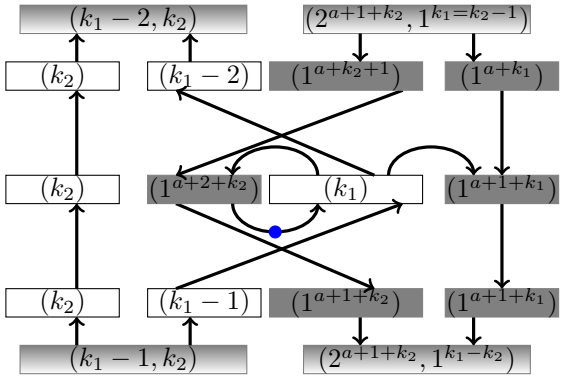
Now we also need to examine (53). Fortunately, things are much simpler here. We rewrite (53) as

$$\bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+1+k_2})}\mathsf{Q}^{(1^{a+2+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+2+k_2})}\mathsf{Q}^{(1^{a+1+k_1})}. \tag{58}$$

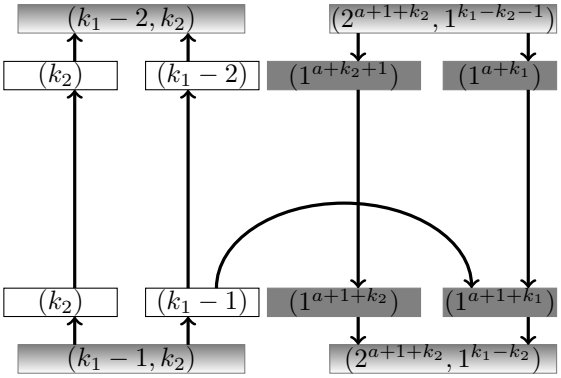
Then the part of the differential which looks like that in (7.3) induces an isomorphism between the two sides. The total differential is upper triangular and hence also induces an isomorphism. Thus all the terms in (53) cancel out.

The differentials. Finally, we need to compute the differentials. In light of lemma 7.6 it suffices to show that the differentials of the form g_1 and g_3 are nonzero.

This is trickier than it looks since the cancellation lemma was applied many times. Let us consider the map g_3. In the original complex for $\mathsf{E}_{i,n-1}\mathsf{E}_{i,n}\mathbf{1}_\lambda$ we see this map show up as the composition



which is equal to a scalar multiple of



The right hand side is clearly equal to the map

$$g_3 : \mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2})} \longrightarrow \mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2-1})}\langle 1\rangle.$$

Thus g_3 shows up as the map induced by the differential

$$\mathsf{P}^{(k_2)}\mathsf{Q}^{(1^{a+1+k_2})}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+1+k_1})} \xrightarrow{\curvearrowright\uparrow\uparrow} \mathsf{P}^{(k_2)}\mathsf{Q}^{(1^{a+1+k_2})}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+k_1})}\langle 1\rangle.$$

However, it is possible that in the cancellation process this map becomes zero. A little bit of reflection convinces one that this can only happen if the other differential

$$\mathsf{P}^{(k_2)}\mathsf{Q}^{(1^{a+1+k_2})}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+1+k_1})} \xrightarrow{\uparrow\uparrow\curvearrowright} \mathsf{P}^{(k_2-1)}\mathsf{Q}^{(1^{a+k_2})}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+1+k_1})}\langle 1\rangle$$

also induces the map g_3 since then in the process of applying the cancellation lemma these two maps could cancel. Fortunately, the right hand side $\mathsf{P}^{(k_2-1)}\mathsf{Q}^{(1^{a+k_2})}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+1+k_1})}$ does not contain any summand $\mathsf{P}^{(k_1-1,k_2)}\mathsf{Q}^{(2^{a+1+k_2},1^{k_1-k_2-1})}$ so this does not happen.

Similarly, one can show that g_1 also occurs in the differential of (49). Thus $\mathsf{E}_{i,n-1}\mathsf{E}_{i,n}\mathbf{1}_\lambda$ is indeed homotopic to the (unique up to homotopy) complex (49) with nonzero multiples of g_1 and g_3 in the differential.

Computation of $\mathsf{E}_{i,n}\mathsf{E}_{i,n-1}\mathbf{1}_\lambda$. This is isomorphic to

$$\left[\cdots \to \mathsf{P}^{(2)}\mathsf{Q}^{(1^{a+5})}\langle -2\rangle \to \mathsf{P}\mathsf{Q}^{(1^{a+4})}\langle -1\rangle \to \mathsf{Q}^{(1^{a+3})}\right]\left[\cdots \to \mathsf{P}^{(2)}\mathsf{Q}^{(1^{a+2})}\langle -2\rangle \to \mathsf{P}\mathsf{Q}^{(1^{a+1})}\langle -1\rangle \to \mathsf{Q}^{(1^{a})}\right]$$

which simplifies to give

$$\bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\left[\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+k_2+3})} \oplus \mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+k_2+2})} \otimes_{\Bbbk} V_1 \oplus \mathsf{P}^{(k_1-2)}\mathsf{Q}^{(1^{a+k_2+2})}\right]\mathsf{Q}^{(1^{a+k_1})}\langle -l\rangle$$

in cohomological degree $-l$. This computation is similar so we just sketch it. Again we collect terms with a grading shift $\langle -l\rangle$ to obtain

$$\bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-2)}\mathsf{Q}^{(1^{a+k_2+1})}\mathsf{Q}^{(1^{a+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+k_2+2})}\mathsf{Q}^{(1^{a+k_1})} \tag{59}$$

and

$$\bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+k_2+2})}\mathsf{Q}^{(1^{a+k_1})} \longrightarrow \bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+k_2+3})}\mathsf{Q}^{(1^{a+k_1})}. \tag{60}$$

It turns out that (59) is an isomorphism and that (60) is injective.

Calculation of (59). We rewrite (59) as

$$\bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+k_2+1})}\mathsf{Q}^{(1^{a+k_1+1})} \longrightarrow \bigoplus_{k_1+k_2=l-1} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(1^{a+k_2+2})}\mathsf{Q}^{(1^{a+k_1})}$$

and then cancel to get

$$\bigoplus_{\substack{k_1+k_2=l-1\\ k_2\geq k_1}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(2^{a+k_1+1},1^{k_2-k_1})} \longrightarrow \bigoplus_{\substack{k_1+k_2=l-1\\ k_2+2\leq k_1}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(2^{a+k_2+2},1^{k_1-k_2-2})}.$$

We then rewrite this as

$$\bigoplus_{\substack{k_1+k_2=l-1\\ k_2\geq k_1}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1-1)}\mathsf{Q}^{(2^{a+k_1+1},1^{k_2-k_1})} \longrightarrow \bigoplus_{\substack{k_1+k_2=l-1\\ k_2\geq k_1}} \mathsf{P}^{(k_1-1)}\mathsf{P}^{(k_2)}\mathsf{Q}^{(2^{a+k_1+1},1^{k_2-k_1})}$$

which turns out to be an isomorphism. So (59) is homotopic to zero.

Calculation of (60). One can check using the same argument as before that the map in equation (60) is surjective. We do not repeat this argument but instead just keep track of the terms left over after cancellation. First we rewrite (60) as

$$\bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+k_2+2})}\mathsf{Q}^{(1^{a+k_1+1})} \longrightarrow \bigoplus_{k_1+k_2=l} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(1^{a+k_2+3})}\mathsf{Q}^{(1^{a+k_1})}$$

which simplifies to

$$\bigoplus_{\substack{k_1+k_2=l\\ k_2\geq k_1-1}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(2^{a+k_1+1},1^{k_2-k_1+1})} \longrightarrow \bigoplus_{\substack{k_1+k_2=l\\ k_2+3\leq k_1}} \mathsf{P}^{(k_2)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(2^{a+k_2+3},1^{k_1-k_2-3})}.$$

We then rewrite both sides to obtain

$$\bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2}} \mathsf{P}^{(k_2+1)}\mathsf{P}^{(k_1)}\mathsf{Q}^{(2^{a+k_2+2},1^{k_1-k_2})} \longrightarrow \bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2}} \mathsf{P}^{(k_2-1)}\mathsf{P}^{(k_1+2)}\mathsf{Q}^{(2^{a+k_2+2},1^{k_1-k_2})}.$$

This in turn simplifies to give

$$\bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2}} \mathsf{P}^{(k_1+1,k_2)}\mathsf{Q}^{(2^{a+k_2+2},1^{k_2-k_1})}\langle -l\rangle \bigoplus_{\substack{k_1+k_2=l-1\\ k_1\geq k_2+1}} \mathsf{P}^{(k_1,k_2+1)}\mathsf{Q}^{(2^{a+k_2+2},1^{k_2-k_1})}\langle -l\rangle$$

in cohomological degree $(-l-1)$ (where we have added back the $\langle -l\rangle$ shift). Replacing l, k_1, k_2 by $l-2, k_1-1, k_2-1$ we see that these are the same terms as those in the complex (49) with the extra shift $\langle 2\rangle$. One can check as before that g_1 and g_3 appear in the differentials and hence $\mathsf{E}_{i,n}\mathsf{E}_{i,n-1}\langle -2\rangle \mathbf{1}_\lambda$ is also homotopic to (49). Thus we are done.

8. The remaining relations

8.1. Commutation of E_i and F_j.

Proposition 8.1. *For any $i \neq j \in I$ and $m, n \in \mathbb{Z}$ we have $\mathsf{F}_{j,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda \cong \mathsf{E}_{i,m}\mathsf{F}_{j,n}\mathbf{1}_\lambda$.*

Proof. There are two cases, depending of whether i and j are connected by an edge. If they are not connected then P_i and Q_i commute with P_j and Q_j and the result follows immediately. So suppose i and j are connected by an edge. There are several cases to consider.

Case 1. Suppose that $\langle \lambda, \alpha_i \rangle + m \geq -1$ and $\langle \lambda, \alpha_j \rangle - n \leq -1$. This means that

$$\begin{aligned}
\mathsf{E}_{i,m}\mathbf{1}_\lambda &\cong \left[\cdots \to \mathsf{P}_i^{(k)}\mathsf{Q}_i^{(1^{a+k})}\langle -k\rangle \to \cdots \to \mathsf{P}_i\mathsf{Q}_i^{(1^{a+1})}\langle -1\rangle \to \mathsf{Q}_i^{(1^a)}\right] \\
\mathsf{F}_{j,n}\mathbf{1}_{\lambda+\alpha_i+m\delta} &\cong \left[\mathsf{Q}_j^{(b)} \to \mathsf{P}_j\mathsf{Q}_j^{(b+1)}\langle 1\rangle \to \cdots \to \mathsf{P}_j^{(1^l)}\mathsf{Q}_j^{(b+l)}\langle l\rangle \to \dots\right]\langle b\rangle[-b] \\
\mathsf{F}_{j,n}\mathbf{1}_\lambda &\cong \left[\mathsf{Q}_j^{(b-1)} \to \mathsf{P}_j\mathsf{Q}_j^{(b)}\langle 1\rangle \to \cdots \to \mathsf{P}_j^{(1^{l'})}\mathsf{Q}_j^{(b-1+l')}\langle l'\rangle \to \dots\right]\langle b-1\rangle[-b+1] \\
\mathsf{E}_{i,m}\mathbf{1}_{\lambda-\alpha_j+n\delta} &\cong \left[\cdots \to \mathsf{P}_i^{(k')}\mathsf{Q}_i^{(1^{a+1+k'})}\langle -k'\rangle \to \cdots \to \mathsf{P}_i\mathsf{Q}_i^{(1^{a+2})}\langle -1\rangle \to \mathsf{Q}_i^{(1^{a+1})}\right]
\end{aligned}$$

where $a = \langle \lambda, \alpha_i \rangle + 1 + m$ and $b = -\langle \lambda, \alpha_j \rangle + n$.

Now, the terms in $\mathsf{F}_{j,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda$ in cohomological degree $(h+b)$ are

$$\bigoplus_{l-k=h} \mathsf{P}_j^{(1^l)}\mathsf{Q}_j^{(b+l)}\mathsf{P}_i^{(k)}\mathsf{Q}_i^{(1^{a+k})}\mathbf{1}_\lambda\langle h+b\rangle.$$

Since $\mathsf{Q}_j^{(b+l)}\mathsf{P}_i^{(k)} \cong \mathsf{P}_i^{(k)}\mathsf{Q}_j^{(b+l)} \oplus \mathsf{P}_i^{(k-1)}\mathsf{Q}_j^{(b+l-1)}$ this simplifies to give

$$\bigoplus_{l-k=h} \left[\mathsf{P}_j^{(1^l)}\mathsf{P}_i^{(k)}\mathsf{Q}_j^{(b+l)}\mathsf{Q}_i^{(1^{a+k})}\mathbf{1}_\lambda \oplus \mathsf{P}_j^{(1^l)}\mathsf{P}_i^{(k-1)}\mathsf{Q}_j^{(b+l-1)}\mathsf{Q}_i^{(1^{a+k})}\mathbf{1}_\lambda\right]\langle h+b\rangle. \tag{61}$$

Likewise, the terms in $\mathsf{E}_{i,m}\mathsf{F}_{j,n}\mathbf{1}_\lambda$ in cohomological degree $(h'+b-1)$ are

$$\bigoplus_{l'-k'=h'} \mathsf{P}_i^{(k')}\mathsf{Q}_i^{(1^{a+1+k'})}\mathsf{P}_j^{(1^{l'})}\mathsf{Q}_j^{(b-1+l')}\mathbf{1}_\lambda\langle h'+b-1\rangle$$

which simplifies to give

$$\bigoplus_{l'-k'=h'} \left[\mathsf{P}_i^{(k')}\mathsf{P}_j^{(1^{l'})}\mathsf{Q}_i^{(1^{a+1+k'})}\mathsf{Q}_j^{(b-1+l')} \oplus \mathsf{P}_i^{(k')}\mathsf{P}_j^{(1^{l'-1})}\mathsf{Q}_i^{(1^{a+k'})}\mathsf{Q}_j^{(b-1+l')}\right]\langle h'+b-1\rangle. \tag{62}$$

Now it is easy to see that the terms in (62) match up with the terms in (61) when $h' = h+1$ and $(k', l') = (k, l+1)$ or $(k', l') = (k-1, l)$. Thus the complexes for $\mathsf{E}_{i,m}\mathsf{F}_{j,n}\mathbf{1}_\lambda$ and $\mathsf{F}_{j,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda$ match up term by term and it is not hard to check that the differentials are the same.

Case 2. Now, consider the case that $\langle \lambda, \alpha_i \rangle + m \geq -1$ but $\langle \lambda, \alpha_j \rangle - n \geq 0$. This means that:

$$\begin{aligned}
\mathsf{E}_{i,m}\mathbf{1}_\lambda &\cong \left[\cdots \to \mathsf{P}_i^{(k)}\mathsf{Q}_i^{(1^{a+k})}\langle -k\rangle \to \cdots \to \mathsf{P}_i\mathsf{Q}_i^{(1^{a+1})}\langle -1\rangle \to \mathsf{Q}_i^{(1^a)}\right] \\
\mathsf{F}_{j,n}\mathbf{1}_{\lambda+\alpha_i+m\delta} &\cong \left[\mathsf{P}_j^{(1^b)} \to \mathsf{P}_j^{(1^{b+1})}\mathsf{Q}_j\langle 1\rangle \to \cdots \to \mathsf{P}_j^{(1^{b+l})}\mathsf{Q}_j^{(l)}\langle l\rangle \to \dots\right] \\
\mathsf{F}_{j,n}\mathbf{1}_\lambda &\cong \left[\mathsf{P}_j^{(1^{b+1})} \to \mathsf{P}_j^{(1^{b+2})}\mathsf{Q}_j\langle 1\rangle \to \cdots \to \mathsf{P}_j^{(1^{b+1+l'})}\mathsf{Q}_j^{(l')}\langle l'\rangle \to \dots\right] \\
\mathsf{E}_{i,m}\mathbf{1}_{\lambda-\alpha_j+n\delta} &\cong \left[\cdots \to \mathsf{P}_i^{(k')}\mathsf{Q}_i^{(1^{a+1+k'})}\langle -k'\rangle \to \cdots \to \mathsf{P}_i\mathsf{Q}_i^{(1^{a+2})}\langle -1\rangle \to \mathsf{Q}_i^{(1^{a+1})}\right]
\end{aligned}$$

where $a = \langle \lambda, \alpha_i \rangle + 1 + m$ and $b = \langle \lambda, \alpha_j \rangle - n$. The terms in $\mathsf{F}_{j,n}\mathsf{E}_{i,m}\mathbf{1}_\lambda$ in cohomological degree h are

$$\bigoplus_{l-k=h} \mathsf{P}_j^{(1^{b+l})}\mathsf{Q}_j^{(l)}\mathsf{P}_i^{(k)}\mathsf{Q}_i^{(1^{a+k})}\langle h\rangle$$

which simplifies to give

$$\bigoplus_{l-k=h} \left[\mathsf{P}_j^{(1^{b+l})} \mathsf{P}_i^{(k)} \mathsf{Q}_j^{(l)} \mathsf{Q}_i^{(1^{a+k})} \mathbf{1}_\lambda \oplus \mathsf{P}_j^{(1^{b+l})} \mathsf{P}_i^{(k-1)} \mathsf{Q}_j^{(l-1)} \mathsf{Q}_i^{(1^{a+k})} \mathbf{1}_\lambda \right] \langle h \rangle. \tag{63}$$

Likewise, the terms in $\mathsf{E}_{i,m}\mathsf{F}_{j,n}\mathbf{1}_\lambda$ in cohomological degree h' are

$$\bigoplus_{l'-k'=h'} \mathsf{P}_i^{(k')} \mathsf{Q}_i^{(1^{a+1+k'})} \mathsf{P}_j^{(1^{b+1+l'})} \mathsf{Q}_j^{(l')} \mathbf{1}_\lambda \langle h' \rangle$$

which simplifies to give

$$\bigoplus_{l'-k'=h'} \left[\mathsf{P}_i^{(k')} \mathsf{P}_j^{(1^{b+1+l'})} \mathsf{Q}_i^{(1^{a+1+k'})} \mathsf{Q}_j^{(l')} \mathbf{1}_\lambda \oplus \mathsf{P}_i^{(k')} \mathsf{P}_j^{(1^{b+l'})} \mathsf{Q}_i^{(1^{a+k'})} \mathsf{Q}_j^{(l')} \mathbf{1}_\lambda \right] \langle h' \rangle. \tag{64}$$

It is easy to see that the terms in (63) match up with those in (64) via the identification $(k', l') = (k, l)$ or $(k', l') = (k-1, l-1)$. Again, it is not hard to check that the differentials also match up.

Case 3 and 4. There are two further cases which to consider, namely when $\langle \lambda, \alpha_i \rangle + m \leq 0$ and either $\langle \lambda, \alpha_j \rangle - n \leq -1$ or $\langle \lambda, \alpha_j \rangle - n \geq 0$. These are proven in exactly the same way as above and so we omit the details. □

8.2. **The $[\mathsf{P}_j, \mathsf{E}_i]$-type commutator.**

Proposition 8.2. *If $\langle i, j \rangle = -1$ then we have the following exact triangles in $\mathsf{Kom}(\mathcal{H})$*

$$\mathsf{E}_{i,m-1}[1]\langle -1 \rangle \mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{P}_j^{[1]}\mathbf{1}_\lambda \to \mathsf{P}_j^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda$$
$$\mathsf{E}_{i,m+1}\mathbf{1}_\lambda \to \mathsf{Q}_j^{[1]}\mathsf{E}_{i,m}\mathbf{1}_\lambda \to \mathsf{E}_{i,m}\mathsf{Q}_j^{[1]}\mathbf{1}_\lambda$$

a as well as the triangles obtained by applying the (left or right) adjoint.

Proof. We prove the first commutation relation (the other case is similar). For simplicity we assume $m = 0$.

Case $\lambda \geq 0$. We have

$$\mathsf{P}_j\mathsf{E}_i\mathbf{1}_\lambda \cong \left[\cdots \to \mathsf{P}_j\mathsf{P}_i^{(2)}\mathsf{Q}_i^{(1^{\lambda+3})}\langle -2 \rangle \to \mathsf{P}_j\mathsf{P}_i\mathsf{Q}_i^{(1^{\lambda+2})}\langle -1 \rangle \to \mathsf{P}_j\mathsf{Q}_i^{(1^{\lambda+1})} \right]$$

and

$$\begin{aligned} \mathsf{E}_i\mathsf{P}_j\mathbf{1}_\lambda &\cong \left[\cdots \to \mathsf{P}_i^{(2)}\mathsf{Q}_i^{(1^{\lambda+3})}\mathsf{P}_j\langle -2 \rangle \to \mathsf{P}_i\mathsf{Q}_i^{(1^{\lambda+2})}\mathsf{P}_j\langle -1 \rangle \to \mathsf{Q}_i^{(1^{\lambda+1})}\mathsf{P}_j \right] \\ &\cong \left[\cdots \to \begin{array}{c} \mathsf{P}_j\mathsf{P}_i^{(2)}\mathsf{Q}_i^{(1^{\lambda+3})}\langle -2 \rangle \\ \oplus \mathsf{P}_i^{(2)}\mathsf{Q}_i^{(1^{\lambda+2})}\langle -2 \rangle \end{array} \to \begin{array}{c} \mathsf{P}_j\mathsf{P}_i\mathsf{Q}_i^{(1^{\lambda+2})}\langle -1 \rangle \\ \oplus \mathsf{P}_i\mathsf{Q}_i^{(1^{\lambda+1})}\langle -1 \rangle \end{array} \to \begin{array}{c} \mathsf{P}_j\mathsf{Q}_i^{(1^{\lambda+1})} \\ \oplus \mathsf{Q}_i^{(1^{\lambda})}. \end{array} \right] \end{aligned}$$

One can check as in the proof of (7.1) that there is a natural map of complexes $\mathsf{E}_i\mathsf{P}_j\mathbf{1}_\lambda \to \mathsf{P}_j\mathsf{E}_i\mathbf{1}_\lambda$ whose cone is isomorphic to

$$\left[\cdots \to \mathsf{P}_i^{(2)}\mathsf{Q}_i^{(1^{\lambda+2})}\langle -2 \rangle \to \mathsf{P}_i\mathsf{Q}_i^{(1^{\lambda+1})}\langle -1 \rangle \to \mathsf{Q}_i^{(1^{\lambda})} \to 0 \right]$$

which is isomorphic to $\mathsf{E}_{i,-1}[1]\mathbf{1}_\lambda$. Thus we get a triangle

$$\mathsf{E}_{i,-1}\mathbf{1}_\lambda \to \mathsf{E}_i\mathsf{P}_j\mathbf{1}_\lambda \to \mathsf{P}_j\mathsf{E}_i\mathbf{1}_\lambda$$

and the result follows since $\mathsf{P}_j^{[1]} = P_j[1]\langle -1 \rangle$.

Case $\lambda \leq 0$. We have

$$\mathsf{P}_j\mathsf{E}_i\mathbf{1}_\lambda \cong \left[\cdots \to \mathsf{P}_j\mathsf{P}_i^{(-\lambda+1)}\mathsf{Q}_i^{(1^2)}\langle -2 \rangle \to \mathsf{P}_j\mathsf{P}_i^{(-\lambda)}\mathsf{Q}_i\langle -1 \rangle \to \mathsf{P}_j\mathsf{P}_i^{(-\lambda-1)} \right] \langle \lambda + 1 \rangle [-\lambda - 1]$$

and

$$\begin{aligned}\mathsf{E}_i\mathsf{P}_j\mathbf{1}_\lambda &\cong \left[\cdots \to \mathsf{P}_i^{(-\lambda+1)}\mathsf{Q}_i^{(1^2)}\mathsf{P}_j\langle -2\rangle \to \mathsf{P}_i^{(-\lambda)}\mathsf{Q}_i\mathsf{P}_j\langle -1\rangle \to \mathsf{P}_i^{(-\lambda-1)}\mathsf{P}_j\right]\langle\lambda+1\rangle[-\lambda-1] \\ &\cong \left[\cdots \to \begin{matrix}\mathsf{P}_j\mathsf{P}_i^{(-\lambda+1)}\mathsf{Q}_i^{(1^2)}\langle -2\rangle \\ \oplus \mathsf{P}_i^{(-\lambda+1)}\mathsf{Q}_i\langle -2\rangle\end{matrix} \to \begin{matrix}\mathsf{P}_j\mathsf{P}_i^{(-\lambda)}\mathsf{Q}_i\langle -1\rangle \\ \oplus \mathsf{P}_i^{(-\lambda)}\langle -1\rangle\end{matrix} \to \mathsf{P}_j\mathsf{P}_i^{(-\lambda-1)}\right]\langle\lambda+1\rangle[-\lambda-1].\end{aligned}$$

Again there is a natural map of complexes $\mathsf{E}_i\mathsf{P}_j\mathbf{1}_\lambda \to \mathsf{P}_j\mathsf{E}_i\mathbf{1}_\lambda$ whose cone is isomorphic to

$$\left[\cdots \to \mathsf{P}_i^{(-\lambda+1)}\mathsf{Q}_i\langle -2\rangle \to \mathsf{P}_i^{(-\lambda)}\langle -1\rangle \to 0 \to 0\right]\langle\lambda+1\rangle[-\lambda-1].$$

Taking into account the shifts this is also isomorphic to $\mathsf{E}_{i,-1}[1]\mathbf{1}_\lambda$ and the result follows as above. □

8.3. **The commutator type relation of E_i and E_j.**

Proposition 8.3. *If $\langle i,j\rangle = -1$ and $m \in \mathbb{Z}$ then there exist unique nonzero maps*

$$\mathsf{E}_{i,m}\mathsf{E}_{j,m+1}\mathbf{1}_\lambda \xrightarrow{\alpha} \mathsf{E}_{j,m+1}\mathsf{E}_{i,m}\langle 1\rangle\mathbf{1}_\lambda \quad \textit{and} \quad \mathsf{E}_{j,m}\mathsf{E}_{i,m+1}\mathbf{1}_\lambda \xrightarrow{\beta} \mathsf{E}_{i,m+1}\mathsf{E}_{j,m}\langle 1\rangle\mathbf{1}_\lambda$$

in $\mathcal{K}$ and we have Cone$(\alpha) \cong$ Cone(β). Meanwhile, if $\langle i,j\rangle = 0$ then $\mathsf{E}_{i,m}$ and $\mathsf{E}_{j,n}$ commute for any $m,n \in \mathbb{Z}$.

The commutation of $\mathsf{E}_{i,m}$ and $\mathsf{E}_{j,n}$ when $\langle i,j\rangle = 0$ is obvious since P_i commutes with Q_j in this case. It remains to prove the first assertion when $\langle i,j\rangle = -1$.

8.3.1. *The definition of α and β.* To simplify notation we assume $m = 0$ (the general case follows in the same way). First we have the following formal result which shows that the maps α and β are unique.

Lemma 8.4. *If $\langle i,j\rangle = -1$ then* $\mathrm{Hom}(\mathsf{E}_i\mathsf{E}_{j,1}\mathbf{1}_\lambda, \mathsf{E}_{j,1}\mathsf{E}_i\langle 1\rangle\mathbf{1}_\lambda) \cong \Bbbk \cong \mathrm{End}(\mathsf{E}_i\mathsf{E}_{j,1}\mathbf{1}_\lambda)$.

Proof. This is a formal consequence of the other relations in a 2-representation, together with the fact that $\mathrm{End}(\mathbf{1}_\lambda) \cong \Bbbk$. More precisely, in [CK] Lemma 4.5 we prove that $\mathrm{Hom}(\mathsf{E}_i\mathsf{E}_j, \mathsf{E}_j\mathsf{E}_i\langle 1\rangle)$ is one-dimensional. To do this we only used that the representation is integrable, biadjointness of E_i and E_j, that E_i and F_j commute and the commutator relations of $[\mathsf{E}_i,\mathsf{F}_i]$ and $[\mathsf{E}_j,\mathsf{F}_j]$. So the same proof applies here if we replace E_j by $\mathsf{E}_{j,1}$. □

We now give an explicit description of α and β as maps of complexes of 1-morphisms in $\mathsf{Kom}(\mathcal{K})$. Let us suppose that $\langle\lambda,\alpha_i\rangle = a-1 \geq 0$ and $\langle\lambda,\alpha_j\rangle = b+1 \geq 0$. Then the general terms of $\mathsf{E}_{j,1}\mathbf{1}_\lambda$ and $\mathsf{E}_i\mathbf{1}_{\lambda+\alpha_j+\delta}$ are

$$\mathsf{P}_j^{(n)}\mathsf{Q}_j^{(1^{b+n+1})}\langle -n\rangle \text{ and } \mathsf{P}_i^{(n)}\mathsf{Q}_i^{(1^{a-1+n})}\langle -n\rangle.$$

This means that $\mathsf{E}_i\mathsf{E}_{j,1}\mathbf{1}_\lambda$ is a complex which looks like

(65)

$$\cdots \to \bigoplus_{k=0}^{n+1} \mathsf{P}_i^{(n+1-k)}\mathsf{Q}_i^{(1^{a+n-k})}\mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+1+k})}\langle -n-1\rangle \to \bigoplus_{k=0}^{n} \mathsf{P}_i^{(n-k)}\mathsf{Q}_i^{(1^{a-1+n-k})}\mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+1+k})}\langle -n\rangle \to \dots$$

where the right hand term is in cohomological degree $-n$.

Similarly, the general terms of $\mathsf{E}_i\mathbf{1}_\lambda$ and $\mathsf{E}_{j,1}\mathbf{1}_{\lambda+\alpha_i}$ are

$$\mathsf{P}_i^{(n)}\mathsf{Q}_i^{(1^{a+n})}\langle -n\rangle \text{ and } \mathsf{P}_j^{(n)}\mathsf{Q}_j^{(1^{b+n})}\langle -n\rangle.$$

This means that $\mathsf{E}_{j,1}\mathsf{E}_i\langle 1\rangle\mathbf{1}_\lambda$ is a complex which looks like

(66)

$$\cdots \to \bigoplus_{k=0}^{n+1} \mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+k})}\mathsf{P}_i^{(n+1-k)}\mathsf{Q}_i^{(1^{a+n+1-k})}\langle -n\rangle \to \bigoplus_{k=0}^{n} \mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+k})}\mathsf{P}_i^{(n-k)}\mathsf{Q}_i^{(1^{a+n-k})}\langle -n+1\rangle \to \dots$$

where the right hand term is in cohomological degree $-n$.

On the other hand, $\mathsf{E}_j\mathsf{E}_{i,1}\mathbf{1}_\lambda$ looks like

(67)
$$\cdots \to \bigoplus_{k=0}^{n+1} \mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b-1+k})}\mathsf{P}_i^{(n+1-k)}\mathsf{Q}_i^{(1^{a+n+2-k})}\langle -n-1\rangle \to \bigoplus_{k=0}^{n} \mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b-1+k})}\mathsf{P}_i^{(n-k)}\mathsf{Q}_i^{(1^{a+n+1-k})}\langle -n\rangle \to \dots$$

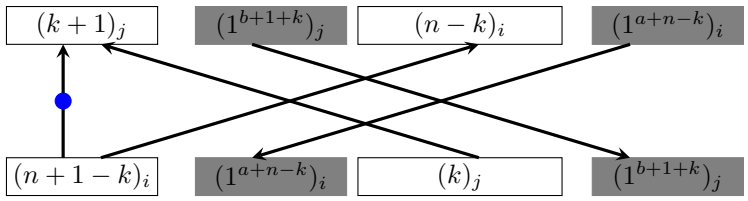
while $\mathsf{E}_{i,1}\mathsf{E}_j\langle 1\rangle\mathbf{1}_\lambda$ looks like

(68)
$$\cdots \to \bigoplus_{k=0}^{n+1} \mathsf{P}_i^{(n+1-k)}\mathsf{Q}_i^{(1^{a+n+1-k})}\mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+k})}\langle -n\rangle \to \bigoplus_{k=0}^{n+1} \mathsf{P}_i^{(n-k)}\mathsf{Q}_i^{(1^{a+n-k})}\mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+k})}\langle -n+1\rangle \to \dots.$$

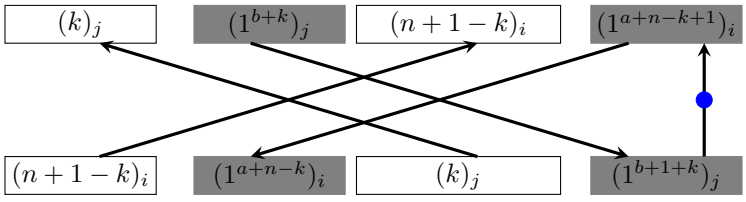
We now write down the pictures which define the map of complexes $\mathsf{E}_i\mathsf{E}_{j,1} \to \mathsf{E}_{j,1}\mathsf{E}_i\langle 1\rangle$. The chain map will take

$$\mathsf{P}_i^{(n+1-k)}\mathsf{Q}_i^{(1^{a+n-k})}\mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+1+k})} \longrightarrow \mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+k})}\mathsf{P}_i^{(n+1-k)}\mathsf{Q}_i^{(1^{a+n+1-k})} \oplus \mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+k})}\mathsf{P}_i^{(n+1-k)}\mathsf{Q}_i^{(1^{a+n+1-k})}.$$

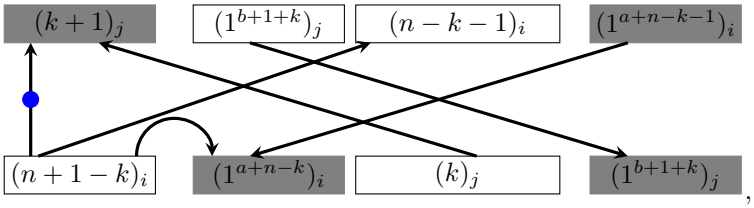
The map to the first summand is

(69)


where the solid dot is a degree one $i-j$ dot. Similarly, the map to the second summand is

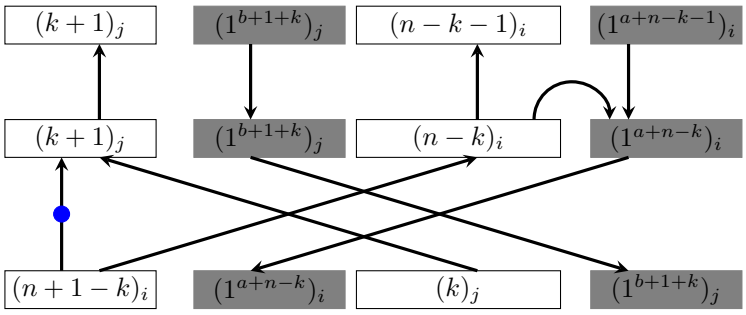
(70)


We need to check that these diagrams define a chain map, meaning that the map above commutes with the differential. Diagrammatically, this amounts to checking that diagrams

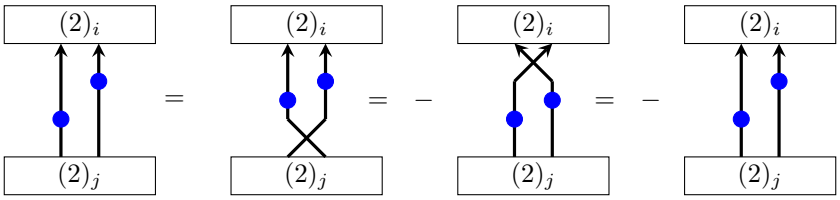


,

and



commute. To see this we simplify the second diagram. The two middle box idempotents in the middle level (the $1^{(b+1+k)}$ and the $(n-k)$) can be absorbed into the idempotents in the bottom level. Now slide the cap downwards. It moves through the first line for free. To pass it through the second line involves creating a sum of two diagrams, the first term of which just moves the cap through; the second term creates a subdiagram



In the above graphical computation the minus sign comes from the two degree one dots passing one another with respect to the horizontal; from the above computation we see that this subdiagram is zero.

We conclude that we have a map of complexes $\mathsf{E}_i\mathsf{E}_{j,1} \to \mathsf{E}_{j,1}\mathsf{E}_i\langle 1\rangle$. A straightforward check shows that this map of complexes is nonzero. This defines the chain map α, and we define the chain map β similarly.

8.3.2. *Proof of proposition 8.3.* Looking at equations (65) and (66) we see that $\mathrm{Cone}(\alpha)$ is a complex where

$$\bigoplus_{k=0}^{n} \mathsf{P}_i^{(n-k)}\mathsf{Q}_i^{(1^{a-1+n-k})}\mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+1+k})}\langle -n\rangle \bigoplus_{k=0}^{n+1} \mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+k})}\mathsf{P}_i^{(n+1-k)}\mathsf{Q}_i^{(1^{a+n+1-k})}\langle -n\rangle \tag{71}$$

is the term appearing in cohomological degree $-n-1$. The second term above can be rewritten as

$$\bigoplus_{k=0}^{n} \mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b-1+k})}\mathsf{P}_i^{(n-k)}\mathsf{Q}_i^{(1^{a+n+1-k})}\langle -n\rangle \bigoplus_{k=0}^{n+1} \mathsf{P}_j^{(k)}\mathsf{P}_i^{(n+1-k)}\mathsf{Q}_j^{(1^{b+k})}\mathsf{Q}_i^{(1^{a+n+1-k})}\langle -n\rangle. \tag{72}$$

On the other hand, looking at equations (67) and (68) we find that $\mathrm{Cone}(\beta)$ is a complex where

$$\bigoplus_{k=0}^{n} \mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b-1+k})}\mathsf{P}_i^{(n-k)}\mathsf{Q}_i^{(1^{a+n+1-k})}\langle -n\rangle \bigoplus_{k=0}^{n+1} \mathsf{P}_i^{(n+1-k)}\mathsf{Q}_i^{(1^{a+n+1-k})}\mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+k})}\langle -n\rangle \tag{73}$$

is the term appearing in cohomological degree $-n-1$. The second term above can be rewritten as

$$\bigoplus_{k=0}^{n} \mathsf{P}_i^{(n-k)}\mathsf{Q}_i^{(1^{a+n-k-1})}\mathsf{P}_j^{(k)}\mathsf{Q}_j^{(1^{b+k+1})}\langle -n\rangle \bigoplus_{k=0}^{n+1} \mathsf{P}_i^{(n+1-k)}\mathsf{P}_j^{(k)}\mathsf{Q}_i^{(1^{a+n+1-k})}\mathsf{Q}_j^{(1^{b+k})}\langle -n\rangle. \tag{74}$$

Let us denote by A^{-n-1} the direct sum of the first term in (71) and all of (72) and likewise by B^{-n-1} the direct sum of the first term in (73) and all of (74). These are the degree $-n-1$ terms in the complexes $\mathrm{Cone}(\alpha)$ and $\mathrm{Cone}(\beta)$. It is straight-forward to check that $\mathsf{A}^{-n-1} \cong \mathsf{B}^{-n-1}$ by just matching terms.

It remains to show that the differentials in $\mathrm{Cone}(\alpha)$ and $\mathrm{Cone}(\beta)$ agree. We do this by applying Lemma 8.5. The first and second conditions in Lemma 8.5 follow from Lemmas 3.5 and 3.6. The third condition follows from a computation almost identical to that used for the proof of the relations in section 7.3. We include the indecomposable 1-morphism $\mathsf{P}_i^{(a+1)}\mathsf{P}_j^{(1^b)}\mathsf{Q}_i^{(1^c)}\mathsf{Q}_j^{(d)}$ into the appropriate term of $\mathrm{Cone}(\alpha)$, apply the boundary map and then project onto each indecomposable summand. The composition is a collection of diagrams which can be simplified. Doing this we find the degree one map from Lemma 3.5. This map can be shown to be nonzero by taking its closure as explained in the example of section 3.5.

Lemma 8.5. *Consider a finite complex*

$$\mathsf{A}^\bullet := \ldots \longrightarrow \oplus_\ell \mathsf{A}_\ell^{-n-1} \longrightarrow \oplus_\ell \mathsf{A}_\ell^{-n} \longrightarrow \ldots \longrightarrow \oplus_\ell \mathsf{A}_\ell^{0}$$

in the homotopy category of some additive category. Suppose that it satisfies the following:

(1) $\mathrm{Hom}(\mathsf{A}_\ell^{-n}, \mathsf{A}_{\ell'}^{-n+1})$ *is either zero or one-dimensional for all* ℓ, ℓ', n
(2) *for any* ℓ, ℓ_1, ℓ_2, n *such that*

$$\mathrm{Hom}(\mathsf{A}_\ell^{-n-1}, \mathsf{A}_{\ell_1}^{-n}) \cong \Bbbk \cong \mathrm{Hom}(\mathsf{A}_\ell^{-n-1}, \mathsf{A}_{\ell_2}^{-n})$$

there exists ℓ' *and a nonzero map in* $\mathrm{Hom}(\mathsf{A}_\ell^{-n-1}, \mathsf{A}_{\ell'}^{-n+1})$ *which factors through* $\mathsf{A}_{\ell_1}^{-n}$ *and* $\mathsf{A}_{\ell_2}^{-n}$
(3) *for any* $\ell, n \neq 0$ *there exists a nonzero map with domain* A_ℓ^{-n}.

Now suppose $\mathsf{B}^\bullet$ *is another complex such that* $\mathsf{A}^{-n} \cong \mathsf{B}^{-n}$ *for all* n *and* $\mathsf{B}^\bullet$ *satisfies the same conditions as* $\mathsf{A}^\bullet$ *above. Then* $\mathsf{A}^\bullet \cong \mathsf{B}^\bullet$.

Proof. The proof is similar to that of lemma 7.4. It is based on the same observation that given a complex

$$A \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} B_1 \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} C$$

where $\mathrm{Hom}(A, B_1) \cong \mathrm{Hom}(A, B_2) \cong \mathrm{Hom}(B_1, C) \cong \mathrm{Hom}(B_2, C) \cong \mathrm{Hom}(A, C) \cong \Bbbk$ are spanned by $\alpha_1, \alpha_2, \beta_1, \beta_2$ and $\beta_1 \circ \alpha_1 = -\beta_2 \circ \alpha_2$ then any three of the maps determines uniquely the fourth.

We now apply this to our problem. Fix an isomorphism $\mathsf{A}^{-n} \xrightarrow{\sim} \mathsf{B}^{-n}$ and proceed by induction starting from the far right. For the base case we note that $\mathsf{A}_\ell^1 = 0$ so there is a unique nonzero map out of each A_ℓ^{-1}. Acting by a multiple of the identity on A_ℓ^{-1} this map can be scaled so that it equals to that in $\mathsf{B}^\bullet$. For the induction step consider a nonzero map $\mathsf{A}_\ell^{-n-1} \to \mathsf{A}_{\ell'}^{-n}$ and rescale A_ℓ^{-n-1} so that this map agrees with that in $\mathsf{B}^\bullet$. Then using the observation above (and induction) it follows that all the other maps out of A_ℓ^{-n-1} must also agree with those in $\mathsf{B}^\bullet$. □

9. Applications, conjectures and further comments

In this section, we set the field $\Bbbk = \mathbb{C}$ to be the complex numbers. Fix $\Gamma \subset SL_2(\mathbb{C})$, a non-trivial finite subgroup. By the McKay correspondence, such subgroups are classified by finite Dynkin diagrams of type A, D or E. To such a diagram there are three associated Lie algebras: a finite dimensional simply-laced Lie algebra $\mathfrak{g}$, the affine Lie algebra $\widehat{\mathfrak{g}}$ and the associated toroidal Lie algebra $\widehat{\widehat{\mathfrak{g}}}$ with $\mathfrak{g} \subset \widehat{\mathfrak{g}} \subset \widehat{\widehat{\mathfrak{g}}}$. We now describe two (essentially equivalent) categorifications of the basic representation of $\dot{U}_q(\widehat{\widehat{\mathfrak{g}}})$ using the finite subgroup $\Gamma \subset SL_2(\mathbb{C})$.

9.1. **2-representations via Hilbert schemes.** Let $X_\Gamma = \widehat{\mathbb{C}^2/\Gamma}$ denote the minimal resolution of the quotient singularity $\mathbb{C}^2/\Gamma$ and denote by $X_\Gamma^{[n]}$ the Hilbert scheme of n points on X_Γ. The diagonal $\mathbb{C}^\times$-action on $\mathbb{C}^2$ induces a $\mathbb{C}^\times$-action on $X_\Gamma^{[n]}$. Let $DCoh^{\mathbb{C}^\times}(X_\Gamma)$ denote the derived category of $\mathbb{C}^\times$-equivariant coherent sheaves on X_Γ. Its Grothendieck group (tensored with $\mathbb{C}(q)$) is denoted $K^{\mathbb{C}^\times}(X_\Gamma)$.

In [CLi1] we constructed a level one integrable 2-representation of $\widehat{\widehat{\mathfrak{h}}}$ on $\bigoplus_{n\in\mathbb{N}} DCoh(X_\Gamma^{[n]})$ where $\widehat{\widehat{\mathfrak{h}}}$ is the toroidal Heisenberg algebra. Corollary 5.3 explains how we obtain a 2-representation of $\widehat{\mathfrak{g}}$ from one of $\widehat{\mathfrak{h}}$. Adding the extra affine vertex to the story this immediately implies the following theorem.

Theorem 9.1. *The action of $\widehat{\widehat{\mathfrak{h}}}$ on $\oplus_{n\in\mathbb{N}} DCoh(X_\Gamma^{[n]})$ induces a 2-representation of $\widehat{\widehat{\mathfrak{g}}}$ on*

$$\bigoplus_{\alpha\in\widehat{Y},n\in\mathbb{N}} DCoh(X_\Gamma^{[n]}).$$

Corollary 9.2. *The quantum toroidal algebra $\dot{U}_q(\widehat{\mathfrak{g}})$ acts on*

$$\bigoplus_{\alpha\in\widehat{Y},n\in\mathbb{N}} K^{\mathbb{C}^\times}(X_\Gamma^{[n]}) = \bigoplus_{n\geq 0} K^{\mathbb{C}^\times}(X_\Gamma^{[n]}) \otimes_{\mathbb{C}(q)} \mathbb{C}(q)[\widehat{Y}].$$

The resulting module is the basic representation.

Theorem 9.1 and Corollary 9.2 were conjectured by Nakajima in [N4]. Moreover, Corollary 9.2 is both an affinization and a q-deformation of the work of Nakajima and Grojnowski which gives affine Lie algebra actions on cohomology of Hilbert schemes.

9.2. **2-representations via wreath products.** A 2-representation of $\widehat{\widehat{\mathfrak{g}}}$ can also be constructed using the representation theory of finite dimensional superalgebras. Let $B_\Gamma := \mathbb{C}[\Gamma] \ltimes \wedge^*(\mathbb{C}^2)$ and set $B_\Gamma(n) = \mathbb{C}[S_n] \ltimes B_\Gamma^{\otimes n}$ (we include $n = 0$ in this definition, setting $B_\Gamma(0) = \mathbb{C}$). The natural $\mathbb{Z}$ grading on $\wedge^*(\mathbb{C}^2)$ makes $B_\Gamma(n)$ into a $\mathbb{Z}$-graded superalgebra. Let $\mathcal{B}_\Gamma(n)$ denote the category of finitely generated, graded $B_\Gamma(n)$ supermodules.

In [CLi1], a level one 2-representation of $\widehat{\widehat{\mathfrak{h}}}$ was constructed on $\bigoplus_{n\geq 0} \mathcal{B}_\Gamma(n)$ categorifying the Fock space representation of $\widehat{\widehat{\mathfrak{h}}}$. We do not need to use derived categories of modules since, in contrast to the action on Hilbert schemes, the Heisenberg generators P_i and Q_i act by exact functors. However, to obtain a 2-representation of $\widehat{\widehat{\mathfrak{g}}}$ we still need to pass to the homotopy category $\mathsf{Kom}(\mathcal{B}_\Gamma(n))$, since E_i and F_i are given by complexes.

Theorem 9.3. *The 2-representation of $\widehat{\widehat{\mathfrak{h}}}$ on $\mathcal{B}_\Gamma(n)$ induces a 2-representation of $\widehat{\widehat{\mathfrak{g}}}$ on*

$$\bigoplus_{\alpha\in\widehat{Y},n\in\mathbb{N}} \mathsf{Kom}(\mathcal{B}_\Gamma(n)).$$

Theorem 9.3 is very similar in spirit to the constructions of toroidal basic representations in [FJW].

9.3. **The Kac-Moody description.** The affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \Bbbk[t, t^{-1}] \oplus \Bbbk c$ is also a Kac-Moody Lie algebra. This Kac-Moody presentation has generators $\{e_i, f_i, h_i\}_{i\in\hat{I}}$. The isomorphism between the Kac-Moody and loop presentations is defined as follows.

Let θ denote the highest root of $\mathfrak{g}$ and $\mathfrak{g}_\theta$ the associated root space. Choose elements $E_\theta \in \mathfrak{g}_{-\theta}$, $F_\theta \in \mathfrak{g}_\theta$ such that $\theta(H_\theta) = -2$, where $H_\theta := [E_\theta, F_\theta]$. Then we define

$$e_0 \mapsto E_\theta \otimes t, \quad f_0 \mapsto F_\theta \otimes t^{-1} \quad \text{and} \quad h_0 \mapsto H_\theta \otimes 1 + c$$

while $e_i \mapsto E_i$ and $f_i \mapsto F_i$ if $i \in I$.

Categorifications of Kac-Moody algebras have been defined by Khovanov-Lauda [KL1, KL2, KL3] and Rouquier [R]. They are given by a 2-category $\mathcal{U}_Q(\widehat{\mathfrak{g}})_{KM}$ which depends on some scalars Q (we use the notation from [CLa]). The 2-category in [KL3] corresponds to a particular choice of such scalars. Although many of these 2-categories are isomorphic this is not always the case. In general the space of isomorphism classes of these 2-categories is the first homology of the associated Dynking diagram.

We next describe a (conjectural) relationship between the categorification of the basic representation constructed in the Kac-Moody setting and the categorification in the current paper. We do this from a geometric and then an algebraic point of view.

9.3.1. *Quiver varieties.* The basic representation of $\widehat{\mathfrak{g}}$ was constructed geometrically by Nakajima using the Γ-equivariant geometry of the Hilbert scheme of points on $\mathbb{C}^2$ (see [N1]). More precisely, $\Gamma \subset SL_2$ acts on $\mathbb{C}^2$ and hence on all the Hilbert schemes $\mathbb{C}^{2^{[n]}}$. The fixed point components $(\mathbb{C}^{2^{[n]}})^\Gamma$ are Nakajima quiver varieties of affine type. Nakajima defines an action of $\widehat{\mathfrak{g}}$ on $\bigoplus_{n\in\mathbb{N}} H_{mid}((\mathbb{A}^{2^{[n]}})^\Gamma, \mathbb{C})$, where H_{mid} denotes the middle cohomology, giving the basic representation. One can also carry out this construction by replacing homology with $\mathbb{C}^\times$-equivariant K-theory, in which case the quantum affine Kac-Moody algebra $\dot{U}_q(\widehat{\mathfrak{g}})$ acts.

This action was lifted to derived categories of coherent sheaves in [CKL3].

Theorem 9.4. [CKL3] *There exists a geometric categorical $\widehat{\mathfrak{g}}$ action on $\bigoplus_{n\in\mathbb{N}} DCoh^{\mathbb{C}^\times}((\mathbb{C}^{2^{[n]}})^\Gamma)$.*

A geometric categorical action is a notion introduced in [CKL2, CK] which enhances that of a naïve categorical action in the context of geometry. In [CKL2] we studied the relationship between geometric categorical $\mathfrak{sl}_2$ actions and 2-representations of $\mathfrak{sl}_2$ in the sense of Khovanov-Lauda and Rouquier. More precisely, we showed that a geometric categorical $\mathfrak{sl}_2$ action induces an action of Rouquier's 2-category (in particular, there is an action of the nilHecke algebra). In [Cau], we plan to prove an analogous result for any simply laced Kac-Moody algebra. In conjunction with [CLa] this will imply that geometric categorical $\widehat{\mathfrak{g}}$ action in Theorem 9.4 extends to give a 2-representation of the 2-category $\mathcal{U}_Q(\widehat{\mathfrak{g}})_{KM}$ (for some choice of scalars Q).

The Grothendieck group of $DCoh^{\mathbb{C}^\times}((\mathbb{C}^{2^{[n]}})^\Gamma)$ contains more than just the basic representation of $\dot{U}_q(\widehat{\mathfrak{g}})$. This is because the entire quantum toroidal algebra $\dot{U}_q(\widehat{\widehat{\mathfrak{g}}})$ acts and this gives its basic representation [N1]. In light of this we conjecture the following.

Conjecture 9.5. *The geometric categorical $\widehat{\mathfrak{g}}$ action in Theorem 9.4 extends to a 2-representation of the toroidal algebra.*

Remark 9.6. The toroidal algebra in the conjecture above should be thought of as the loop algebra of the Kac-Moody quantum affine algebra. In particular, $\mathcal{U}_Q(\widehat{\mathfrak{g}})_{KM}$ should act (for some choice of Q) together with the loop algebra part which acts like in the definition in section 4.2 but at level zero rather than level one. This conjecture was not proven in [CKL3] in part because there was no such definition of a 2-representation of toroidal algebras.

So there are two possible categorifications of the basic representation of $\dot{U}_q(\widehat{\widehat{\mathfrak{g}}})$ using derived categories of coherent sheaves, that of Theorem 9.1 and that of Conjecture 9.5. These categorifications are in some ways quite different. For example, in the categorification involving $(\mathbb{C}^{2^{[n]}})^\Gamma$, the Kac-Moody generators E_i and F_i are explicitly described, while the homogeneous Heisenberg generators P_i and Q_i are not as easily visible. On the other hand, in the categorification involving $\widehat{\mathbb{C}^2/\Gamma}^{[n]}$ the Heisenberg generators P_i and Q_i acquire a simpler geometric interpretation while the Kac-Moody generators E_i and F_i are given by more complicated categorified vertex operators.

However, the varieties $(\mathbb{C}^{2[n]})^\Gamma$ and $\widehat{\mathbb{C}^2/\Gamma}^{[k]}$ are closely related. Both can be realized as Nakajima quiver varieties, but for different stability conditions and hence are derived equivalent. Subsequently one can conjecture the following.

Conjecture 9.7. *There is an equivalence between the 2-representations of $\widehat{\widehat{\mathfrak{g}}}$ from Theorem 9.1 and Conjecture 9.5.*

The above conjecture is complicated by the fact that the isomorphism between the loop and Kac-Moody presentations of the quantum affine algebra is somewhat subtle. In particular, to prove conjecture 9.7 one should assign endofunctors to the Kac-Moody 1-morphism e_0 lifting the relation $e_0 = E_\theta \otimes t$ between the Kac-Moody and loop descriptions of this operator. Such an assignment is not given in the current paper.

9.3.2. *Cyclotomic KLR algebras and wreath products.* There is a parallel algebraic version of these categorifications. In [KL1, KL3] a family of cyclotomic quiver Hecke algebras – also known as cyclotomic KLR algebras – were defined. In particular, the following theorem was conjectured in [KL1, KL3] and subsequently proven in [KK, W].

Theorem 9.8. *The 2-categories $\mathcal{U}_Q(\widehat{\mathfrak{g}})_{KM}$ act on $\oplus_\lambda R^{\Lambda_0}_{Q,\lambda} - mod$ where $R^{\Lambda_0}_{Q,\lambda}$ is the cyclotomic KLR algebra for the weight space λ in the basic representation V_{Λ_0} (and for some choice of scalars Q).*

In light of the connections between KLR algebras and quiver varieties [VV, R] this theorem is an algebraic analogue of Theorem 9.4. Subsequently we expect that Theorem 9.8 can be extended to give a 2-representation of quantum toroidal algebras. Then there should be an equivalence between the categorification of the basic representation of $\dot{U}_q(\widehat{\widehat{\mathfrak{g}}})$ using toroidal cyclotomic KLR algebras and the categorification of Theorem 9.3.

However, since toroidal cyclotomic KLR algebras have not been defined, we now restrict the construction of Theorem 9.3 from the quantum toroidal algebra to the quantum affine algebra in order to formulate a precise conjecture relating cyclotomic KLR algebras to the algebras $B_\Gamma(n)$ of section 9.

The isomorphism classes of indecomposable projective B_Γ modules, $\{\mathsf{P}_i\}_{i\in\hat{I}}$ are in bijection with the nodes of the affine Dynkin diagram. The endomorphism algebra

$$B'_\Gamma := \mathrm{End}_{B_\Gamma}(\oplus_{i\in I}\mathsf{P}_i)$$

where we omit the projective module correponding to the affine node, is a subalgebra of B_Γ. Let $B'_\Gamma(n) = \Bbbk[S_n] \ltimes {B'_\Gamma}^{\otimes n}$.

Conjecture 9.9. *For each $n \in \mathbb{N}$ there is some weight λ of the form $\lambda = w\cdot\Lambda_0 - n\delta$ (and some choice of scalars Q) such that $R^{\Lambda_0}_{Q,\lambda}$ is Morita equivalent to $B'_\Gamma(n)$.*

Remark 9.10. Note that there is a braid group acting and hence, assuming the conjecture above, there is a derived Morita equivalence between $R^{\Lambda_0}_{Q,\lambda}$ and the algebra $B'_\Gamma(n)$ for any $\lambda = w\cdot\Lambda_0 - n\delta$. Part of the content of Conjecture 9.9 is that this equivalence is non-derived for an appropriate λ.

9.4. **Braid groups.** An application of 2-representations of quantum groups in the Kac-Moody setting is that they induce actions of the associated braid group [CR, CK]. More precisely, in [CK] we defined the concept of a geometric categorical action and showed that it induces braid group actions. In the last section of that paper we give a version of a strong categorical $\mathfrak{g}$ action and explain that the same proof implies that such an action also induces a braid group action. More generally, we expect the following.

Conjecture 9.11. *A 2-representation of $\widehat{\widehat{\mathfrak{g}}}$ (in the sense of this paper) induces an action of the double affine braid group $B_{\widehat{\widehat{\mathfrak{g}}}}$.*

9.5. **Work of Carlsson-Okounkov.** In [CO, Car] Carlsson and Okounkov describe an operator, denoted W, on the cohomology of Hilbert schemes of points on a surface. This operator is defined using the Chern class of a virtual bundle over these Hilbert schemes, and their main theorem states that W can be expressed as a vertex operator. It would be interesting to understand their result at the categorified level, and to define an analogue of their operator W as a functor between derived categories of coherent sheaves on Hilbert schemes. For ALE spaces, the resulting functor should be related to the higher vertex operators of this paper.

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