FLOW BY GAUSS CURVATURE TO
THE ALEKSANDROV AND DUAL MINKOWSKI PROBLEMS

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ABSTRACT. In this paper we study a contracting flow of closed, convex hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ with speed $fr^\alpha K$, where K is the Gauss curvature, r is the distance from the hypersurface to the origin, and f is a positive and smooth function. If $\alpha \geq n + 1$, we prove that the flow exists for all time and converges smoothly after normalisation to a soliton, which is a sphere centred at the origin if $f \equiv 1$. Our argument provides a parabolic proof in the smooth category for the classical Aleksandrov problem, and resolves the dual q-Minkowski problem introduced by Huang, Lutwak, Yang and Zhang [30], for the case $q < 0$. If $\alpha < n+1$, corresponding to the case $q > 0$, we also establish the same results for even function f and origin-symmetric initial condition, but for non-symmetric f, counterexample is given for the above smooth convergence.

1. INTRODUCTION

Flow generated by the Gauss curvature was first studied by Firey [21] to model the shape change of tumbling stones. Since then the evolution of hypersurfaces by their Gauss curvature has been studied by many authors [2]-[6], [11]-[15], [20, 26, 29]. A main interest is to understand the asymptotic behavior of the flows. It was conjectured that the α-power of the Gauss curvature, for $\alpha > \frac{1}{n+2}$, deforms a convex hypersurface in $\mathbb{R}^{n+1}$ into a round point. This is a difficult problem and has been studied by many authors in the last three decades. The first result was by Chow [18] who provided a proof for the case $\alpha = 1/n$. In [4] Andrews proved the conjecture for the case $n = 2$ and $\alpha = 1$. Very recently, Choi and Daskalopoulos [13] gave an affirmative answer to the conjecture for $1/n < \alpha < 1 + 1/n$, and with Brendle [11] they resolved the conjecture for all $\alpha > \frac{1}{n+2}$, in all dimensions.

As a natural extension, anisotropic flows have also attracted much attention and have been extensively investigated [17, 22, 23]. They provide alternative proofs for the existence of solutions to elliptic PDEs arising in geometry and physics. For example a proof based on the logarithmic Gauss curvature flow was given in [15] for the classical Minkowski problem, and in [43] for a prescribing Gauss curvature problem. Expansion of

convex hypersurfaces by their Gauss curvature has also been studied by several authors [24, 25, 35, 39, 42].

Let $\mathcal{M}_0$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^n$ enclosing the origin. In this paper we study the following anisotropic Gauss curvature flow,

$$(1.1) \qquad \begin{cases} \dfrac{\partial X}{\partial t}(x,t) = -f(\nu)r^\alpha K(x,t)\nu, \\ X(x,0) = X_0(x), \end{cases}$$

where $K(\cdot,t)$ is the Gauss curvature of hypersurface $\mathcal{M}_t$, parametrized by $X(\cdot,t) : \mathbb{S}^n \to \mathbb{R}^{n+1}$, $\nu(\cdot,t)$ is the unit outer normal at $X(\cdot,t)$, and f is a given positive smooth function on $\mathbb{S}^n$. We denote by $r = |X(x,t)|$ the distance from the point $X(x,t)$ to the origin, and regard it as a function of $\xi = \xi(x,t) := X(x,t)/|X(x,t)| \in \mathbb{S}^n$. We call it the radial function of $\mathcal{M}_t$.

When $\alpha \geq n+1$, we prove that if $f \equiv 1$, the hypersurface $\mathcal{M}_t$ converges smoothly after normalisation to a sphere. For general positive and smooth function f, we prove that $\mathcal{M}_t$ converges smoothly after normalisation to a hypersurface which is a solution to the classical Aleksandrov problem [1] ($\alpha = n+1$) and to the dual q-Minkowski problem [30] for $q \leq 0$ ($\alpha > n+1$). Our proof of the smooth convergence consists of two parts:

  (i) uniform positive upper and lower bounds for the radial function of $\widetilde{\mathcal{M}}_t$; and
  (ii) uniform positive upper and lower bounds for the principal curvatures of $\widetilde{\mathcal{M}}_t$,

where $\widetilde{\mathcal{M}}_t$ is the normalised solution given in (1.6) below. Once the upper and lower bounds for the principal curvatures are established, higher order regularity of $\widetilde{\mathcal{M}}_t$ follows from Krylov's regularity theory. We then infer the smooth convergence by using the functional (1.9). Our proof of part (ii) applies to the flow (1.1) for all $\alpha \in \mathbb{R}^1$, as long as part (i) is true. In particular it also applies to the original Gauss curvature flow (namely the case $\alpha = 0$) for which the estimates (ii) were established for $f \equiv 1$ in [26].

When $\alpha < n+1$, we establish the smooth convergence for even f, provided the initial hypersurface is symmetric with respect to the origin. We also give examples to show that, without the symmetry assumption, part (i) above fails and so the smooth convergence does not hold.

As a result we also obtain the existence of smooth symmetric solutions to the dual q-Minkowski problem for all $q \in \mathbb{R}^1$, assuming the function f is smooth, positive, and f is even when $q > 0$. The dual q-Minkowski problem was recently introduced by Huang, Lutwak, Yang, and Zhang [30] where they proved the existence of symmetric weak solutions for the case $q \in (0, n+1)$ under some conditions. Their conditions were recently improved by Zhao [45]. For $q < 0$ the existence and uniqueness of weak solution were obtained in [44]. When $q = n+1$ it is the logarithm Minkowski problem studied in

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[9]. In [9] and [30], the existence of weak solutions was proved when the inhomogeneous term is a non-negative measure not concentrated in any sub-spaces. For other related results, we refer the readers to [8, 10, 38] and the references therein.

Let us state our first main result as follows.

Theorem 1.1. *Let $\mathcal{M}_0$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$ enclosing the origin. If $f \equiv 1$ and $\alpha \geq n + 1$, then the flow (1.1) has a unique smooth solution $\mathcal{M}_t$ for all time $t > 0$, which converges to the origin. After a proper rescaling $X \to \phi^{-1}(t)X$, the hypersurface $\widetilde{\mathcal{M}}_t = \phi^{-1}(t)\mathcal{M}_t$ converges exponentially fast to the unit sphere centred at the origin in the C^∞ topology.*

Our choice for the rescaling factor $\phi(t)$ is motivated by the following calculation. Assume

$$(1.2) \qquad\qquad X(\cdot, t) = \phi(t)X_0(\cdot)$$

evolves under the flow (1.1) with initial data $\phi_0 X_0$, where ϕ is a positive function and $\phi_0 = \phi(0)$. Since the normal vector is unchanged by the homothety, we obtain, by differentiating (1.2) in t and multiplying $\nu_0 = \nu(\cdot, t)$ to both sides,

$$(1.3) \qquad\qquad \phi'(t)\langle X_0, \nu_0 \rangle = -\phi^{\alpha-n}(t)fr_0^\alpha K_0,$$

where K_0 is the Gauss curvature of $\mathcal{M}_0 = X_0(\mathbb{S}^n)$, and r_0 is the radial function of $\mathcal{M}_0$. By (1.3) we have

$$\phi'(t) = -\lambda\phi^{\alpha-n}(t)$$

for some constant $\lambda > 0$. We may suppose $\lambda = 1$. Then

$$(1.4) \qquad\qquad \begin{aligned} \phi(t) &= \phi_0 e^{-t}, &&\text{if } \alpha = n + 1, \\ \phi(t) &= [\phi_0^q - qt]^{\frac{1}{q}}, &&\text{if } \alpha \neq n + 1, \end{aligned}$$

where $q = n + 1 - \alpha$, $\phi_0 = \phi(0) > 0$. By (1.3), one sees that $\mathcal{M}_0$ satisfies the following elliptic equation

$$(1.5) \qquad\qquad \frac{u(x)}{r^\alpha(\xi)K(p)} = f(x) \quad \forall\, x \in \mathbb{S}^n,$$

where $p \in \mathcal{M}_0$ is the point such that the unit outer normal $\nu(p) = x$, $\xi = p/|p| \in \mathbb{S}^n$, and u is the support function of $\mathcal{M}_0$, given by

$$u(x) = \sup\{\langle x, y \rangle :\ y \in \mathcal{M}_0\}.$$

The above calculation suggests that if we expect that our flow converges to a soliton which satisfies (1.5), it is reasonable to rescale the flow by a time-dependent factor $\phi(t)$ which is in the form of (1.4).

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Let us introduce the normalised flow for (1.1). Let

$$\widetilde{\mathcal{M}_t} = \phi^{-1}(t)\mathcal{M}_t,$$

(1.6)

$$\widetilde{X}(\cdot, \tau) = \phi^{-1}(t)X(\cdot, t),$$

where

$$\tau = \begin{cases} t & \text{if } \alpha = n+1, \\ \dfrac{1}{q}\log\dfrac{\phi_0^q}{\phi_0^q - qt} & \text{if } \alpha \neq n+1. \end{cases}$$

Then $\widetilde{X}(\cdot, \tau)$ satisfies the following normalised flow

(1.7)

$$\begin{cases} \dfrac{\partial X}{\partial t}(x, t) = -f(\nu)r^\alpha K(x, t)\nu + X(x, t), \\ X(\cdot, 0) = \phi_0^{-1}X_0. \end{cases}$$

For convenience we still use t instead of τ to denote the time variable and omit the "tilde" if no confusions arise.

The asymptotic behavior of (1.1) is equivalent to the long time behaviour of the normalised flow (1.7). Indeed, in order to prove Theorem 1.1, we shall establish the a priori estimates for (1.7), and show that $|X| \to 1$ smoothly as $t \to \infty$, provided $f \equiv 1$ and ϕ_0 is chosen such that

(1.8)

$$\phi_0 = \exp(\frac{1}{o_n}\int_{\mathbb{S}^n}\log r_0(\xi)d\xi), \text{ if } \alpha = n+1,$$

$$\min_{\mathbb{S}^n} r_0(\cdot) \leq \phi_0 \leq \max_{\mathbb{S}^n} r_0(\cdot), \text{ if } \alpha > n+1,$$

where $o_n = |\mathbb{S}^n|$ denotes the area of the sphere $\mathbb{S}^n$.

The following functional plays an important role in our argument,

(1.9) $$\mathcal{J}_\alpha(\mathcal{M}_t) = \begin{cases} \displaystyle\int_{\mathbb{S}^n} f(x)\log u(x, t)dx - \int_{\mathbb{S}^n}\log r(\xi, t)d\xi, \text{ if } \alpha = n+1, \\ \displaystyle\int_{\mathbb{S}^n} f(x)\log u(x, t)dx - \frac{1}{q}\int_{\mathbb{S}^n} r^q(\xi, t)d\xi, \text{ if } \alpha \neq n+1, \end{cases}$$

where $q = n+1-\alpha$ as above, $u(\cdot, t)$ and $r(\cdot, t)$ are respectively the support function and radial function of $\mathcal{M}_t$. This functional was introduced in [30]. We will show in Lemma 2.1 below that $\mathcal{J}_\alpha(\mathcal{M}_t)$ is strictly decreasing unless $\mathcal{M}_t$ solves the elliptic equation (1.5). By this functional and the a priori estimates for the normalised flow (1.7), we obtain the following convergence result for the anisotropic flow (1.1).

Theorem 1.2. *Let $\mathcal{M}_0$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$ which contains the origin in its interior. Let f be a smooth positive function on $\mathbb{S}^n$. If $\alpha > n+1$, then the flow (1.1) has a unique smooth solution $\mathcal{M}_t$ for all time $t > 0$.*

When $t \to \infty$, the rescaled hypersurfaces $\widetilde{\mathcal{M}}_t$ converge smoothly to the unique smooth solution of (1.5), which is a minimiser of the functional (1.9).

When $\alpha = n + 1$, in order that the solution of (1.1) converges to a solution of (1.5), we assume that $f \in C^\infty(\mathbb{S}^n; \mathbb{R}_+)$ and satisfies the following conditions

$$(1.10) \qquad\qquad \int_{\mathbb{S}^n} f = o_n := |\mathbb{S}^n|,$$

$$(1.11) \qquad\qquad \int_\omega f < |\mathbb{S}^n| - |\omega^*|$$

for any spherically convex subset $\omega \subset \mathbb{S}^n$. Here $|\cdot|$ denotes the n-dimensional Hausdorff measure, and $\omega^* \subset \mathbb{S}^n$ is the dual set of ω, namely $\omega^* = \{\xi \in \mathbb{S}^n : \ x \cdot \xi \leq 0, \ \ \forall\, x \in \omega\}$.

Theorem 1.3. *Let $\mathcal{M}_0$ be as in Theorem 1.2. Assume $\alpha = n + 1$ and (1.10), (1.11) hold. Then (1.1) has a unique smooth solution $\mathcal{M}_t$ for all time $t > 0$. When $t \to \infty$, the rescaled hypersurfaces $\widetilde{\mathcal{M}}_t$ converge smoothly to the smooth solution of (1.5), which is a minimiser of the functional (1.9).*

Theorem 1.3 gives a proof for the classical Aleksandrov problem in smooth category by a curvature flow approach. We point out that conditions (1.10) and (1.11) are necessary for Aleksandrov's problem [1], but for the flow (1.1), condition (1.10) is satisfied by any bounded positive function f provided we make a scaling of the time t. At the end of the paper we will show that Theorem 1.3 does not hold if (1.11) is violated.

Let $\mathcal{M}$ be a convex hypersurface in $\mathbb{R}^{n+1}$ with the origin $\mathcal{O}$ in its interior. Then $\mathcal{M}$ is a spherical radial graph via the mapping

$$\vec{r} : \xi \in \mathbb{S}^n \mapsto r(\xi)\xi \in \mathcal{M}.$$

Let $\mathscr{A} = \mathscr{A}_{\mathcal{M}}$ be a set-valued mapping given by

$$\mathscr{A}(\omega) = \cup_{\xi \in \omega}\{\nu(\vec{r}(\xi))\},$$

where ν is the Gauss map of $\mathcal{M}$. Aleksandrov raised the following problem: given a finite nonnegative Borel measure μ on $\mathbb{S}^n$, whether there exists a convex hypersurface $\mathcal{M}$ such that

$$(1.12) \qquad\qquad |\mathscr{A}(\omega)| = \mu(\omega) \ \ \forall\, \text{Borel sets } \omega \subset \mathbb{S}^n.$$

The left hand side of (1.12) defines a measure on $\mathbb{S}^n$, which is called the integral Gauss curvature of $\mathcal{M}$. The existence and uniqueness (up to a constant multiplication) of weak solution to this problem were obtained by Aleksandrov [1], assuming that μ is nonnegative, $\mu(\mathbb{S}^n) = o_n$ and $\mu(\mathbb{S}^n \setminus \omega) > |\omega^*|$ for any convex $\omega \subset \mathbb{S}^n$. These conditions

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are equivalent to (1.10) and (1.11), if μ has a density function f. If $\mathcal{M}$ is a hypersurface with prescribed integral Gauss curvature μ, then its polar dual

$$(1.13) \qquad \mathcal{M}^* = \partial\{z \in \mathbb{R}^{n+1} : z \cdot y \leq 1 \ \forall \, y \in \mathcal{M}\}$$

solves (1.5) for $\alpha = n + 1$.

For general α, the limiting hypersurface of the flow (1.1) is related to the dual Minkowski problem introduced most recently in [30]. Given a real number q and a finite Borel measure μ on the sphere $\mathbb{S}^n$, the authors asked if there exists a convex body Ω with the origin inside such that its q-th dual curvature measure

$$(1.14) \qquad \widetilde{C}_q(\Omega, \cdot) = \mu(\cdot).$$

Denote by $\mathcal{M}$ the boundary of Ω, and by $\mathscr{A}^* = \mathscr{A}_{\mathcal{M}}^*$ the "inverse" of $\mathscr{A}_{\mathcal{M}}$, namely

$$\mathscr{A}^*(\omega) = \{\xi \in \mathbb{S}^n : \nu(\vec{r}(\xi)) \in \omega\}.$$

The q-th dual curvature measure is defined by

$$(1.15) \qquad \widetilde{C}_q(\Omega, \omega) = \int_{\mathscr{A}^*(\omega)} r^q(\xi) d\xi.$$

Hence the dual Minkowski problem (1.14) is equivalent to the equation

$$(1.16) \qquad r^q |\mathrm{Jac}\mathscr{A}^*| = f \ \text{ on } \mathbb{S}^n,$$

provided μ has a density function f. Here $|\mathrm{Jac}\mathscr{A}^*|$ denotes the determinant of the Jacobian of the mapping $x \mapsto \xi = \mathscr{A}_{\mathcal{M}}^*(x)$. By (2.16) below, we see that the dual Minkowski problem is equivalent to the solvability of the equation (1.5) with $\alpha = n+1-q$. Noting that

$$(1.17) \qquad \begin{aligned} \mathscr{A}_{\mathcal{M}}^*(\omega) &= \{\xi \in \mathbb{S}^n : \nu(\vec{r}(\xi)) \in \omega\} \\ &= \{\nu^*(\vec{r}^*(x)) : x \in \omega\} \\ &= \mathscr{A}_{\mathcal{M}^*}(\omega), \end{aligned}$$

where ν and ν^* denote the unit outer normal of $\mathcal{M}$ and $\mathcal{M}^*$ respectively, we also see that if $\mathcal{M}^*$ solves the Aleksandrov problem (1.12), then $\mathcal{M}$ solves the dual Minkowski problem (1.14) for $q = 0$, and so is a solution to (1.5) for $\alpha = n + 1$.

When $\alpha < n + 1$, we consider the behaviour of origin-symmetric hypersurfaces under the flow (1.1), assuming that f is an even function, namely $f(x) = f(-x)$ for all $x \in \mathbb{S}^n$. In this case the solution $\mathcal{M}_t$ shrinks to a point in finite time, namely as $t \to T$ for some $T < \infty$. Our next theorem shows that the normalised solution converges smoothly if f is smooth and positive.

Theorem 1.4. *Let $\mathcal{M}_0$ be a smooth, closed, uniformly convex, and origin-symmetric hypersurface in $\mathbb{R}^{n+1}$. Let $\alpha < n + 1$. If f is a smooth, positive, even function on $\mathbb{S}^n$, then the flow (1.1) has a unique smooth solution $\mathcal{M}_t$. After normalisation, the rescaled hypersurfaces $\widetilde{\mathcal{M}}_t$ converge smoothly to a smooth solution of (1.5), which is a minimiser of the functional (1.9). Moreover, if $f \equiv 1$ and $0 \le \alpha < n + 1$, then $\widetilde{\mathcal{M}}_t$ converge smoothly to a sphere.*

In the proof of Theorem 1.4, we will choose the constant ϕ_0 in the rescaling (1.4) by

$$(1.18) \qquad \phi_0 = \left(\int_{\mathbb{S}^n} r_0^q(\xi) d\xi \Big/ \int_{\mathbb{S}^n} f(x) dx \right)^{\frac{1}{q}},$$

where r_0 is the radial function of the initial convex hypersurface $\mathcal{M}_0$. This choice is such that the functional $\mathcal{I}_q$ in (3.7) is a constant. This property is crucial for the uniform positive upper and lower bounds for the support function in the normalised flow (1.7).

Without the symmetry assumption, Theorem 1.4 is not true. In fact, when $\alpha < n+1$, we find that the hypersurfaces evolving by (1.1) may reach the origin in finite time, before the hypersurface shrinks to a point. Therefore the smooth convergence does not hold in general.

Theorem 1.5. *Suppose $n \ge 1$ and $\alpha < n + 1$. There exists a smooth, closed, uniformly convex hypersurface $\mathcal{M}_0$, such that under the flow (1.1),*

$$(1.19) \qquad \mathcal{R}(X(\cdot, t)) := \frac{\max_{\mathbb{S}^n} r(\cdot, t)}{\min_{\mathbb{S}^n} r(\cdot, t)} \to \infty \ \ as \ t \to T$$

for some $T > 0$.

Equation (1.5) can be written, in terms of the support function u, as a Monge-Ampère equation on the sphere,

$$(1.20) \qquad \det(\nabla^2 u + uI) = \frac{f(x)}{u(x)}(|\nabla u|^2 + u^2)^{\alpha/2} \ \ on \ \mathbb{S}^n.$$

By Theorems 1.1-1.4, we have the following existence results for equation (1.20).

Theorem 1.6. *Let f be a smooth and positive function on the sphere $\mathbb{S}^n$.*
(i) If $\alpha > n + 1$, there is a unique smooth, uniformly convex solution to (1.20).
(ii) If $\alpha = n + 1$ and f satisfies (1.10), (1.11), there is a smooth, uniformly convex solution to (1.20). The solution is unique up to dilation.
(iii) If $\alpha < n + 1$ and f is even, there is an origin-symmetric solution to (1.20).
(iv) If $f \equiv 1$, then the solution must be a sphere when $\alpha \ge n + 1$, and the origin-symmetric solution must be a sphere when $0 \le \alpha < n + 1$.

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In case (ii) of Theorem 1.6, the existence and uniqueness (up to dilation) of the solution were proved by [1], and the regularity of the solution was obtained in [36, 37]. In this paper we use the generalised solution to the Aleksandrov problem as a barrier to establish the uniform estimate for the corresponding Gauss curvature flow. Our main concern of this paper is the smooth convergence of the flow, which also provides an alternative proof for the regularity of the solution.

It is interesting to compare equation (1.20) with the L_p-Minkowski problem

$$(1.21) \qquad\qquad \det(\nabla^2 u + uI) = \frac{f(x)}{u^{1-p}(x)} \quad \text{on } \mathbb{S}^n.$$

For equation (1.21), there is a solution if $p > -n - 1$ [16] and no solution in general if $p \le -n - 1$. In Theorem 1.6 we proved the existence of solutions to (1.20) for all $\alpha \in \mathbb{R}^1$, which looks stronger. This is due to the associated functional (1.9), in which the first integral $\int_{\mathbb{S}^n} f \log u$ is bounded for our solution. This property, together with (3.7), enables us to establish a uniform bound for the support function u (Lemma 3.3).

This paper is organised as follows. In Section 2 we collect some properties of convex hypersurfaces, and show that the flow (1.1) can be reduced to a scalar parabolic equation of Monge-Ampère type, via the support function or the radial function. We will also show in Section 2 that (1.7) is a descending gradient flow of the functional (1.9). In Section 3 we establish the uniform positive upper and lower bounds for the support function of the normalised flow (1.7). The uniform positive upper and lower bounds for the principal curvatures are proved in Section 4. The a priori estimates ensure the longtime existence and the convergence of the normalised flow. The proofs of Theorems 1.1-1.4 will be presented in Section 5. Finally in Section 6 we prove Theorem 1.5.

2. Preliminaries

Let us first recall some basic properties of convex hypersurfaces. Let $\mathcal{M}$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$. Assume that $\mathcal{M}$ is parametrized by the inverse Gauss map $X : \mathbb{S}^n \to \mathcal{M}$. The support function $u : \mathbb{S}^n \to \mathbb{R}$ of $\mathcal{M}$ is defined by

$$(2.1) \qquad\qquad u(x) = \sup\{\langle x, y \rangle : y \in \mathcal{M}\}.$$

The supremum is attained at a point y such that x is the outer normal of $\mathcal{M}$ at y. It is easy to check that

$$(2.2) \qquad y = u(x)x + \nabla u(x),$$

where ∇ is the covariant derivative with respect to the standard metric e_{ij} of the sphere $\mathbb{S}^n$. Hence

$$(2.3) \qquad r = |y| = \sqrt{u^2 + |\nabla u|^2}.$$

The second fundamental form of $\mathcal{M}$ is given by, see e.g. [5, 42],

$$(2.4) \qquad h_{ij} = u_{ij} + ue_{ij},$$

where $u_{ij} = \nabla^2_{ij}u$ denotes the second order covariant derivative of u with respect the spherical metric e_{ij}. By Weingarten's formula,

$$(2.5) \qquad e_{ij} = \left\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial \nu}{\partial x_j} \right\rangle = h_{ik}g^{kl}h_{jl},$$

where g_{ij} is the metric of $\mathcal{M}$ and g^{ij} its inverse. It follows from (2.4) and (2.5) that the principal radii of curvature of $\mathcal{M}$, under a smooth local orthonormal frame on $\mathbb{S}^n$, are the eigenvalues of the matrix

$$(2.6) \qquad b_{ij} = u_{ij} + u\delta_{ij}.$$

In particular the Gauss curvature is given by

$$(2.7) \qquad K = 1/\det(u_{ij} + u\delta_{ij}) = S_n^{-1}(u_{ij} + u\delta_{ij}),$$

where

$$S_k = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

denotes the k-th elementary symmetric polynomial.

Let $X(\cdot, t)$ be a smooth solution to the normalised flow (1.7) and let $u(\cdot, t)$ be its support function. From the above discussion we see that the flow (1.7) can be reduced to the initial value problem for the support function u:

$$(2.8) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) = -f(x)r^\alpha S_n^{-1}(u_{ij} + u\delta_{ij})(x,t) + u(x,t) & \text{on } \mathbb{S}^n \times [0, \infty), \\ u(\cdot, 0) = \widetilde{u}_0 := \phi_0^{-1}u_0, \end{cases}$$

where $r = \sqrt{u^2 + |\nabla u|^2}(x,t)$ as in (2.3), u_0 is the support function of the initial hypersurface $\mathcal{M}_0$, and ϕ_0 is the dilation constant in (1.7).

As $\mathcal{M}$ encloses the origin, it can be parametrized via the radial function $r : \mathbb{S}^n \to \mathbb{R}_+$,

$$\mathcal{M} = \{r(\xi)\xi : \xi \in \mathbb{S}^n\}.$$

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The following formulae are well-known, see e.g. [25],

$$(2.9) \qquad \nu = \frac{r\xi - \nabla r}{\sqrt{r^2 + |\nabla r|^2}},$$

$$(2.10) \qquad \begin{aligned} g_{ij} &= r^2 e_{ij} + r_i r_j, \\ h_{ij} &= \frac{r^2 e_{ij} + 2r_i r_j - r r_{ij}}{\sqrt{r^2 + |\nabla r|^2}}. \end{aligned}$$

Set

$$(2.11) \qquad v = \frac{r}{u} = \sqrt{1 + |\nabla \log r|^2},$$

where the last equality follows by multiplying ξ to both sides of (2.9). The normalised flow (1.7) can be also described by the following scalar equation for $r(\cdot, t)$,

$$(2.12) \qquad \begin{cases} \dfrac{\partial r}{\partial t}(\xi, t) = -vfr^\alpha K(\xi, t) + r(\xi, t) \quad \text{on } \mathbb{S}^n \times [0, \infty), \\ r(\cdot, 0) = \widetilde{r}_0 := \phi_0^{-1} r_0, \end{cases}$$

where r_0 is the radial function of $\mathcal{M}_0$, and $K(\xi, t)$ denotes the Gauss curvature at $r(\xi, t)\xi \in \mathcal{M}_t$. Note that in (2.12) f takes value at $\nu = \nu(\xi, t)$ given by (2.9). By (2.10) we have, under a local orthonormal frame on $\mathbb{S}^n$,

$$(2.13) \qquad K = \frac{\det h_{ij}}{\det g_{ij}} = v^{-n-2} r^{-3n} \det(r^2 \delta_{ij} + 2r_i r_j - r r_{ij}).$$

Given any $\omega \subset \mathbb{S}^n$, let $\mathcal{C} = \mathcal{C}_{\mathcal{M}, \omega}$ be the "cone-like" region with the vertex at the origin and the base $\nu^{-1}(\omega) \subset \mathcal{M}$, namely

$$\mathcal{C} := \{ z \in \mathbb{R}^{n+1} : z = \lambda \nu^{-1}(x), \ \lambda \in [0, 1], x \in \omega \}.$$

It is well-known that the volume element of $\mathcal{C}$ can be expressed by

$$(2.14) \qquad d\mathrm{Vol}(\mathcal{C}) = \frac{1}{n+1} \frac{u(x)}{K(p)} dx = \frac{1}{n+1} r^{n+1}(\xi) d\xi,$$

where $p = \nu^{-1}(x) \in \mathcal{M}$, and ξ and x are associated by

$$(2.15) \qquad r(\xi)\xi = u(x)x + \nabla u(x),$$

namely $p = \nu^{-1}(x) = \vec{r}(\xi)$. By the second equality in (2.14), we find that the determinant of the Jacobian of the mapping $x \mapsto \xi = \mathscr{A}_{\mathcal{M}}^*(x)$ is given by

$$(2.16) \qquad |\mathrm{Jac}\mathscr{A}^*| = \left| \frac{d\xi}{dx} \right| = \frac{u(x)}{r^{n+1}(\xi)K(p)}.$$

Lemma 2.1. *The functional* (1.9) *is non-increasing along the normalised flow* (1.7). *Namely* $\frac{d}{dt}\mathcal{J}_\alpha(\mathcal{M}_t) \leq 0$, *and the equality holds if and only if* $\mathcal{M}_t$ *satisfies the elliptic equation* (1.5).

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Proof. For $\alpha \neq n+1$, it is easy to see

(2.17)
$$\frac{d}{dt}\mathcal{J}_\alpha(\mathcal{M}_t) = \int_{\mathbb{S}^n} f(x)\frac{u_t}{u}dx - \int_{\mathbb{S}^n} \frac{r_t}{r^{1-q}}d\xi.$$

Let $x = x(\xi, t) = \nu(r(\xi, t)\xi)$. By (2.15) we have

$$\log r(\xi, t) = \log u(x, t) - \log(x \cdot \xi).$$

Differentiate the above identity and denote $\dot{x} = \partial_t x(\xi, t)$. We obtain

(2.18)
$$\begin{aligned}
\frac{r_t}{r}(\xi, t) &= \frac{u_t + \nabla u \cdot \dot{x}}{u} - \frac{\dot{x} \cdot \xi}{x \cdot \xi} \\
&= \frac{u_t + (\nabla u - r\xi) \cdot \dot{x}}{u} \\
&= \frac{u_t}{u}(x, t).
\end{aligned}$$

This identity can be also seen from (2.8), (2.11) and (2.12). Plugging (2.18) in (2.17) and then using (2.16) to change the variables, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{J}_\alpha(\mathcal{M}_t) &= \int_{\mathbb{S}^n} \frac{u_t}{u}\left(f - \frac{u}{r^{n+1-q}K}\right)dx \\
&= -\int_{\mathbb{S}^n} \frac{(fr^\alpha K - u)^2}{ur^\alpha K}dx \\
&\leq 0.
\end{aligned}$$

Clearly $\frac{d}{dt}\mathcal{J}_\alpha(\mathcal{M}_t) = 0$ if and only if

$$f(x)r^\alpha(\xi, t)K(p) - u(x, t) = 0.$$

Namely $\mathcal{M}_t$ satisfies (1.5).

When $\alpha = n+1$, we have by differentiating (1.9)

$$\frac{d}{dt}\mathcal{J}_{n+1}(\mathcal{M}_t) = \int_{\mathbb{S}^n} f(x)\frac{u_t}{u}dx - \int_{\mathbb{S}^n} \frac{r_t}{r}d\xi.$$

By (2.16) and (2.18) we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{J}_{n+1}(\mathcal{M}_t) &= \int_{\mathbb{S}^n} \frac{u_t}{u}\left(f(x) - \left|\frac{d\xi}{dx}\right|\right)dx \\
&= -\int_{\mathbb{S}^n} \frac{(fr^{n+1}K - u)^2}{ur^{n+1}K}dx \\
&\leq 0.
\end{aligned}$$

This completes the proof. $\qquad\square$

The next lemma shows that the functional $\mathcal{J}_{n+1}$ is also monotone along the Gauss curvature flow for origin-symmetric solutions. This lemma is of interest itself, though it is not needed in the proof of our main theorems.

Lemma 2.2. *Let $\mathcal{M}_t$ be a family of smooth, closed, uniformly convex and origin-symmetric hypersurfaces which evolve under the normalised Gauss curvature flow (1.7) with $\alpha = 0$ and $f \equiv 1$. Assume $\mathrm{Vol}(\mathcal{M}_0) = \mathrm{Vol}(B_1)$. Then $\frac{d}{dt}\mathcal{J}_{n+1}(\mathcal{M}_t) \leq 0$, and the equality holds if and only if $\mathcal{M}_t$ is the unit sphere.*

Proof. Let Ω_t denote the convex body whose boundary is $\mathcal{M}_t$. Note that the functional $\mathcal{J}_{n+1}$ is unchanged under dilation. The volume $\mathrm{Vol}(\Omega_t)$ is preserved under the normalised Gauss curvature flow

$$(2.19) \qquad \partial_t u = -K + u,$$

where $u(\cdot, t)$ is the support function of $\mathcal{M}_t$. This can be easily seen from the following evolution equation

$$
\begin{aligned}
\frac{d}{dt}\mathrm{Vol}(\Omega_t) &= \frac{1}{n+1}\frac{d}{dt}\int_{\mathbb{S}^n}\frac{u}{K}dx \\
&= \int_{\mathbb{S}^n}\frac{u_t}{K}dx \\
&= (n+1)\big(-\mathrm{Vol}(\Omega_0) + \mathrm{Vol}(\Omega_t)\big).
\end{aligned}
$$

Hence

$$(2.20) \qquad \int_{\mathbb{S}^n}\frac{u}{K}dx = |\mathbb{S}^n|, \quad \forall\, t \geq 0.$$

By the Hölder inequality

$$\left(\int_{\mathbb{S}^n}dx\right)^2 \leq \left(\int_{\mathbb{S}^n}\frac{K}{u}dx\right)\left(\int_{\mathbb{S}^n}\frac{u}{K}dx\right).$$

This together with (2.20) shows

$$(2.21) \qquad \int_{\mathbb{S}^n}\frac{K}{u}dx \geq \int_{\mathbb{S}^n}dx, \quad \forall\, t \geq 0.$$

Recall that Blaschke-Santoló inequality of origin-symmetric convex body gives

$$(2.22) \qquad \mathrm{Vol}(\Omega_t)\mathrm{Vol}(\Omega_t^*) \leq \mathrm{Vol}^2(B_1),$$

where Ω^* is the polar dual of Ω. Therefore

$$\left(\int_{\mathbb{S}^n}dx\right)^2 \geq \int_{\mathbb{S}^n}\frac{u}{K}dx\int_{\mathbb{S}^n}(r^*)^{n+1}d\xi^* = \int_{\mathbb{S}^n}\frac{u}{K}dx\int_{\mathbb{S}^n}\frac{1}{u^{n+1}}dx.$$

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This together with (2.20) implies that

$$(2.23) \qquad \int_{\mathbb{S}^n} \frac{1}{r^{n+1}} dx \leq \int_{\mathbb{S}^n} \frac{1}{u^{n+1}} dx \leq \int_{\mathbb{S}^n} dx.$$

Combining (2.21) and (2.23), we conclude that, under the flow (2.19),

$$
\begin{aligned}
\frac{d}{dt} \mathcal{J}_{n+1} &= \int_{\mathbb{S}^n} \frac{u_t}{u} dx - \int_{\mathbb{S}^n} \frac{r_t}{r} d\xi \\
&= -\int_{\mathbb{S}^n} \frac{K}{u} dx + \int_{\mathbb{S}^n} \frac{K}{u} \left| \frac{d\xi}{dx} \right| dx \\
&= -\int_{\mathbb{S}^n} \frac{K}{u} dx + \int_{\mathbb{S}^n} \frac{1}{r^{n+1}} dx \\
&\leq 0.
\end{aligned}
$$

The last equality holds if and only if the equality in (2.21) and in (2.23) holds, by (2.22) it occurs when $\mathcal{M}_t = \mathbb{S}^n$ only. $\qquad \square$

3. A priori estimates I

In this section we establish the uniform positive upper and lower bounds for the support function of the normalised flow (1.7).

Lemma 3.1. *Let $u(\cdot, t)$, $t \in (0, T]$, be a smooth, uniformly convex solution to (2.8). If $\alpha > n + 1$, then there is a positive constant C depending only on α, and the lower and upper bounds of f and $u(\cdot, 0)$ such that*

$$(3.1) \qquad 1/C \leq u(\cdot, t) \leq C \quad \forall\, t \in (0, T].$$

If $\alpha = n + 1$ and $f \equiv 1$, then

$$(3.2) \qquad \min_{\mathbb{S}^n} u(\cdot, 0) \leq u(\cdot, t) \leq \max_{\mathbb{S}^n} u(\cdot, 0) \quad \forall\, t \in (0, T].$$

Proof. Let $u_{\min}(t) = \min_{x \in \mathbb{S}^n} u(x, t)$. By (2.8) we have

$$(3.3) \qquad \frac{d}{dt} u_{\min} \geq -(f u_{\min}^{-q} - 1) u_{\min}.$$

where $q = n + 1 - \alpha \leq 0$.

If $\alpha > n + 1$, we may assume that $u_{\min}(t) < (\max_{\mathbb{S}^n} f)^{\frac{1}{q}}$, otherwise we are through. Hence $\frac{d}{dt} u_{\min} \geq 0$. This implies

$$u(\cdot, t) \geq \min \left\{ (\max_{\mathbb{S}^n} f)^{\frac{1}{q}}, \min_{\mathbb{S}^n} u(\cdot, 0) \right\}.$$

Similarly we have

$$u(\cdot, t) \leq \max \left\{ (\min_{\mathbb{S}^n} f)^{\frac{1}{q}}, \max_{\mathbb{S}^n} u(\cdot, 0) \right\}.$$

This proves (3.1)

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If $\alpha = n+1$ and $f \equiv 1$, then (3.3) gives $\frac{d}{dt}u_{\min} \geq 0$. Similarly we have $\frac{d}{dt}u_{\max} \leq 0$. Therefore (3.2) follows. $\qquad\square$

When $\alpha = n+1$, for general positive function f which satisfies (1.10) and (1.11), we can use a barrier argument to derive the L^∞-norm estimates.

Lemma 3.2. *Let u be as in Lemma 3.1. If $\alpha = n+1$, and f satisfies (1.10) and (1.11), then there is a positive constant C depending only on $\min_{\mathbb{S}^n} u(\cdot,0)$, $\max_{\mathbb{S}^n} u(\cdot,0)$ and f such that*

$$(3.4) \qquad\qquad 1/C \leq u(\cdot,t) \leq C \quad \forall\, t \in (0,T].$$

Before proving Lemma 3.2, we recall the existence of generalised solutions to Aleksandrov's problem, of which the proof consists of two steps. The first one is to prove the existence of polyhedron $\mathcal{N}_k^*$ whose integral Gauss curvature is a discrete measure converging weakly to f. Noting that the integral Gauss curvature is invariant under dilation, one may assume that the diameter of $\mathcal{N}_k^*$ is equal to 1. Hence by convexity $\mathcal{N}_k^*$ converges to a limit $\mathcal{N}^*$. In the second step one uses condition (1.11) to show that $\mathcal{N}^*$ contains nonempty interior and the origin is an interior point. The proof of the second step is also elementary, see page 520, lines 17-27, [37].

Proof of Lemma 3.2. Let $\mathcal{N}$ be the polar dual of the generalised solution $\mathcal{N}^*$, defined in (1.13). We use $\mathcal{N}$ as barrier to prove (3.4). Let $\mathcal{M}_t$ be a smooth convex solution to the normalised flow (1.7). Let $\mathcal{N}_0 = s_0\mathcal{N}$ and $\mathcal{N}_1 = s_1\mathcal{N}$, where the constants $s_1 > s_0 > 0$ are chosen such that $\mathcal{N}_0$ is strictly contained in $\mathcal{M}_0$ and $\mathcal{M}_0$ is strictly contained in $\mathcal{N}_1$. Let r_t, ρ_0, ρ_1 be respectively the radial functions of $\mathcal{M}_t, \mathcal{N}_0, \mathcal{N}_1$. Note that for any constant $s > 0$, $s\mathcal{N}$ is a stationary solution to (1.7) in the generalised sense.

We claim that $\mathcal{M}_t$ is contained in $\mathcal{N}_1$ for all $t > 0$. For if not, there exists a time $t_0 > 0$ such that $\sup_{\xi \in \mathbb{S}^n} r_{t_0}(\xi)/\rho_1(\xi) = 1$. Denote $G = \mathcal{M}_{t_0} \cap \mathcal{N}_1$ (G can be a point or a closed set). Since $\frac{\partial}{\partial t}r_t(\xi)$ is smooth in both ξ and t, replacing ρ_1 by $(1+a)\rho_1$ for a small constant a, we may assume that the velocity of $\mathcal{M}_t$ is positive on $G \times \{t_0\}$, and so also in a neighbourhood of $G \times \{t_0\}$. Therefore there exist sufficiently small constants $\epsilon, \delta > 0$, such that the velocity of $\mathcal{M}_t$ is greater than δ at $\xi r_{t_0}(\xi) \in \mathcal{M}_{t_0}$, for all $\xi \in \omega$, where $\omega = \{\xi \in \mathbb{S}^n : r_{t_0}(\xi) > (1-\epsilon)\rho_1(\xi)\}$. By equation (2.12), it means the Gauss curvature of $\mathcal{M}_{t_0}$ is strictly smaller than that of $\mathcal{N}_1$ for all $\xi \in \omega$. Applying the comparison principle for generalised solutions of the elliptic Monge-Ampère equation to the functions r_{t_0} and $(1-\epsilon)\rho_1$, we reach a contradiction.

Similarly we can prove that $\mathcal{N}_0$ is contained in $\mathcal{M}_t$ for all $t > 0$. $\qquad\square$

For $\alpha < n+1$, we consider the origin-symmetric hypersurfaces and give the following L^∞-norm estimates.

Lemma 3.3. *Let $\mathcal{M}_t$, where $t \in (0, T]$, be an origin-symmetric, uniformly convex solution to the normalised flow (1.7), and $u(\cdot, t)$ be its support function. For $\alpha < n+1$, there is a positive constant C depending only on α, $\mathcal{M}_0$ and f, such that*

$$(3.5) \qquad\qquad 1/C \le u(\cdot, t) \le C \quad \forall\, t \in (0, T].$$

Proof. Let us denote by $\mathcal{I}_q(\mathcal{M}_t)$ the L^q integral of the radial function $r(\xi, t)$, i.e.,

$$\mathcal{I}_q(\mathcal{M}_t) = \int_{\mathbb{S}^n} r^q(\xi, t) d\xi,$$

where $q = n + 1 - \alpha$. By (2.12), we have

$$\begin{aligned}
\frac{d}{dt} \mathcal{I}_q(\mathcal{M}_t) &= q \int_{\mathbb{S}^n} r^{q-1}\Big(-\frac{r}{u} f(x) r^\alpha K + r \Big) d\xi \\
&= -q \int_{\mathbb{S}^n} f(x) \frac{r^{n+1} K}{u} d\xi + q \int_{\mathbb{S}^n} r^q d\xi,
\end{aligned}$$

where f takes value at $x = \nu(\xi, t)$ given by (2.9). By the variable change formula (2.16), we obtain

$$\frac{d}{dt} \mathcal{I}_q(\mathcal{M}_t) = q\Big(-\int_{\mathbb{S}^n} f(x) dx + \mathcal{I}_q(\mathcal{M}_t) \Big).$$

Solving this ODE, one sees

$$(3.6) \qquad \mathcal{I}_q(\mathcal{M}_t) = e^{qt}\Big(\mathcal{I}_q(\mathcal{M}_0) - \int_{\mathbb{S}^n} f \Big) + \int_{\mathbb{S}^n} f$$

It follows that, by our choice of the rescaling factor ϕ_0 in (1.18),

$$(3.7) \qquad \mathcal{I}_q(\mathcal{M}_t) \equiv \int_{\mathbb{S}^n} f(x) dx, \quad \forall\, t \in (0, T].$$

Let $r_{\min}(t) = \min_{\mathbb{S}^n} r(\cdot, t)$ and $r_{\max}(t) = \max_{\mathbb{S}^n} r(\cdot, t)$. By a rotation of coordinates we may assume that $r_{\max}(t) = r(e_1, t)$. Since Ω_t is origin-symmetric, the points $\pm r_{\max}(t) e_1 \in \mathcal{M}_t$. Hence

$$u(x, t) = \sup\{ p \cdot x : \ p \in \mathcal{M}_t \} \ge r_{\max}(t) |x \cdot e_1|, \ \forall\, x \in \mathbb{S}^n.$$

Therefore

$$\begin{aligned}
\int_{\mathbb{S}^n} f(x) \log u(x, t) dx &\ge \Big(\int_{\mathbb{S}^n} f(x) dx \Big) \log r_{\max}(t) + \int_{\mathbb{S}^n} f(x) \log |x \cdot e_1| dx \\
(3.8) \qquad &\ge |\mathbb{S}^n| (\min_{\mathbb{S}^n} f) \log r_{\max}(t) - C \max_{\mathbb{S}^n} f.
\end{aligned}$$

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By Lemma 2.1 and (3.7), we conclude

$$\mathcal{J}_\alpha(\mathcal{M}_0) \geq \mathcal{J}_\alpha(\mathcal{M}_t) = \int_{\mathbb{S}^n} f(x) \log u(x,t) dx - \frac{1}{q} \int_{\mathbb{S}^n} f.$$

This together with (3.8) implies

(3.9) $$r_{\max}(t) \leq C_1 e^{C_2 \mathcal{J}_\alpha(\mathcal{M}_0)} \leq C.$$

This proves the upper bound in (3.5).

Next we derive a positive lower bound for $u(\cdot, t)$. We divide it into two cases.

Case (i), $q \in (0, n+1]$. By Hölder inequality,

$$\mathcal{I}_q(\mathcal{M}_t) \leq \mathcal{I}_{n+1}^{\frac{q}{n+1}}(\mathcal{M}_t) |\mathbb{S}^n|^{\frac{\alpha}{n+1}}.$$

Hence

(3.10) $$\frac{|\mathbb{S}^n|^{-\frac{\alpha}{q}}}{n+1} \mathcal{I}_q^{\frac{n+1}{q}}(\mathcal{M}_t) \leq \frac{1}{n+1} \mathcal{I}_{n+1}(\mathcal{M}_t) = \mathrm{Vol}(\Omega_t),$$

where Ω_t denotes the convex body enclosed by $\mathcal{M}_t$. Assume by a rotation if necessary $r(e_{n+1}, t) = r_{\min}(t)$. Since Ω_t is origin-symmetric, we find that Ω_t is contained in a cube

$$Q_t = \{z \in \mathbb{R}^{n+1} : \ -r_{\max}(t) \leq z_i \leq r_{\max}(t) \text{ for } 1 \leq i \leq n, \ -r_{\min}(t) \leq z_{n+1} \leq r_{\min}(t)\}.$$

Therefore by (3.10)

$$\frac{|\mathbb{S}^n|^{-\frac{\alpha}{q}}}{n+1} \mathcal{I}_q^{\frac{n+1}{q}}(\mathcal{M}_t) \leq 2^{n+1} r_{\max}^n(t) r_{\min}(t).$$

By (3.7), the left hand side of the above inequality is a positive constant. Using (3.9), we get $\min_{\mathbb{S}^n} u(\cdot, t) = r_{\min}(t) \geq 1/C$.

Case (ii), $q > n+1$. We have

$$
\begin{aligned}
\mathcal{I}_q(\mathcal{M}_t) &= r_{\max}^q(t) \int_{\mathbb{S}^n} \left(\frac{r(\xi, t)}{r_{\max}(t)} \right)^q d\xi \\
&\leq r_{\max}^q(t) \int_{\mathbb{S}^n} \left(\frac{r(\xi, t)}{r_{\max}(t)} \right)^{n+1} d\xi \\
&= (n+1) r_{\max}^{q-n-1}(t) \mathrm{Vol}(\Omega_t) \\
&\leq C r_{\max}^{q-1}(t) r_{\min}(t).
\end{aligned}
$$

The lower bound of $r_{\min}(t)$ now follows from (3.7) and (3.9). $\qquad \square$

For convex hypersurface, the gradient estimate is a direct consequence of the L^∞-norm estimate.

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Lemma 3.4. *Let* $u(\cdot, t)$, $t \in (0, T]$, *be a smooth, uniformly convex solution to* (2.8). *Then we have the gradient estimate*

(3.11)
$$|\nabla u(\cdot, t)| \leq \max_{\mathbb{S}^n \times (0, T]} u, \quad \forall\, t \in (0, T].$$

Proof. This is due to convexity. □

Similarly we have the estimates for the radial function r.

Lemma 3.5. *Let* $X(\cdot, t)$, $t \in (0, T]$, *be a uniformly convex solution to* (1.7). *Let* u *and* r *be its support function and radial function, respectively. Then*

(3.12)
$$\min_{\mathbb{S}^n \times (0, T]} u \leq r(\cdot, t) \leq \max_{\mathbb{S}^n \times (0, T]} u \quad \forall\, t \in (0, T],$$

and

(3.13)
$$|\nabla r(\cdot, t)| \leq C \quad \forall\, t \in (0, T],$$

where $C > 0$ *depends only on* $\min_{\mathbb{S}^n \times (0, T]} u$ *and* $\max_{\mathbb{S}^n \times (0, T]} u$

Proof. Estimates (3.12) follow from (3.4) as one has $\min_{\mathbb{S}^n} u(\cdot, t) = \min_{\mathbb{S}^n} r(\cdot, t)$ and $\max_{\mathbb{S}^n} u(\cdot, t) = \max_{\mathbb{S}^n} r(\cdot, t)$. Estimate (3.13) follows from (3.12) because by (2.11) we have $|\nabla r| \leq \frac{r^2}{u}$. □

4. A priori estimates II

In this section we establish uniform positive upper and lower bounds for the principal curvatures for the normalised flow (1.7). We point out that the curvature estimates in this section are for any $\alpha \in \mathbb{R}^1$.

We first derive an upper bound for the Gauss curvature $K(\cdot, t)$.

Lemma 4.1. *Let* $X(\cdot, t)$ *be a uniformly convex solution to the normalised flow* (1.7) *which encloses the origin for* $t \in (0, T]$. *Then there is a positive constant* C *depending only on* α, f, $\min_{\mathbb{S}^n \times (0, T]} u$ *and* $\max_{\mathbb{S}^n \times (0, T]} u$, *such that*

(4.1)
$$K(\cdot, t) \leq C, \quad \forall\, t \in (0, T].$$

Proof. Consider the auxiliary function

$$Q = \frac{-u_t}{u - \epsilon_0} = \frac{f r^\alpha K - u}{u - \epsilon_0},$$

where

$$\epsilon_0 = \frac{1}{2} \min_{x \in \mathbb{S}^n,\, t \in (0, T]} u(x, t) > 0.$$

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At the point where Q attains its spatial maximum, we have

(4.2)
$$0 = \nabla_i Q = \frac{-u_{ti}}{u - \epsilon_0} + \frac{u_t u_i}{(u - \epsilon_0)^2},$$

and

$$0 \geq \nabla_{ij}^2 Q = \frac{-u_{tij}}{u - \epsilon_0} + \frac{u_{ti} u_j + u_{tj} u_i + u_t u_{ij}}{(u - \epsilon_0)^2} - \frac{2 u_t u_i u_j}{(u - \epsilon_0)^3}$$

(4.3)
$$= \frac{-u_{tij}}{u - \epsilon_0} + \frac{u_t u_{ij}}{(u - \epsilon_0)^2},$$

where (4.2) was used in the second equality above. The first inequality in (4.3) should be understood in the sense of negative-semidefinite matrix. By (4.3) and (2.6) we infer that

(4.4)
$$- u_{tij} - u_t \delta_{ij} \leq (b_{ij} - \epsilon_0 \delta_{ij}) Q.$$

Using the equation (2.8), we then have

$$
\begin{aligned}
Q_t &= \frac{-u_{tt}}{u - \epsilon_0} + Q^2 \\
&= \frac{f r^\alpha S_n^{-2}}{u - \epsilon_0} S_n^{ij}(-u_{tij} - u_t \delta_{ij}) + \frac{\alpha f r^{\alpha-1}}{(u - \epsilon_0) S_n} r_t + Q + Q^2 \\
\end{aligned}
$$

(4.5)
$$\leq \frac{f r^\alpha K}{u - \epsilon_0}(n - \epsilon_0 H) Q + \frac{\alpha f r^{\alpha-1} K}{u - \epsilon_0} r_t + Q + Q^2,$$

where H denotes the mean curvature of $X(\cdot, t)$.

By (2.3) and (4.2),

(4.6)
$$r_t = \frac{u u_t + \sum u_k u_{kt}}{r} = \frac{\epsilon_0 u - r^2}{r} Q.$$

Without loss of generality we assume that $K \approx Q \gg 1$. Plugging (4.6) into (4.5) and noticing that $H \geq n K^{\frac{1}{n}}$, we obtain

$$Q_t \leq C_0 Q^2 \big(C_1 - \epsilon_0 Q^{\frac{1}{n}} \big),$$

for some C_0, C_1 only depending on α, f and the L^∞-norm of u. From the ode we infer that $Q \leq C$ for some $C > 0$ depending on $Q(0)$, C_1 and ϵ_0. Our a priori bound (4.1) follows consequently. $\qquad \square$

Next we prove that the principal curvatures of $\mathcal{M}_t$ are bounded by positive constants from both above and below. To obtain the positive lower bound for the principal curvatures of $\mathcal{M}_t$, we will study an expanding flow by Gauss curvature for the dual hypersurface of $\mathcal{M}_t$. This technique was previously used in [12, 31, 32, 33]. Expanding flows by Gauss curvature have been studied in [24, 25, 39, 41, 42]. Our estimates are also inspired by these works.

Lemma 4.2. *Let $X(\cdot, t)$ be the solution of the normalised flow (1.7) for $t \in (0, T]$. Then there is a positive constant C depending only on α, f, $\min_{\mathbb{S}^n \times (0,T]} u$ and $\max_{\mathbb{S}^n \times (0,T]} u$, such that the principal curvatures of $X(\cdot, t)$ are bounded from above and below*

$$
(4.7) \qquad\qquad C^{-1} \leq \kappa_i(\cdot, t) \leq C, \quad \forall \, t \in (0, T], \ and \ i = 1, \ldots, n.
$$

Proof. To prove the lower bound in (4.7), we employ the dual flow of (1.7), and establish an upper bound of principal curvature for the dual flow. This, together with Lemma 4.1, also implies the upper bound in (4.7).

We denote by $\mathcal{M}_t^*$ the polar set of $\mathcal{M}_t = X(\mathbb{S}^n, t)$, see (1.13) for the definition of the polar set. It is well-known that

$$
(4.8) \qquad\qquad r(\xi, t) = \frac{1}{u^*(\xi, t)},
$$

where $u^*(\cdot, t)$ denotes the support function of $\mathcal{M}_t^*$. Hence by (2.13), we obtain the following relation

$$
(4.9) \qquad\qquad \frac{u^{n+2}(x, t)(u^*(\xi, t))^{n+2}}{K(p)K^*(p^*)} = 1,
$$

where $p \in \mathcal{M}_t$, $p^* \in \mathcal{M}_t^*$ are the two points satisfying $p \cdot p^* = 1$, and x, ξ are respectively the unit outer normals of $\mathcal{M}_t$ and $\mathcal{M}_t^*$ at p and p^*. Therefore by equation (2.12) we obtain the equation for u^*,

$$
(4.10) \qquad \partial_t u^*(\xi, t) = \frac{(u^*(\xi, t))^{n+3-\alpha} f}{(r^*)^{n+1} K^*} - u^*(\xi, t), \ \xi \in \mathbb{S}^n, \ t \in (0, T],
$$

where

$$
K^* = S_n^{-1}(\nabla^2 u^* + u^* I)(\xi, t)
$$

is the Gauss curvature of $\mathcal{M}_t^*$ at the point $p^* = \nabla u^*(\xi, t) + u^*(\xi, t)\xi$, and

$$
r^* = |p^*| = \sqrt{|\nabla u^*|^2 + (u^*)^2}(\xi, t)
$$

is the distance from p^* to the origin. Note that f takes value at

$$
x = p^*/|p^*| = \frac{\nabla u^* + u^*\xi}{\sqrt{|\nabla u^*|^2 + (u^*)^2}} \in \mathbb{S}^n.
$$

By (4.8), $1/C \leq u^* \leq C$ and $|\nabla u^*| \leq C$ for some C only depending on $\max_{\mathbb{S}^n \times (0,T]} u$, $\min_{\mathbb{S}^n \times (0,T]} u$.

Let $b_{ij}^* = u_{ij}^* + u^*\delta_{ij}$, and h_*^{ij} be the inverse matrix of b_{ij}^*. As discussed in Section 2, the eigenvalues of b_{ij}^* and h_*^{ij} are respectively the principal radii and principal curvatures of $\mathcal{M}_t^*$. Consider the function

$$
(4.11) \qquad\qquad w = w(\xi, t, \tau) = \log h_*^{\tau\tau} - \beta \log u^* + \frac{A}{2}(r^*)^2,
$$

where τ is a unit vector in the tangential space of $\mathbb{S}^n$, while β and $A = A(\beta)$ are large constants to be specified later on. Assume w attains its maximum at (ξ_0, t_0), along the direction $\tau = e_1$. By a rotation, we also assume h_*^{ij} and b_{ij}^* are diagonal at this point.

It is direct to see, at the point where w attains its maximum,

$$
\begin{aligned}
0 \leq \partial_t w &= b_{11}^* \partial_t h_*^{11} - \beta \frac{u_t^*}{u^*} + Ar^* r_t^* \\
&= -h_*^{11} \partial_t b_{11}^* - \beta \frac{u_t^*}{u^*} + Ar^* r_t^*,
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
0 &= \nabla_i w = -h_*^{11} \nabla_i b_{11}^* - \beta \frac{u_i^*}{u^*} + Ar^* r_i^* \\
&= -h_*^{11} u_{i11}^* - h_*^{11} u_1^* \delta_{1i} - \beta \frac{u_i^*}{u^*} + Ar^* r_i^*,
\end{aligned}
\tag{4.13}
$$

where $u_{ijk}^* = \nabla_k u_{ij}^*$ throughout this paper. We also have

$$
\begin{aligned}
0 \geq \nabla_{ij}^2 w &= -h_*^{11} \nabla_{ij}^2 b_{11}^* + 2h_*^{11} \sum h_*^{kk} \nabla_1 b_{ik}^* \nabla_1 b_{kj}^* \\
&\quad -(h_*^{11})^2 \nabla_i b_{11}^* \nabla_j b_{11}^* - \beta \frac{u_{ij}^*}{u^*} + \beta \frac{u_i^* u_j^*}{(u^*)^2} \\
&\quad +A(r^* r_{ij}^* + r_i^* r_j^*),
\end{aligned}
\tag{4.14}
$$

where the first inequality means that $\nabla_{ij} w$ is a negative-semidefinite matrix. Note that $\nabla_k b_{ij}^*$ is symmetric in all indices.

The equation (4.10) can be written as

$$
\log(u_t^* + u^*) - \log S_n = \log \left(\frac{(u^*)^{n+3-\alpha}}{(r^*)^{n+1}} f \right) =: \psi(\xi, u^*, \nabla u^*).
\tag{4.15}
$$

Differentiating (4.15) gives

$$
\begin{aligned}
\frac{u_{tk}^* + u_k^*}{u_t^* + u^*} &= \sum h_*^{ij} \nabla_k b_{ij}^* + \nabla_k \psi \\
&= \sum h_*^{ij} u_{kij}^* + \sum h_*^{ij} u_j^* \delta_{ik} + \nabla_k \psi,
\end{aligned}
\tag{4.16}
$$

and

$$
\frac{u_{t11}^* + u_{11}^*}{u_t^* + u^*} - \frac{(u_{t1}^* + u_1^*)^2}{(u_t^* + u^*)^2} = \sum h_*^{ij} \nabla_{11}^2 b_{ij}^* - \sum h_*^{ii} h_*^{jj} (\nabla_1 b_{ij}^*)^2 + \nabla_{11}^2 \psi.
\tag{4.17}
$$

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Dividing (4.12) by $u_t^* + u^*$ and using (4.17), we have

$$
\begin{aligned}
0 \;\leq\;& -h_*^{11}\left(\frac{u_{11t}^* + u_{11}^*}{u_t^* + u^*} - \frac{b_{11}^*}{u_t^* + u^*} + 1\right) - \frac{\beta u_t^*}{u^*(u_t^* + u^*)} + \frac{Ar^*r_t^*}{u_t^* + u^*} \\
\;=\;& -h_*^{11}\frac{u_{11t}^* + u_{11}^*}{u_t^* + u^*} - h_*^{11} + \frac{1+\beta}{u_t^* + u^*} - \frac{\beta}{u^*} + \frac{Ar^*r_t^*}{u_t^* + u^*}
\end{aligned}
$$

$$
\begin{aligned}
\text{(4.18)} \qquad \;\leq\;& -h_*^{11}\sum h_*^{ij}\nabla_{11}^2 b_{ij}^* + h_*^{11}\sum h_*^{ii}h_*^{jj}(\nabla_1 b_{ij}^*)^2 \\
& -h_*^{11}\nabla_{11}^2\psi + \frac{1+\beta}{u_t^* + u^*} + \frac{Ar^*r_t^*}{u_t^* + u^*}.
\end{aligned}
$$

By the Ricci identity, we have

$$
\nabla_{11}^2 b_{ij}^* = \nabla_{ij}^2 b_{11}^* - \delta_{ij}b_{11}^* + \delta_{11}b_{ij}^* - \delta_{i1}b_{1j}^* + \delta_{1j}b_{1i}^*.
$$

Plugging this identity in (4.18) and employing (4.14), we obtain

$$
\begin{aligned}
0 \;\leq\;& h_*^{11}\sum\left(h_*^{11}h_*^{ii}(\nabla_i b_{11}^*)^2 - h_*^{ii}h_*^{jj}(\nabla_1 b_{ij}^*)^2\right) + (H^* - nh_*^{11}) \\
& -\beta H^* + C\beta - \beta\sum h_*^{ij}\frac{u_i^*u_j^*}{(u^*)^2} - h_*^{11}\nabla_{11}^2\psi \\
& +\frac{1+\beta}{u_t^* + u^*} + \frac{Ar^*r_t^*}{u_t^* + u^*} - A\sum h_*^{ij}(r^*r_{ij}^* + r_i^*r_j^*)
\end{aligned}
$$

$$
\begin{aligned}
\text{(4.19)} \qquad \;\leq\;& -\beta H^* + C\beta - h_*^{11}\nabla_{11}^2\psi \\
& +\frac{1+\beta}{u_t^* + u^*} + \frac{Ar^*r_t^*}{u_t^* + u^*} - A\sum h_*^{ij}(r^*r_{ij}^* + r_i^*r_j^*),
\end{aligned}
$$

where $H^* = \sum h_*^{ii}$ is the mean curvature of $\mathcal{M}_t^*$.

It is direct to calculate

$$
r_t^* = \frac{u^*u_t^* + \sum u_k^*u_{kt}^*}{r^*},
$$

$$
\text{(4.20)} \qquad r_i^* = \frac{u^*u_i^* + \sum u_k^*u_{ki}^*}{r^*} = \frac{u_i^*b_{ii}^*}{r^*},
$$

and

$$
r_{ij}^* = \frac{u^*u_{ij}^* + u_i^*u_j^* + \sum u_k^*u_{kij}^* + \sum u_{ki}^*u_{kj}^*}{r^*} - \frac{u_i^*u_j^*b_{ii}^*b_{jj}^*}{(r^*)^3}.
$$

Hence, by (4.16),

$$
\begin{aligned}
\frac{r^*r_t^*}{u_t^* + u^*} - \sum h_*^{ij}(r^*r_{ij}^* + r_i^*r_j^*) \;=\;& \frac{u^*u_t^*}{u_t^* + u^*} - u^*\sum h_*^{ij}u_{ij}^* \\
& -\sum h_*^{ii}(u_{ii}^*)^2 - \frac{|\nabla u^*|^2}{u_t^* + u^*} + \sum u_k^*\nabla_k\psi.
\end{aligned}
$$

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Since

$$\frac{u^* u_t^*}{u_t^* + u^*} - \frac{|\nabla u^*|^2}{u_t^* + u^*} = u^* - \frac{(r^*)^2}{u_t^* + u^*},$$

and

$$
\begin{aligned}
-u^* \sum h_*^{ij} u_{ij}^* - \sum h_*^{ii}(u_{ii}^*)^2 &= -u^* \sum h_*^{ii}(b_{ii}^* - u^* \delta_{ii}) - \sum h_*^{ii}(b_{ii}^* - u \delta_{ii})^2 \\
&= nu^* - \sum b_{ii}^*,
\end{aligned}
$$

we further deduce

(4.21) $$\frac{r^* r_t^*}{u_t^* + u^*} - \sum h_*^{ij}(r^* r_{ij}^* + r_i^* r_j^*) \leq C - \frac{(r^*)^2}{u_t^* + u^*} + \sum u_k^* \nabla_k \psi.$$

Plugging (4.21) in (4.19), we get

$$
\begin{aligned}
0 &\leq -\beta H^* + C\beta + CA - h_*^{11} \nabla_{11}^2 \psi + \frac{1 + \beta - A(r^*)^2}{u_t^* + u^*} + A \sum u_k^* \nabla_k \psi \\
\text{(4.22)} \quad &\leq -\beta H^* + C\beta + CA - h_*^{11} \nabla_{11}^2 \psi + A \sum u_k^* \nabla_k \psi,
\end{aligned}
$$

provided $A > 2(1 + \beta)/\min_{\mathbb{S}^n \times (0,T]}(r^*)^2 \geq C(1 + \beta)$, for some $C > 0$ only depending on $\max_{\mathbb{S}^n \times (0,T]} u$.

By (4.13) and (4.20), we have

$$
\begin{aligned}
-h_*^{11} \nabla_{11}^2 \psi + A \sum u_k^* \nabla_k \psi &\leq C h_*^{11}(1 + (u_{11}^*)^2) + CA \\
&\quad - h_*^{11} \sum \psi_{u_k^*} u_{k11}^* + A \sum \psi_{u_k^*} u_k^* u_{kk}^* \\
&\leq C h_*^{11} + C/h_*^{11} + CA + C\beta.
\end{aligned}
$$

Hence (4.22) can be further estimated as

$$
\begin{aligned}
0 &\leq -\beta H^* + C h_*^{11} + C\beta + CA \\
&\leq -\frac{1}{2}\beta h_*^{11} + C\beta + CA,
\end{aligned}
$$

by choosing β large. This inequality tells us the principal curvature of $\mathcal{M}^*$ are bounded from above, namely

$$\max_{\xi \in \mathbb{S}^n} \kappa_i^*(\xi, t) \leq C, \quad \forall\, t \in (0, T] \text{ and } i = 1, \ldots, n.$$

By Lemma 4.1 and (4.9), we have $K^*(\cdot, t) \geq 1/C$. Therefore

$$1/C \leq \kappa_i^*(\cdot, t) \leq C, \quad \forall\, t \in (0, T] \text{ and } i = 1, \ldots, n.$$

By duality, (4.7) follows. $\qquad\square$

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As a consequence of the above a priori estimates, one sees that the convexity of the hypersurface $\mathcal{M}_t$ is preserved under the flow (1.7) and the solution $X(\cdot, t)$ is uniformly convex.

By estimates (4.7), equation (2.8) is uniformly parabolic. By the L^∞-norm estimates and gradient estimates in Lemmas 3.1–3.4, one obtains the Hölder continuity of $\nabla^2 u$ and u_t by Krylov's theory [34]. Estimates for higher derivatives then follows from the standard regularity theory of uniformly parabolic equations. Hence we obtain the long time existence and regularity of solutions for the normalised flow (1.7). The uniqueness of smooth solutions to (2.8) follows from the comparison principle, see Lemma 6.1 below. We obtain the following theorem.

Theorem 4.1. *Let $\mathcal{M}_0$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$ which encloses the origin. Let f be a positive smooth function on $\mathbb{S}^n$. Then the normalised flow (1.7) has a unique smooth, uniformly convex solution $\mathcal{M}_t$ for all time, if one of the following is satisfied*

    (i) $\alpha > n + 1$;
    (ii) $\alpha = n + 1$, *and f satisfies* (1.10), (1.11);
    (iii) $\alpha < n + 1$, *and $\mathcal{M}_t$ is origin-symmetric as long as the flow exists.*

Moreover we have the a priori estimates

$$(4.23) \qquad \qquad \|u\|_{C_{x,t}^{k,m}\left(\mathbb{S}^n \times [0,\infty)\right)} \leq C_{k,m},$$

where $C_{k,m} > 0$ depends only on k, m, f, α and the geometry of $\mathcal{M}_0$.

5. Proofs of Theorems 1.1 - 1.4

In this section we prove the asymptotical convergence of solutions to the normalised flow (1.7). First we prove Theorem 1.1.

Proof of Theorem 1.1. Case i): $\alpha > n + 1$.

Let $u(\cdot, t)$ be the solution of (2.8). By our choice of ϕ_0 in (1.8), we have

$$a := \min_{\mathbb{S}^n} u(\cdot, 0) \leq 1 \leq \max_{\mathbb{S}^n} u(\cdot, 0) =: b.$$

Let us introduce two time-dependent functions

$$u_1 = [1 - (1 - a^q)e^{qt}]^{1/q},$$
$$u_2 = [1 - (1 - b^q)e^{qt}]^{1/q},$$

where $q = n + 1 - \alpha < 0$. It is easy to check that both u_1 and u_2 satisfy equation (2.8), and the spheres of radii u_1 and u_2 are solutions of (1.7). By the comparison principle,

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$u_1 \le u \le u_2$. Hence

$$(b^q - 1)e^{qt} \le u^q - 1 \le (a^q - 1)e^{qt}.$$

Thus u converges to 1 exponentially.

To obtain the exponential convergence of u to 1 in the C^k norms, we use the following interpolation inequality, see e.g. [28],

$$(5.1) \qquad \int_{\mathbb{S}^n} |\nabla^k T|^2 \le C_{m,n} \Big(\int_{\mathbb{S}^n} |\nabla^m T|^2 \Big)^{\frac{k}{m}} \Big(\int_{\mathbb{S}^n} |T|^2 \Big)^{1 - \frac{k}{m}}$$

where T is any smooth tensor field on $\mathbb{S}^n$, and k, m are any integers such that $0 \le k \le m$. Applying this to $u - 1$ and using the fact that all derivatives of u are bounded independently of t, we conclude

$$\int_{\mathbb{S}^n} |\nabla^k u|^2 \le C_{k,\gamma} e^{-\gamma t}$$

for any $\gamma \in (0, \tilde{\gamma})$ and any positive integer k, where $\tilde{\gamma} > 0$ is a constant depending only on q. By the Sobolev embedding theorem on $\mathbb{S}^n$, see [7], we have

$$\|u - 1\|_{C^l(\mathbb{S}^n)} \le C_{k,l} \Big(\int_{\mathbb{S}^n} |\nabla^k u|^2 + \int_{\mathbb{S}^n} |u - 1|^2 \Big)^{\frac{1}{2}}$$

for any $k > l + n/2$. It follows that $\|u(\cdot, t) - 1\|_{C^l(\mathbb{S}^n)} \to 0$ exponentially as $t \to \infty$ for all integers $l \ge 1$. Namely $u(\cdot, t)$ converges to 1 in C^∞ topology as $t \to \infty$.

Case ii): $\alpha = n + 1$. We first prove the following lemma.

Lemma 5.1. *There exist positive constants C and γ such that if $X(\cdot, t)$ is a solution to the normalised flow (1.7), we have the estimate*

$$(5.2) \qquad \max_{\mathbb{S}^n} \frac{|\nabla r(\cdot, t)|}{r(\cdot, t)} \le C e^{-\gamma t} \quad \forall\, t > 0,$$

where $r(\cdot, t)$ is the radial function of $X(\cdot, t)$.

Proof. Denote $w = \log r$. By (2.10) and (2.13), we have, under a local orthonormal frame,

$$g_{ij} = e^{2w}(\delta_{ij} + w_i w_j),$$
$$h_{ij} = e^w (1 + |\nabla w|^2)^{-\frac{1}{2}} (\delta_{ij} + w_i w_j - w_{ij}),$$

and

$$(5.3) \qquad K = \frac{\det h_{ij}}{\det g_{ij}} = (1 + |\nabla w|^2)^{-\frac{n+2}{2}} e^{-nw} \det a_{ij},$$

where

$$a_{ij} = \delta_{ij} + w_i w_j - w_{ij}.$$

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By (2.11), (2.12) and (5.3), it is not hard to verify that w satisfies the following PDE

(5.4)
$$w_t = -(1 + |\nabla w|^2)^{-\frac{n+1}{2}} \det a_{ij} + 1.$$

Consider the auxiliary function
$$Q = \frac{1}{2}|\nabla w|^2.$$

At the point where Q attains its spatial maximum, we have
$$0 = \nabla_k Q = \sum w_i w_{ik},$$

and $\nabla^2_{ij} Q$ is a non-positive matrix
$$0 \geq \nabla^2_{ij} Q = \sum w_k w_{kij} + \sum w_{ik} w_{kj}.$$

Denote $\varrho = (1 + |\nabla w|^2)^{-\frac{n+1}{2}}$. By differentiating (5.4), we obtain, at the point where Q achieves its spatial maximum,

$$
\begin{aligned}
\partial_t Q &= \sum w_k w_{kt} = -\det a_{ij} \sum w_k \varrho_k - \varrho \sum w_k \nabla_k \det a_{ij} \\
&= \varrho \sum S_n^{ij} \nabla_k w_{ij} w_k.
\end{aligned}
$$

By the Ricci identity, we have
$$\nabla_k w_{ij} = \nabla_j w_{ik} + \delta_{ik} w_j - \delta_{ij} w_k.$$

Hence
$$
\begin{aligned}
\partial_t Q &= \varrho \sum S_n^{ij} \left(Q_{ij} - w_{ik} w_{kj} + w_i w_j - \delta_{ij} |\nabla w|^2 \right) \\
&\leq \varrho \left(\max_i S_n^{ii} - \sum S_n^{ii} \right) |\nabla w|^2.
\end{aligned}
$$

If $n \geq 2$, we get

(5.5)
$$\partial_t Q \leq -\gamma Q,$$

for some positive constant γ, where we have used the estimates $\varrho \geq C^{-1}$ and $C^{-1} \leq \kappa(\cdot, t) \leq C$, which are established in Section 3. Estimate (5.2) follows from (5.5) immediately.

For $n = 1$, the equation (5.4) becomes quasi-linear

(5.6)
$$w_t = \frac{w_{xx}}{1 + w_x^2} \quad \text{on } \mathbb{S}^1 \times [0, \infty).$$

Let
$$\bar{w} := \frac{1}{2\pi} \int_{\mathbb{S}^1} w(x, t) dx$$

be the average of w. By the divergence theorem,
$$\frac{d}{dt} \bar{w} = \frac{1}{2\pi} \int_{\mathbb{S}^1} (\arctan(w_x))_x dx = 0.$$

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Hence $\bar{w}$ is a constant. Then it is simple to calculate

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{S}^1}(w-\bar{w})^2\right) = \int_{\mathbb{S}^1}(w-\bar{w})(\arctan w_x)_x dx$$

$$= -\int_{\mathbb{S}^1} w_x \arctan w_x dx.$$

Note that, $w_x \arctan w_x \geq \delta_0 w_x^2$ for some $\delta_0 > 0$ depending only on the upper bound of $|w_x|$. We deduce that, by the Poincaré inequality,

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{S}^1}(w-\bar{w})^2\right) \leq -\delta_0 \int_{\mathbb{S}^1} w_x^2 dx \leq -C\int_{\mathbb{S}^1}(w-\bar{w})^2.$$

This implies w exponentially converges to a constant in L^2-norm at $t \to \infty$. The exponential decay (5.2) follows by applying the interpolation inequality (5.1) to $w - \bar{w}$. $\qquad\square$

Now we prove Case ii) of Theorem 1.1. Lemma 5.1 implies $|\nabla r(\cdot, t)| \to 0$ exponentially as $t \to \infty$.

As in Case i), we infer by the interpolation inequality and the a priori estimates in Section 3, that r exponentially converges to a constant in the C^∞ topology as $t \to \infty$. Let us show that the constant must be 1.

By (2.12), we get

$$\frac{d}{dt}\left(\int_{\mathbb{S}^n}\log r(\xi, t)d\xi\right) = \int_{\mathbb{S}^n}\left(-\frac{r^{n+1}K}{u}+1\right)d\xi.$$

By (2.16),

$$\frac{d}{dt}\left(\int_{\mathbb{S}^n}\log r(\xi, t)d\xi\right) = 0.$$

Therefore by our choice of ϕ_0 in (1.8)

$$\int_{\mathbb{S}^n}\log r(\xi, t)d\xi = \int_{\mathbb{S}^n}\log r(\xi, 0)d\xi = 0.$$

This implies $r(\cdot, t) \to 1$ as $t \to \infty$. $\qquad\square$

Recall that the normalised flow (1.7) is a gradient flow of the functional $\mathcal{J}_\alpha$ (see (1.9) for the definition). We next complete the proofs of Theorem 1.2 -1.4.

Proof of Theorem 1.2. By our a priori estimates Lemmas 3.1 and 3.5, there is a constant $C > 0$, independent of t, such that

(5.7) $$|\mathcal{J}_\alpha(X(\cdot, t))| \leq C \ \ \forall \, t \in [0, \infty).$$

By Lemma 2.1, we obtain

$$\mathcal{J}_\alpha(X(\cdot,T)) - \mathcal{J}_\alpha(X(\cdot,0)) \;=\; -\int_0^T \int_{\mathbb{S}^n} \frac{(fr^\alpha K - u)^2}{ur^\alpha K} dxdt$$

$$\leq\; -\delta_0 \int_0^T \int_{\mathbb{S}^n} (fr^\alpha K - u)^2 dxdt.$$

By (5.7), the above inequality implies there exists a subsequence of times $t_j \to \infty$ such that $\mathcal{M}_{t_j}$ converges to a limiting hypersurface which satisfies (1.5).

To complete the proof of Theorem 1.2, it suffices to show that the solution of (1.5) is unique. Using (2.3) and (2.7), the equation (1.5) can be written as

(5.8) $$\frac{u}{(u^2 + |\nabla u|^2)^{\frac{\alpha}{2}}} \det(\nabla^2 u + uI) = f.$$

Let u_1 and u_2 be two smooth solutions of (5.8). Suppose $G = u_1/u_2$ attains its maximum at $x_0 \in \mathbb{S}^n$. Then at x_0,

$$0 = \nabla \log G = \frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2},$$

and $\nabla^2 \log G$ is a negative-semidefinite matrix at x_0

$$0 \;\geq\; \nabla^2 \log G$$

$$=\; \frac{\nabla^2 u_1}{u_1} - \frac{\nabla u_1 \otimes \nabla u_1}{u_1^2} - \frac{\nabla^2 u_2}{u_2} + \frac{\nabla u_2 \otimes \nabla u_2}{u_2^2}$$

$$=\; \frac{\nabla^2 u_1}{u_1} - \frac{\nabla^2 u_2}{u_2}.$$

By (5.8) we get at x_0,

$$1 \;=\; \frac{u_2^{n+1-\alpha}}{u_1^{n+1-\alpha}} \frac{(1 + |\nabla u_1|^2/u_1^2)^{\frac{\alpha}{2}}}{(1 + |\nabla u_2|/u_2^2)^{\frac{\alpha}{2}}} \frac{\det(u_2^{-1}\nabla^2 u_2 + I)}{\det(u_1^{-1}\nabla^2 u_1 + I)}$$

$$\geq\; G^{\alpha-n-1}(x_0).$$

Since $\alpha > n + 1$, $G(x_0) = \max_{\mathbb{S}^n} G \leq 1$. Similarly one can show $\min_{\mathbb{S}^n} G \geq 1$. Therefore $u_1 \equiv u_2$. $\qquad\square$

Proof of Theorem 1.3. The long time existence of the flow (1.7) follows from Theorem 4.1. As in the proof of Theorem 1.2, $\mathcal{M}_t$ converges by a subsequence to a homothetic limit. To prove the convergence of $\mathcal{M}_t$ along $t \to \infty$, it suffices to show the limiting hypersurface is unique.

First we claim that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two smooth solutions to (1.5) for $\alpha = n + 1$, then $\mathcal{M}_1$ and $\mathcal{M}_2$ differ only by a dilation. This is a well known result [1, 37], we sketch the proof in [37] for reader's convenience. Assume not, then there is a $\lambda > 0$ such that

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(i) the set $\omega := \{\xi \in \mathbb{S}^n : r_{\lambda \mathcal{M}_2}(\xi) \geq r_{\mathcal{M}_1}(\xi)\}$ is a proper subset of $\mathbb{S}^n$ with positive measure.

(ii) the set $\omega_1 := \mathscr{A}_{\mathcal{M}_1}(\omega)$ is contained in $\omega_2 := \mathscr{A}_{\lambda \mathcal{M}_2}(\omega)$, and $|\omega_2| > |\omega_1|$.

But on the other hand, by (2.16), we have

$$\int_{\omega_1} f = |\mathscr{A}^*_{\mathcal{M}_1}(\omega_1)| = |\omega|,$$

$$\int_{\omega_2} f = |\mathscr{A}^*_{\mathcal{M}_2}(\omega_2)| = |\mathscr{A}^*_{\lambda \mathcal{M}_2}(\omega_2)| = |\omega|.$$

Hence $\int_{\omega_1} f = \int_{\omega_2} f$, which is in contradiction with (ii) above.

Next we show that

(5.9)
$$\int_{\mathbb{S}^n} \log r(\xi, t) d\xi = \int_{\mathbb{S}^n} \log r(\xi, 0) d\xi = \text{const.}.$$

This formula and the above claim imply that $\mathcal{M}_t$ converges to a unique limit.

To prove (5.9), dividing (2.12) by r and integrating over $\mathbb{S}^n$, we obtain, by (2.11),

$$\frac{d}{dt}\left(\int_{\mathbb{S}^n} \log r(\xi, t) d\xi\right) = -\int_{\mathbb{S}^n} f(x) \frac{r^{n+1} K}{u} d\xi + o_n.$$

By the variable change (2.16) and using (1.10), we have

$$\frac{d}{dt}\left(\int_{\mathbb{S}^n} \log r(\xi, t) d\xi\right) = -\int_{\mathbb{S}^n} f(x) dx + o_n = 0.$$

Hence we obtain (5.9). □

Proof of Theorem 1.4. Since f is even and $\mathcal{M}_0$ is origin-symmetric, the solution remains origin-symmetric for $t > 0$. The long time existence of the flow (1.7) now follows from Theorem 4.1. As in the proof of Theorem 1.2, $\mathcal{M}_t$ converges by a subsequence to a homothetic limit. We conclude that $\mathcal{M}_t$ converges in C^∞-topology to a smooth solution of (1.5) as $t \to \infty$ by using the argument of [3, 27]. A tractable proof for this was presented in Section 4 of [27].

It remains to show, when $f \equiv 1$ and $\alpha \geq 0$, the only origin-symmetric solitons are spheres. By (1.5), a soliton to the flow (2.8) satisfies

(5.10)
$$uS_n = (u^2 + |\nabla u|^2)^{\frac{\alpha}{2}} \geq u^\alpha.$$

While using (4.9), the polar body of our soliton satisfies

(5.11)
$$u^* S_n^* = \left(\frac{((u^*)^2 + |\nabla u^*|^2)^{\frac{1}{2}}}{u^*}\right)^{n+1} (u^*)^\alpha \geq (u^*)^\alpha.$$

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Let us denote by Ω and Ω^* the convex bodies whose support functions are u and u^* respectively. Integrating (5.10) and (5.11) over $\mathbb{S}^n$ and then multiplying yield

$$\mathrm{Vol}(\Omega)\mathrm{Vol}(\Omega^*) \ \geq \ \frac{1}{(n+1)^2}\Big(\int_{\mathbb{S}^n} u^\alpha\Big)\Big(\int_{\mathbb{S}^n} (u^*)^\alpha\Big)$$

(5.12)
$$\geq \ \frac{1}{(n+1)^2}\Big(\int_{\mathbb{S}^n} (uu^*)^{\frac{\alpha}{2}}\Big)^2.$$

Note that $uu^* = \frac{1}{rr^*}$, and by definition the polar dual,

$$0 < r(\xi)r^*(\xi) = \langle r(\xi)\xi, r^*(\xi)\xi\rangle \leq 1.$$

Hence $uu^* \geq 1$. It then follows from (5.12) that

(5.13) $$\mathrm{Vol}(\Omega)\mathrm{Vol}(\Omega^*) \geq \mathrm{Vol}^2(B_1),$$

where B_1 denotes the unit ball in $\mathbb{R}^{n+1}$. The Blaschke-Stanló inequality tells us

$$\mathrm{Vol}(\Omega)\mathrm{Vol}(\Omega^*) \leq \mathrm{Vol}^2(B_1).$$

Therefore by the characterisation of equality cases, Ω must be an ellipsoid. By (5.10) and (5.11), we infer that $\Omega = B_1$, otherwise the inequality in (5.13) would become strict, which is not possible.

$\square$

6. Proof of Theorem 1.5

In this section we show that if $\alpha < n+1$ the flow (1.1) may have unbounded ratio of radii, namely

(6.1) $$\mathcal{R}(X(\cdot,t)) = \frac{\max_{\mathbb{S}^n} r(\cdot,t)}{\min_{\mathbb{S}^n} r(\cdot,t)} \to \infty \ \text{ as } \ t \to T$$

for some $T > 0$. To prove (6.1), we show that $\min_{\mathbb{S}^n} r(\cdot,t) \to 0$ in finite time while $\max_{\mathbb{S}^n} r(\cdot,t)$ remains positive. In contrast, it is worth mentioning that in [40], the author obtained an a priori bound for the ratio $\max_{\mathbb{S}^n} r / \min_{\mathbb{S}^n} r$ if r is the radial function of the solution to the Aleksandrov problem.

Let $X(\cdot,t)$ be a convex solution to (1.1). Then its support function u satisfies the equation

(6.2) $$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = -fr^\alpha S_n^{-1}(\nabla_{ij}^2 u + u\delta_{ij})(x,t), \\ u(\cdot,0) = u_0. \end{cases}$$

Given a smooth, closed, uniformly convex hypersurface $\mathcal{M}_0$, our a priori estimates in Section 3 imply the existence of a smooth, closed, uniformly convex solution to the flow

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(1.1) for small $t > 0$. The solution remains smooth until either the solution shrinks to the origin, or (6.1) occurs at some time $T > 0$.

Definition 6.1. *A time dependent family of convex hypersurfaces $Y(\cdot, t)$ is a sub-solution to* (6.2) *if its support function w satisfies*

(6.3)
$$\begin{cases} \dfrac{\partial w}{\partial t}(x, t) \geq -fr^{\alpha}S_n^{-1}(\nabla_{ij}^2 w + w\delta_{ij})(x, t), \\ w(\cdot, 0) \geq u_0, \end{cases}$$

where r is the radial function of the associated hypersurface.

By definition, the hypersurface $\mathcal{M}_0$ (independent of t), whose support function is u_0, is a sub-solution to (6.2). We will use the following comparison principle.

Lemma 6.1. *Let $X(\cdot, t)$ be a solution to* (1.1) *and $Y(\cdot, t)$ a sub-solution. Suppose $X(\cdot, 0)$ is contained in the interior $Y(\cdot, 0)$. Then $X(\cdot, t)$ is contained in the interior $Y(\cdot, t)$ for all $t > 0$, as long as the solutions exist.*

Proof. Let $u(\cdot, t)$ and $w(\cdot, t)$ be the support functions of $X(\cdot, t)$ and $Y(\cdot, t)$. Then u and w satisfy (6.2) and (6.3) respectively with $u(x, 0) \leq w(x, 0)$ for all $x \in \mathbb{S}^n$. For $\lambda > 0$, let us denote $u^{\lambda}(x, t) = \lambda u(x, \lambda^{\beta} t)$, where $\beta = \alpha - n - 1$. It is easily seen that u^{λ} solves (6.2) with $u^{\lambda}(\cdot, 0) = \lambda u_0$. Let $\lambda < 1$. Then $u^{\lambda}(\cdot, 0) < w(\cdot, 0)$. By the comparison principle for parabolic equation,

(6.4) $$u^{\lambda}(x, t) < w(x, t), \forall\ x \in \mathbb{S}^n \text{ and } t > 0,$$

as long as the solutions exist. Sending $\lambda \to 1$, we obtain $u(x, t) \leq w(x, t)$. $\qquad\square$

Note that in Lemma 6.1, we do not require that $Y(\cdot, t)$ is shrinking. Moreover, it suffices to assume that $Y(\cdot, t)$ is a sub-solution in the viscosity sense. In particular Lemma 6.1 applies if $Y(\cdot, t)$ is $C^{1,1}$ smooth.

To prove Theorem 1.5, by the comparison principle (Lemma 6.1), it suffices to construct a sub-solution $Y(\cdot, t)$ such that $\min_{\mathbb{S}^n} w(\cdot, t) \to 0$ but $\max_{\mathbb{S}^n} w(\cdot, t)$ remains positive, as $t \to T$ for some finite time $T > 0$. By a translation of time, we show below that there is a sub-solution $Y(\cdot, t)$ for $t \in (-1, 0)$ such that (6.1) holds as $t \nearrow 0$.

Lemma 6.2. *For any given a positive function f, there is a sub-solution $Y(\cdot, t)$, where $t \in (-1, 0)$, to*

(6.5)
$$\begin{cases} \dfrac{\partial u}{\partial t}(x, t) = -afr^{\alpha}S_n^{-1}(\nabla_{ij}^2 u + u\delta_{ij})(x, t), \\ u(\cdot, 0) = u_0. \end{cases}$$

for a sufficiently large constant $a > 0$, such that $\min_{\mathbb{S}^n} w(\cdot, t) \to 0$ but $\max_{\mathbb{S}^n} w(\cdot, t)$ remains positive, as $t \nearrow 0$.

Proof. The sub-solution we constructed is a family of closed convex hypersurfaces $\widehat{\mathcal{M}}_t := Y(\mathbb{S}^n, t)$. First note that it suffices to prove Lemma 6.2 for $q = n + 1 - \alpha > 0$ is small. Indeed, if $Y(\mathbb{S}^n, t)$ is a sub-solution to (6.5) for some α, it is also a sub-solution to (6.5) for $\alpha' < \alpha$, provided we replace a by $a \sup\{|p|^{\alpha - \alpha'}; \ p \in \widehat{\mathcal{M}}_t, t \in (-1, 0)\}$.

Near the origin, let $\widehat{\mathcal{M}}_t$ be the graph of a function on $\mathbb{R}^n$, $\phi(\rho, t)$ $(\rho = |x|)$, given by

$$(6.6) \qquad \phi(\rho, t) = \begin{cases} -|t|^\theta + |t|^{-\theta + \sigma\theta} \rho^2, & \text{if } \rho < |t|^\theta, \\ -|t|^\theta - \dfrac{1-\sigma}{1+\sigma}|t|^{\theta(1+\sigma)} + \dfrac{2}{1+\sigma}\rho^{1+\sigma}, & \text{if } |t|^\theta \le \rho \le 1, \end{cases}$$

where $\sigma = \frac{q\theta - 1}{n\theta}$ and $\theta > \frac{1}{q}$ is a constant. It is easy to verify that ϕ is strictly convex, and $\phi \in C^{1,1}(B_1(0))$.

By direct computation, we have,

(i) if $0 \le \rho \le |t|^\theta$, then

$$r^\alpha K \ge |t|^{\alpha\theta} |t|^{n\theta(\sigma - 1)} = |t|^{\theta - 1},$$

$$(6.7) \qquad |\frac{\partial}{\partial t} Y(p, t)| \le \theta |t|^{\theta - 1}.$$

where $p = (x, \phi(|x|, t))$ is a point on the graph of ϕ and K is the Gauss curvature of the graph of ϕ at p.

(ii) if $|t|^\theta \le \rho \le 1$, then

$$r^\alpha K \ge \rho^\alpha K \ge C\rho^\alpha \rho^{(\sigma - 1)n} \ge C\rho^{1 - \frac{1}{\theta}} \ge C|t|^{\theta - 1},$$

$$(6.8) \qquad |\frac{\partial}{\partial t} Y(p, t)| \le \theta |t|^{\theta - 1}.$$

Hence the graph of $\phi(\cdot, t)$ is a sub-solution to (6.5), provided a is sufficiently large.

Next we extend the graph of ϕ to a closed convex hypersurface $\widehat{\mathcal{M}}_t$, such that it is $C^{1,1}$ smooth, uniformly convex, rotationally symmetric, and depends smoothly on t. Moreover we may assume that the ball $B_1(z)$ is contained in the interior of $\widehat{\mathcal{M}}_t$, for all $t \in (-1, 0)$, where $z = (0, \cdots, 0, 10)$ is a point on the x_{n+1}-axis. Then $\widehat{\mathcal{M}}_t$ is a sub-solution to (6.5), for sufficiently large a. $\qquad\square$

We are in position to prove Theorem 1.5. For a given $\tau \in (-1, 0)$, let $\mathcal{M}_0$ be a smooth, closed, uniformly convex hypersurface inside $\widehat{\mathcal{M}}_\tau$ and enclosing the ball $B_1(z)$. Let $\mathcal{M}_t$ be the solution to the flow (6.5) with initial data $\mathcal{M}_0$. By Lemma 6.1, $\mathcal{M}_t$ touches the origin at $t = t_0$, for some $t_0 \in (\tau, 0)$. We choose τ very close to 0, so that t_0 is sufficiently small.

On the other hand, let $\tilde{X}(\cdot, t)$ be the solution to

$$(6.9) \qquad \frac{\partial X}{\partial t} = -\beta \tilde{f} \tilde{r}^\alpha K \nu,$$

with initial condition $\tilde{X}(\cdot, \tau) = \partial B_1(z)$, where $\beta = 2^\alpha \sup\{|p|^\alpha : p \in \mathcal{M}_t, \tau < t < t_0\}$, $\tilde{f} = a \max_{\mathbb{S}^n} f$, and $\tilde{r} = |X - z|$ is the distance from z to X. We can choose τ close enough to 0 that the ball $B_{1/2}(z)$ is contained in the interior of $\tilde{X}(\cdot, t)$ for all $t \in (\tau, t_0)$. Since $\mathcal{M}_t$ is a sub-solution to (6.9), by the comparison principle, we see that the ball $B_{1/2}(z)$ is contained in the interior of $\mathcal{M}_t$ for all $t \in (\tau, t_0)$. Hence as $t \nearrow t_0$, we have $\min r(\cdot, t) \to 0$ and $\max r(\cdot, t) > |z| = 10$. Hence (6.1) is proved for $\mathcal{M}_t$.

We have proved Theorem 1.5 when f is replaced by af, for large constant $a > 0$. Making the rescaling $\widetilde{\mathcal{M}}_t = a^{-1/q} \mathcal{M}_t$, one easily verifies that $\widetilde{\mathcal{M}}_t$ solves the flow (1.1) for the function f. Theorem 1.5 is proved.

Finally we point out that if f does not satisfy (1.11), then (1.19) holds for $\alpha = n+1$. Indeed, assume to the contrary that the ratio $\frac{\max_{\mathbb{S}^n} r(\cdot, t)}{\min_{\mathbb{S}^n} r(\cdot, t)}$ is uniformly bounded, then by (5.9), the radial function $r(\cdot, t)$ is uniformly bounded from both above and below. Hence by the a priori estimates (Lemmas 4.1 and 4.2), the flow converges smoothly to a limit which solves (1.5). It means the Aleksandrov problem has a smooth solution without condition (1.11). But this is impossible as (1.11) is necessary for the solvability of the Aleksandrov problem.

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