INFINITELY MANY SOLUTIONS FOR CENTRO-AFFINE MINKOWSKI PROBLEM

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ABSTRACT. We study the multiplicity result for the centro-affine Minkowski problem. It is well-known that all ellipsoids with constant volume have the same centro-affine curvature. In this paper, we construct a positive, Hölder continuous function \( f \in C^\alpha(\mathbb{S}^n) \) such that there are infinitely many \( C^{2,\alpha} \) hypersurfaces which are not affine-equivalent, but have the same centro-affine curvature \( 1/f \).

1. Introduction

The \( L_p \)-Minkowski problem is to find a convex body \( K \) in \( \mathbb{R}^{n+1} \) with the prescribed \( p \)-area measure \([19]\). It is closely related to the self-similar solution of Gauss curvature flows \([3, 4, 23, 26]\), and has attracted much attention over the last two decades \([8, 13, 14, 15, 16, 20, 21, 22, 25]\). When \( p = -n - 1 \), the problem is equivalent to the solvability of a Monge-Ampère type equation

\[
\det(\nabla^2 H + HI) = \frac{f}{H^{n+2}} \quad \text{on } \mathbb{S}^n,
\]

where \( H \in C^0(\mathbb{S}^n) \) is the support function of \( K \), and \( \nabla^2 H \) denotes the second covariant derivative with respect to an orthonormal frame on \( \mathbb{S}^n \). This problem is also called the centro-affine Minkowski problem \([14]\), as each solution of (1.1) determines a hypersurface whose centro-affine curvature is equal to \( 1/f \).

The problem corresponds to the critical case of the Blaschke-Santaló inequality,

\[
\sup_K \inf_{\xi \in K} \frac{\text{Vol}(K)}{\text{Vol}(B_1)^2} \int_{\mathbb{S}^n} \frac{dx}{(H(x) - x \cdot \xi)^{n+1}} \leq (n + 1)^2.
\]

The equality holds if and only if \( K \) is an ellipsoid. This means that all ellipsoids centred at the origin of the unit ball volume solve the centro-affine Minkowski problem for \( f = 1 \). In fact it has been known for a long time that the solutions of (1.1) with \( f = 1 \) are all such ellipsoids \([9, 17]\). If \( f \) is not a constant, the centro-affine Minkowski problem is

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rather complicated. An obstruction condition was found in [14], and the existence was obtained in [21, 22] under symmetry assumptions.

The centro-affine Minkowski problem can be compared with the problem of prescribing scalar curvature on the sphere, which involves critical exponent of the Sobolev inequality. Let $(S^n, g_0)$, $n \geq 3$, be a Riemannian manifold, where $g_0$ is not necessarily the standard spherical metric $g_{S^n}$. The prescribing scalar curvature problem asks if one can find a conformal metric $g = u^{4/n-2}g_0$, $u > 0$ on $S^n$, such that its scalar curvature $R_g$ equals to a given function $f$. It is equivalent to solving the semi-linear equation

$$(1.2) \quad -\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u = f(x) u^{\frac{n+2}{n-2}} \quad \text{on } S^n.\quad \tag{1.2}$$

This problem has been studied extensively, see [2, 10, 12, 18, 24] and the references therein. Solutions of (1.2) are not unique in general. One case is rather trivial: if $g_0 = g_{S^n}$ and $f = 1$, then equation (1.2) is invariant under the action of the conformal group on $S^n$, and therefore has infinitely many solutions. It is natural to ask if the multiplicity result is still true when $g_0 \neq g_{S^n}$, or $f \neq \text{const}$. In [1, 5] the authors constructed a metric $g_0$ (not necessarily smooth), which is a small perturbation of $g_{S^n}$, such that (1.2), with $f = 1$, admits an infinite number of solutions with unbounded $L^\infty$-norm. This result was further extended to the smooth category for $n \geq 25$ [6, 7], which implies the set of solutions to the Yamabe problem can fail to be compact in such dimensions. On the other hand, in [2, 11, 27], the authors constructed non-constant function $f$ such that (1.2) with $g_0 = g_{S^n}$ has many distinct solutions.

The purpose of this paper is to answer an analogue question for the centro-affine Minkowski problem: whether equation (1.1) has infinitely many solutions which are not affine-equivalent when $f$ is not a constant; or equivalently, if there are infinitely many distinct hypersurfaces, not affine-related, with the same centro-affine curvature.

**Theorem 1.1.** There is a positive function $f \in C^\alpha(S^n)$ such that (1.1) admits infinitely many $C^{2,\alpha}$ solutions which are not affine-equivalent.

Let $H$ be a solution to (1.1) and $K$ be the associated convex body in $\mathbb{R}^{n+1}$. After making a unimodular linear transformation $A \in SL(n+1)$, the convex body $K$ is changed to $K_A$ with support $H_A$. We have, see e.g. [14, 21],

$$(1.3) \quad H_A(x) = |Ax| H\left(\frac{Ax}{|Ax|}\right), \quad x \in S^n,\quad \tag{1.3}$$
and $H_A$ solves the equation

$$\det(\nabla^2 H_A + H_A I) = \frac{f_A}{H_A^{n+2}}, \quad f_A(x) = f\left(\frac{Ax}{|Ax|}\right).$$

We say that two solutions of (1.1), $H_1$ and $H_2$, are affine-equivalent, if there is a unimodular linear transformation $A \in SL(n+1)$ such that $H_2 = (H_1)_A$. When $f = \text{const}$, we know that the solutions of (1.1) are ellipsoids with a constant volume, hence must be affine-equivalent. This means that the solutions are in fact unique up to the volume-preserving affine transformations. However, our Theorem 1.1 shows that, when $f$ is not a constant, the situation becomes quite different, as the multiple solutions that we will construct are not affine-equivalent.

As stated above, Theorem 1.1 is, in some aspects, paralleled to the multiplicity result of the semi-linear equation with critical exponent on $S^n$ [1, 2, 5, 6, 7, 11, 27]. In our paper [16], infinitely many $C^{1,1}$ non-ellipsoidal solutions have been constructed. The contribution here is to refine our previous analysis and get more regular solutions. The existence of infinite number of solutions here is obtained by a careful comparison between the solutions of (1.1) for non-constant $f$ and the ellipsoidal solutions for $f = \text{const}$. The technique in this paper is different from [16], and is more involved.

The organisation of this paper is as follows. We consider the rotationally symmetric solutions of (1.1). By Legendre transform it is equivalent to solving a nonlinear ordinary differential equation coupled with an asymptotic condition at infinity. In Section 2 we prove an auxiliary proposition, which describes a relation between $f$ and the asymptotic behaviour of the ODE. We then complete the proof of Theorem 1.1 in Section 3.

2. Auxiliary proposition

We consider the symmetric solutions for (1.1). Let $K$ be a convex body which is rotationally symmetric with respect to the $x_{n+1}$-axis in $\mathbb{R}^{n+1}$. Assume that $K$ is also symmetric with respect to the plane $\{x_{n+1} = 0\}$. Namely $(x, x_{n+1}) \in K$ if and only if $(x, -x_{n+1}) \in K$, where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Then there exists a one-dimensional convex function $v = v(\rho)$ such that

$$\partial K \cap \{x_{n+1} \leq 0\} = \{(x, v(|x|)) : |x| \leq r\}. $$
Suppose that $K$ is a smooth solution to (1.1). Then $v$ satisfies

$$
\begin{aligned}
\begin{cases}
  v''(v')^{n-1}f = \rho^{n-1}(\rho v' - v)^{n+2}, \\
v'(0) = 0, v(0) = -h, \\
v'(r) = +\infty,
\end{cases}
\end{aligned}
$$

(2.1)

where $h$ is a positive constant. Indeed it is not hard to see

$$
(2.2)
\text{Gauss curvature of } K = \frac{v''v^{n-1}}{\rho^{n-1}(1 + v'^2)^{n+2}},
$$

and the support function of $K$ can be calculated by

$$
(2.3)
H = \frac{\rho v' - v}{\sqrt{1 + v'^2}}.
$$

Since the eigenvalues of $\{\nabla^2 H + HI\}$ are principal radii of $K$, the ODE in (2.1) is derived from (1.1), (2.2) and (2.3). Conditions $v'(0) = 0$ and $v'(r) = +\infty$ follow by the symmetry of $K$. Note that $f$ in (2.1) is computed at the unit outer normal of $K$

$$
\frac{(v', 0, \cdots, 0, -1)}{\sqrt{1 + v'^2}}.
$$

We shall simply write $f = f(v')$ in the sequel if no confusion arises.

Let $u(s)$ be the Legendre transform of $v(\rho)$, namely

$$
u(s) = \sup_{\rho \in [0,r]} \{s\rho - v(\rho)\}.
$$

It is not hard to see the supremum above is achieved at $\rho$ such that $s = v'(\rho)$, and $u''(s) = 1/v''(\rho)$. By (2.1), $u$ solves the ODE

$$
u''(u')^{n-1}u^{n+2} = s^{n-1}f(s), \text{ for } s \geq 0,
$$

(2.4)

and satisfies the initial value condition

$$
u'(0) = 0 \text{ and } u(0) = h > 0,
$$

(2.5)

and the asymptotic condition

$$
|u(s) - rs| \to 0 \text{ as } s \to \infty.
$$

(2.6)

Equation (2.4) is invariant under the rescaling

$$
s = h^{\frac{n+1}{n}}t \quad \text{and} \quad w(t) = h^{-1}u(h^{\frac{n+1}{n}}t).
$$
Namely $w$ solves the rescaled problem

\[
\begin{cases}
  w''(w')^{n-1}w^{n+2} = t^{n-1}g(t), & \text{for } t \geq 0, \\
  w'(0) = 0, w(0) = 1,
\end{cases}
\]  

where $g(t) = f(h^{\frac{n+1}{n}}t)$.

Recall that the centro-affine Gauss curvature of the ellipsoids of the volume $c_0^{\frac{1}{n}}\Vol(B_1)$ equals to $1/c_0$. Hence for each $h > 0$,

\[v(\rho) = -h\sqrt{1 - c_0^{-\frac{1}{n}}h^{\frac{2}{n}}\rho^2}\]

solves (2.1) for $f = c_0$. By Legendre transform and the rescaling, it is not hard to see

\[w(t) = \sqrt{1 + c_0^{\frac{1}{n}}t^2}\]

is the solution to the initial value problem (2.7) with $g = c_0$. Note that

\[w(t) \to c_0^{\frac{1}{n}}t \text{ as } t \to \infty.\]

Namely the asymptotic line of $w$ always passes through the origin whenever $g$ is a constant. Keeping this observation in mind, we prove the following proposition.

**Proposition 2.1.** Suppose that $w$ is the solution to (2.7). There is a universal constant $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, letting $\delta = \varepsilon^2$, $R = \varepsilon^{-1}$ and $T = \varepsilon^{-3}$, we have

(i) if $1 \leq g(t) \leq 1 + \varepsilon$ on $[0, T]$,

\[g(t) = \begin{cases} 
  1 + \varepsilon, & t \in [\delta, R] \\
  1 & t \in [R + \delta, T],
\end{cases}\]

then $|w(t) - \ell_\infty(t)| \to 0$ as $t \to \infty$ for an affine function $\ell_\infty$ satisfying $\ell_\infty(0) > 0$;

(ii) if $1 \leq g(t) \leq 1 + \varepsilon$ on $[0, T]$,

\[g(t) = \begin{cases} 
  1 & t \in [\delta, R] \\
  1 + \varepsilon, & t \in [R + \delta, T],
\end{cases}\]

then $|w(t) - \ell_\infty(t)| \to 0$ as $t \to \infty$ for an affine function $\ell_\infty$ satisfying $\ell_\infty(0) < 0$.

Note that $g$ can take any value between 1 and $1 + \varepsilon_0$ for $t \in (T, \infty)$.
This proposition plays a key role in the construction of multiple solutions to (2.4)-(2.6) as will be shown in next section. The proof follows by carefully comparing the solution of (2.7) with the function in the form of \((1+c_0^{1/n}t^2)^{1/2}\) for some suitable \(c_0\), and therefore is different from the analysis in [16].

We first prove the following lemma.

**Lemma 2.2.** Assume that \(c_1 \leq g \leq c_2\) for two positive constants \(c_1, c_2\). Let \(w(t)\) be the solution of the initial value problem (2.7). Then there exists a constant \(C > 0\), only depending on \(c_1, c_2\) and \(n\), such that

\[
 w'(t) \leq C, \quad \forall \ t > 0
\]

and

\[
 C^{-1}t - C \leq w(t) \leq Ct + 1 \quad \forall \ t > 0.
\]

**Proof.** It follows from (2.7) that

\[
 (w^n)'(t) = nt^{n-1}w^{n+2}g(t).
\]

This implies

\[
 w^n \left( \frac{1}{2} \right) \geq nc_1 \int_0^{\frac{1}{2}} t^{n-1} \frac{dt}{w^{n+2}} \geq \frac{c_1}{w^{n+2}(1)} \left( \frac{1}{2} \right)^n.
\]

By convexity we obtain

\[
 w(1) \geq \frac{1}{2} w'(1) + w(\frac{1}{2}) \geq \frac{1}{4} c_1^{\frac{1}{n}} w^{-\frac{n+2}{n}}(1) + 1.
\]

This implies

\[
 w(1) - 1 \geq \delta_0
\]

where \(\delta_0 > 0\) only depends on \(n\) and \(c_1\). Since \(w(t) \geq (w(1) - 1)t\) for \(t \geq 1\), we have, by (2.9) and (2.10),

\[
 w^n(t) \leq nc_2 \left( \int_0^1 t^{n-1}dt + \frac{1}{(w(1) - 1)^{n+2}} \int_1^\infty t^{-3}dt \right) \leq C, \quad \forall \ t \geq 0.
\]
Let \( b(t) = w(t) - tw'(t) \). Then \( b'(t) = -tw''(t) \). Hence by (2.7)

\[
|b(t)| \leq |b(1)| + \int_1^t |b'(s)|ds \leq w(1) + w'(1) + c_2 \int_1^t \frac{s^n}{(w'(s))^{n-1}w^{n+2}}ds \leq 1 + 2w'(1) + \frac{c_2}{(w(1) - 1)^{2n+1}} \int_1^t s^{-2}ds,
\]

where \( w'(1) \geq w(1) - 1 \) is used in the last inequality. By (2.10) and (2.11), one gets

\[
|b(t)| \leq C, \quad \forall t \geq 0.
\]

This completes the proof.

\[\Box\]

For any \( t_0 > 0 \), let \( \ell_{t_0}(t) \) be the support function of \( w(t) \) at \( t = t_0 \). Lemma 2.2 says \( \ell'_{t_0} \) and \( \ell_{t_0}(0) \) are uniformly bounded. Hence \( \ell_{t_0}(t) \) converges to an affine function \( \ell_\infty(t) \) as \( t_0 \to \infty \), which is the asymptotic line of \( w(t) \). In fact, it has been shown in [16] that the initial value problem (2.7) has a unique strictly convex solution \( w(t) \), and there is an affine function \( \ell_\infty \) such that \( \lim_{t \to \infty} |w(t) - \ell_\infty(t)| = 0 \). To prove Proposition 2.1 it remains to determine the sign of \( \ell_\infty(0) \).

**Proof of the statement (i) of Proposition 2.1** Let us set

\[
(2.12) \quad b(t) = w(t) - tw'(t).
\]

It is not hard to see

\[
\ell_\infty(0) = \lim_{t \to \infty} b(t).
\]

Let

\[
\tilde{w}(t) = \sqrt{1 + c_0^\frac{1}{n} t^2} \quad \text{with} \quad c_0 = 1 + \varepsilon.
\]

Clearly \( \tilde{w}(t) \) solves (2.7) for \( g = 1 + \varepsilon \). Direct computation shows

\[
\tilde{b}(t) = \tilde{w}(t) - t\tilde{w}'(t) = (1 + c_0^\frac{1}{n} t^2)^{-\frac{1}{2}}.
\]

Hence \( \tilde{b}(t) > 0 \) for all \( t \geq 0 \), while \( \tilde{b}(t) \to 0 \) as \( t \to \infty \). To see \( \ell_\infty(0) > 0 \), our strategy is to compare \( b(t) \) and \( \tilde{b}(t) \).

By (2.7) we have

\[
(2.13) \quad b'(t) = -tw'' = -\frac{t^n g(t)}{(w'(s))^{n-1}w^{n+2}}.
\]
Therefore
\[ \ell_{\infty}(0) = \int_R b'(t) dt + b(R) \]
(2.14)
\[ \geq - \int_R \frac{s^n g(s)}{(w')^{n+1}w^{n+2}} ds + (1 + c_0^2 R^2)^{-\frac{1}{2}} - |b(R) - \bar{b}(R)|.\]

Let us estimate \(|b(R) - \bar{b}(R)|\). To this end, we first estimate \(|w' - \bar{w}'|\) and \(|w - \bar{w}|\) respectively in the sequel.

By (2.7) we have
(2.15)
\[ w^n(t) = \int_0^t n g(s) s^{n-1} \frac{1}{w^{n+2}(s)} ds \]
and \(\tilde{w}^n(t) = \int_0^t n(1 + \varepsilon) s^{n-1} \frac{1}{\tilde{w}^{n+2}(s)} ds.\)

For \(t \geq 1\), we obtain
(2.16)
\[ w^n(t) \geq \int_0^1 n g(s) s^{n-1} \frac{1}{w^{n+2}(s)} ds \geq \frac{1}{w^{n+2}(1)} \geq \frac{1}{(w(0) + w'(1))^{n+2}} \geq 1/C, \]
where the last inequality follows from \(w' \leq C\) as shown in Lemma 2.2. Hence, for \(t \in [1, R]\),
\[ |w'(t) - \bar{w}'(t)| \leq C |w^n(t) - \tilde{w}^n(t)| \]
\[ \leq C \int_0^t g(s) s^{n-1} \frac{1}{w^{n+2}(s)} ds \]
\[ \leq C \int_0^t s^{n-1} \frac{1}{w^{n+2}} |g - (1 + \varepsilon)\frac{1}{\tilde{w}^{n+2}} ds + C \int_0^t s^{n-1} \frac{1}{\tilde{w}^{n+2}} - \frac{1}{w^{n+2}} ds. \]

Note that \(g = 1 + \varepsilon\) in \([\delta, R]\). We then obtain by the mean value theorem, for \(t \in [1, R]\),
\[ |w'(t) - \bar{w}'(t)| \leq C \varepsilon \delta^n + C \int_0^t \frac{s^{n-1}}{\zeta^{n+3}} |w(s) - \tilde{w}(s)| ds, \]
(2.17)
\[ \leq C \varepsilon \delta^n + C \int_0^t \frac{s^{n-1}}{\zeta^{n+3}} \left[ \int_0^s |w'(\tau) - \tilde{w}'(\tau)| d\tau \right] ds \]
for some \(\xi = \xi(s)\), a function taking value between \(w(s)\) and \(\tilde{w}(s)\). Denote
\[ K(t) = \sup_{0 \leq t' \leq t} |w'(t') - \tilde{w}'(t')|. \]

Then (2.17) implies
\[ K(t) \leq C \varepsilon \delta^n + C \int_0^t \frac{s^n}{\xi^{n+3}} K(s) ds. \]
Applying Grönwall’s inequality to \(K(t)\), and noting that by Lemma 2.2
\[ \int_0^\infty \frac{s^n}{\xi^{n+3}} ds \leq C, \]
we obtain
\begin{equation}
|w'(t) - \tilde{w}'(t)| \leq K(t) \leq C\varepsilon\delta^n \quad \text{for} \quad 1 \leq t \leq R.
\end{equation}

Set
\[
\psi(t) = \frac{n}{t^n} \int_0^t g(s) s^{n-1} \, ds \quad \text{and} \quad \tilde{\psi}(t) = \frac{n}{t^n} \int_0^t (1 + \varepsilon) s^{n-1} \, ds
\]
Then by (2.15) \( w'(t) = t\psi(t) \) and \( \tilde{w}'(t) = t\tilde{\psi}(t) \). Hence by the mean value theorem
\[
|w(t) - \tilde{w}(t)| = \left| \int_0^t s(\psi(s) - \tilde{\psi}(s)) \, ds \right|
\leq \int_0^t s^{1-n} \eta^{\frac{1}{n}-1} \left( \int_0^s \frac{\tau^{n-1}}{w^{n+2}} \, d\tau \right) g - (1 + \varepsilon) \, d\tau \right) \, ds
\]
\[
+ (1 + \varepsilon) \int_0^t s^{1-n} \eta^{\frac{1}{n}-1} \left( \int_0^s \frac{\tau^{n-1}}{w^{n+2}} - \frac{1}{\tilde{w}^{n+2}} \right) \, d\tau \right) \, ds,
\]
where \( \eta = \eta(s) \) taking value between \( \psi(s) \) and \( \tilde{\psi}(s) \). Denote
\[
W(t) = \sup_{0 \leq s \leq t} |w'(s) - \tilde{w}'(s)|.
\]
Another use of the mean value theorem gives, for \( t \in [0, 1] \),
\begin{equation}
W(t) \leq \left( \int_0^\delta + \int_{\delta}^1 \right) \left[ s^{1-n} \eta^{\frac{1}{n}-1} \left( \int_0^s \frac{\tau^{n-1}}{w^{n+2}} \, d\tau \right) g - (1 + \varepsilon) \, d\tau \right) \, ds
\]
\[
+ C \int_0^t s^{1-n} \eta^{\frac{1}{n}-1} \left( \int_0^s \frac{\tau^{n-1}}{\xi^{n+3}} \, d\tau \right) W(s) \, ds,
\]
where \( \xi = \xi(\tau) \) is a value between \( w(\tau) \) and \( \tilde{w}(\tau) \). Note that by Lemma 2.2
\[
C^{-1} \leq \eta(s) \leq C, \quad 0 \leq s \leq 1,
\]
\[
1 \leq \xi(s) \leq C, \quad 0 \leq s \leq 1.
\]
Hence we further deduce from (2.19), for \( t \in [0, 1] \),
\[
W(t) \leq C\varepsilon \int_0^\delta s^{1-n} \int_0^s \tau^{n-1} \, d\tau \, ds + C\varepsilon\delta^n \int_\delta^1 s^{1-n} \, ds + C \int_0^t sW(s) \, ds
\]
\[
\leq C\varepsilon\delta^2 + C \int_0^1 sW(s) \, ds.
\]
By Grönwall’s inequality again, we get
\begin{equation}
|w(t) - \tilde{w}(t)| \leq W(t) \leq C\varepsilon\delta^2 \quad \text{for} \quad 0 \leq t \leq 1.
\end{equation}
It follows from (2.12) and (2.13) that, for $t \in [1, R]$,

$$|b(t) - \tilde{b}(t)| \leq |b(1) - \tilde{b}(1)| + \int_1^t |b'(s) - \tilde{b}'(s)| ds$$

$$\leq |w(1) - \tilde{w}(1)| + |w'(1) - \tilde{w}'(1)| + \int_1^t \frac{s^n}{(w')^{n-1} w^{n+2}} |g - (1 + \varepsilon)| ds$$

$$+ C \int_1^t \frac{s^n}{(w')^{n-1}} \frac{1}{w^{n+2}} |ds| + C \int_1^t \frac{s^n}{(\tilde{w}')^{n-1}} \frac{1}{(w')^{n-1}} |ds|.$$ 

By (2.16), (2.18) and (2.20) and Lemma 2.2, we further obtain, for $t \in [1, R]$,

$$\delta$$

$$\tilde{b}$$

By (2.16), (2.18) and (2.20) and Lemma 2.2, we further obtain, for $t \in [1, R]$,

$$|b(t) - \tilde{b}(t)| \leq C \varepsilon \delta + C \int_1^t s^{-3} |w(s) - \tilde{w}(s)| ds$$

$$\leq C \varepsilon \delta + C \int_1^t s^{-3} |b(s) - \tilde{b}(s)| ds + C \int_1^t s^{-2} |w'(s) - \tilde{w}'(s)| ds$$

Another use of Grönwall’s inequality gives

$$(2.21) \quad |b(t) - \tilde{b}(t)| \leq C \varepsilon \delta \exp \left( \int_1^\infty s^{-3} ds \right) \leq C \varepsilon \delta \quad \text{for } 1 \leq t \leq R.$$ 

Let us go back to (2.14) and continue the estimates. Note that $\tilde{b}(R) \geq R^{-1}/C$ for some universal $C > 0$. By (2.21) we can choose

$$(2.22) \quad \delta = \delta(\varepsilon, R) = o(R^{-1} \varepsilon^{-1})$$

so that $b(R) > 0$. Then by convexity

$$(2.23) \quad w'(t) \geq w'(R) \quad \text{and} \quad w(t) \geq w'(R) t, \quad \forall \ t \geq R.$$ 

Plugging (2.21) in (2.14), and employing Lemma 2.2 and (2.23), we obtain

$$\ell_\infty(0) \geq (1 + c_0^{\frac{1}{n}} R^2)^{-\frac{1}{2}} - \left( \int_R^{R+\delta} + \int_{R+\delta}^T \frac{s^n g}{w^{n-1} w^{n+2}} ds \right) - C \varepsilon \delta - CT^{-1}$$

$$\geq (1 + c_0^{\frac{1}{n}} R^2)^{-\frac{1}{2}} - (w'(R))^{-2n-1} R^{-1} - C \delta R^{-2} - C \varepsilon \delta - CT^{-1}$$

$$\geq (1 + c_0^{\frac{1}{n}} R^2)^{-\frac{1}{2}} - (\tilde{w}'(R))^{-2n-1} R^{-1} - C \delta R^{-2} - C \varepsilon \delta - CT^{-1},$$

where we have used (2.18) in the last inequality. Direct computation shows

$$(\tilde{w}'(R))^{-2n-1} = c_0^{-\frac{2n+1}{2n}} \left( 1 + c_0^{-\frac{1}{n}} R^{-2} \right)^{\frac{2n+1}{2n}}.$$
Therefore
\[
\ell_{\infty}(0) \geq R^{-1}c_0^{-\frac{1}{n}}(1 + c_0^{-\frac{1}{n}}R^{-2})^{-\frac{1}{2}} \left[1 - c_0^{-1}(1 + c_0^{-\frac{1}{n}}R^{-2})^{n+1}\right] - C\delta R^{-2} - C\varepsilon \delta - CT^{-1}
\]
\[
\geq \frac{1}{2}R^{-1}\left[1 - (1 - \varepsilon + o(\varepsilon))(1 + O(R^{-2})) - C\delta/R - C\varepsilon \delta R - CR/T\right].
\]
Hence if we choose
(2.24) \quad R = \varepsilon^{-1}, \quad \delta = \varepsilon^2 \quad \text{and} \quad T = \varepsilon^{-3}
then
\[
\ell_{\infty}(0) \geq \frac{1}{2}R^{-1}(\varepsilon + o(\varepsilon)) > 0
\]
for \(\varepsilon\) small. Note that (2.24) is compatible with (2.22).

\[\square\]

**Proof of the statement (ii) of Proposition 2.1.** We shall study the asymptotic behaviour of the solution by comparing it with \(\tilde{w}(t) = \sqrt{1 + t^2}\).

We still use the function \(b(t)\) given by (2.12), and consider
\[
\tilde{b}(t) = \tilde{w}(t) - t\tilde{w}'(t) = (1 + t^2)^{-\frac{1}{2}}.
\]
Applying the above argument to the current situation, we find that (2.21) still holds. Hence
(2.25) \quad |b(R) - \tilde{b}(R)| \leq C\varepsilon \delta.
Consequently
\[
\ell_{\infty}(0) = \lim_{t \to \infty} b(t) = b(R) + \int_R^\infty b'(s)ds
\]
(2.26) \quad \leq \tilde{b}(R) + C\varepsilon \delta + \int_R^\infty b'(s)ds.

To see \(\ell_{\infty}(0) < 0\), we need to estimate \(b'(s)\). By (2.13) one gets
(2.27) \quad \frac{b'(t)}{b'(t)} = (1 + \varepsilon)(w'(t))^{n-1}w^{n+2}(t), \quad \text{for} \quad R + \delta \leq t \leq T.
Clearly
(2.28) \quad (w'(t))^{n-1}w^{n+2}(t) \geq (w'(R))^{2n+1}t^{n+2}, \quad \text{for} \quad t \geq R.
By (2.7) and Lemma 2.2, we obtain
\[
(w'(t))^n \leq (w'(R))^n + \int_R^t (w'(R)s + b(R))^{n+2}ds
\]
(2.29)
\[
\leq (w'(R))^n(1 + CR^{-2}), \quad \text{for } t \geq R.
\]
Plugging (2.28) and (2.29) in (2.27), we get, for \( R + \delta \leq t \leq T \),
\[
\frac{b'(t)}{b'(t)} \geq (1 + \varepsilon)\frac{\tilde{w}'^{2n+1}(R)}{w'^{n-1}(R)} (\frac{w}{t})^{n-2} (1 - C)R^{-2}.
\]
(2.30)
By (2.13), \( b(t) \) is non-increasing. Hence for \( t \geq R \),
\[
\frac{w}{t} = w'(t) + \frac{b(t)}{t}
\leq w'(t) + \frac{b(R)}{R}
\leq w'(t) + \frac{\tilde{b}(R)}{R} + \frac{|b(R) - \tilde{b}(R)|}{R}
\leq w'(R)(1 + \frac{C}{R^2} + \frac{C\varepsilon\delta}{R}).
\]
In the last inequality above we use (2.25), (2.29), as well as
\[
\tilde{w}'^{n}(R) \geq w'(1) \geq 1/C
\]
which was shown in (2.16). Hence (2.30) can be further estimated as
\[
\frac{b'(t)}{b'(t)} \geq (1 + \varepsilon)(1 - C\varepsilon\delta^n)(1 - CR^{-2} - C\varepsilon\delta R^{-1}), \quad \text{for } R + \delta \leq t \leq T.
\]
It is direct to check that (2.18) is still true in our current case. Therefore
\[
\frac{b'(t)}{b'(t)} \geq (1 + \varepsilon)(1 - C\varepsilon\delta)(1 - CR^{-2} - C\varepsilon\delta R^{-1}), \quad \text{for } R + \delta \leq t \leq T.
\]
(2.31)
Note that \( b'(t), \tilde{b}'(t) \leq 0 \). By (2.26) and (2.31), we obtain
\[
\ell_{\infty}(0) \leq \tilde{b}(R) + C\varepsilon\delta + \int_R^{R+\delta} b'(t) + \int_T^{\infty} b'(t)
\]
\[
+ (1 + \varepsilon)(1 - C\varepsilon\delta^n)(1 - CR^{-2} - C\varepsilon\delta R^{-1}) \int_{R+\delta}^T \tilde{b}'(t)dt
\]
\[
\leq \tilde{b}(R)[1 - (1 + \varepsilon)(1 - C\varepsilon\delta)(1 - CR^{-2} - C\varepsilon\delta R^{-1})]
\]
\[
+ (1 + o(1))\tilde{b}(T) + C\varepsilon\delta + CR^{-2}\delta + C/T
\]
\[
\leq \tilde{b}(R)[1 - (1 + \varepsilon)(1 + o(\varepsilon) + O(R^{-2})) + C\varepsilon\delta R + C\delta/R + CR/T].
\]
Hence if we choose \( R, \delta \) and \( T \) satisfying (2.24), then
\[
\ell_{\infty}(0) \leq \tilde{b}(R)(-\varepsilon + o(\varepsilon)) < 0.
\]
3. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. Let $f$ be a function in the following form

\[
   f(s) = \begin{cases} 
   1 + \varepsilon_{i+1} & \text{for } s \in [P_{4i+2}, P_{4i+1}], \\
   1 & \text{for } s \in [P_{4i+4}, P_{4i+3}], \\
   1 + \varepsilon_1 & \text{for } s \in (P_1, \infty), \\
   1 & \text{for } s = 0,
   \end{cases}
\]

and

\[
   1 \leq f(s) \leq 1 + \varepsilon_{i+1} \quad \text{for } s \in (P_{4i+3}, P_{4i+2}),
\]

\[
   1 \leq f(s) \leq 1 + \varepsilon_{i+2} \quad \text{for } s \in (P_{4i+5}, P_{4i+4}),
\]

where the index $i$ goes from zero to infinity, and $\{\varepsilon_j\}_{j=1}^{\infty}$ (satisfying $\varepsilon_j \to 0$), $\{P_j\}_{j=1}^{\infty}$ (satisfying $P_j \to 0$) are two decreasing sequences to be determined later. Set

\[
   R_j = \varepsilon_j^{-1}, \quad \delta_j = \varepsilon_j^2 \quad \text{and} \quad T_j = \varepsilon_j^{-3}
\]

as in Proposition 2.1. Let $\{h_j\}_{j=1}^{\infty}$ (satisfying $h_j \to 0$) be another decreasing sequence to be specified below. Our goal is to suitably choose $\{\varepsilon_j\}$, $\{P_j\}$ and $\{h_j\}$ so that

\[
   f(h_{2i+1}^{\frac{n+1}{n}} t) = \begin{cases} 
   1 & \text{for } t \in [\delta_{i+1}, R_{i+1}], \\
   1 + \varepsilon_{i+1} & \text{for } t \in [R_{i+1} + \delta_{i+1}, T_{i+1}],
   \end{cases}
\]

and

\[
   f(h_{2i+2}^{\frac{n+1}{n}} t) = \begin{cases} 
   1 + \varepsilon_{i+2} & \text{for } t \in [\delta_{i+2}, R_{i+2}], \\
   1 & \text{for } t \in [R_{i+2} + \delta_{i+2}, T_{i+2}].
   \end{cases}
\]

Then, by Proposition 2.1 and rescaling back, there is a sequence of convex solutions $u_j$ to (2.4) and (2.5) with initial values $h_j$ whose asymptotic lines lie below and above the origin alternatively. By [16, Lemma 3.2], we can pick up a $\hat{h}_i \in (h_{2i+2}, h_{2i+1})$ such that the convex solution $\hat{u}_i$ to (2.4) and (2.5), for $h = \hat{h}_i$, satisfies the asymptotic condition (2.6). Since $\hat{h}_i$ is a decreasing sequence, these solutions $\hat{u}_i$ are distinct. We therefore obtain an infinite number of solutions to the centro-affine Minkowski problem by pulling back to equation (1.1).
Once \( \{ \varepsilon_j \} \) is chosen, we construct \( \{ P_j \} \) and \( \{ h_j \} \) accordingly as follows. Note that \( \{ \delta_j, R_j, T_j \} \) is determined by (3.3). We start with

\[
P_1 = T_1, \quad P_2 = R_1 + \delta_1, \quad P_3 = R_1, \quad P_4 = \delta_1 \text{ and } h_1 = 1.
\]

Thus (3.4) holds trivially for \( i = 0 \) through (3.1). We next assign the value for \( h_2 \) by

\[
h_2 = \left( \frac{P_4}{R_2 + \delta_2} \right)^{\frac{n}{n+1}}.
\]

Then we set

\[
P_5 = h_2^{\frac{n+1}{n}} R_2 \text{ and } P_6 = h_2^{\frac{n+1}{n}} \delta_2.
\]

If \( \varepsilon_2 = \varepsilon_1^{\beta_2} \), with \( \beta_2 = \frac{3}{2} > 1 \), then

\[
h_2^{\frac{n+1}{n}} T_2 = \frac{\delta_1 T_2}{R_2 + \delta_2} \leq R_1 = P_3.
\]

Hence we conclude that (3.5) holds for \( i = 0 \) via (3.1). We next define \( h_3 \) by

\[
h_3 = \left( \frac{P_6}{R_2 + \delta_2} \right)^{\frac{n}{n+1}},
\]

and determine \( P_7, P_8 \) by

\[
P_7 = h_3^{\frac{n+1}{n}} R_2 \text{ and } P_8 = h_3^{\frac{n+1}{n}} \delta_2.
\]

As

\[
h_3^{\frac{n+1}{n}} T_2 = h_2^{\frac{n+1}{n}} \frac{\delta_2 T_2}{R_2 + \delta_2} \leq h_2^{\frac{n+1}{n}} R_2 = P_5,
\]

we have (3.4) for \( i = 1 \) by (3.1).

In general we assign the values for \( \{ h_j \} \) and \( \{ P_j \} \) recursively. Assume that we have determined \( h_j \) for \( 1 \leq j \leq 2i + 1 \), and \( P_l \) for \( 1 \leq l \leq 4i + 4 \). Firstly let

\[
(3.6) \quad h_{2i+2} = \left( \frac{P_{4i+4}}{R_{i+2} + \delta_{i+2}} \right)^{\frac{n}{n+1}}.
\]

Then we set

\[
(3.7) \quad P_{4i+5} = h_{2i+2}^{\frac{n+1}{n}} R_{i+2} \text{ and } P_{4i+6} = h_{2i+2}^{\frac{n+1}{n}} \delta_{i+2}.
\]

Once \( P_{4i+6} \) is given, we further take

\[
(3.8) \quad h_{2i+3} = \left( \frac{P_{4i+6}}{R_{i+2} + \delta_{i+2}} \right)^{\frac{n}{n+1}},
\]

and

\[
(3.9) \quad P_{4i+7} = h_{2i+3}^{\frac{n+1}{n}} R_{i+2} \text{ and } P_{4i+8} = h_{2i+3}^{\frac{n+1}{n}} \delta_{i+2}.
\]
Let \( \{\varepsilon_j\} \) be the sequence such that \( \varepsilon_1 = \min \{\frac{1}{2}, \varepsilon_0\} \), where \( \varepsilon_0 \) is the universal constant given in Proposition 2.1 and

\( (3.10) \)
\[
\varepsilon_{j+1} = \varepsilon_j^\beta, \quad \text{with} \quad \beta = \frac{3}{2} > 1.
\]

Then one can check by (3.6)–(3.10) that

\[
h_{2i+2} T_{i+2} = \frac{T_{i+2}}{R_{i+2} + \delta_{i+2}} P_{4i+4}
\]
\[
= \frac{T_{i+2}}{R_{i+2} + \delta_{i+2}} \frac{\delta_{i+1}}{R_{i+1}} P_{4i+3}
\]
\[
\leq P_{4i+3}
\]

and

\[
h_{2i+3} T_{i+2} = \frac{T_{i+2}}{R_{i+2} + \delta_{i+2}} P_{4i+6}
\]
\[
= \frac{T_{i+2}}{R_{i+2} + \delta_{i+2}} \frac{\delta_{i+1}}{R_{i+1}} P_{4i+5}
\]
\[
\leq P_{4i+5}
\]

Consequently (3.4) and (3.5) follow.

We next show that \( f \) can be extended to a Hölder continuous function. By (3.6)–(3.9), it is direct to calculate

\[
h_{2i+1} \geq \frac{\delta_{i+1}}{R_{i+1} + \delta_{i+1}} h_{2i}^{\frac{\gamma}{2}}
\]
\[
= \frac{\delta_{i+1}}{(R_{i+1} + \delta_{i+1})^2} P_i
\]
\[
= \frac{\delta_{i+1} \delta_i}{(R_{i+1} + \delta_{i+1})^2} h_{2i-1}^{\frac{\gamma}{2}}
\]
\[
\geq \frac{1}{2} \varepsilon_1^{\gamma} h_{2i-1}^{\frac{\gamma}{2}}
\]

where \( \gamma = 2b + d(\beta^{-1} - 1) \), and \( b > 1, d > 0 \) are the constants such that

\[
\frac{\delta_i}{R_i} = \varepsilon_i^b \quad \text{and} \quad \delta_i = \varepsilon_i^d.
\]

By (3.3), \( b = 3 \) and \( d = 2 \). Consequently

\[
h_{2i+1} \geq \frac{1}{4} \varepsilon_1^{\gamma} h_{2i-1}^{\frac{\gamma}{2}} \geq \cdots \geq \frac{1}{2i} \left( \prod_{j=2}^{i+1} \varepsilon_j \right) \gamma h_{1}^{\frac{\gamma}{2}} = \frac{1}{2i} \varepsilon_1^{\gamma} \gamma h_{1}^{\frac{\gamma}{2}}.
\]
Using (3.8) and (3.9), we then obtain
\[ |P_{4i+2} - P_{4i+3}| = \delta_{i+1} h_{2i+1} \geq \frac{1}{2} \varepsilon_{i+1} \gamma^{\frac{d+1}{d-1}}. \]
Hence, by (3.2), we have
\[
\frac{\text{osc}_{[P_{4i+3}, P_{4i+2}]} f(s)}{|P_{4i+3} - P_{4i+2}|^\alpha} \leq \frac{\varepsilon_{i+1}}{|P_{4i+3} - P_{4i+2}|^\alpha} \leq 2^{i\alpha} \varepsilon_1 \beta(1 - \alpha d - \alpha \gamma^{\frac{d+1}{d-1}} - i \alpha) \leq C,
\]
for any \( \alpha \) such that
\[
0 < \alpha < \left(d + \frac{\gamma \beta}{\beta - 1}\right)^{-1} = \frac{\beta - 1}{2b \beta}. \tag{3.12}
\]
Note that \( C \) depends on \( \alpha \) but is independent of \( i \). Similarly one can check that
\[
\frac{\text{osc}_{[P_{4i+1}, P_{4i}]} f(s)}{|P_{4i+1} - P_{4i}|^\alpha} \leq C, \quad \text{independent of } i,
\]
for such \( \alpha \) in (3.12). Employing (3.11) and (3.13), by a suitable modification, the function \( f \) can be \( C^\alpha \) in \( \cup_{i \geq 0} \{(P_{4i+5}, P_{4i+4}) \cup (P_{4i+3}, P_{4i+2})\} \). By (3.11) and (3.2), \( f \) is Hölder continuous in \( s > 0 \).

The above calculation also implies that, after the even extension, \( f \) is \( C^\alpha \) across \( \{s = 0\} \). In fact we have by (3.6)–(3.9),
\[
P_{4k} = \frac{\delta_k}{R_k + \delta_k} P_{4k-2} = \left(\frac{\delta_k}{R_k + \delta_k}\right)^2 P_{4(k-1)} \geq \frac{1}{2} \varepsilon_k 2^b P_{4(k-1)} \geq \cdots \geq \frac{1}{2k} \left[\prod_{j=1}^k \varepsilon_j\right] 2^b P_0,
\]
where \( P_0 := (\delta_1/R_1)^{-1} P_4 \) for convenience. By (3.10),
\[
P_{4k} \geq \frac{1}{2^k} \varepsilon_1^{2b} P_0 \geq \frac{1}{2^k} \varepsilon_1^{2b} \frac{\gamma^{k-1}}{\beta^k} P_0.
\]
Consequently by (3.1)
\[
\frac{\text{osc}_{[0, P_{4k}]} f(s)}{P_{4k}^\alpha} \leq \frac{2^{ak} \varepsilon_k}{\varepsilon_1^{2a_k \sum_{j=0}^{k-1} \beta_j}} P_0^\alpha \leq \frac{1}{P_0^\alpha} \frac{\gamma^{k-1}(1 - \alpha d - \alpha \gamma^{d+1} - \alpha k)}{\varepsilon_1^{2a_k \sum_{j=0}^{k-1} \beta_j} P_0^\alpha}.
\]
Hence
\[
\lim_{k \to \infty} \frac{\text{osc}_{[0, P_{2k}]} f(s)}{|P_{2k}|^\alpha} \leq C,
\]
for any \( \alpha \) satisfying (3.12).
Therefore $f$ is globally $C^\alpha$ continuous on $\mathbb{R}$. Hence the ODE \((2.4)-(2.6)\) has infinitely many different $C^{2,\alpha}$ solutions (as they have different initial values $h$), which by symmetry yield infinitely many distinct $C^{2,\alpha}$ solutions for the centro-affine Minkowski problem. In fact we obtained an infinite number of symmetric solutions to the equation

\begin{equation}
H^{n+2} \det(\nabla^2 H + HI) = \bar{f}(x), \ x \in \mathbb{S}^n,
\end{equation}

with

\[
\bar{f}(x) = \begin{cases} 
  f(\tan \theta), & \text{if } \theta \in [0, \pi/2) \\
  1 + \varepsilon_1, & \text{if } \theta = \pi/2 \\
  f(\tan(\pi - \theta)), & \text{if } \theta \in (\pi/2, \pi]
\end{cases}, \ x_{n+1} = -\cos \theta,
\]

while $f$ is the $C^\alpha$ function on $\mathbb{R}$ as constructed above which satisfies \((3.1)\). It remains to show that the symmetric solutions we obtained are not affine-equivalent. Let $H_1 \neq H_2$ be two such solutions. Assume there exists a unimodular linear transformation $A \in SL(n+1)$ such that $H_2 = (H_1)_A$. By the symmetry, $A = \text{diag}\{\mu, \cdots, \mu, \mu^{-n}\}$ for some $1 \neq \mu > 0$. By \((1.4)\), $(H_1)_A$ satisfies \((3.14)\) with RHS

\[
\bar{f}_A(x) = \begin{cases} 
  f\left(\mu^{n+1} \tan \theta\right), & \text{if } \theta \in [0, \pi/2) \\
  1 + \varepsilon_1, & \text{if } \theta = \pi/2 \\
  f\left(\mu^{n+1} \tan(\pi - \theta)\right), & \text{if } \theta \in (\pi/2, \pi]
\end{cases}, \ x_{n+1} = -\cos \theta.
\]

By \((3.1)\), it is easy to see that there is no $\mu \neq 1$ such that $\bar{f} \equiv \bar{f}_A$. Consequently it is not possible that $H_2 = (H_1)_A$. We therefore complete the proof of Theorem 1.1.

References


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