ON THE PLANAR DUAL MINKOWSKI PROBLEM

SHIBING CHEN AND QI-RUI LI

ABSTRACT. In this paper, we resolve the planar dual Minkowski problem, proposed by Huang-Lutwak-Yang-Zhang (Acta Math. 216 (2016): 325–388) for all positive indices without any symmetry assumption. More precisely, given any \( q > 0 \), and function \( f \) on \( S^1 \), bounded by two positive constants, we show that there exists a convex body \( \Omega \) in the plane, containing the origin in its interior, whose dual curvature measure \( \tilde{C}_q(\Omega, \cdot) \) has density \( f \). In particular, if \( f \) is smooth, then \( \partial \Omega \) is also smooth.

1. Introduction

The Minkowski type problems in convex geometry are characterisation problems of the differentials of geometric functions of convex bodies. The fundamental geometric functionals in the Brunn-Minkowski theory are the quermassintegrals, which include volume and surface area as special cases, while the area measures and the curvature measures can be viewed as the differentials of these quermassintegrals [47]. The associated Minkowski type problems then arise: that is to prescribe area measures and curvature measures. There is a large number of papers devoted to the study of these problems, see e.g. [2, 16, 25, 27, 42, 46] and the references therein.

The dual Brunn-Minkowski theory emerged in the mid-1970s when Lutwak introduced the dual mixed volume [40]. The main geometric functionals in this theory are the dual quermassintegrals. Arising from the dual quermassintegrals is a new family of geometric measures, dual curvature measures \( \tilde{C}_q \) for \( q \in \mathbb{R} \), discovered by Huang-Lutwak-Yang-Zhang in their recent seminal work [31]. Denote by \( \mathcal{K}_0^n \) the set of all convex bodies (i.e., compact convex sets that have non-empty interior) in \( \mathbb{R}^{n+1} \) containing the origin in their interiors. Associated to each convex body \( \Omega \in \mathcal{K}_0^n \) are the support function \( u = u_\Omega : S^n \to \mathbb{R} \) and the radial function \( r = r_\Omega : S^n \to \mathbb{R} \), which are respectively

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defined by
\[ u(x) = \max \{x \cdot z : z \in \Omega\}, \]
and
\[ r(\xi) = \max \{\lambda : \lambda \xi \in \Omega\} \]
Let \( \bar{r}(\xi) = \bar{r}_\Omega(\xi) := r_\Omega(\xi)\xi \). Then obviously \( \partial \Omega = \{\bar{r}(\xi) : \xi \in S^n\} \). For \( x \in S^n \), the hyperplane
\[ \ell_\Omega (x) = \{z \in \mathbb{R}^{n+1} : z \cdot x = u_\Omega(x)\} \]
is called the supporting hyperplane of \( \Omega \) with the unit normal \( x \). The spherical image \( \nu = \nu_\Omega : \partial \Omega \to S^n \) is then given by
\[ \nu(z) = \{x \in S^n : z \in \ell_\Omega(x)\}. \]
With the help of these notions, we can introduce two set-valued mappings, namely the radial Gauss mapping \( \mathcal{A} = \mathcal{A}_\Omega \) and the reverse radial Gauss mapping \( \mathcal{A}^* = \mathcal{A}_\Omega^* \) as follows: for any \( \omega \subseteq S^n \),
\begin{align*}
\mathcal{A}(\omega) &= \{\nu(\bar{r}(\xi)) : \xi \in \omega\}, \\
\mathcal{A}^*(\omega) &= \{\xi \in S^n : \nu(\bar{r}(\xi)) \in \omega\}. 
\end{align*}
The \( q \)-th dual curvature measure in [31] is defined as
\[ \tilde{C}_q(\Omega, \omega) = \int_{\mathcal{A}^*(\omega)} r^q(\xi) d\xi, \text{ for } \omega \subseteq S^n \]
where \( d\xi \) denotes the standard measure of \( S^n \).

The dual curvature measures (1.2) contain significant geometric information in the dual Brunn-Minkowski theory. For example, it was shown by Huang-Lutwak-Yang-Zhang [31] that, for \( t \geq 0 \) and a Borel set \( \omega \subseteq S^n \), the local dual parallel bodies \( \tilde{A}_t(\Omega, \omega) \) have a Steiner type formula with the dual curvature measures as their coefficients, namely
\[ \text{Volume}(\tilde{A}_t(\Omega, \omega)) = \frac{1}{n+1} \sum_{i=0}^{n+1} \binom{n+1}{i} t^{n+1-i} \tilde{C}_i(\Omega, \omega). \]
Here the local dual parallel bodies are given by
\[ \tilde{A}_t(\Omega, \omega) = \{z \in \mathbb{R}^{n+1} : 0 \leq |z| \leq r_\Omega(\xi) + t \text{ with } \xi \in \mathcal{A}_\Omega^*(\omega)\}. \]
For other geometric meanings and properties of the dual curvature measures, we refer the readers to the comprehensive paper [31].

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1This definition differs a dimension constant with that in [31].
The associated Minkowski type problem was also posed by Huang-Lutwak-Yang-Zhang [31]. Given a real number $q$ and a finite Borel measure $\mu$ on the sphere $S^n$, the authors asked if there exists a convex body $\Omega \in K_0^n$ such that its $q$-th dual curvature measure
\begin{equation}
\tilde{C}_q(\Omega, \cdot) = \mu(\cdot).
\end{equation}
This problem is called the dual Minkowski problem. By using the variable change formula, this is in smooth category equivalent to solving the following equation
\begin{equation}
\begin{aligned}
 r^q(\mathcal{A}^*(x))|\text{Jac} \mathcal{A}^*| &= f(x) \quad \text{on } S^n,
\end{aligned}
\end{equation}
provided $\mu$ has a density function $f$. Here $|\text{Jac} \mathcal{A}^*|$ denotes the determinant of the Jacobian of the mapping $x \mapsto \xi = \mathcal{A}^*(x)$. It is known that $r(\mathcal{A}^*(x))$ and $|\text{Jac} \mathcal{A}^*|$ can be written in terms of the support function, see e.g. [31, 37],
\begin{equation}
\begin{aligned}
 r(\mathcal{A}^*(x)) &= \sqrt{u^2(x) + |\nabla u|^2} \\
 |\text{Jac} \mathcal{A}^*| &= u \det(\nabla^2 u + uI) \\
 &= \frac{u(x)}{r^{n+1}(\mathcal{A}^*(x))K} \\
 &= \frac{u \det(\nabla^2 u + uI)}{\sqrt{u^2 + |\nabla u|^{2n+1}}},
\end{aligned}
\end{equation}
where $K$ in the above formula is the Gauss curvature of $\partial \Omega$ at $\bar{r}(\mathcal{A}^*(x))$, and $\nabla$ denotes the covariant derivative with respect to an orthonormal frame on $S^n$. The second equality in (1.6) is because the principal radii of curvature of $\partial \Omega$ are the eigenvalues of the matrix $\nabla^2 u + uI$. Hence (1.4) turns out to be a Monge-Ampère type equation
\begin{equation}
\begin{aligned}
 \det(\nabla^2 u + uI) &= \frac{\sqrt{u^2 + |\nabla u|^{2n+1}} f}{u} \quad \text{on } S^n.
\end{aligned}
\end{equation}

The dual Minkowski problem includes the logarithmic Minkowski problem for cone-volume measure ($q = n + 1$) and the Aleksandrov problem for integral Gauss curvature ($q = 0$) as special cases. It also connects the fields of convex geometry and differential equations, and thus has attracted much attention recently. In [31] the authors proved the existence of origin-symmetric solutions to the problem when $q \in (0, n + 1]$, and $\mu$ is even and satisfies a condition which reflects how much concentration $\mu$ can have in subspaces. The dual Minkowski problem for $q > 0$ is further studied in a series of subsequent papers [8, 11, 34, 37, 52], but symmetry assumptions on $\mu$ are always
assumed in these mentioned work. Our purpose in the current paper is to solve the planar dual Minkowski problem \((n = 1)\) for non-symmetric measures. For the planar case, equation (1.7) becomes a second-order nonlinear ordinary differential equation
\[
(1.8) \quad u_{\theta\theta} + u = \frac{\sqrt{u^2 + u^2_{\theta}}}{u} f \text{ on } S^1,
\]
where \(\theta\) is the arc-length parameter\(^2\) on \(S^1\). In what follows, we say \(u \in C^2(S^1)\) is strictly convex, if \(\kappa = (u_{\theta\theta} + u)^{-1} > 0\) everywhere on \(S^1\). This means the corresponding convex body \(\Omega_u\) (see (2.1) below) has positive boundary curvature \(\kappa\). Our main theorem is stated as follows.

**Theorem 1.1.** Consider (1.3) and (1.8) for \(q > 0\).

(a) Let \(f\) be a positive smooth function on \(S^1\). Then (1.8) has a positive, smooth, strictly convex solution. Equivalently there is a smooth and strictly convex body in \(K_0^1\) solving the dual Minkowski problem (1.3).
(b) Let \(\mu\) be the measure with density \(f \in L^\infty(S^1)\), and \(f \geq \lambda_0\) for some constant \(\lambda_0 > 0\). There exists a planar convex body in \(K_0^1\) solving the dual Minkowski problem (1.3).

Our method can be also applied if \(f\) is a function of multi-variables. See Theorem 5.1 & 5.2 in Section 5.2. We point out that even for the planar case Theorem 1.1 is the first existence result for the dual Minkowski problem \((q > 0)\) without any symmetry assumptions on the prescribed measure \(\mu\).

For \(q \leq 0\), the dual Minkowski problem is well-studied for all dimensions. The following theorem is obtained in [37, 51].

**Theorem 1.2 ([37, 51]).** Consider (1.3) and (1.7) for \(q < 0\).

(a) Let \(f\) be a positive smooth function. Then (1.7) has a unique positive, smooth, strictly convex solution. Equivalently there is a smooth and strictly convex body in \(K_0^n\) solving the dual Minkowski problem (1.3).
(b) Let \(\mu\) be a non-zero finite Borel measure on \(S^n\), not concentrated in any closed hemisphere. There exists a unique convex body in \(K_0^n\) solving the dual Minkowski problem (1.3).

\(^2\)In the sequel, we also use \(\theta \in S^1\) to denote the outer normal vector of the convex body associated to \(u\). We will keep this ambiguity to avoid more notations.
For $q = 0$, the dual Minkowski problem is equivalent to the Aleksandrov problem \cite{2}, i.e. the characterisation problem of integral Gauss curvature. In fact $\Omega \in \mathcal{K}_0^n$ satisfies (1.3) if and only if its polar body
\[
\Omega^* = \{ z \in \mathbb{R}^{n+1} : z \cdot y \leq 1 \text{ for all } y \in \Omega \}
\]
solves the Aleksandrov problem \cite{31}. This problem may not have a solution in general. Aleksandrov \cite{2} found the necessary and sufficient conditions on $\mu$ for the solvability of the problem. Later on Oliker \cite{45} also provided a variational proof for the existence of solutions. The regularity was obtained by Pogorelov \cite{46} and Oliker \cite{44}.

**Theorem 1.3** (\cite{2,44,45,46}). Consider (1.3) and (1.7) for $q = 0$.

(a) Let $f$ be a positive smooth function. Then (1.7) has a unique (up to a dilation) positive, smooth, strictly convex solution if and only if $f$ satisfies
\[
\int_{S^n} f = |S^n|, \text{ and } \int_\omega f < |S^n| - |\omega^*|
\]
for any spherically convex subset set $\omega \subseteq S^n$. Here $|\cdot|$ denotes the $n$-dimensional Hausdorff measure, and $\omega^* = \{ \xi \in S^n : x \cdot \xi \leq 0, \forall x \in \omega \}$ is the dual set of $\omega$.

(b) Let $\mu$ be a non-zero finite Borel measure on $S^n$. Then there exists a unique (up to a dilation) convex body in $\mathcal{K}_0^n$ solving (1.3) if and only if $\mu$ satisfies
\[
\mu(S^n) = |S^n|, \text{ and } \mu(\omega) < |S^n| - |\omega^*|
\]
for any spherically convex subset set $\omega \subseteq S^n$.

We can also employ a flow method to prove the regularity of the solution to Aleksandrov problem by using the weak solutions as barriers, see \cite{37}.

By Theorem 1.1-1.3, the dual Minkowski problem for the planar case is quite understood if $\mu$ is absolutely continuous with respect to the standard measure of $S^1$.

Our idea of the proof of Theorem 1.1 is to use a degree-theoretic argument. The key aspect of this theory is that the degree remains invariant under continuous deformations of the equation as long as there are uniform a priori estimates for all solutions of the equation. For a given $q \in (0, \infty)$, we consider a family of equations
\[
u_{\theta\theta} + u = \frac{\sqrt{u^2 + u_\theta^2}^{2-q}}{u} f_t \text{ on } S^1,
\]
where $t \in [0, 1]$, and

\begin{equation}
(1.11)\quad f_t = 1 + t(f - 1) \quad \text{and} \quad q_t = 2 + t(q - 2).
\end{equation}

Fix $\alpha \in (0, 1)$, and denote by $C^2_c(S^1)$ the subset of functions from $C^{2,\alpha}(S^1)$ which are positive and strictly convex (i.e. $\kappa = (u_{\theta\theta} + u)^{-1} > 0$). We define a family of operators $\Psi_t : C^2_c(S^1) \to C^\alpha(S^1)$ by

\begin{equation}
(1.12)\quad \Psi_t[u] = u_{\theta\theta} + u - \frac{\sqrt{u^2 + u_{\theta}^{2q}}}{u} f_t.
\end{equation}

The crucial ingredient of using the degree theory is to obtain the a priori estimates

\begin{equation}
(1.13)\quad C_0 \leq u_t \leq C_1
\end{equation}

for two positive constants $C_0$ and $C_1$ which are allowed to depend on $q$ and the lower and upper bounds of $f$, but independent of $t$, where $u_t$ are solutions of equation (1.10). Since $q > 0$, our key estimate (1.13) does not follow from the maximum principle. Estimate (1.13) will be shown in Section 2 & 3 by a careful analysis, and is the core of this paper. Once this estimate is established, it is easy to construct an open bounded subset $O$ of $C^2_c(S^1)$ such that

$$\Psi_t^{-1}(0) \cap \partial O = \emptyset \quad \text{for} \quad 0 \leq t \leq 1.$$ 

Hence the degree $\deg(\Psi_t, O, 0)$ is well-defined for all $0 \leq t \leq 1$ and is independent of $t$, see e.g. [38, 43]. We now need to do a calculation verifying that for $t = 1$ the degree $\deg(\Psi_1, O, 0) \neq 0$. Note that, when $t = 0$, equation (1.10) is exactly the equation of self-similar solutions of the curve shortening flow. Gage-Hamilton [23] proved that any convex curve evolving by the curve shortening flow must converge to a circle after a proper rescaling. Shortly after that, Grayson [24] showed Gage-Hamilton’s theorem is still true for any embedded curve without the convexity assumption, while an alternative proof is also given by Huisken [32]. Therefore equation (1.10) for $t = 0$ has only one solution $u_0 \equiv 1$. One can check that the linearised operator $L_{0, u_0} : v \in C^2_c(S^1) \mapsto \delta_u \Psi_0|_{u_0} [v] \in C^\alpha(S^1)$ has a trivial kernel. It follows

$$\deg(\Psi_1, O, 0) = \deg(\Psi_0, O, 0) \neq 0.$$ 

This shows that (1.8) admits at least one solution.

It is worth comparing the dual Minkowski problem with the $L_p$-Minkowski problem, which was introduced by Lutwak [41] when developing the $L_p$ Brunn-Minkowski theory.
It amounts to solve the following equation

\[(1.14) \quad \det(\nabla^2 u + aI) = u^{p-1}f \text{ on } \mathbb{S}^n.\]

This problem was intensively investigated in the last two decades, see e.g. [10, 13, 14, 17, 26, 29, 30, 33, 36, 39, 41] and the references therein. The $L_p$-Minkowski problem is variational, namely \((1.14)\) is the Euler-Lagrangian equation of some functional. Hence variational argument can be employed to show the existence of solutions, see e.g. [17]. As shown in [31], equation \((1.7)\) is also variational. For $q > 0$, the variational argument is effective for seeking origin-symmetric solutions to the dual Minkowski problem [31], but seems difficult to be used for non-symmetric situation. This is different from $L_p$-Minkowski problem, where the variational argument is still available to obtain non-symmetric solutions. A brief discussion about this can be found in Section 5.3.

For the planar case, equation \((1.14)\) has attracted much attention not only from convex geometry and differential equations but also from curvature flow viewpoint, as it is the equation of self-similar solutions of a class of anisotropic curve flows. There is a vast body of literature concerning the planar $L_p$-Minkowski problem [1, 3, 5, 15, 18, 19, 20, 21, 48, 49, 50]. However, even for this simplest case ($n = 1$), a neat result analogous to Theorem 1.1-1.3 is not possible for the $L_p$-Minkowski problem when $p \leq -2$, as for $p$ in this range $f$ has to satisfy some necessary condition (“obstruction”) in order that the problem is solvable [1, 15, 21, 49]. See also the discussion for higher dimensions when $p = -n - 1$ in [17, 39].

The $L_0$-Minkowski problem, also called the logarithmic Minkowski problem [7, 10, 53], is highly interesting. It is exactly the dual Minkowski problem with $q = n + 1$, and is also equivalent to finding the self-similar solutions of Gauss curvature flow. The problem is closely related to the Firey conjecture [22] which was resolved by [4, 12, 28]. The solutions to the logarithmic Minkowski problem may not be unique in general. It was proved by Yagisita [50] that there exists non-constant function $f$ such that the planar logarithmic Minkowski problem, or equivalently equation \((1.8)\) with $q = 2$, admits at least two solutions. However it is not clear that if the set of all solutions is compact. Our estimate \((1.13)\) (when $t = 1$) implies that, for any given $q$ and function $f \in C^{k,\alpha}(\mathbb{S}^1)$ pinched between two positive constants, the set of solutions to \((1.8)\) lies in a compact subset of $C^{k+2,\alpha}(\mathbb{S}^1)$. See a discussion for this in Section 5.1. In particular it implies a compactness result for the logarithmic Minkowski problem in the plane.
The paper is organised as follows. In Section 2, we prove that if a positive convex function \( u \) solves equation (1.8) for a given \( q \), then the convex body \( \Omega = \Omega_u \) which is uniquely determined by \( u \) has a good shape. Namely the ratio of circumradius and inradius of \( \Omega \) is bounded by a constant that only depends on \( q \) and the upper and lower bounds of \( f \). Once the good shape of \( \Omega \) is obtained, it is not hard to conclude that \( u \) has a uniform upper bound, again depending only on \( q \) and \( f \). The upper bound estimate will be given in Section 3. We shall also show that \( u \) is uniformly away from zero in Section 3. These estimates imply (1.13). Section 4 is then devoted to the proof of Theorem 1.1 by using the degree theory. Several remarks will be presented in Section 5, and further applications and perspectives will be also discussed there. In particular, our method can be applied to more general nonlinear ordinary differential equations, which will be discussed in Section 5.2.

2. Good Shape estimate

Let \( \Omega \) be a convex body in \( \mathbb{R}^2 \) and \( u \) be its support function. Then
\[
\Omega = \Omega_u := \bigcap_{\theta \in \mathbb{S}^1} \{ z \in \mathbb{R}^2 : \theta \cdot z \leq u(\theta) \}.
\]
If \( u \) is a strictly convex function on \( \mathbb{S}^1 \), then convex body \( \Omega_u \), defined by the RHS of (2.1), has support function \( u \). Associated to \( \Omega \) is a width function \( w = w_\Omega : \mathbb{S}^1 \rightarrow \mathbb{R} \) which is defined as \( w(\theta) = u(\theta) + u(-\theta) \). The width function is determined by the geometry of \( \Omega \) and is clearly independent of the choice of the origin. The maximum \( w_\Omega^+ \) and \( w_\Omega^- \) values of \( w(\theta) \) are called the diameter and minimal width respectively. We say that \( \Omega \) is of good shape if \( w_\Omega^+ \leq C^*w_\Omega^- \) for some constant \( C^* \) under the control. In this section we aim to obtain the good shape estimate as follows.

**Proposition 2.1.** For a fixed \( q > 0 \), assume that \( u \) is a positive, smooth, strictly convex solution to (1.8). Let \( \Omega \in \mathbb{K}_0^1 \) be the convex body given by (2.1). If \( f \geq \lambda_0 \) for some constant \( \lambda_0 > 0 \), then there is a constant \( C_q^* \) depending only on \( q \), \( \lambda_0 \) and \( |f|_{L^\infty(S^1)} \), such that
\[
w_\Omega^+ \leq C_q^*w_\Omega^-.
\]
Moreover, our constant \( C_q^* \) is uniformly bounded when \( q \) varies in a compact sub-interval of \((0, +\infty)\).
We first show some lemmas that will be used in the proof of Proposition 2.1.

**Lemma 2.1.** Let $f \in L^\infty(S^1)$ with $f \geq \lambda_0$ for some constant $\lambda_0 > 0$. Given $q > 0$, assume that $u$ is a positive, smooth, strictly convex solution to (1.8). If $u(\theta)$ is monotone in a connected sub-arc $I \subseteq S^1$, then for any $\theta_0, \theta_1 \in I$

\begin{align}
|f|_{L^\infty(S^1)} \ln \frac{u(\theta_1)}{u(\theta_0)} &\geq \frac{1}{q} \left(r^q(\mathcal{A}^*(\theta_1)) - r^q(\mathcal{A}^*(\theta_0))\right), \text{ provided } u(\theta_1) \geq u(\theta_0),
\end{align}

and

\begin{align}
\lambda_0 \ln \frac{u(\theta_1)}{u(\theta_0)} &\leq \frac{1}{q} \left(r^q(\mathcal{A}^*(\theta_1)) - r^q(\mathcal{A}^*(\theta_0))\right), \text{ provided } u(\theta_1) \geq u(\theta_0),
\end{align}

where $r(\mathcal{A}^*(\theta))$ is given in (1.5). Note that $u(\theta)$ is monotone in $I$ if and only if $r(\mathcal{A}^*(\theta))$ is monotone in $I$.

Let $\Omega = \Omega_u$ and $\gamma = \partial\Omega$. For any $\theta \in S^1$, it is well-known that $r(\mathcal{A}^*(\theta))$ is exactly the distance from the origin to the point in $\gamma$ at where the unit outer normal of $\gamma$ is $\theta$. Hence the inequality (2.3) contains useful geometric information.

**Proof of Lemma 2.1.** It is easy to see

\[\frac{d}{d\theta} r(\mathcal{A}^*(\theta)) = d\frac{d}{d\theta} \sqrt{u^2 + u^2_\theta} = \frac{u_\theta(u_\theta + u)}{r(\mathcal{A}^*(\theta))} = \frac{u_\theta}{u} r^{1-q}(\mathcal{A}^*(\theta)) f,\]

which gives

\begin{align}
\frac{1}{q} \frac{d}{d\theta} r^q(\mathcal{A}^*(\theta)) = f \frac{d}{d\theta} (\ln u).
\end{align}

Suppose $u$ is non-increasing in $I$, then $u_\theta \leq 0$ in $I$. If $u(\theta_1) \geq u(\theta_0)$, then integrating (2.5) from $\theta_1$ to $\theta_0$ gives

\[\frac{1}{q} \left(r^q(\mathcal{A}^*(\theta_0)) - r^q(\mathcal{A}^*(\theta_1))\right) \geq |f|_{L^\infty(S^1)} \ln \frac{u(\theta_0)}{u(\theta_1)}.
\]

This is (2.3). When $u$ is non-decreasing in $I$, then $u_\theta \geq 0$ in $I$. Hence we get (2.3) by integrating (2.5) from $\theta_0$ to $\theta_1$. A similar argument yields (2.4).

From (2.5), one can see that $u(\theta)$ is monotone in $I$ if and only if $r(\mathcal{A}^*(\theta))$ is monotone in $I$.  

\[\square\]
Lemma 2.2. Assume that $u$ is a positive, smooth, strictly convex solution to (1.8). Then

$$\int_{\mathcal{A}^*(\omega)} r^q(\xi) d\xi = \int_{\omega} f, \text{ for any Borel subset } \omega \subseteq \mathbb{S}^1,$$

where $r : \mathbb{S}^1 \rightarrow \mathbb{R}$ is the radial function of $\Omega_u$.

Proof. By (1.6), our equation (1.8) implies that

$$r^q(\mathcal{A}^*(\theta))|\text{Jac}\mathcal{A}^*(\theta)| = f(\theta).$$

Therefore (2.6) follows from the variable change formula. □

This lemma shows that in smooth category the dual Minkowski problem (1.3) is equivalent to our equation (1.8) when $n = 1$.

Lemma 2.3. Assume that $u$ is a positive, smooth, strictly convex solution to (1.8). Suppose $f \geq \lambda_0$ for some constant $\lambda_0 > 0$.

(i) If $0 < q \leq 2$, then

$$w_{\Omega_u}^- w_{\Omega_u}^+ \geq \frac{|S^1|}{8} \lambda_0^\frac{2}{q}.$$

(ii) If $q > 2$, then

$$w_{\Omega_u}^- (w_{\Omega_u}^+)^{q-1} \geq \frac{|S^1|}{8} \lambda_0.$$

Proof. By (2.6),

$$\int_{S^1} r^q = \int_{S^1} f \geq \lambda_0 |S^1|.$$

If $0 < q \leq 2$, then by Hölder inequality

$$\int_{S^1} r^q \leq |S^1|^\frac{2-q}{q} \left( \int_{S^1} r^2 \right)^\frac{q}{2}.$$

Hence

$$|S^1|^{-\frac{2-q}{q}} \left( \int_{S^1} r^2 \right)^\frac{q}{2} \leq \frac{1}{2} \int_{S^1} r^2 = \text{Volume}(\Omega_u).$$

By John’s lemma [35], there is an ellipsoid $E$ centred at the origin and a point $z_E \in \mathbb{R}^2$ such that $E \subseteq \Omega_u - z_E \subseteq 2E$. Clearly

$$w_E^+ \leq w_{\Omega_u}^+ \text{ and } w_E^- \leq w_{\Omega_u}^-,$$
and therefore

\[ (2.11) \quad \text{Volume}(\Omega_u) \leq \text{Volume}(2E) \leq 4w_E^- w_E^+ \leq 4w_{\Omega_u}^- w_{\Omega_u}^+. \]

We then infer (2.7) by plugging (2.9) and (2.11) in (2.10).

If \( q > 2 \), we have, letting \( r_{\max} = \max_{\xi \in S^1} r(\xi) \),

\[
\int_{S^1} r^q = r_{\max}^q \int_{S^1} \left( \frac{r(\xi)}{r_{\max}} \right)^q d\xi \leq r_{\max}^q \int_{S^1} \left( \frac{r(\xi)}{r_{\max}} \right)^2 d\xi = 2r_{\max}^{q-2} \text{Volume}(\Omega_u).
\]

By (2.11), and noting that \( r_{\max} \leq w_{\Omega_u}^+ \), we get

\[
\int_{S^1} r^q \leq 8w_{\Omega_u}^- (w_{\Omega_u}^+)^{q-1}.
\]

This together with (2.9) shows (2.8).

\[ \square \]

We are now able to finish the good shape estimate.

**Proof of Proposition 2.1.** In this proof, we use \( C_q^* \) to denote a constant which depends only on \( q, \lambda_0 \) and \( |f|_{L^\infty(S^1)} \) and is positive, uniformly bounded when \( q \) varies in a compact sub-interval of \((0, +\infty)\) as required in Proposition 2.1. We will also use \( C \) to denote a universal constant.

Our idea is as follows. We find a suitable sub-arc \( I \) of \( S^1 \) on which \( u(\theta) \) is monotone, and so (2.3) in Lemma 2.1 can be applied. If the L.H.S. and the R.H.S. of (2.3) can be estimated in terms of \( C_q^* \ln(w_{\Omega_u}^+/w_{\Omega_u}^-) \) and \( (w_{\Omega_u}^+)^q/C_q^* \) respectively, then by Lemma 2.3 we conclude that the L.H.S. of (2.3) grows like \( \ln w_{\Omega_u}^+ \) while the R.H.S. of (2.3) grows like \( (w_{\Omega_u}^+)^q \), which is not possible if \( w_{\Omega_u}^+ \) is too large. The good shape estimate (2.2) then follows from Lemma 2.3. However some bad situations may occur, which will be ruled out by making use of Lemma 2.2.

Let us first fix a coordinate. By John’s lemma we have an ellipsoid centred at the origin and a point \( z_E \in \mathbb{R}^2 \) such that

\[ (2.12) \quad E + z_E \subseteq \Omega_u \subseteq 2E + z_E. \]
Let us assume \( E = \{ z = (z_1, z_2) \in \mathbb{R}^2 : z_1^2/a_1^2 + z_2^2/a_2^2 \leq 1 \} \) with
\[
(2.13) \quad a_1 \geq a_2 > 0.
\]

One still has some freedom to choose the positive direction for the \( z_i \)-axis, denote by \( e_i \).
We choose \( e_1 \) such that
\[
(2.14) \quad u(e_1) \geq u(-e_1).
\]

For convenience, we denote
\[
d = u(-e_1) \quad \text{and} \quad L = u(e_1).
\]

Note that \( z_E \in \{ z_1 < \frac{4}{5}L \} \). Otherwise \( a_1 \leq \frac{1}{5}L \), which contradicts with \( \Omega_u \subseteq 2E + z_E \) (as this inclusion implies \( L \leq 4a_1 \)). Consider the segment \( AB = \Omega_u \cap \{ z_1 = \frac{4}{5}L \} \).

Let \( C \) be the intersection of \( \Omega_u \) and \( \{ z_1 = L \} \), \( \ell_A \) (resp. \( \ell_B \)) be the straight line passing through \( A, C \) (resp. \( B, C \)), and \( \ell_{z_E} \) be the vertical line passing through \( z_E \). Denote \( A' = \ell_A \cap \ell_{z_E}, B' = \ell_B \cap \ell_{z_E} \) and \( A''B'' = (E + z_E) \cap \ell_{z_E} \) (see Figure 1).

As \( A, B, C \) are on the boundary of \( \Omega_u \), we have by convexity \( |A'B'| \geq |A''B''| = 2a_2 \).

Hence
\[
|AB| = \frac{\frac{4}{5}L}{\text{dist}(C, \ell_{z_E})} |A'B'| \geq \frac{\frac{4}{5}L}{2L} \cdot 2a_2 = \frac{1}{5}a_2.
\]

We now choose \( e_2 \) such that if \( A, B \) are chosen to satisfy \( \overrightarrow{BA} // e_2 \) then
\[
(2.15) \quad A \text{ lies above } \left( \frac{4}{5}L, \frac{1}{10}a_2 \right).
\]

Assumptions (2.14) and (2.15) are needed in our later discussion.

![Figure 1](image-url)
Denote \( l = u(e_2) \). Note that by (2.12), (2.14) and (2.15),
\[
\frac{1}{10}a_2 \leq l \leq 4a_2 \quad \text{and} \quad a_1 \leq L \leq 4a_1.
\]
Consequently,
\[
C^{-1}w^+_{\Omega_u} \leq \frac{L}{l} \leq Cw^+_{\Omega_u}.
\]
Hence estimate (2.2) is equivalent to
\[
\frac{L}{l} \leq C_q^*.
\]
Under the coordinate chosen above, we denote by \( m^* \) the \( z_1 \)-component of the point \( \vec{r}_{\Omega_u}(\mathcal{A}^*_u(e_2)) \in \Omega_u \). In what follows we sometimes omit the subscript “\( \Omega_u \)” if no confusion arises. Let us split our argument into two cases.

**CASE I**: \( m^* \leq \frac{3}{4}L \).

Let \( \mathcal{U} = \{ \vec{r}(\mathcal{A}^*(\theta)) : \theta \cdot e_2 \geq 0 \} \), i.e., the upper half portion of \( \partial \Omega_u \), squeezed between vertical lines \( \{ z_1 = -d \} \) and \( \{ z_1 = L \} \), connecting \( \vec{r}(\mathcal{A}^*(-e_1)) \) with \( \vec{r}(\mathcal{A}^*(e_1)) \). Clearly \( \mathcal{U} \) can be represented as the graph
\[
\mathcal{U} = \{ \gamma(s) = (s, \varphi(s)) \in \mathbb{R}^2 : -d \leq s \leq L \}
\]
for a concave function \( \varphi \).

We denote by \( \phi(s) \) the angle between \( e_2 \) and the unit outer normal of \( \Omega_u \) at \( \gamma(s) \), by \( \tau(s) \) the angle between \( e_1 \) and \( \overrightarrow{O\gamma(s)} \). It is not hard to see
\[
|\tan \tau(s)| \leq \frac{4a_2}{3L} \leq C\frac{w^-_{\Omega_u}}{w^+_{\Omega_u}}, \quad \text{for} \quad s \in \left[ \frac{3}{4}L, \frac{4}{5}L \right],
\]
and by convexity
\[
|\tan \phi(s)| \leq \frac{4a_2}{5L} \leq C\frac{w^-_{\Omega_u}}{w^+_{\Omega_u}}, \quad \text{for} \quad s \in \left[ \frac{3}{4}L, \frac{4}{5}L \right].
\]
We always assume
\[
\frac{w^+_{\Omega_u}}{w^-_{\Omega_u}} \gg 1 \quad \text{or equivalently} \quad \frac{L}{l} \gg 1,
\]
only otherwise we are done. Then (2.18) and (2.19) show
\[
\begin{align*}
\{ \tau(s) : \frac{3}{4}L \leq s \leq \frac{4}{5}L \} & \subseteq [-\frac{\pi}{100}, \frac{\pi}{100}], \\
\{ \phi(s) : \frac{3}{4}L \leq s \leq \frac{4}{5}L \} & \subseteq [-\frac{\pi}{100}, \frac{\pi}{100}],
\end{align*}
\]
We remark that to conclude (2.18), (2.19) and (2.21), the conditions “\( m^* \leq \frac{3}{4}L \)” and (2.15) are not used. Note that (2.21) implies the distance from the origin to \( \gamma(s) \), namely \( \sqrt{s^2 + \varphi^2(s)} \) is a monotone function when \( s \) varies in \( [\frac{3}{4}L, \frac{4}{5}L] \). Hence we can apply (2.3) of Lemma 2.1 to conclude that

\[
(2.22) \quad r^q(\mathcal{A}^*(\theta_1)) - r^q(\mathcal{A}^*(\theta_0)) \leq C_q^* \ln \frac{u(\theta_1)}{u(\theta_0)},
\]

where \( \theta_0 \) and \( \theta_1 \) denote the unit outer normal of \( \Omega_u \) at \( \gamma(\frac{3}{4}L) \) and \( \gamma(\frac{4}{5}L) \) respectively.

By (2.20),

\[
(2.23) \quad r^q(\mathcal{A}^*(\theta_1)) - r^q(\mathcal{A}^*(\theta_0)) \geq L^q / C_q^*.
\]

Since \( m^* \leq \frac{3}{4}L \), it then follows from (2.15) and (2.16) that

\[
\varphi(\frac{3}{4}L) \geq \varphi(\frac{4}{5}L) \geq \frac{a_2}{10}.
\]

Consequently, by (2.19),

\[
u(\theta_0) \geq C^{-1}a_2 \geq C^{-1}w_{\Omega_u}^-.\]

Using Lemma 2.3 we deduce that

\[
(2.24) \quad \ln \frac{u(\theta_1)}{u(\theta_0)} \leq \ln \frac{w_{\Omega_u}^+}{w_{\Omega_u}} + C_q^* \leq C_q^*(\ln w_{\Omega_u}^+ + 1) \leq C_q^*(\ln L + 1).
\]

Plugging (2.23) and (2.24) in (2.22), we obtain

\[
(2.25) \quad w_{\Omega_u}^+ \leq CL \leq C_q^*.
\]

By Lemma 2.3 we further infer that

\[
\frac{w_{\Omega_u}^+}{w_{\Omega_u}^-} \leq \begin{cases} \frac{C_q^*(w_{\Omega_u}^+)^2}{w_{\Omega_u}^-}, & \text{if } 0 < q \leq 2, \\ \frac{C_q^*(w_{\Omega_u}^+)^q}{w_{\Omega_u}^-}, & \text{if } q > 2. \end{cases}
\]

This together with (2.25) proves the good shape estimate.

**CASE II:** \( m^* > \frac{3}{4}L \).

Denote by \( \theta(s) \) the unit outer normal of \( \Omega_u \) at \( \gamma(s) \). We assume that \( \delta := u(\theta(\frac{1}{2}L)) \) is very small, namely

\[
(2.26) \quad \delta \leq lL^{-\beta}
\]

for some \( \beta = \beta_q > 1 \), to be determined, which only depends on \( q \) and is uniformly bounded when \( q \) varies in a compact sub-interval of \( (0, +\infty) \). Note that \( \varphi(s) \) is a non-decreasing function when \( s \in [0, m^*] \). Hence \( \varphi(\mathcal{A}^*(\theta(s))) \) is monotone in \( s \in [0, m^*] \). If
δ does not satisfy (2.26), Lemma 2.1 tells us

\begin{equation}
(2.27) \quad r^q(\mathcal{A}^*(e_2)) - r^q(\mathcal{A}^*(\theta(\frac{1}{2}L))) \leq C_q^* \ln \frac{u(e_2)}{u(\theta(\frac{1}{2}L))} \leq \beta C_q^* \ln L.
\end{equation}

The left hand side above can be estimated as

\begin{equation}
(2.28) \quad r^q(\mathcal{A}^*(e_2)) - r^q(\mathcal{A}^*(\theta(L))) \geq L \frac{q}{C_q^*}.
\end{equation}

By (2.27) and (2.28), we get an upper bound of \( L \), and therefore, by an analogous argument as in CASE I, the good shape estimate is proved.

In what follows we show (2.17) in case of (2.26). Consider the sub-arc

\[ \omega = \{ \theta(s) : -d \leq s \leq \frac{1}{2}L \} \subset S^1. \]

As above, let \( \phi(s) \) be the angle between \( e_2 \) and \( \theta(s) \). By convexity,

\begin{equation}
(2.29) \quad \tan(\frac{1}{2}L) \leq \frac{1}{\frac{1}{2}L} \ll 1.
\end{equation}

Then \( \{ (\cos \vartheta, \sin \vartheta) : 51\pi/100 \leq \vartheta \leq \pi \} \subset \omega \). Consequently

\begin{equation}
(2.30) \quad \int_{\omega} f \geq \frac{49\pi}{100} \lambda_0.
\end{equation}

We next estimate \( \int_{\mathcal{A}^*(\omega)} r^q \) and use Lemma 2.2 to derive the upper bound for \( L \).

To this end, we shall also consider the lower half part of \( \partial \Omega_u \), namely \( \hat{U} = \{ \mathcal{A}^*(\theta) : \theta \cdot e_2 \leq 0 \} \). Let \( \hat{\varphi} \) be the convex function on \([-d, L] \) such that

\[ \hat{U} = \{ \hat{\gamma}(s) = (s, \hat{\varphi}(s)) \in \mathbb{R}^2 : -d \leq s \leq L \}. \]

In the sequel we use \( \hat{\theta}(s) \) to denote the unit outer normal of \( \Omega_u \) at \( \hat{\gamma}(s) \).
Denote $P = \gamma(\frac{1}{2}L)$. Let $\ell_P$ be the supporting line of $\Omega_u$ at $P$, $\ell_P^+ = \{t\theta(\frac{1}{2}L) : t \in \mathbb{R}^1\}$ be the straight line which passes through the origin and is perpendicular to $\ell_P$. Denote $Q = \ell_P^+ \cap \ell_P$. Let $C_1$ be the planar cone delimited by rays $\{t\overrightarrow{OQ} : t \geq 0\}$ and $\{t\overrightarrow{OP} : t \geq 0\}$, and $C_2 = \{z = (z_1, z_2) : z_1 \leq 0\}$ be the left half plane. Denote $\tilde{\omega}_1 = C_1 \cap S^1$ and $\tilde{\omega}_2 = C_2 \cap S^1$.

Then $\mathscr{A}^*(\omega) \subset \tilde{\omega}_1 \cup \tilde{\omega}_2$.

For any $\xi \in \tilde{\omega}_1$, let $P_\xi = \{t\xi : t \geq 0\} \cap \ell_P$ and $\tilde{r}_1(\xi) = |P_\xi|$. We have, by convexity (see $\triangle OPQ$ in Figure 2),

\[(2.31) \quad \int_{\tilde{\omega}_1} r_{\tilde{\omega}_1}^q \leq \int_{\tilde{\omega}_1} \tilde{r}_1^q \leq \int_{\{\xi \in \tilde{\omega}_1 : \tilde{r}_1(\xi) \leq 1\}} \tilde{r}_1^q + \int_{\{\xi \in \tilde{\omega}_1 : \tilde{r}_1(\xi) > 1\}} \tilde{r}_1^q\]

where $q' = \min\{\frac{1}{2} q, 1\}$. It is straightforward to see

\[
\int_{\{\xi \in \tilde{\omega}_1 : \tilde{r}_1(\xi) > 1\}} \tilde{r}_1^q \leq |\overrightarrow{OP}|^q \cdot \{\xi \in \tilde{\omega}_1 : \tilde{r}_1(\xi) > 1\} \leq C_q^* L^q (\delta - \delta/L) \leq C_q^* L^q \delta.
\]

On the other hand,

\[
\int_{\{\xi \in \tilde{\omega}_1 : \tilde{r}_1(\xi) \leq 1\}} \tilde{r}_1^q \leq \int_0^{\frac{\pi}{2}} \arcsin \frac{\delta}{\cos \vartheta} (\frac{\delta}{\cos \vartheta})^{q'} d\vartheta \leq C_q^* \int_{\arcsin \delta}^{\frac{\pi}{2}} (\frac{\delta}{\vartheta})^{q'} d\vartheta \leq C_q^* \delta^{q'}.
\]

Plugging the above two inequalities in (2.31), and using (2.26), we obtain

\[(2.32) \quad \int_{\tilde{\omega}_1} r_{\tilde{\omega}_1}^q \leq C_q^* \delta^{q'} (1 + \delta^{1-q'}L^q) \leq C_q^* L^{-1},\]

by suitably choosing $\beta = \beta_q$.

Let $h = \varphi(0)$ and $\hat{h} = -\hat{\varphi}(0)$. By (2.26) and (2.29) we have

\[(2.33) \quad h \leq 2|\overrightarrow{OQ}| = 2\delta \leq C L^{-\beta + 1}.
\]

We claim that

\[(2.34) \quad \hat{h} \leq C L^{-\beta + 1}.
\]

For this, let $\hat{\varphi}(s)$ be the angle between $\theta(s)$ and $-\varepsilon_2$, and $\hat{\tau}(s)$ be the angle between $\overrightarrow{O\hat{\gamma}(s)}$ and $e_1$. We observe that (2.21) is still true if $\phi(s)$ and $\tau(s)$ are replaced by $\hat{\phi}(s)$
and \( \tilde{\tau}(s) \) respectively. Hence, similar to CASE I, we can apply Lemma 2.1 to the portion 
\( \{ \tilde{\gamma}(s) : \frac{3}{4}L \leq s \leq \frac{4}{5}L \} \) and conclude that
\[
L^q \leq C_q^* \ln \frac{u(\tilde{\theta}(\frac{4}{5}L))}{u(\tilde{\theta}(\frac{3}{4}L))} \leq C_q^* \ln \frac{w_{\tilde{\tau}}^+}{u(\tilde{\theta}(\frac{3}{4}L))}.
\]
This shows
\[
(2.35) \quad u(\tilde{\theta}(\frac{3}{4}L)) \leq CL^{-\beta+1}.
\]
Since \( \tilde{\theta}(\frac{3}{4}L) \) lies in a small neighbourhood of \(-e_2\) in \( S^1 \), we obtain (2.34) from (2.35).

Let us look at the triangle \( \triangle_0 \) whose vertices are \( z_0 + e_2, z_0 - e_2, \) and \( \gamma (-d) \). By convexity, \( \triangle_0 \subseteq \Omega_u \). Hence, by (2.33), (2.34) and Lemma 2.3, we obtain
\[
(2.36) \quad d \leq \frac{(h + \tilde{\gamma})}{2a_2} 4a_1 \leq L^{-\frac{\beta}{2}},
\]
provided that one properly chooses \( \beta = \beta_q \).

Now consider the rectangle (see Figure 2)
\[
\mathcal{R} = \{(z_1, z_2) \in \mathbb{R}^2 : -d \leq z_1 \leq 0, -2a_2 \leq z_2 \leq 2a_2 \}.
\]
For any \( \xi \in \tilde{\omega}_2 \setminus \{\pm e_2\} \), let \( Q_\xi = \{t\xi : t \geq 0\} \cap \partial \mathcal{R} \), and \( \tilde{r}_2(\xi) = |OQ_\xi| \). Obviously
\[
(2.37) \quad \int_{\tilde{\omega}_2} \tilde{r}_2^q \leq \int_{\tilde{\omega}_2} \tilde{r}_2^q + \int_{\{\xi \in \tilde{\omega}_2 : \tilde{r}_2 \geq 1\}} \tilde{r}_2^q \leq C_q^* a_2 \frac{d}{\sin \vartheta} \left( \frac{d}{\sin \vartheta} \right)^q \vartheta \leq C_q^* L^{-1},
\]
by choosing suitable \( \beta = \beta_q \).

If \( a_2 \) is bounded, the situation is better, and the above estimate is certainly valid. Hence (2.37) can be estimated as
\[
\int_{\tilde{\omega}_2} r_{\tilde{\omega}}^q \leq C_q^* L^{-1}.
\]
This together with (2.30) and (2.32) shows
\[
\frac{\pi}{3} \lambda_0 \leq \int \int_{\tilde{\omega}_1} r_{\tilde{\omega}_u}^q + \int \int_{\tilde{\omega}_2} r_{\tilde{\omega}_u}^q \leq C_q^* L^{-1}.
\]
So \( L \leq C_q^* \). Thus we complete the proof by the same argument as in CASE I.

\[ \square \]

3. Upper/Lower bound estimates on support function

We first show that solutions to (1.8) have an upper bound. This is a direct consequence of Proposition 2.1.

**Proposition 3.1.** For a fixed \( q > 0 \), assume that \( u \) is a positive, smooth, strictly convex solution to (1.8). If \( f \geq \lambda_0 \) for some constant \( \lambda_0 > 0 \), then there is a constant \( C_q^* \) depending only on \( q, \lambda_0 \) and \( |f|_{L^\infty(S^1)} \), such that

\[
1/C_q^* \leq w_{\Omega_u}^+ \leq C_q^*.
\]

The constant \( C_q^* \) is uniformly bounded when \( q \) varies in a compact sub-interval of \((0, +\infty)\).

**Proof.** Denote by \( r \) the radial function of \( \Omega_u \). If \( q \geq 2 \), then by Lemma 2.2 and Hölder inequality

\[
\int_{S^1} f = \int_{S^1} r^q \geq C_q^* \left( \int_{S^1} r^2 \right)^{\frac{q}{2}} = C_q^* (\text{Volume}(\Omega_u))^{\frac{q}{2}} \geq C_q^* (w_{\Omega_u}^+)^q.
\]

The last inequality is due to Proposition 2.1. This implies \( w_{\Omega_u}^+ \leq C_q^* \). If \( 0 < q < 2 \), we have, by using \( r/w_{\Omega_u}^+ \leq 1 \),

\[
\int_{S^1} f = (w_{\Omega_u}^+)^q \int_{S^1} \left( \frac{r}{w_{\Omega_u}^+} \right)^q \geq (w_{\Omega_u}^+)^q \int_{S^1} \left( \frac{r}{w_{\Omega_u}^+} \right)^2 \geq C_q^* (w_{\Omega_u}^+)^{q-2} \text{Volume}(\Omega_u) \geq C_q^* (w_{\Omega_u}^+)^q.
\]

The last inequality is again by using Proposition 2.1. Hence the RHS inequality in (3.1) holds. Another inequality in (3.1) follows from Lemma 2.3.

\[ \square \]

The main goal in this section is to prove the following lower bound estimate.

**Proposition 3.2.** For a fixed \( q > 0 \), assume that \( u \) is a positive, smooth, strictly convex solution to (1.8). If \( f \geq \lambda_0 \) for some constant \( \lambda_0 > 0 \), then there is a constant \( C_q^* \) depending only on \( q, \lambda_0 \) and \( |f|_{L^\infty(S^1)} \), such that

\[
\min_{S^1} u(\theta) \geq 1/C_q^*.
\]

The constant \( C_q^* \) is uniformly bounded when \( q \) varies in a compact sub-interval of \((0, +\infty)\).
Proof. Let us choose the coordinate such that
\[ u(-e_1) = \min_{\theta} u(\theta) \text{ and } u(e_2) \geq u(-e_2), \]
where \( e_i \) are the directions of the \( z_i \)-axis. We denote, as in Proposition 2.1,
\[ d = u(-e_1), \quad L = u(e_1), \quad \text{and} \quad l = u(e_2). \]

It follows from Propositions 2.1 & 3.1 that
\[ \frac{1}{C_q^*} \leq l \leq L \leq C_q^*. \]

Let \( U = \{ \vec{r}(\mathcal{A}^*(\theta)) : \theta \cdot e_2 \geq 0 \} \subset \partial \Omega \). Then there is a convex function \( \varphi : [-d, L] \to \mathbb{R} \) such that
\[ U = \{ \gamma(s) = (s, \varphi(s)) \in \mathbb{R}^2 : -d \leq s \leq L \}. \]

Let \( m^* \in [-d, L] \) be the value such that \( (m^*, \varphi(m^*)) = \vec{r}(\mathcal{A}^*(e_2)) \). There are two possibilities: either \( m^* > 0 \), or \( m^* \leq 0 \). We divide our argument into two cases correspondingly.

**Case I:** \( m^* > 0 \).

Denote by \( \theta(s) \) the unit outer normal of \( \Omega_u \) at \( \gamma(s) \). Let \( P = \gamma(0), \) and \( \delta = u(\theta(0)) \).
Clearly \( r(\mathcal{A}^*(\theta(s))) \) is non-decreasing when \( s \) varies in \([0, m^*]\). We infer from (2.4) in Lemma 2.1 that
\[ \ln \frac{l}{\delta} = \ln \frac{u(\theta(m^*))}{u(\theta(0))} \leq C_q^* \left( r^q(\mathcal{A}^*(\theta(m^*))) - r^q(\mathcal{A}^*(\theta(0))) \right) \leq C_q^*. \]
Consequently

(3.4) \[ \delta \geq 1/C_q^*. \]

Denote \( h = \varphi(0) \). It follows from (3.4) that

(3.5) \[ 1/C_q^* \leq h \leq l \leq C_q^*. \]

Since \( u(\theta(0)) = \tilde{r}(\mathcal{A}^*(\theta(0))) \cdot \theta(0) \), we have \( \delta = he_2 \cdot \theta(0) \). Hence, by (3.4) and (3.5),

(3.6) \[ \theta(0) \cdot e_2 \geq 1/C_q^*. \]

Let \( \omega = \{ \theta(s) \in \mathbb{S}^1 : -d < s < 0 \} \). Then (3.6) implies

(3.7) \[ \int_{\omega} f \geq \lambda_0 |\omega| \geq 1/C_q^*. \]

Clearly

\[ \mathcal{A}^*(\omega) = \{ (\cos \vartheta, \sin \vartheta) : \vartheta \in (\frac{\pi}{2}, \pi) \}. \]

We consider the rectangle (see Figure 3)

\[ \mathcal{R} = \{ (z_1, z_2) \in \mathbb{R}^2 : -d \leq z_1 \leq 0, 0 \leq z_2 \leq h \}. \]

For any \( \xi \in \mathcal{A}^*(\omega) \), \{ \{ t\xi : t \geq 0 \} \cap \partial \mathcal{R} \) is a single point, which we denote by \( P_\xi \). Let \( \tilde{r}(\xi) = |\overrightarrow{OP_\xi}| \). Since \( d = u(-e_1) = \min_{\mathbb{S}^1} u \), we have \( d \leq h \). Hence \( \tilde{r}(\xi) \leq 2h \) for all \( \xi \in \mathcal{A}^*(\omega) \). Let \( q' = \min\{\frac{1}{2}q, \frac{1}{2}\} \). By (3.5), one infers that

\[ \int_{\mathcal{A}^*(\omega)} \tilde{r}^{q'}_{\Omega_u} \leq \int_{\mathcal{A}^*(\omega)} \tilde{r}^{q'} \]

\[ \leq (2h)^{q'} \int_{\mathcal{A}^*(\omega)} (\frac{\tilde{r}}{2h})^{q'} \]

\[ \leq C_q^* \int_{\mathcal{A}^*(\omega)} \tilde{r}^{q'} \]

\[ = C_q^* \int_{\tilde{\omega}_1} \tilde{r}^{q'} + C_q^* \int_{\tilde{\omega}_2} \tilde{r}^{q'} \]

where \( \tilde{\omega}_1 = \{ (\cos \vartheta, \sin \vartheta) : \vartheta \in (\frac{\pi}{2}, \frac{\pi}{2} + \arctan \frac{d}{h}) \} \) and \( \tilde{\omega}_2 = \{ (\cos \vartheta, \sin \vartheta) : \vartheta \in [\frac{\pi}{2} + \arctan \frac{d}{h}, \pi) \} \). Without loss of generality, we assume \( d \ll 1 \). We have by (3.5)

\[ \int_{\tilde{\omega}_1} \tilde{r}^{q'} \leq (2h)^{q'} \arctan \frac{d}{h} \leq C_q^* d, \]

and, by \( q' < 1 \),

\[ \int_{\tilde{\omega}_2} \tilde{r}^{q'} = \int_{\arctan \frac{d}{h}}\frac{d}{\sin \vartheta} \left( \frac{d}{\sin \vartheta} \right)^{q'} d\vartheta \leq C_q^* d^{q'}. \]
Therefore

\[(3.8) \quad \int_{\partial^* (\omega)} r_q^q \sigma_{\omega} \leq C_q^* d' \cdot \]

Using Lemma 2.2, we conclude from (3.7) and (3.8) that \( d \geq 1/C_q^* \), thus achieving (3.2).

**CASE II**: \( m^* \leq 0 \).

Recall that \( l = u(e_2) = \varphi(m^*) \). Let \( R \) be a rectangle given by (see Figure 4)

\[\mathcal{R} = \{(z_1, z_2) \in \mathbb{R}^2 : -d \leq z_1 \leq 0, \ 0 \leq z_2 \leq l\},\]

and let \( \tilde{\omega} \) be an open sub-arc in \( S^1 \) defined as

\[\tilde{\omega} = \{(\cos \vartheta, \sin \vartheta) : \vartheta \in (\frac{\pi}{2}, \pi)\} .\]

Since \(-d < m^* \leq 0\), we deduce

**CASHE (3.9) \[|\mathcal{A}(\tilde{\omega})| \geq \frac{\pi}{2}.\]

For any \( \xi \in \tilde{\omega} \), let \( P_\xi = \{t \xi : t \geq 0\} \cap \partial \mathcal{R} \), and \( \tilde{r}(\xi) = |\overrightarrow{OP_\xi}|\). Making use of Lemma 2.2, we have by (3.9)

**CASHE (3.10) \[\frac{\pi}{2} \lambda_0 \leq \int_{\mathcal{A}(\tilde{\omega})} f = \int_{\tilde{\omega}} r_q^q \leq \int_{\tilde{\omega}} \tilde{r}^q \leq C_q^* d' \cdot\]

where \( q' = \min\{\frac{1}{2} q, \frac{1}{2}\} \). The last inequality above is derived by a similar way as for (3.8). Hence \( d \geq 1/C_q^* \). We finish the proof.

4. **Proof of Theorem 1.1**

In this section, we use the degree theory to finish the proof of Theorem 1.1.

For fixed \( q \), we consider equations (1.10). When \( t \in [0, 1] \), \( q_t \) varies between \( q \) and 2. Hence the uniform estimate (1.13) follows from Proposition 3.1 & 3.2. The uniform \( C^1 \) estimate for \( u_t \) follows by the convexity. By (1.5), one sees that \( \max_{S^1} |\nabla u_t| \leq w^+_t \). It is straightforward to deduce from equation (1.10) that

**CASE (4.1) \[|u_t|_{C^2, \alpha(S^1)} \leq C_2 ,\]

where the positive constant \( C_2 \) depends only on \( q \), \( \lambda_0 \) and \( |f|_{C^0(S^1)} \), but is independent of \( t \).
Recall that $C^2_c(S^1)$ is defined as a subset of $C^{2,\alpha}(S^1)$, consisting of all strictly convex functions on $S^1$. We now introduce an open bounded subset $O$ of $C^2_c(S^1)$,

$$O = \{ v \in C^2_c(S^1) : \frac{1}{2}C_0 < v < 2C_1, |v|_{C^{2,\alpha}(S^1)} < 2C_2 \}.$$ 

Clearly, the operator $\Psi_t$, given by (1.12), satisfies

$$\Psi_t^{-1}(0) \cap \partial O = \emptyset$$

for all $t \in [0,1]$. Therefore the degree $\deg(\Psi_t, O, 0)$ is well-defined for all $0 \leq t \leq 1$ and is independent of $t$; see [38, 43].

As mentioned before, for $t = 0$, equation (1.10) has a unique solution $u_0 \equiv 1$. This is because the equation describes the self-similar solution of curve shortening flow, and by Gage-Hamilton’s Theorem [23] it must be a circle; see also [5, 9, 24, 32]. It is not hard to see that the linearised operator of $\Psi_0$ at $u_0$ is given by

$$\mathcal{L}_{0,u_0}[v] = v_{\theta\theta} + 2v.$$ 

If $\mathcal{L}_{0,u_0}(v) = 0$, then $v$ satisfies

$$(4.2) \quad v_{\theta\theta} + 2v = 0 \text{ on } S^1 = \mathbb{R}^1/2\pi\mathbb{Z}.$$ 

If $v$ satisfies (4.2) in $\mathbb{R}^1$, then $v$ must be in the form of

$$v(\theta) = a \sin(\sqrt{2}\theta) + b \cos(\sqrt{2}\theta),$$

where $a, b$ are two constants. So $v$ cannot be $2\pi$-periodic. This implies that $\mathcal{L}_{0,u_0}$ has a trivial kernel. Hence

$$\deg(\Psi_t, O, 0) = \deg(\Psi_0, O, 1) \neq 0,$$

which implies that equation (1.10) has at least one solution for all $t \in [0,1]$, in particular for $t = 1$. The higher order regularity of the solutions follows from differentiating (1.8) repeatedly. The assertion (a) in Theorem 1.1 is proved.

It remains to show (b) in Theorem 1.1. Given a function $f \in L^\infty(S^1)$ such that $f \geq \lambda_0$ for some constant $\lambda_0 > 0$. We take a sequence of smooth functions $f_i \in C^\infty(S^1)$ such that $f_i$ converges to $f$ in $L^1$ sense and $f_i \geq \lambda_0$ for all $i$. By (a) of Theorem 1.1 there is a convex body $\Omega_i \in \mathcal{K}_1$ such that

$$\widetilde{C}_q(\Omega_i, \omega) = \int_\omega f_i, \text{ for any Borel set } \omega \subseteq S^1.$$
Our a priori estimates in Proposition 3.1 & 3.2 imply that, by Blaschke selection theorem, there is a convergent subsequence of \(\{\Omega_i\}\) such that
\[
\Omega_{i_j} \to \Omega \in \mathcal{K}_0^1
\]
in the Hausdorff metric. By the weak convergence of the dual curvature measure (see Lemma 3.6 in [31]), we deduce from (4.3) that
\[
\tilde{C}_q(\Omega, \omega) = \hat{\omega}_f, \quad \text{for any Borel set } \omega \subseteq S^1,
\]
which proves (b) of Theorem 1.1.

5. Remarks, applications and perspectives

This section is devoted to the remarks and comments. We shall also discuss some further applications and perspectives.

5.1. Comments on the a priori estimates.

We would like to point out that it is not necessary to require \(f\) has a positive lower bound in order to obtain the good shape estimate (Proposition 2.1) and the upper bound estimate (Proposition 3.1). The main ingredients in the proof of Proposition 2.1 are:

(i) using (2.3) of Lemma 2.1 to control \(w_{\Omega u}^+\);
(ii) employing Lemma 2.2 to rule out some bad situations.

We also use Lemma 2.3 in the proof of Proposition 2.1 but estimates (2.7) and (2.8) are valid if \(\lambda_0\) is replaced by \(f\). It is clear that (2.3) of Lemma 2.1 only requires the upper bound of \(f\). For (ii), the condition \(f \geq \lambda_0\) is used only in (2.30). Given \(\xi \in S^1\) and \(\rho\), we define
\[
\omega_{\xi, \rho} = \{\eta \in S^1 : \text{dist}_{S^1}(\eta, \xi) \leq \rho\},
\]
where \(\text{dist}_{S^1}(\cdot, \cdot)\) denotes the spherical distance of two points in \(S^1\). It is not hard to see that our proof of Proposition 2.1 still works if we assume that there is a constant \(\rho < \frac{\pi}{4}\) such that
\[
\int_{\omega_{\xi, \rho}} f > 0, \quad \text{for all } \xi \in S^1.
\]
This condition is certainly weaker than \(f \geq \lambda_0 > 0\).

Our second remark is that Propositions 3.1 & 3.2 yield the compactness of the set of solutions. Let \(f\) be a positive function in \(C^{k, \alpha}(S^1)\) for some \(\alpha \in (0, 1)\) By part (a)
of Theorem 1.1 and an approximation if necessary, it is easy to see that our equation (1.8) has a positive solution in \( C^{2+k,\alpha}(S^1) \). Denote by \( S_{q,f} \) all the solutions to equation (1.8). By [50], \( S_{q,f} \) may not be a singleton set in general. Our a priori estimates in Propositions 3.1 & 3.2 imply that even if \( S_{q,f} \) contains infinite many elements, there are positive constant \( \varepsilon \) and \( C \) only depending on \( q \) and \( f \) such that \( S_{q,f} \) lies in a compact subset \( K_{q,f} \) of \( C^{2+k,\alpha}(S^1) \),

\[
S_{q,f} \subseteq K_{q,f} := \{ v \in C^{2+k,\alpha}(S^1) : |v|_{C^{2+k,\alpha}(S^1)} \leq C \}.
\]

and

\[
u \geq \varepsilon, \ \forall \ u \in S_{q,f}.
\]

Estimate (5.2) implies that the origin is uniformly away from \( \partial \Omega_u \) provided \( \Omega_u \) solves the dual Minkowski problem. This uniform lower bound is, however, no longer true when dimension \( n \geq 2 \). See the counter-example in section 6 of [17] for the logarithmic Minkowski problem.

For the \( L_p \)-Minkowski problem, it has been shown that the problem may admit multiple solutions when \( p < 0 \), and the compactness does not hold in general [29, 33, 36].

5.2. Further applications.

Our argument allows us to deal with more general \( f \). The following theorem can be proved.

**Theorem 5.1.** Let \( f : S^1 \times \mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}^1 \) be a smooth function. For \( q > 0 \), consider the nonlinear ordinary differential equation

\[
u_{\theta\theta} + \nu + \sqrt{u^2 + u^2_{\theta} - q} f(\theta, u, u_{\theta}) \text{ on } S^1.
\]

If there are two positive constants \( \lambda, \Lambda \) such that

\[
\lambda \leq f(\theta, h, z) \leq \Lambda, \ \forall \ (\theta, h, z) \in S^1 \times \mathbb{R}^1 \times \mathbb{R}^1,
\]

then (5.3) has a positive, smooth, strictly convex solution.

**Proof.** Note that any positive, strictly convex smooth function \( u \) determines a unique convex body \( \Omega_u \) via (2.1).

The proofs of Lemmata 2.1 & 2.3 and the proofs of Proposition 2.1, 3.1 & 3.2 require only the lower and upper bounds of \( f \). Lemma 2.2 is also available if \( f \) is a function
of more variables. Hence the same arguments used in previous sections yield estimates (3.1) and (3.2) for solutions of (5.3), once condition (5.4) is assumed.

The existence of a smooth solution to (5.3) follows by the same manner as presented in Section 4.

□

By applying a duality argument to the above theorem, we obtain the following result.

**Theorem 5.2.** Let \( f : S^1 \times S^1 \to \mathbb{R}^1 \) be a smooth function. If there are two positive constants \( \lambda, \Lambda \) such that

\[
\lambda \leq f(\theta, \xi) \leq \Lambda, \quad \forall (\theta, \xi) \in S^1 \times S^1,
\]

then for any given \( q > 0 \), there is a positive, smooth, strictly convex solution \( u \in C^\infty(S^n) \) solving the following equation

\[
u_{\theta \theta} + u = u^2 + u_\theta^2 f(\theta, \mathcal{A}^*(\theta)) \quad \text{on} \ S^1,
\]

where \( \mathcal{A}^* \) is the mapping defined in (1.1) for the convex body \( \Omega_u \).

**Proof.** By Theorem 5.1 there is a positive, smooth, strictly convex function \( v \in C^\infty(S^1) \) solving the equation

\[
v''(\xi) + v(\xi) = \frac{\sqrt{v^2 + (v')^2} - q}{v(\xi)f\left(\frac{v\xi + v'\xi^\perp}{\sqrt{v^2 + (v')^2}}, \xi\right)}, \quad \text{for} \ \xi \in S^1,
\]

where “prime” means the derivative with respect to an arc-length parameter \( s \) of \( S^1 \). This arc-length parameter gives a tangential vector along \( S^1 \), which we denote by \( \xi^\perp \).

Let \( \Omega = \Omega_v \) be the smooth convex body associated to \( v \) as given by (2.1). Let \( \Omega^* \) be the polar body of \( \Omega \), see (1.9) for the definition. Denote by \( \varrho : S^1 \to \mathbb{R} \) the radial function of \( \Omega \) and by \( u : S^1 \to \mathbb{R} \) the support function of \( \Omega^* \). It is well-known by definition (1.9) that, see e.g. [31, 47],

\[
u(\theta) = \frac{1}{\varrho(\theta)} = \frac{1}{\sqrt{v^2(\xi) + (v')^2}},
\]

where \( \theta \) and \( \xi \) are related by

\[
\xi = \frac{u\theta + u_\theta\theta^\perp}{\sqrt{u^2 + u_\theta^2}} \quad \text{and} \quad \theta = \frac{v\xi + v'\xi^\perp}{\sqrt{v^2 + (v')^2}}.
\]
Note that $\xi = \mathcal{A}^*(\theta)$. Next formula describes the relation between curvatures of $\Omega$ and $\Omega^*$, and can be found in e.g. [37],

$$(v''(\xi) + v(\xi))v^3(\xi) = \frac{1}{(u_{\theta\theta}(\theta) + u(\theta))u^3(\theta)}.$$ 

Plugging (5.7), (5.8) and (5.9) in (5.6), we find that $u$ satisfies equation (5.5). The smoothness of $u$ can be seen from (5.7), as $v \in C^\infty(S^1)$. This completes the proof.

\[\Box\]

5.3. Variational method.

Variational method has been used both in the dual Minkowski problem and the $L_p$-Minkowski problem. In [31], the authors showed that (1.7) is the Euler equation of the following functional (up to a rescaling)

$$(5.10) \quad \mathcal{J}(u) = \begin{cases} -\int_{S^n} \log u(x) d\mu(x) + \frac{1}{q} \log \left( \int_{S^n} r^q(\xi) d\xi \right), & \text{for } q \neq 0, \\ -\int_{S^n} \log u(x) d\mu(x) + \int_{S^n} \log r(\xi) d\xi, & \text{for } q = 0, \end{cases}$$

where $r$ is the radial function of the convex body $\Omega_u$ given by (2.1), $d\mu = f dx$, and

$$\int_{S^n} g d\mu := \frac{1}{\mu(S^n)} \int_{S^n} g d\mu$$

and

$$\int_{S^n} g d\xi := \frac{1}{|S^n|} \int_{S^n} g d\xi.$$

For the case $q = 0$, we assume $|\mu(S^n)| = |S^n|$. The solution to (1.3) can be obtained by maximising (5.10). This approach is successful for $q < 0$ [51], for $q = 0$ [45], and for $0 < q \leq n + 1$ and $\mu$ is even [10, 31]. For non-symmetric case, it seems hard to derive the needed a priori estimates for maximising sequences of (5.10) when $q > 0$.

For the $L_p$-Minkowski problem, following the argument of Chou-Wang [17], it can be seen that (1.14) is the Euler equation of the functional below (again up to a rescaling)

$$(5.11) \quad \Phi(u) = \mathcal{I}(u) + \frac{1}{n+1} \log \left( \text{Volume}(\Omega_u) \right),$$

where

$$(5.12) \quad \mathcal{I}(u) = \begin{cases} -\frac{1}{p} \log \left( \int_{S^n} u^p(x) d\mu(x) \right), & \text{for } p \neq 0, \\ -\int_{S^n} \log u(x) d\mu(x), & \text{for } p = 0, \end{cases}$$

Given $u$, let $z \in \Omega_u$. One can consider more general $\tilde{\mathcal{I}}(u, z)$ by replacing $u$ in (5.12) with

$$u_z(x) = u(x) - z \cdot x.$$
This is equivalent to translate the centre of the convex body $\Omega_u$ from the origin to $z$.

We observe the following:

(P1) $I$ is homogeneous of degree zero, i.e. $I(tu) = I(u)$ for all $t > 0$;

(P2) The second term on the right hand side of (5.11) is only related to the volume of $\Omega_u$ and hence is independent of the choice of centre;

(P3) Suppose that $u$ is smooth and positive. Given any $\eta \in \mathbb{R}^{n+1}$, then for $p = 0$

$$
\frac{d^2}{dt^2}|_{t=0} I(u, z + t\eta) = \int_{\mathbb{S}^n} \frac{(\eta \cdot x)^2}{u_z^2} d\mu(x) \geq 0,
$$

and for $p \neq 0$

$$
\frac{d^2}{dt^2}|_{t=0} I(u, z + t\eta) = p \left( \frac{\int_{\mathbb{S}^n} \frac{\eta \cdot x}{u_z^{2-p}} d\mu(x)}{\int_{\mathbb{S}^n} u_z^p d\mu(x)} \right)^2 + (1 - p) \left( \frac{\int_{\mathbb{S}^n} \eta \cdot x}{\int_{\mathbb{S}^n} u_z^{1-p} d\mu(x)} \right)^2.
$$

Hence, if $p < 1$, we have as in [6],

$$
\frac{d^2}{dt^2}|_{t=0} I(u, z + t\eta) \geq \frac{1 - p}{\left( \frac{\int_{\mathbb{S}^n} u_z^p d\mu(x)}{\int_{\mathbb{S}^n} u_z d\mu(x)} \right)^2} \left[ \left( \frac{\int_{\mathbb{S}^n} u_z^p d\mu(x)}{\int_{\mathbb{S}^n} u_z^{2-p} d\mu(x)} \right) - \left( \frac{\int_{\mathbb{S}^n} \eta \cdot x}{\int_{\mathbb{S}^n} u_z^{1-p} d\mu(x)} \right)^2 \right]
$$

$$
= \frac{1 - p}{\left( \frac{\int_{\mathbb{S}^n} u_z^p d\mu(x)}{\int_{\mathbb{S}^n} u_z d\mu(x)} \right)^2} \cdot \frac{1}{2} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{(u_z(x)(y \cdot \eta) - u_z(y)(x \cdot \eta))^2}{u_z^{2-p}(x)u_z^{2-p}(y)} d\mu(x) d\mu(y) \geq 0.
$$

Finding a maximiser for $\Phi(u)$ is more or less equivalent to solving the following min-max problem introduced by [17]

$$
(5.13) \quad \sup_u \left\{ \inf_{\xi \in \Omega_u} I(u, \xi) : \text{Volume}(\Omega_u) = 1 \right\}.
$$

This can be seen from Properties (P1) and (P2). Property (P3) tells us $I(u, z)$ is convex with respect to $z$ when $p < 1$. Hence it is possible to solve the minimisation problem $\inf_{\xi \in \Omega_u} I(u, \xi)$, namely there is a unique $\xi_{\Omega_u}$ such that the infimum is achieved. Moreover it has been shown that $\xi_{\Omega_u}$ varies in a Lipschitz way when we perturb $\Omega_u$ smoothly, which enables one to show if $\Omega$ solves the min-max problem (5.13) then it also solves the $L_p$-Minkowski problem [17]. By considering a modification of (5.13), and using the Blaschke-Santaló inequality, Chou-Wang [17] established the needed a priori estimates and thus resolved the $L_p$-Minkowski problem for $-n - 1 < p < 1$. 

27
There are some similarities between (5.10) and (5.11). For the dual Minkowski problem, one may hope to introduce an analogous min-max problem in order to find non-symmetric solutions for \( q > 0 \). However, property (P2) is no longer true for functional (5.10), as the second term there certainly depends on the choice of centre unless \( q = n+1 \).

Given \( z \in \Omega_u \), one may consider the following \( q \)-volume, which is a little bit general than the second term appearing on the right hand side of (5.10),

\[
\tilde{V}_q(u, z) := \frac{1}{q} \log \left( \int_{S^n} r_z^q(\xi) d\xi \right),
\]

where \( r_z \) is the radial function of \( \Omega_u \) with respect to the centre \( z \). But Property (P3) is not true for \( \tilde{V}_q(u, z) \), namely \( \tilde{V}_q(u, z) \) may not be convex or concave with respect to \( z \) in general. The variational method seems difficult to be used to find non-symmetric solutions to the dual Minkowski problem when \( q > 0 \).

5.4. Remark on Theorem 1.2

Part (b) of Theorem 1.2 is obtained in [51] by a variational argument. An alternative proof is to use an approximation argument. The idea of approximation was also used by Chou and Wang for \( L_p \)-Minkowski problem (see the proof for Part (b) of Theorem A in [17]). Let \( \{\mu_j\}, d\mu_j = f_j(x)dx, f_j \) positive and smooth, be a sequence of measures converging weakly to \( \mu \). By the result in (a) of Theorem 1.2 let \( u_j(x) \) be the solution of (1.7), \( \Omega_j = \Omega_{u_j} \) be the associated convex body, and \( r_j(\xi) \) be the radial function of \( \Omega_j \). Obviously

\[
(5.14) \quad \tilde{C}_q(\Omega_j, \cdot) = \mu_j.
\]

Let \( r_{j \min} = \min_{\xi \in S^n} r_j(\xi) \), and assume that \( \xi_0 \in S^n \) is the direction such that \( r_j \) attains its minimum. By convexity

\[
r_j(\xi) \leq r_{j \min} / (\xi \cdot \xi_0), \quad \text{provided} \ \xi \cdot \xi_0 \geq 1/2.
\]

Since \( q < 0 \), the above inequality yields the following estimate

\[
\int_{S^n} f_j = \int_{S^n} r_j^q \geq r_{j \min}^q \int_{\{\xi \in S^n : \xi \cdot \xi_0 \geq 1/2\}} (\xi \cdot \xi_0)^{-q} d\xi \geq r_{j \min}^q \delta_n,
\]

where \( \delta_n > 0 \) is a universal constant only depending on the dimension \( n \). Consequently, there is a \( \delta_0 > 0 \), only depending on \( n \) and \( \mu(S^n) \), such that

\[
(5.15) \quad r_{j \min} \geq \delta_0.
\]
On the other hand, letting $\alpha = n + 1 - q$, we have by (1.7),

$$\text{Volume}(\Omega_j) = \frac{1}{n+1} \int_{\mathbb{S}^n} u_j \det(\nabla^2 u_j + u_j I) \geq \frac{1}{n+1} \int u_j^\alpha d\mu_j.$$ 

Let $x_0 \in \mathbb{S}^n$ be the direction such that $u_j$ achieves its maximum $u_j \max$. Then $u_j(x) \geq u_j \max x_0 \cdot x$ for all $x$ such that $x \cdot x_0 > 0$. Therefore

$$u_j \max^{n+1} \text{Volume}(B_1) \geq \text{Volume}(\Omega_j) \geq \int_{\{x \in \mathbb{S}^n : x \cdot x_0 > 0\}} u_j^\alpha d\mu_j \geq u_j \max^\alpha \delta_1$$

where $B_1$ is the unit ball in $\mathbb{R}^{n+1}$ and $\delta_1 > 0$ is a constant only depending on $\mu$. Since $\alpha > n + 1$, we have

(5.16) \quad u_j \max \leq C = C(n, \mu).

Making use of (5.15) and (5.16), we have by Blaschke selection theorem that $\Omega_j$ subsequently converges to a convex body $\Omega \in K_0^n$. It follows from the weak convergence of the dual curvature measure [31] and (5.14) that $\Omega$ satisfies (1.3).

References


CENTRE FOR MATHEMATICS AND ITS APPLICATIONS, THE AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT 2601, AUSTRALIA.

E-mail address: shibing.chen@anu.edu.au

CENTRE FOR MATHEMATICS AND ITS APPLICATIONS, THE AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT 2601, AUSTRALIA.

E-mail address: qi-rui.li@anu.edu.au