VARIATIONS OF A CLASS OF MONGE-AMPE`RE TYPE FUNCTIONALS AND THEIR APPLICATIONS

HAODI CHEN, SHIBING CHEN, AND QI-RUI LI

ABSTRACT. In this paper, we study a class of Monge-Amp`ere type functionals arising from the $L_p$ dual Minkowski problem in convex geometry. As an application, we obtain some existence and non-uniqueness results for the problem.

1. Introduction

The characterisation problem of geometric measures in convex geometry has a long history and strong influence on fully nonlinear PDEs. A best known example is the classical Minkowski problem. For a full discussion on this problem and its resolution, one may consult Cheng-Yau [18] and Pogorelov [34]. Other important geometric measures in Brunn-Minkowski theory include curvature measures and area measures, and the associated problems of prescribing curvature and area measures were also intensively studied. See Schneider’s book [35] for a comprehensive introduction.

Most recently Lutwak-Yang-Zhang [33] introduced the $L_p$ dual curvature measures and proposed the associated Minkowski type problem. Let $K_0$ be the set of all convex bodies (i.e., compact convex sets that have non-empty interior) in $\mathbb{R}^{n+1}$ containing the origin in their interiors. Associated to each $\Omega \in K_0$ are the support function $u = u_\Omega : S^n \to \mathbb{R}$ and the radial function $r = r_\Omega : S^n \to \mathbb{R}$, which are respectively defined by $u(x) = \max\{x \cdot z : z \in \Omega\}$, and $r(\xi) = \max\{\lambda : \lambda \xi \in \Omega\}$. Let $\tilde{r}(\xi) = \tilde{r}_\Omega(\xi) := r_\Omega(\xi)\xi$. Then $\partial \Omega = \{\tilde{r}(\xi) : \xi \in S^n\}$. Denote by $\nu = \nu_\Omega : \partial \Omega \to S^n$ the spherical image, namely $\nu(z) = \{x \in S^n : z \cdot x = u_\Omega(x)\}$. With these notions in hand, the radial Gauss mapping $\mathcal{A} = \mathcal{A}_\Omega$ and the reverse radial Gauss mapping $\mathcal{A}^* = \mathcal{A}_\Omega^*$ are defined as follows: for any $\omega \subseteq S^n$,

\begin{align}
\mathcal{A}(\omega) &= \{\nu(\tilde{r}(\xi)) : \xi \in \omega\}, \\
\mathcal{A}^*(\omega) &= \{\xi \in S^n : \nu(\tilde{r}(\xi)) \in \omega\}.
\end{align}

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In [33] the $L_p$ dual curvature measures $\tilde{C}_{p,q}(\Omega, \cdot)$, where $p, q \in \mathbb{R}$, are a two-parameter family of Borel measures on $\mathbb{S}^n$, defined by

$$\tilde{C}_{p,q}(\Omega, \omega) = \int_{\mathbb{S}^n(\omega)} \frac{r^q(\xi)}{u^p(\mathcal{S}(\xi))} d\sigma_{\mathbb{S}^n}(\xi).$$

The associated Minkowski type problem was posed by Lutwak-Yang-Zhang [33]: Given a finite Borel measure $\mu$ on $\mathbb{S}^n$, find necessary and sufficient conditions on $\mu$ so that it is the $L_p$ dual curvature measure of a convex body. If $\mu$ is absolutely continuous w.r.t. $\sigma_{\mathbb{S}^n}$ and $f = \frac{d\mu}{d\sigma_{\mathbb{S}^n}}$ is the Radon-Nikodym derivative, then, in terms of the support function $u$, the problem reduces to the following Monge-Ampère equations

$$\det(\nabla^2 u + u I) = \left( u^2 + |\nabla u|^2 \right)^{n+1-q} u^{p-1} f(x) \quad \text{on } \mathbb{S}^n,$$

where $\nabla$ is the covariant derivative w.r.t. an orthonormal frame on $\mathbb{S}^n$.

The $L_p$ dual Minkowski problem includes the classical Minkowski problem as a special case, and unifies the $L_p$-Minkowski problem and dual Minkowski problem introduced in [25, 31]. There is a large number of papers devoted to these problems, see e.g. [4, 7, 16, 17, 19, 21, 30, 32] for the $L_p$-Minkowski problem, and [5, 15, 24, 25, 29, 38] for the dual Minkowski problem.

The $L_p$-Minkowski problem amounts to solve (1.3) with $q = n + 1$. It is of particular interest, as the problem describes the self-similar solutions to the flows by powers of the Gauss curvature [3, 22]:

$$\partial_t X(x, t) = -K^\alpha(x, t)\nu(x, t),$$

where $X(\cdot, t)$ is a time-dependent embedding of a family of convex hypersurfaces $\mathcal{M}_t$, $K(\cdot, t)$ and $\nu(\cdot, t)$ denote the Gauss curvature and unit outer normal of $\mathcal{M}_t$ respectively. In fact the self-similar solutions to (1.4) satisfy (1.3) with $f \equiv 1$ and $p = 1 - 1/\alpha$. For $\alpha = 1$, flow (1.4) was first studied by Firey [20] to model the shape change of tumbling stones. It was conjectured that, when $\alpha > 1/(n + 2)$, flow (1.4) deforms each convex hypersurface in $\mathbb{R}^{n+1}$ into a round point. Andrews proved the conjecture for the case $n = 1$ in [2], and for the case $n = 2$ and $\alpha = 1$ in [1]. Very recently Brendle-Choi-Daskalopoulos [8] resolved this conjecture for all dimensions $n \geq 2$. This shows that $u \equiv 1$ is the only solution to (1.3) when $q = n + 1$, $p \in (-n - 1, 1)$ and $f \equiv 1$. However

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1 Lutwak-Yang-Zhang's $L_p$ dual curvature measure [33] is more general than (1.2), as their definition allows a dependence of a fixed star body $Q$ (i.e. a compact star-shaped set about the origin). If $Q$ is taken as the unit ball $B_1 \subseteq \mathbb{R}^{n+1}$, then their conception is formulated by (1.2).
for non-constant \( f \), the \( L_p \)-Minkowski problem admits multiple solutions when \( p \leq 0 \) \cite{23, 27, 28, 37}.

For general \( p, q \in \mathbb{R} \), the existence of solutions to (1.3) was partially addressed in \cite{6, 12, 14, 26}, and the uniqueness of solutions was proved for \( p > q \) \cite{26, 33}. The main goal of this paper is to show a non-uniqueness result for the \( L_p \) dual Minkowski problem.

We say \( u \in C^2(S^n) \) is uniformly convex if \( u \) is the support function of a convex body whose boundary has uniformly positive principal curvatures. Our main result is the following.

**Theorem 1.1.** Let \( f \equiv 1 \). Then equation (1.3) admits an even, smooth, uniformly convex, positive solution \( u \neq 1 \), provided that \( p, q \in \mathbb{R} \) satisfy one of the following

(A1) \( q - 2n - 2 > p \geq 0 \),
(A2) \( q > 0 \) and \(-q^* < p < \min\{0, q - 2n - 2\}\), where \( q^* \) is given in (1.9) below.
(A3) \( p + 2n + 2 < q \leq 0 \).

Clearly \( u \equiv 1 \) is a solution to (1.3) for \( f \equiv 1 \). Hence our Theorem 1.1 shows that if one of (A1)-(A3) holds, then besides the unit ball \( B_1 \) there is another origin-symmetric convex body \( \Omega \) whose \( L_p \) dual curvature measure coincides with the standard spherical measure \( \sigma_{S^n} \). Since (1.5) is not affine-invariant unless \( q = -p = n + 1 \), ellipsoids are not solutions to the problem in general. In \cite{24} the authors showed that, if \( f \equiv 1, n = 1, p = 0 \), and \( q \) is an even integer no less than 6, then (1.3) has a non-constant solution. Our Theorem 1.1 (A1) extends their result.

Theorem 1.1 follows from Theorems 1.2 & 1.3 below. Both theorems are proved by studying a Monge-Ampère type functional (1.5). For any finite Borel measure \( \sigma \) on \( S^n \) and integrable function \( g \), we use the following convention:

\[
\int_{S^n} g d\sigma := \frac{1}{\sigma(S^n)} \int_{S^n} g d\sigma.
\]

Let \( \mu \) and \( \mu^* \) be two finite Borel measures on \( S^n \). Consider the functional:

\[
J_{p,q,\mu,\mu^*}(\Omega) = \Phi_{p,\mu}(\Omega) + \Psi_{q,\mu^*}(\Omega), \quad \text{for } \Omega \in \mathcal{K}_0,
\]

where

\[
\Phi_{p,\mu}(\Omega) = \begin{cases} 
-\frac{1}{p} \log \int_{S^n} u_\Omega^p(x) d\mu(x), & \text{if } p \neq 0, \\
-\int_{S^n} \log u_\Omega(x) d\mu(x), & \text{if } p = 0,
\end{cases}
\]

and

\[
\Psi_{q,\mu^*}(\Omega) = \int_{S^n} \frac{1}{q} \sigma_{S^n}(\Omega) d\mu^*(x).
\]
and

\begin{equation}
\Psi_{q,\mu^*}(\Omega) = \begin{cases} \\
\frac{1}{q} \log \int_{\mathbb{S}^n} r_\Omega^q(\xi) d\mu^*(\xi), & \text{if } q \neq 0, \\
\int_{\mathbb{S}^n} \log r_\Omega(\xi) d\mu^*(\xi), & \text{if } q = 0.
\end{cases}
\end{equation}

Observe that $J_{p,q,\mu,\mu^*}$ is homogeneous degree zero, namely

\begin{equation}
J_{p,q,\mu,\mu^*}(t\Omega) = J_{p,q,\mu,\mu^*}(\Omega), \quad \forall t > 0.
\end{equation}

For convenience, we shall omit sometimes the subscript $\mu$ (or $\mu^*$) in (1.5)-(1.7) if $\mu$ (or $\mu^*$) is exactly the standard spherical measure. We will see that, up to a rescaling, (1.3) is the Euler equation of functional (1.5) for $d\mu = f d\sigma^n$ and $d\mu^* = d\sigma^n$.

Let $\mathcal{K}_0 \subset \mathcal{K}_0$ be the set of all origin-symmetric convex bodies. By a Blaschke-Santaló type inequality [13], we are able to use a variational argument to prove Theorem 1.2 below, which shows the existence of origin-symmetric solutions to the $L_p$ dual Minkowski problem.

To our best knowledge, even for the symmetric measures, Theorem 1.2 under condition (B2) gives the first existence result for the problem when $p < 0$, $q > 0$ and $q \neq n + 1$, hence is of particular interest. We point out that, under condition (B1) or (B3), the existence of origin-symmetric solutions was obtained in [12, 26] and in [29] for $p = 0$. As this existence result is needed in our main result Theorem 1.1, we still include a proof in this paper for reader’s convenience.

**Theorem 1.2.** Let $d\mu^* = d\sigma^n$, $d\mu = f d\sigma^n$, $f$ be an even function on $\mathbb{S}^n$, and $1/C \leq f \leq C$ for some constant $C > 0$. Assume that $p, q \in \mathbb{R}$ satisfy one of the following

(B1) $p \geq 0$ and $q \geq 0$;

(B2) $q > 0$ and $-q^* < p < 0$, where $q^* > 0$ is defined as

\begin{equation}
q^* = \begin{cases} \\
\frac{q}{q-n} & \text{if } q > n + 1, \\
n + 1 & \text{if } q = n + 1, \\
nq & \text{if } 1 < q < n + 1, \\
\frac{q-1}{q-1} & \text{if } 0 < q \leq 1.
\end{cases}
\end{equation}

(B3) $p \leq 0$ and $q \leq 0$. 

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Then there is a convex body \( \Omega_0 \in \mathcal{K}_0 \) such that
\[
J_{p,q,\mu}(\Omega_0) = \max\{J_{p,q,\mu}(\Omega) : \Omega \in \mathcal{K}_0\}.
\]
Moreover \( \partial \Omega_0 \) is strictly convex and is \( C^{1,\gamma} \) for some \( \gamma \in (0,1) \), and satisfies
\[
\tilde{C}_{p,q}(\Omega_0,\omega) = \lambda_{\Omega_0} \int_{\omega} f d\sigma_{S^n}, \quad \text{for any Borel set } \omega \subseteq S^n,
\]
where
\[
\lambda_{\Omega_0} = \frac{\int_{S^n} r^q d\sigma_{S^n}}{\int_{S^n} \omega d\mu}.
\]
If \( f \) is additionally smooth, then the support function \( u_{\Omega_0} \) is an even, smooth, uniformly convex, positive solution to (1.3) with \( f \) replaced by \( \lambda_{\Omega_0} f \).

For \( p \neq q \), \( \tilde{\Omega}_0 := \frac{1}{\lambda_{\Omega_0}} \Omega_0 \) solves the \( L_p \) dual Minkowski problem, namely \( \tilde{\Omega}_0 \) satisfies
\[
\tilde{C}_{p,q}(\tilde{\Omega}_0,\omega) = \int_{\omega} f d\sigma_{S^n}, \quad \text{for any Borel set } \omega \subseteq S^n,
\]
a and \( u_{\tilde{\Omega}_0} \) is an even, smooth, uniformly convex, positive solution to (1.3), provided \( f \) is additionally smooth.

Remark 1.1. In [26], the existence of solutions to (1.3) when \( f \) is not necessarily even was obtained for \( p > q \). When \( p \leq q \), the existence result, without evenness assumption on \( f \), becomes much more difficult. It was available for \( p > -n - 1 \) and \( q = n + 1 \) [19], and for \( p > 1 \) and \( q > 0 \) [6]. In a subsequent paper [14], we will prove that, for \( p > 0 \) and all \( q \in \mathbb{R} \), the problem admits a weak solution if the prescribed measure \( \mu \) is not concentrated on any closed hemisphere, while the evenness of \( \mu \) is not required.

Remark 1.2. As in [4], we are able to prove by approximation the \( L_p \) dual Minkowski problem admits an origin-symmetric solution when \( p, q \) satisfy condition (B2) in Theorem 1.2 and \( f \) is an even and nonnegative function on \( S^n \), \( \int_{S^n} f d\sigma_{S^n} > 0 \), and \( L^{q^*} \)-integrable (when \( q^* \neq +\infty \)) or \( L^s \)-integrable for some \( s > 1 \) (when \( q^* = +\infty \)). See Theorem 3.2 in Section 3 below.

We then show that, if \( \mu = \mu^* = \sigma_{S^n} \) and \( q > p + 2n + 2 \), then the unit ball \( B_1 \) is not a maximiser of (1.10). This together with Theorem 1.2 proves Theorem 1.1.

Theorem 1.3. Let \( \mu = \mu^* = \sigma_{S^n} \). If \( q > p + 2n + 2 \), then there is an even function \( \eta \in C^\infty(S^n) \), and a small \( \varepsilon > 0 \), such that the convex body \( \Omega_{\varepsilon} \in \mathcal{K}_0 \), whose support
function is \( u(x, t) = 1 + t\eta(x) \), satisfies

\[
(1.14) \quad J_{p,q}(B_1) < J_{p,q}(\Omega_t), \quad \text{for } t \in (0, \varepsilon).
\]

This paper is organised as follows. In Section 2, we calculate the first and second variations of functional \((1.5)\). We show that \( B_1 \) is an unstable critical point of the functional \( J_{p,q} \) provided \( q > p + 2n + 2 \), which consequently proves Theorem \(1.3\). In Section 3, we prove Theorem \(1.2\) via variational argument, and then complete the proof of Theorem \(1.1\). The Poincaré inequality on \( S^n \) is related to the stability of \( B_1 \) under the functional \((1.5)\). It can be obtained by studying the eigenvalues of the spherical Laplace operator \([36]\). In Section 4, we provide an alternative proof for the Poincaré inequality with sharp constant via the uniqueness of the self-similar solution to the flow \((1.4)\) when \( \alpha > \frac{1}{n+2} \) \([1, 2, 8]\), which makes our paper self-contained.

2. Second variation for Monge–Ampère type functional \((1.5)\)

Let \( u \) and \( r \) be respectively the support function and radial function of \( \Omega \in \mathcal{K}_0 \). Given any \( \eta \in C^0(S^n) \), there is an \( \varepsilon > 0 \), depending on \( \min_{S^n} u \) and \( \max_{S^n} |\eta| \), such that \( u(x) + t\eta(x) > 0 \) for all \( x \in S^n \) and \( |t| < \varepsilon \). Consider a family of convex bodies

\[
(2.1) \quad \Omega_t = \{ z : z \cdot x \leq u(x) + t\eta(x), \; x \in S^n \}, \quad \text{for } |t| < \varepsilon.
\]

Let \( u(x, t) \) and \( r(x, t) \) be the support function and radial function of \( \Omega_t \).

**Lemma 2.1.** Suppose that \( \partial \Omega \) is \( C^1 \) and strictly convex at \( z_0 \in \partial \Omega \). Then the limits below exist

\[
\dot{u}(x_0) := \lim_{t \to 0} \frac{u(x_0, t) - u(x_0, 0)}{t},
\]

\[
\dot{r}(\xi_0) := \lim_{t \to 0} \frac{r(\xi_0, t) - r(\xi_0, 0)}{t},
\]

where \( x_0 \) is the unit outer normal of \( \partial \Omega \) at \( z_0 \) and \( \xi_0 = z_0/|z_0| = \mathcal{A}_1^*(x_0) \). Furthermore

\[
(2.2) \quad \dot{u}(x_0) = \eta(x_0),
\]

and

\[
(2.3) \quad \frac{\dot{r}(\xi_0)}{r(\xi_0)} = \frac{\dot{u}}{u}(x_0).
\]

**Proof.** By \((2.1)\) and the definition of support function, we have

\[
(2.4) \quad u(x, t) \leq u(x) + t\eta(x), \quad \text{for all } x \in S^n, \; |t| < \varepsilon.
\]
Therefore

\[(2.5) \quad \limsup_{t \to 0^+} \frac{u(x_0, t) - u(x_0, 0)}{t} \leq \eta(x_0).\]

On the other hand, let \( u_{z_0}(x) = u(x) - z_0 \cdot x \) and \( u_{z_0}(x, t) = u(x, t) - z_0 \cdot x \). Since \( \partial \Omega \) is \( C^1 \) at \( z_0 \), one infers that there exists a \( x_1 \in S^n \) so that

\[(2.6) \quad u_{z_0}(x_0, t) = (u_{z_0} + t\eta)(x_1) \quad \text{with} \quad x_t \to x_0 \quad \text{as} \quad t \to 0.\]

For this, as \( u_{z_0}(x_0, 0) = u_{z_0}(x_0) = 0 \), if \( x_{t_k} \to x_1 \) then \( u_{z_0}(x_1) = 0 \) and so \( x_1 \) is a unit outer normal at \( z_0 \). Hence \( x_1 \) must coincide with \( x_0 \). Therefore, by \( u_{z_0}(x_1) \geq 0 \),

\[(2.7) \quad \liminf_{t \to 0^+} \frac{u(x_0, t) - u(x_0, 0)}{t} = \liminf_{t \to 0^+} \frac{u_{z_0}(x_0, t) - u_{z_0}(x_0)}{t} = \liminf_{t \to 0^+} \frac{u_{z_0}(x_t) + t\eta(x_t)}{t} \geq \eta(x_0).\]

For \( t \to 0^- \), \( (2.4) \) and \( (2.6) \) give respectively

\[\liminf_{t \to 0^-} \frac{u(x_0, t) - u(x_0, 0)}{t} \geq \eta(x_0), \quad \text{and} \quad \limsup_{t \to 0^-} \frac{u(x_0, t) - u(x_0, 0)}{t} \leq \eta(x_0).\]

Hence \( (2.2) \) follows.

Next we prove \( (2.3) \). For this, let \( h(x) = (\eta/u)(x) \in C^0(S^n) \). By \( (2.2) \), we have

\[(2.8) \quad u(x_0, t) = u(x_0) + t\eta(x_0) + o(t).\]

It follows that

\[0 = -\log r(\xi_0) + \log u(x_0) - \log(\xi_0 \cdot x_0)\]

\[= -\log r(\xi_0) + \log u(x_0, t) - \log(\xi_0 \cdot x_0) + \left( \log u(x_0) - \log u(x_0, t) \right)\]

\[(2.9) \quad \geq -\log r(\xi_0) + \log r(\xi_0, t) - th(x_0) + o(t).\]

On the other hand, since \( \partial \Omega \) is strictly convex at \( z_0 \), there is a \( \xi_t \in S^n \) such that

\[-\log r(\xi_t, t) = \log(\xi_t \cdot x_0) - \log u(x_0, t) \quad \text{with} \quad \xi_t \to \xi_0 \quad \text{as} \quad t \to 0.\]

This together with \( (2.8) \) shows that

\[0 \leq -\log r(\xi_t) + \log u(x_0) - \log(\xi_t \cdot x_0)\]

\[= -\log r(\xi_t) + \log u(x_0) - \left( \log u(x_0, t) - \log r(\xi_t, t) \right)\]

\[(2.10) \quad = -\log r(\xi_0) + \log r(\xi_0, t) - th(x_0) + o(t).\]

We complete the proof by \( (2.9) \) and \( (2.10) \).
In the rest of this section, we always assume that $u$ is uniformly convex. Then the radial Gauss mapping $A$ and the reverse radial Gauss mapping $A^*$, defined by (1.1), are one-to-one mappings. Given $\omega \subset \mathbb{S}^n$, we consider the “cone-like” region inside $\Omega$

$$\mathcal{C} := \{ z \in \mathbb{R}^{n+1} : z = \lambda \nu^{-1}(x), \, \lambda \in [0, 1], \, x \in \omega \},$$

where $\nu$ denotes the spherical image of $\Omega$. It is well-known that the volume element of $\mathcal{C}$ can be expressed by

$$d \text{Vol}(\mathcal{C}) = \frac{1}{n+1} \frac{u(x)}{K(\nu^{-1}(x))} d\sigma_{\mathbb{S}^n}(x) = \frac{1}{n+1} r^{n+1}(\xi) d\sigma_{\mathbb{S}^n}(\xi),$$

where $K(z)$ is the Gauss curvature of $\partial \Omega$ at $z$. It follows that the determinants of the Jacobian of the mappings $A$ and $A^*$ are given by

$$|\text{Jac} A^*|(x) = \frac{u(x)}{r^{n+1}(A^*(x))K(\nu^{-1}(x))},$$

$$|\text{Jac} A|(\xi) = \frac{r^{n+1}(\xi)K(\tilde{r}(\xi))}{u(A(\xi))}. \quad (2.11)$$

Let $\eta \in C^\infty(\mathbb{S}^n)$. By the uniform convexity of $u$, there is a small $\varepsilon > 0$ such that, for all $|t| < \varepsilon$, (i) $\Omega_t$ defined by (2.1) lies in $K_0$; (ii) $u(x,t) := u(x) + t\eta$ is the support function of $\Omega_t$; and (iii) $u(x,t)$ is uniformly convex. Let us compute the first and second variations of functional (1.5).

**Proposition 2.1.** Let $\Omega \in K_0$ be a convex body whose support function $u$ is uniformly convex. Given $\eta \in C^\infty(\mathbb{S}^n)$, let $\Omega_t$ be the convex bodies defined by (2.1). Let $d\mu = f d\sigma_{\mathbb{S}^n}$ and $d\mu^* = f^* d\sigma_{\mathbb{S}^n}$. Denote by $\alpha = n+1-q$, $\beta = p-1$. Then

$$\left. \frac{d}{dt} \right|_{t=0} J_{p,q,\mu,\mu^*}(\Omega_t) = \frac{1}{\int_{\mathbb{S}^n} r^{q} d\mu^*} \int_{\mathbb{S}^n} J_{p,q,\mu,\mu^*}(x)\eta(x) d\sigma_{\mathbb{S}^n}(x), \quad (2.12)$$

where

$$J_{p,q,\mu,\mu^*}(x) = \frac{f^* \circ A^*}{(r \circ A^*_{\Omega})^\alpha K} - \lambda u^\beta f(x), \quad \text{with} \quad \lambda = \frac{\int_{\mathbb{S}^n} r^q d\mu^*}{\int_{\mathbb{S}^n} u^\beta d\mu},$$

and $K$ is the Gauss curvature of $\partial \Omega$ calculated at $\nu^{-1}_\Omega(x)$.\n
If \( \Omega \) is a convex body satisfying \( J_{p,q,\mu,\sigma_S^n} \equiv 0 \), then

\[
\frac{d^2}{dt^2} \bigg|_{t=0} J_{p,q,\mu,\sigma_S^n}(\Omega_t) = \frac{1}{\int_{S^n} u^p d\mu} \left\{ \int_{S^n} \left( \sum h^{ij} \eta_{ij} + H \eta \right) u^3 \eta d\mu - \alpha \int_{S^n} u^\beta u^n + \nabla u \cdot \nabla \eta \right\} + \frac{1}{\int_{S^n} w^q d\mu} \left\{ \int_{S^n} \left( \sum h^{ij} \eta_{ij} + H \eta \right) u^3 \eta d\mu - \alpha \int_{S^n} u^\beta u^n + \nabla u \cdot \nabla \eta \right\} - \beta \int_{S^n} u^{\beta - 1} \eta^2 d\mu + \frac{p - q}{\int_{S^n} w^q d\mu} \left( \int_{S^n} u^3 \eta d\mu \right) \}
\]

where \( \{h^{ij}\} \) is the inverse matrix of \( \{u_{ij} + u \delta_{ij}\} \), and \( H = \sum h^{ii} \) is the mean curvature of \( \partial \Omega \).

**Proof.** Note that, for all \( |t| < \varepsilon \), \( \partial \Omega_t \) are \( C^2 \) and strictly convex, with uniformly convex support function \( u(x,t) = u(x) + t\eta \). Denote by \( r = r(\xi, t) \) the radial function of \( \Omega_t \). Hence, by (2.11) and Lemma 2.1 we compute the first variation of (1.5) as follows

\[
\frac{d}{dt} J_{p,q,\mu,\sigma_S^n}(\Omega_t) = -\frac{1}{\int_{S^n} u^p d\mu} \int_{S^n} u^3 \eta d\mu + \frac{1}{\int_{S^n} w^q d\mu} \int_{S^n} r^q \frac{\dot{r}}{r} d\mu^* = -\frac{1}{\int_{S^n} u^p d\mu} \int_{S^n} u^3 \eta d\mu + \frac{1}{\int_{S^n} w^q d\mu} \int_{S^n} \frac{f^* \circ \sigma^*_{\Omega_t}}{(r \circ \sigma^*_{\Omega_t})^\alpha K} \eta \sigma_S^n \]

(2.14)

where the geometric quantities above are of \( \Omega_t \). Taking \( t = 0 \), we get (2.12).

Note that \( r \circ \sigma^*_{\Omega_t} = \sqrt{u_{\Omega_t}^2 + |\nabla u_{\Omega_t}|^2} \). Letting \( \mu^* = \sigma_S^n \) in (2.14) and then differentiating (2.14) w.r.t. \( t \) again, we further calculate, by the assumption \( J_{p,q,\mu,\sigma_S^n} \equiv 0 \),

\[
\frac{d^2}{dt^2} \bigg|_{t=0} J_{p,q,\mu,\sigma_S^n}(\Omega_t) = \frac{1}{\int_{S^n} u^p d\mu} \left\{ \int_{S^n} \sum S^{ij}_{\eta,ij} \eta \frac{\eta}{(r \circ \sigma^*_{\Omega_t})^\alpha K} d\sigma_S^n \right\} - \alpha \int_{S^n} \frac{u^n + \nabla u \cdot \nabla \eta}{(r \circ \sigma^*_{\Omega_t})^\alpha K} d\sigma_S^n - \lambda \beta \int_{S^n} u^{\beta - 1} \eta^2 d\sigma_S^n - \frac{q}{\int_{S^n} u^p d\mu} \left\{ \int_{S^n} \sum h^{ij} \eta_{ij} + H \eta \right\} \eta d\mu - \alpha \int_{S^n} u^\beta u^n + \nabla u \cdot \nabla \eta \right\} - \beta \int_{S^n} u^{\beta - 1} \eta^2 d\mu + \frac{p - q}{\int_{S^n} u^p d\mu} \left( \int_{S^n} u^3 \eta d\mu \right) \}
\]

This finishes the proof.
By virtue of Proposition 2.1, we are able to prove Theorem 1.3.

**Proof of Theorem 1.3.** Let \( \eta \in C^\infty(S^n) \) be an even function. As the unit ball \( B_1 \) is uniformly convex, there is a small \( \varepsilon = \varepsilon_\eta > 0 \), depending on \( \eta \), such that, for all \( |t| < \varepsilon \), \( \Omega^\eta_t := \{ z \in \mathbb{R}^{n+1} : x \cdot z \leq 1 + t\eta(x), \ x \in S^n \} \) has support function \( u(x,t) = 1 + t\eta(x) \), which is positive and uniformly convex. Clearly \( \Omega^\eta_t \in \mathcal{K}^e_0 \).

By Proposition 2.1,

\[
\frac{d}{dt} \bigg|_{t=0} J_{p,q}(\Omega^\eta_t) = 0,
\]

and

\[
\frac{d^2}{dt^2} \bigg|_{t=0} J_{p,q}(\Omega^\eta_t) = \int_{S^n} (\eta \Delta \eta + (n - \alpha - \beta)\eta^2) d\sigma_{S^n} + (p - q) \left( \int_{S^n} \eta d\sigma_{S^n} \right)^2
\]

(2.17)

\[
= (q - p) \int_{S^n} (\eta - \bar{\eta})^2 d\sigma_{S^n} - \int_{S^n} |\nabla \eta|^2 d\sigma_{S^n},
\]

where \( \bar{\eta} := \int_{S^n} \eta d\sigma_{S^n} \) is the mean value of \( \eta \).

By (i) in Theorem 4.1 below, there is an \( \eta_0 \in C^\infty(S^n) \), with \( \bar{\eta}_0 = 0, \eta_0 \not\equiv 0 \), such that

\[
(2n + 2 + \frac{1}{2} \delta_{p,q}) \int_{S^n} \eta_0^2 d\sigma_{S^n} \geq \int_{S^n} |\nabla \eta_0|^2 d\sigma_{S^n},
\]

where

\[
\delta_{p,q} := q - p - 2n - 2 > 0.
\]

Then for \( \Omega_t := \Omega^\eta_0_t \), whose support function is \( 1 + t\eta_0 \), one has by (2.17)

(2.18)

\[
\frac{d^2}{dt^2} \bigg|_{t=0} J_{p,q}(\Omega_t) \geq \frac{1}{2} \delta_{p,q} \int_{S^n} \eta_0^2 d\sigma_{S^n} > 0.
\]

For \( \varepsilon_0 = \varepsilon_{\eta_0} > 0 \) very small, one knows that \( \partial \Omega_t \) is smooth and uniformly convex for all \( |t| < \varepsilon_0 \). Hence by (2.17) and (2.18)

\[
J_{p,q}(\Omega_t) = J_{p,q}(B_1) + t \frac{d}{dt} \bigg|_{t=0} J_{p,q}(\Omega_t) + \frac{1}{2} t^2 \frac{d^2}{dt^2} \bigg|_{t=0} J_{p,q}(\Omega_t) + o(t^2)
\]

\[
> J_{p,q}(B_1), \text{ for } t \in (0, \varepsilon_0'),
\]

provided \( \varepsilon_0' \), depending on \( \eta_0 \), is sufficiently small.

\( \square \)

**Remark 2.1.** When \( p = 0 \) and \( q = n + 1 \), the second variation of the functional was obtained in [22].
3. Proof of Theorems 1.1 & 1.2

In this section, we first prove Theorem 1.2. This together with Theorem 1.3 shows Theorem 1.1.

Given $\Omega \in \mathcal{K}^e_0$, the polar set of $\Omega$ is defined as follows

$$\Omega^* = \{ y \in \mathbb{R}^{n+1} : y \cdot x \leq 1 \ \forall \ x \in \Omega \}.$$ 

The following generalised Blaschke-Santaló inequality was proved in [13].

**Theorem 3.1 (Blaschke-Santaló type inequality [13]).** Given $q > 0$, let $q^* > 0$ be the number given by (1.9). For $\gamma \in (0, q^*]$, $\gamma \neq +\infty$, there is a constant $C_{n,q,\gamma} > 0$ such that,

$$\left( \int_{S^n} r^q \sigma_{S^n} \right)^{\frac{1}{q}} \left( \int_{S^n} r^{\gamma q} \sigma_{S^n} \right)^{\frac{1}{\gamma}} \leq C_{n,q,\gamma}, \ \forall \ \Omega \in \mathcal{K}^e_0. $$

Theorem 3.1 enables us to solve the optimisation problem (1.10).

**Proposition 3.1.** Under the assumptions of Theorem 1.2, there is a convex body $\Omega_0 \in \mathcal{K}^e_0$ solving the maximisation problem (1.10).

**Proof.** We prove Proposition 3.1 when either (B1) or (B2) holds. For the case (B3), we use a dual argument.

Assume that either (B1) or (B2) is satisfied. We have $p \leq 0$ and $q > 0$. By the homogeneity (1.8), it suffices to show there is a $\Omega_0 \in \mathcal{K}^e_0$, with $\Psi_{q,\sigma_{S^n}}(\Omega_0) = 0$, such that

$$\tilde{\Phi}_{p,\mu}(\Omega_0) = \max_{\Omega \in \mathcal{K}^e_0} \left\{ \tilde{\Phi}_{p,\mu}(\Omega) : \Psi_{q,\sigma_{S^n}}(\Omega) = 0 \right\},$$

where

$$\tilde{\Phi}_{p,\mu}(\Omega) := \begin{cases} -\frac{1}{p} \int_{S^n} u^p \mu, & \text{if } p \neq 0, \\ - \int_{S^n} \log u \mu, & \text{if } p = 0. \end{cases}$$

For (3.2), let $\{\Omega_j\} \subset \mathcal{K}^e_0$, with $\Psi_{q,\sigma_{S^n}}(\Omega_j) = 0$, be a maximising sequence. We denote $u_j = u_{\Omega_j}$ and $r_j = r_{\Omega_j}$ for convenience. We claim

$$\max_{S^n} u_j \leq C,$$

for some $C > 0$, independent of $j$. We next prove (3.3) (under the assumption (B1) or (B2)) case by case: (i) $p < 0$; (ii) $p = 0$; (iii) $p > 0$. 
**Case I:** \( p < 0 \). We follow an argument in [19]. Let \( \delta > 0 \) be a fixed small constant. Set \( S_1^j = \mathbb{S}^n \cap \{ u_j \leq \delta \} \), \( S_2^j = \mathbb{S}^n \cap \{ \delta < u_j < 1/\delta \} \) and \( S_3^j = \mathbb{S}^n \cap \{ u_j \geq 1/\delta \} \). Since \( \Omega_j \) is origin-symmetric, we conclude that

\[
|S_1^j| \to 0, \quad |S_2^j| \to 0, \quad \text{as } L_j := \max_{\mathbb{S}^n} u_j \to \infty.
\]

Let \( \Omega^* \) be the polar set of \( \Omega \), and \( r_j^* = r_{\Omega_j^*} \). It is well known that \( r_j^* = 1/u_j \), see e.g. [35]. Denote \( \gamma = -p > 0 \). By condition (B2), \( \gamma < q^* \). We have

\[
\tilde{\Phi}_{p,\mu}(\Omega_j) = \frac{1}{\gamma} \int_{S_1^j \cup S_2^j \cup S_3^j} u_j^{-\gamma} f d\sigma_{\mathbb{S}^n} \\
\leq C \int_{S_1^j} r_j^{*\gamma} d\sigma_{\mathbb{S}^n} + C\delta|S_2^j| + C\delta^\gamma \\
\leq C \left( \int_{\mathbb{S}^n} r_j^{*q'} d\sigma_{\mathbb{S}^n} \right)^{\frac{\gamma}{q'}} |S_1^j|^{1-\frac{\gamma}{q'}} + C\delta|S_2^j| + C\delta^\gamma,
\]

for any \( \gamma < q' < q^* \). Since \( \int_{\mathbb{S}^n} r_j^q d\sigma_{\mathbb{S}^n} = 1 \), if \( L_j \to \infty \), then by (3.1) and (3.4)

\[
\limsup_{j \to \infty} \tilde{\Phi}_{p,\mu}(\Omega_j) \leq C\delta^\gamma.
\]

As \( \tilde{\Phi}_{p,\mu}(\Omega_j) \geq \tilde{\Phi}_{p,\mu}(B_1) > 0 \), we arrive a contradiction by letting \( \delta \to 0 \).

**Case II:** \( p = 0 \). Let \( l_j = \min_{\mathbb{S}^n} r_j \) and \( L_j = \max_{\mathbb{S}^n} r_j \). By a rotation of coordinates we may assume that \( L_j = r_j(e_1) \). Since \( \Omega_j \) is origin-symmetric, the points \( \pm L_j e_1 \in \partial \Omega_j \). Hence

\[
u_j(x) = \max\{ z \cdot x : x \in \Omega_j \} \geq L_j |x \cdot e_1|, \quad \forall x \in \mathbb{S}^n.
\]

Therefore

\[
\tilde{\Phi}_{p,\mu}(\Omega_j) \leq - (\log L_j)/C - \int_{\mathbb{S}^n} \log |x \cdot e_1| f(x) d\sigma_{\mathbb{S}^n}(x) \\
\leq - (\log L_j)/C + C,
\]

which implies that \( \tilde{\Phi}_{p,\mu}(\Omega_j) \to -\infty \), if \( L_j \to \infty \). This cannot occur as \( \{ \Omega_j \} \) is a maximising sequence.

**Case III:** \( p > 0 \). Again let \( L_j = \max_{\mathbb{S}^n} r_j \). As in Case II, we have (3.5). Hence

\[
\tilde{\Phi}_{p,\mu}(\Omega_j) \leq - \frac{1}{p} \int_{\{ x \in \mathbb{S}^n : x \cdot e_1 \geq \frac{1}{L_j} \}} u_j^p d\mu \leq - L_j^p/C \to -\infty \quad \text{if } L_j \to \infty.
\]

Since \( \{ \Omega_j \} \) is a maximising sequence, one infers \( L_j \leq C \).

Combining Case I-III, we have proved (3.3) under the assumption (B1) or (B2).
Let \( w_j^+ = \max_{x \in \mathbb{S}^n} \left( u_j(x) + u_j(-x) \right) \) and \( w_j^- = \min_{x \in \mathbb{S}^n} \left( u_j(x) + u_j(-x) \right) \) be the maximum and the minimum of the width of \( \Omega_j \). We next show that

\[
(3.6) \quad w_j^- \geq 1/C,
\]

for some \( C > 0 \), independent of \( j \). This estimate together with (3.3) means that \( \Omega_j \) is of uniformly good shape.

For \( 0 < q < n + 1 \), we have

\[
1 = \left( \int_{\mathbb{S}^n} r_j^q d\sigma_{\mathbb{S}^n} \right)^{\frac{n+1}{q}} \leq \int_{\mathbb{S}^n} r_j^{n+1} d\sigma_{\mathbb{S}^n} \leq C \text{Volume}(\Omega_j) \leq (w_j^+)^n w_j^-,
\]

which shows (3.6) by using (3.3).

For \( q \geq n + 1 \), we have

\[
1 = \int_{\mathbb{S}^n} r_j^q d\sigma_{\mathbb{S}^n} = (w_j^+)^q \int_{\mathbb{S}^n} \left( \frac{r_j}{w_j^+} \right)^q d\sigma_{\mathbb{S}^n} \leq C (w_j^+)^{q-n-1} \text{Volume}(\Omega_j) \leq C (w_j^+)^{q-1} w_j^-.
\]

Again, (3.6) follows from (3.3).

As above, \( l_j = \min_{\mathbb{S}^n} r_j \). Assume without loss of generality that \( l_j = r_j(e_1) \). By the symmetry of \( \Omega_j \),

\[
l_j \geq r_j(\xi)|\xi \cdot e_1|, \quad \forall \xi \in \mathbb{S}^n.
\]

For \( q = 0 \), we thus have

\[
0 = \int_{\mathbb{S}^n} \log r_j d\sigma_{\mathbb{S}^n} \leq \log l_j - \int_{\mathbb{S}^n} \log |\xi \cdot e_1| d\sigma_{\mathbb{S}^n} \leq \log l_j + C.
\]

This shows that \( l_j \geq \delta \) for some \( \delta > 0 \) uniformly, and so (3.6) follows.

In virtue of (3.3) and (3.6), we conclude by the Blaschke selection theorem that \( \Omega_j \), after passing to a subsequence, converges to a \( \Omega_0 \in \mathcal{K}_0^e \) in Hausdorff distance, thus completing the proof under the assumption (B1) or (B2).

For case (B3), let \( \{\Omega_j\} \subset \mathcal{K}_0^e \) be a maximising sequence of functional \( J_{p,q,\mu} \). Let \( p' = -q \) and \( q' = -p \), and \( \Omega_j^* \) be the polar set of \( \Omega_j \). One easily sees that

\[
(3.7) \quad J_{p,q,\mu,\sigma_{\mathbb{S}^n}}(\Omega) = J_{p',q',\sigma_{\mathbb{S}^n},\mu}(\Omega^*) \quad \forall \Omega \in \mathcal{K}_0^e.
\]

It then follows that \( \{\Omega_j^*\} \) is a maximising sequence of \( J_{p',q',\sigma_{\mathbb{S}^n},\mu} \). Observe that if \( p, q \) satisfy (B3), then \( p', q' \) satisfy (B1). Hence, by our previous argument for (B1), it is not hard to conclude that, after a proper rescaling, \( t_j \Omega_j^* \) converges to a \( \Omega_0^* \in \mathcal{K}_0^e \) such that

\[
J_{p',q',\sigma_{\mathbb{S}^n},\mu}(\Omega_0^*) = \max \{ J_{p',q',\sigma_{\mathbb{S}^n},\mu}(\Omega) : \Omega \in \mathcal{K}_0^e \}
\]
By (3.7), we conclude that $\Omega_0 = \Omega_0^*$ satisfies (1.10). Note that if $p', q'$ satisfy (B2), then $p, q$ also satisfy (B2). Hence we would not get more by applying the above dual argument to (B2).

We then show that, after a dilation, the maximiser $\Omega_0$ obtained in Proposition 3.1 is a solution to the $L_p$ dual Minkowski problem, under an additional assumption: $\partial \Omega_0$ is $C^1$ and strictly convex.

**Proposition 3.2.** If $\partial \Omega_0$ is $C^1$ and strictly convex, then $\Omega_0$ satisfies (1.11).

**Proof.** Let $u$ and $r$ be respectively the support function and radial function of $\Omega_0$. For any even function $\eta \in C^0(S^n)$, let $\Omega_t \in K_e^c$ be the convex bodies given by (2.1), with $u$ replaced by $u_t$. Denote by $u(x, t)$ and $r(x, t)$ the support function and radial function of $\Omega_t$. By Lemma 2.1, we have as in proof of Proposition 2.1
\[
\frac{d}{dt} \bigg|_{t=0} J_{p,q,\mu}(\Omega_t) = \frac{1}{\int_{S^n} r^q d\sigma_{S^n}} \left( -\lambda_{\Omega_0} \int_{S^n} u^{p-1} \eta d\mu + \int_{S^n} r^q \eta \circ \nu_{\Omega_0} d\sigma_{S^n} \right).
\]
By [33, Lemma 5.1], we further calculate
\[
(3.8) \quad \frac{d}{dt} \bigg|_{t=0} J_{p,q,\mu}(\Omega_t) = \frac{1}{\int_{S^n} r^q d\sigma_{S^n}} \left( -\lambda_{\Omega_0} \int_{S^n} u^{p-1} \eta d\mu + \int_{S^n} u^{p-1} \eta d\tilde{C}_{p,q}(\Omega_0, \cdot) \right).
\]
Since $\Omega_0$ is the maximiser and $\eta$ is arbitrary, we deduce that
\[
\int_{S^n} g d\tilde{C}_{p,q}(\Omega_0, \cdot) = \lambda_{\Omega_0} \int_{S^n} g d\mu, \quad \forall \text{ even function } g \in C^0(S^n),
\]
thus completing the proof by the evenness of $f$.

\[\square\]

**Proposition 3.3.** Let $\Omega_0$ be the maximiser obtained in Proposition 3.1. Then $\partial \Omega_0$ is strictly convex and is $C^{1, \gamma}$ for some $\gamma \in (0, 1)$.

**Proof.** Let $u$ be the support function of $\Omega_0$ and $\bar{u} = \bar{u}_{\Omega_0}$ be its homogeneous degree one extension, namely $\bar{u} : \mathbb{R}^{n+1} \to \mathbb{R}$, defined by
\[
\bar{u}(Y) = \sup_{Z \in \Omega_0} Y \cdot Z.
\]
The face of $\Omega_0$ with outer normal $Y \in \mathbb{R}^{n+1}$ is then given by
\[
F_{\Omega_0}(Y) = \{ Z \in \Omega_0 : \bar{u}(Y) = Y \cdot Z \},
\]
which lies in \( \partial \Omega_0 \) provided \( Y \neq 0 \), and

\[
(3.9) \quad \partial \bar{u}(Y) = F_{\Omega_0}(Y),
\]

where \( \partial \bar{u}(Y) := \{ X \in \mathbb{R}^{n+1} : \bar{u}(Z) \geq \bar{u}(Y) + \langle X, Z - Y \rangle, \forall Z \in \mathbb{R}^{n+1} \} \) is the subgradient of \( \bar{u} \) at \( Y \). See Schneider’s book [35] for all this.

For \( e \in \mathbb{S}^n \), let \( L_e \) be the hyperplane in \( \mathbb{R}^{n+1} \) which is tangential to \( \mathbb{S}^n \) at \( e \). Denote by \( \pi = \pi_e : \mathbb{R}^n \to \mathbb{S}^n \) the radial projection from \( L_e \) to \( \mathbb{S}^n \),

\[
\pi(y) = \frac{y + e}{\sqrt{1 + |y|^2}}.
\]

Let \( v = v_e : \mathbb{R}^n \to \mathbb{R} \) be the restriction of \( \bar{u} \) on \( L_e \), that is

\[
(3.10) \quad v(y) = \bar{u}(y + e) = \sqrt{1 + |y|^2} u(\pi(y)).
\]

It is not hard to check by (3.9) and (3.10) that

\[
(3.11) \quad \partial v(y) = \{ X - (X \cdot e)e : X \in \partial \bar{u}(y + e) \}.
\]

Let \( \mathcal{H}^n \) denotes the \( n \)-dimensional Hausdorff measure. Recall that the surface area measure \( \mathcal{S}(\Omega_0, \cdot) \) is defined as

\[
(3.12) \quad \mathcal{S}(\Omega_0, \omega) = \mathcal{H}^n(\nu_{\Omega_0}^{-1}(\omega)), \quad \text{for Borel set } \omega \subset \mathbb{S}^n.
\]

It follows from (3.9)-(3.11) that for any \( D \subset \mathbb{R}^n \)

\[
(3.13) \quad \mathcal{M}_v(D) = \int_{\pi(D)} \langle x, e \rangle \, d\mathcal{S}(\Omega_0, x),
\]

where \( \mathcal{M}_v(D) := \mathcal{H}^n(\partial v(D)) \) is the Monge-Ampère measure associated to \( v \). We claim that, \( \mathcal{S}(\Omega_0, \cdot) \) is absolutely continuous w.r.t. \( \sigma_{\mathbb{S}^n} \), and there is a \( C > 0 \), such that

\[
(3.14) \quad \frac{1}{C} \leq \varrho_{\Omega_0} := \frac{d\mathcal{S}(\Omega_0, \cdot)}{d\sigma_{\mathbb{S}^n}} \leq C.
\]

Note that \( \varrho_{\Omega_0} \) is the reciprocal Gauss curvature if \( \Omega_0 \) is \( C^2 \) smooth. Once (3.14) is proved, we deduce by (3.13) and \( \det D\pi(y) = (1 + |y|^2)^{-\frac{n+1}{2}} \) that

\[
(3.15) \quad d\mathcal{M}_v = \frac{\varrho_{\Omega_0} \circ \pi}{(1 + |y|^2)^{\frac{n+1}{2}}} dy.
\]

For (3.15), one may consult [18, 35] for a full discussion. By (3.14) and (3.15), the density of the Monge-Ampère measure of \( v \) in a compact set is bounded between two constants. For a given \( y_0 \in \mathbb{R}^n \), let \( \ell_{y_0} \) be the support function of \( v(y) \) at \( y_0 \). In view of (3.11), the contact set \( C_{y_0} := \{ y \in \mathbb{R}^n : v(y) = \ell_{y_0}(y) \} \) cannot contain a straight line in \( \mathbb{R}^n \). Hence we conclude by [9, 11] that \( v \) is strictly convex and \( C^{1,\gamma'}_{\text{loc}} \) for some
γ′ ∈ (0, 1). See also [19]. This implies that ∂Ω_0 is strictly convex. Let φ : D′ → ℝ be the convex function such that \{(x, φ(x)) : x ∈ D′\} ⊆ ∂Ω_0, where D′ is a closed convex domain, containing the origin, lying in Ω_0 ∩ \{X ∈ ℝ^{n+1} : X · e = 0\}. One can check that φ is exactly the Legendre transform of v. Therefore, by (3.14) and (3.15), dM_φ/dx is bounded between two positive constants. By [9, 11], φ is C^{1,γ} for some γ ∈ (0, 1).

It remains to show (3.14). For η ∈ C_0(S^n), consider

\[ Ω_t = \{ z ∈ ℝ^{n+1} : z · x ≤ e^{tη(x)}u(x), \ ∀ x ∈ S^n \}, \]

which is a perturbation of Ω_0. Denote by u^t = u(x,t) and r^t = r(ξ,t) the support and radial functions of Ω_t. It follows from [33, Theorem 6.4] that

\[ \frac{d}{dt}\left|_{t=0}\right. Ψ_q(r^t) = \frac{1}{\int_{S^n} r^q dσ_{S^n}} \int_{S^n} η dC_q(Ω_0, ·). \]

Since u^t ≤ e^{tη}u, one has

\[ \lim_{t→0^+} \frac{u^t(x) - u(x)}{t} ≤ ηu(x). \]

By (3.16) and (3.17), we obtain

\[ 0 ≥ \lim_{t→0^+} \frac{J_{p,q}(Ω_t) - J_{p,q}(Ω_0)}{t} \]

\[ ≥ \frac{1}{\int_{S^n} r^q dσ_{S^n}} \left\{ -λΩ_0 \int_{S^n} u^p η dμ + \int_{S^n} η dC_q(Ω_0, ·) \right\} \]

\[ = \frac{1}{\int_{S^n} r^q dσ_{S^n}} \left\{ -λΩ_0 \int_{S^n} u^p f η dσ_{S^n} + \int_{S^n} (r ∘ s^*_{Ω_0}) q-n-1 u η dS(Ω_0, ·) \right\}, \]

where λΩ_0 is given by (1.12), and the last equality is due to [25, Lemma 3.7]. Since η is arbitrary, u and r are bounded between two positive constants, we get

\[ Ω_0 ≤ C. \]

Let Ω_0* be the polar set of Ω_0. Then r^* = 1/u, see e.g. [35]. For η ∈ C_0(S^n), consider

\[ Ω_t^* = \text{conv}\{e^{tη(x)}r^*(x) : x ∈ S^n\}, \]

and Ω_t = (Ω_t^*)*. Denote by u^{st} = u^*(ξ,t) and r^{st} = r^*(x,t) the support and radial function of Ω_t^*, by u^t = u(x,t) and r^t = r(ξ,t) the support and radial function of Ω_t. Since r^{st} ≥ e^{tη}r^*, one gets u^t ≤ e^{-tη}u. Therefore

\[ \lim_{t→0^+} \frac{u^t(x) - u(x)}{t} ≤ -ηu(x). \]
By [33, Theorem 6.1],

\[ (3.19) \quad \frac{d}{dt} \bigg|_{t=0} \Psi_q(r^t) = -\frac{1}{\int_{S^n} r^q d\sigma_{S^n}} \int_{S^n} \eta d\tilde{C}_q(\Omega_0, \cdot). \]

It follows by (3.18) and (3.19)

\[ 0 \geq \lim_{t \to 0^+} \frac{\mathcal{J}_{p,q}(\Omega_t) - \mathcal{J}_{p,q}(\Omega_0)}{t} \geq \frac{1}{\int_{S^n} r^q d\sigma_{S^n}} \left\{ \lambda_{\Omega_0} \int_{S^n} u^p f \eta d\sigma_{S^n} - \int_{S^n} (r \circ \mathcal{A}_{\Omega_0}^*)^{q-n-1} u \eta dS(\Omega_0, \cdot) \right\}, \]

which shows that

\[ \varrho_{\Omega_0} \geq \frac{1}{C}. \]

This completes the proof. \qed

Remark 3.1. To see the maximiser of the optimisation problem (1.10) is a solution to the \(L_p\) dual Minkowski problem (1.11), one can also follow the argument in [25, Lemma 3.7].

We are at the position to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. By Proposition 3.1-3.3, it remains to show \(u = u_{\Omega_0}\) is smooth and uniformly convex. Note that for \(p \neq q\), it is not hard to see \(\tilde{\Omega}_0 := \lambda_{\Omega_0}^{\frac{1}{p-q}}\Omega_0\) satisfies (1.13), and \(u_{\tilde{\Omega}_0}\) solves (1.3). By the homogeneity (1.8), \(\tilde{\Omega}_0\) is also a maximiser for (1.10).

By [25, Lemma 3.7] and [33, Proposition 5.4], it follows from (3.8) that, \(\forall \eta \in C^0(S^n),\)

\[ \frac{d}{dt} \bigg|_{t=0} \mathcal{J}_{p,q,\mu}(\Omega_t) = \frac{1}{\int_{S^n} r^q d\sigma_{S^n}} \left\{ -\lambda_{\Omega_0} \int_{S^n} u^{p-1} \eta f d\sigma_{S^n} + \int_{S^n} (r \circ \mathcal{A}_{\Omega_0}^*)^{q-n-1} u \eta dS(\Omega_0, \cdot) \right\}. \]

Since \(\Omega_0\) is the maximiser of (1.10), we obtain

\[ (3.20) \quad \frac{dS(\Omega_0, \cdot)}{d\sigma_{S^n}} = \lambda_{\Omega_0} (r \circ \mathcal{A}_{\Omega_0}^*)^{n+1-q} u^{p-1} f. \]

Given any \(e \in S^n\), let \(v\) and \(\varphi = v^*\) (the Legendre transform of \(v\)) be as in Proposition 3.3. Then

\[ (3.21) \quad \det D^2 v = \lambda_{\Omega_0} (1 + |y|^2)^{-\frac{n+1}{2}} v^{p-1} (|Dv|^2 + (Dv \cdot y - v)^2)^{-\frac{n+1-q}{2}} f \circ \pi, \]
and
\begin{equation}
\det D^2 \varphi = \lambda_{\Omega_0}^{-1}(1 + |D \varphi|^2)^{\frac{n+1}{2}}(D \varphi \cdot x - \varphi)^{1-p}(|x|^2 + \varphi^2)^{-\frac{n+1}{2}} / f\left(\frac{D \varphi, -1}{\sqrt{1 + |D \varphi|^2}}\right),
\end{equation}
in the Aleksandrov sense. By Proposition 3.3, \(v\) and \(\varphi\) are strictly convex and are \(C^{1,\gamma}\) for some \(\gamma \in (0, 1)\).

If \(f\) is Hölder, then the right hand sides of (3.21) and (3.22) are both Hölder continuous. By [10], \(v\) and \(\varphi\) are both \(C^{2,\gamma'}\) for some \(\gamma' \in (0, 1)\). Smoothness of \(v\) and \(v \varphi\) then follows from the standard theory of uniformly elliptic equations, provided \(f \in C^\infty(S^n)\).

Hence \(\partial \Omega_0\) is smooth. By (3.20), \(u \in C^\infty(S^n)\) solves (1.3) with \(f\) replaced by \(\lambda_{\Omega_0} f\).

\[ \] 

The following result improves Theorem 1.2 under condition (B2).

**Theorem 3.2.** Let \(p, q\) satisfy condition (B2) in Theorem 1.2 and \(d\mu = f d\sigma_{S^n}\), where \(f\) is an even, non-negative function and \(\int_{S^n} f d\sigma_{S^n} > 0\). Assume that \(f \in L^{q^*/(q^* + p)}(S^n)\) if \(q^* \neq +\infty\), or \(f \in L^s(S^n)\) for some \(s > 1\) if \(q^* = +\infty\). Then there is a convex body \(\Omega \in K_0^e\) such that \(\tilde{C}_{p,q}(\Omega, \omega) = \mu(\omega)\) for all Borel set \(\omega \subseteq S^n\).

**Proof.** We use an approximation argument similar to [4]. For positive integers \(j\), let \(d\mu_j = f_j d\sigma_{S^n}\) be a sequence of measures, where \(f_j\) is a truncation of \(f\);

\[ f_j(x) = \begin{cases} 
\frac{1}{j} & \text{if } f(x) \geq j, \\
\frac{1}{j} & \text{if } 1/j < f(x) < j, \\
1/j & \text{if } f(x) \leq 1/j.
\end{cases} \]

Recall that \(J_{\mu, \mu_j}\) satisfies (1.8). Hence by Theorem 1.2 there is a \(\tilde{\Omega}_j \in K_0^e\) such that

\[ \tilde{C}_{p,q}(\tilde{\Omega}_j, \omega) = \mu_j(\omega), \text{ for any Borel set } \omega \subseteq S^n, \]

and if \(\tilde{r}_j = r_{\tilde{\Omega}_j}\) and \(\tilde{u}_j = u_{\tilde{\Omega}_j}\), then

\begin{equation}
\left( \int_{S^n} \tilde{u}_j^p \, d\mu_j \right)^{\frac{1}{p}} \left( \int_{S^n} \tilde{r}_j^q d\sigma_{S^n} \right)^{\frac{1}{q}} = \exp J_{\mu, \mu_j}(\tilde{\Omega}_j) \geq \exp J_{\mu, \mu_j}(B_1) \geq 1/C_{f,n,p},
\end{equation}

for a positive constant \(C_{f,n,p} > 0\), independent of \(j\).

Let \(\Omega_j = \lambda_j \tilde{\Omega}_j\), where \(\lambda_j = \left( \int_{S^n} \tilde{r}_j^q d\sigma_{S^n} \right)^{-\frac{1}{q}}\) so that

\begin{equation}
\int_{S^n} \tilde{r}_j^q d\sigma_{S^n} = 1.
\end{equation}
Let $u_j = u_{\Omega_j}$, $r_j = r_{\Omega_j}$ and $L_j := \max_{S^n} u_j$. As in the proof of Proposition 3.1 for a small constant $\delta > 0$, let $S_1^j = S^n \cap \{u_j \leq \delta\}$, $S_2^j = S^n \cap \{\delta < u_j < 1/\delta\}$ and $S_3^j = S^n \cap \{u_j \geq 1/\delta\}$. It is not hard to see that

\[(3.26) \quad |S_1^j| \to 0 \text{ and } |S_2^j| \to 0, \quad \text{if } L_j \to \infty.\]

First let us consider the case $q^* \neq \infty$. Denote $\gamma = -p > 0$ and $r_j^* = r_{\Omega^*_j}$, the radial function of $\Omega^*_j$ (the polar set of $\Omega_j$). We have

\[
\int_{S^n} u_j^q d\mu_j = \int_{S_1^j \cup S_2^j \cup S_3^j} u_j^{-\gamma} f_j d\sigma_{S^n}
\]

\[
\leq C \left( \int_{S^n} r_j^{*q^*} d\sigma_{S^n} \right)^{\frac{\gamma}{q^*}} \left( \int_{S_1^j} f_j r_j^{-\gamma} d\sigma_{S^n} \right)^{\frac{q^* - \gamma}{q^*}} + C_\delta \int_{S_2^j} f_j d\sigma_{S^n} + C\delta^\gamma
\]

\[(3.28) \quad \to C\delta^\gamma, \quad \text{if } L_j \to \infty,
\]

where $f \in L^{q^*/(p+\gamma)}(S^n)$, (3.1), (3.25) and (3.26) are used for the last line. As the LHS of (3.24) is rescaling invariant, its value is unchanged if $\tilde{u}_j, \tilde{r}_j$ are replaced by $u_j, r_j$. We conclude that $L_j$ are uniformly bounded, by (3.25), (3.28) and sending $\delta \to 0$. As in the proof of Proposition 3.1 we also deduce from (3.25) that $l_j := \min_{S^n} u_j$ stay uniformly away from zero.

In view of (3.23), for $q^* \neq +\infty$.

\[(3.29) \quad \int_{S^n} \tilde{r}_j^q d\sigma_{S^n} = \int_{S^n} \tilde{u}_j^q d\mu_j = \int_{S^n} \tilde{r}_j^{*q^*} f_j d\sigma_{S^n} \leq \left( \int_{S^n} \tilde{r}_j^{*q^*} d\sigma_{S^n} \right)^{\frac{q^*}{q^*}} \left( \int_{S^n} f_j \tilde{r}_j^{-\gamma} d\sigma_{S^n} \right)^{\frac{q^* - \gamma}{q^*}}.
\]

The first equality in (3.29) together with (3.24) shows that

\[
\int_{S^n} \tilde{r}_j^q d\sigma_{S^n} \geq 1/C_{f,n,p,q} > 0.
\]

While the inequality in (3.29), (3.1) and $f \in L^{q^*/(p+\gamma)}(S^n)$ give

\[
\int_{S^n} \tilde{r}_j^q d\sigma_{S^n} \leq C_{f,n,p,q}.
\]

Hence $1/C_{f,n,p,q} \leq \lambda_j \leq C_{f,n,p,q}$, for a constant $C_{f,n,p,q} > 0$ only depending on $f, n, p, q$.

The above estimates for $L_j, l_j, \lambda_j$ imply $\max_{S^n} u_{\tilde{\Omega}_j}$ and $\min_{S^n} u_{\tilde{\Omega}_j}$ are uniformly bounded from above and below. By the Blaschke selection theorem, $\tilde{\Omega}_j$ converges, after passing to a subsequence, to a $\Omega \in K^n_0$ in Hausdorff distance. By the weak convergence of $L_p$ dual curvature measures [33, Proposition 5.2], it follows from (3.23) that $\tilde{C}_{p,q}(\Omega, \omega) = \mu(\omega)$, thus completing the proof for $q^* \neq +\infty$. 

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When \( q^* = +\infty \), (3.27) and (3.29) still hold if \( q^* \) is replaced by \( \alpha = \frac{s\gamma}{s-1} \). Hence we can finish the proof by the same discussion as above.

Next we prove Theorem 1.1.

**Proof of Theorem 1.1.** By Theorem 1.2 and by the homogeneity (1.8), there is a \( \Omega_0 \in K_0^e \) such that \( u_{\Omega_0} \) is a solution to the equation (1.3) with \( f \equiv 1 \) and \( J_{p,q}(\Omega_0) = \max\{J_{p,q}(\Omega) : \Omega \in K_0^e\} \). We deduce from Theorem 1.3 that \( J_{p,q}(B_1) < J_{p,q}(\Omega_0) \).

Therefore \( u_{\Omega_0} \neq u_{B_1} \). While \( u_{B_1} \equiv 1 \) and \( u_{\Omega_0} \) both solve (1.3) when \( f \equiv 1 \).

\( \square \)

## 4. Sharp Poincaré inequality on \( S^n \)

This section is devoted to the Poincaré inequality on \( S^n \). This inequality is well-known and has many applications. It can be proved by studying the eigenvalues of the spherical Laplace operator [36]. We prove it by the stability of the unit ball \( B_1 \) under the functional (1.5) and the uniqueness of the self-similar solution to the powered Gauss curvature flow (1.4).

**Theorem 4.1.** We have

(i) \[
\inf \left\{ \frac{\int_{S^n} |\nabla \eta|^2 d\sigma_{S^n}}{\int_{S^n} \eta^2 d\sigma_{S^n}} : \eta \in C^\infty(S^n) \text{ is even, } \int_{S^n} \eta d\sigma_{S^n} = 0, \eta \not\equiv 0 \right\} = 2n + 2;
\]

(ii) \[
\inf \left\{ \frac{\int_{S^n} |\nabla \eta|^2 d\sigma_{S^n}}{\int_{S^n} \eta^2 d\sigma_{S^n}} : \eta \in C^\infty(S^n), \int_{S^n} \eta d\sigma_{S^n} = 0, \eta \not\equiv 0 \right\} = n.
\]

**Remark 4.1.** By approximation, Theorem 4.1 holds for \( \eta \in W^{1,2}(S^n) \).

**Proof of (i) in Theorem 4.1.** Let \( \Omega_t = \{z \in \mathbb{R}^{n+1} : z \cdot x \leq 1 + t\eta(x)\} \in K_0^e \). Consider

\[
J_{p}(\Omega_t) := J_{p,n+1}(\Omega_t).
\]
By Proposition 2.1, or more precisely by letting $q = n + 1$ in (2.17), we have

$$d^2 |_{t=0} J_p(\Omega) = (n + 1 - p) \int_{\mathbb{S}^n} (\eta - \bar{\eta})^2 d\sigma_{\mathbb{S}^n} - \int_{\mathbb{S}^n} |\nabla \eta|^2 d\sigma_{\mathbb{S}^n}. \tag{4.1}$$

When $f \equiv 1$ and $q = n + 1$, (1.3) is the equation of the self-similar solutions to the flow (1.4) with $p = 1 - 1/\alpha$. By [1, 2, 8], $u \equiv 1$ is the only solution for $p \in (-n - 1, 1)$. By [3, 19], the equation (1.3) with $p > -n - 1$ and $q = n + 1$ admits a solution which maximises the functional $J_p$. We therefore conclude that

$$d^2 |_{t=0} J_p(\bar{\Omega}) = 0, \forall p > -n - 1.$$

This together with (4.1) implies, by letting $p \to -n - 1$,

$$\int_{\mathbb{S}^n} (\eta - \bar{\eta})^2 d\sigma_{\mathbb{S}^n} \leq \frac{1}{2n + 2} \int_{\mathbb{S}^n} |\nabla \eta|^2 d\sigma_{\mathbb{S}^n}. \tag{4.2}$$

We next show (4.2) is sharp. Assume not, then for sufficiently small $\varepsilon > 0$, there is an even $\eta_\varepsilon \neq 0$, $\bar{\eta}_\varepsilon = 0$, such that

$$\int_{\mathbb{S}^n} |\nabla \eta_\varepsilon|^2 d\sigma_{\mathbb{S}^n} = (2n + 2 - 2\varepsilon) \int_{\mathbb{S}^n} \eta_\varepsilon^2 d\sigma_{\mathbb{S}^n}.$$

Let $\Omega^\varepsilon_t = \{ z \in \mathbb{R}^{n+1} : z \cdot x \leq 1 + t\eta_\varepsilon \}$. Then for $p = -n - 1 + \varepsilon$ we have by (4.1),

$$\frac{d^2}{dt^2} |_{t=0} J_{-n-1+\varepsilon}(\Omega^\varepsilon_t) = \varepsilon \int_{\mathbb{S}^n} \eta_\varepsilon^2 d\sigma_{\mathbb{S}^n} > 0.$$

This means, by virtue of [3, 19], there is another convex body $\Omega' \neq B_1$ maximising $J_{-n-1+\varepsilon}$ among $K_0$, and $u_{\Omega'}$ solves (1.3) with $f \equiv 1$, $p = -n - 1 + \varepsilon$, and $q = n + 1$, contradicting with the uniqueness of the solution.

\[\Box\]

**Proof of (ii) in Theorem 4.1.** Consider the functional

$$\tilde{J}_p(\Omega, z) = -\frac{1}{p} \log \int_{\mathbb{S}^n} u_z^p d\sigma_{\mathbb{S}^n} + \frac{1}{n+1} \log \int_{\mathbb{S}^n} r_z^{n+1} d\sigma_{\mathbb{S}^n}, \tag{4.3}$$

where $z \in \text{Int} \ \Omega$ and $u_z, r_z$ are the support and radial function of $\Omega$ w.r.t. the centre $z$, namely $u_z(x) = \max \{(y - z) \cdot x : y \in \Omega \}$ and $r_z(\xi) = \max \{ \lambda : \lambda \xi + z \in \Omega \}$. This functional was used by Andrews-Guan-Ni [3] in the study of the flow (1.4) with $\alpha = (1 - p)^{-1}$. Note that the second term on the RHS of (4.3) is independent of $z$, as

$$\frac{1}{n+1} \int_{\mathbb{S}^n} r_z^{n+1} d\sigma_{\mathbb{S}^n} = \text{Volume}(\Omega).$$

In fact $u \equiv 1$ is also the unique solution for $p > 1$ and $p \neq n + 1$, see e.g. [32].
Given $\Omega$, let $z_\varepsilon = z_\varepsilon(\Omega)$ be the entropy point of $\Omega$, namely $z_\varepsilon$ minimises

$$z \mapsto \mathcal{J}_p(\Omega, z),$$

among all $z \in \text{Int} \Omega$.

For $p < 1$, it was proved in [3] that for each bounded convex $\Omega$ with $\text{Int} \Omega \neq \emptyset$, there exists a unique entropy point $z_\varepsilon \in \text{Int} \Omega$, and one readily sees

$$\int_{S^n} \frac{x}{u_{-p}^z(x)} d\sigma_{S^n}(x) = 0. \quad (4.4)$$

Let $\Omega_t = \{z \in \mathbb{R}^{n+1} : z \cdot x \leq 1 + t\eta \} \in \mathcal{K}_0$. Denote by $z(t) = z_\varepsilon(\Omega_t)$, the entropy point of $\Omega_t$. By Lemma 2.1, we compute, for $|t|$ very small,

$$\frac{d}{dt} \mathcal{J}_p(\Omega_t, z(t)) = \frac{1}{\int_{S^n} r_z^{n+1} d\sigma_{S^n}} \left( \int_{S^n} \eta K d\sigma_{S^n} - \int_{S^n} r_z^{n+1} d\sigma_{S^n} \int_{S^n} u_z^{p-1}(\eta - \dot{z} \cdot x) d\sigma_{S^n} \right), \quad (4.5)$$

where $u_z, r_z$ are support and radial function of $\Omega_t$ w.r.t. $z = z(t)$, and $K$ is the Gauss curvature of $\Omega_t$. Differentiating (4.5) again, we obtain

$$\frac{d^2}{dt^2} \mathcal{J}_p(\Omega_t, z(t))$$

$$= \frac{1}{\int_{S^n} r_z^{n+1} d\sigma_{S^n}} \left\{ \int_{S^n} h^{ij}(\eta_{ij} + \eta \delta_{ij}) \eta d\sigma_{S^n} - \beta \int_{S^n} r_z^{n+1} d\sigma_{S^n} \int_{S^n} u_z^{p-2}(\eta - \dot{z} \cdot x) d\sigma_{S^n} \right. $$

$$- \frac{n + 1}{\int_{S^n} u_z^p d\sigma_{S^n}} \int_{S^n} \eta K d\sigma_{S^n} \int_{S^n} u_z^{p-1} \eta d\sigma_{S^n} + p \frac{\int_{S^n} r_z^{n+1} d\sigma_{S^n}}{\left(\int_{S^n} u_z^p d\sigma_{S^n}\right)^2} \left( \int_{S^n} u_z^{p-1} \eta d\sigma_{S^n} \right)^2 \left\}, \quad (4.6)$$

where $\beta = p - 1$ and $h^{ij}$ is the inverse matrix of $u_{ij} + u \delta_{ij}$. By (4.4), one has

$$\int_{S^n} u_z^{p-1} \dot{z} \cdot x d\sigma_{S^n} = 0, \quad \forall \ t.$$ 

Differentiate this identity w.r.t. $t$ to get

$$\int_{S^n} u_z^{p-2} \eta \dot{z} \cdot x d\sigma_{S^n} = \int_{S^n} u_z^{p-2}(\dot{\eta} \cdot x)^2 d\sigma_{S^n}. \quad (4.7)$$

Hence one infers by plugging (4.7) in (4.6) and by $\Omega_0 = B_1, z(0) = 0$,

$$\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{J}_p(\Omega_t, z(t)) = \int_{S^n} (\Delta \eta + n \eta) \eta d\sigma_{S^n} - \beta \int_{S^n} \eta^2 d\sigma_{S^n} + \beta \int_{S^n} (\dot{\eta} \cdot x) \eta d\sigma_{S^n}$$

$$- (n + 1 - p) \left( \int_{S^n} \eta d\sigma_{S^n} \right)^2$$

$$= - \int_{S^n} |\nabla \eta|^2 d\sigma_{S^n} + (n + 1 - p) \int_{S^n} (\eta - \bar{\eta})^2 d\sigma_{S^n} + \beta \int_{S^n} (\dot{\eta} \cdot x)^2 d\sigma_{S^n}. \quad (4.8)$$
It is straightforward to see
\[ (4.9) \quad \int_{S^n} (\dot{z} \cdot x)^2 d\sigma_{S^n} = |\dot{z}|^2 \int_{S^n} (\dot{\zeta} / |\dot{\zeta}| \cdot x)^2 d\sigma_{S^n} = |\dot{z}|^2 \int_{S^n} x_1^2 d\sigma_{S^n}. \]

Since \( z(t) \) is determined by (4.4), hence depends on \( p \). For clarification, let us denote it by \( z_p(t) \). Then we have by differentiating (4.4)
\[
0 = (p - 1) \int_{S^n} u_{z_p(t)}^{p-2} (\eta - \dot{z}_p \cdot x) x d\sigma_{S^n}, \quad \text{for all } p < 1.
\]

Sending \( t = 0 \), we have
\[
\int_{S^n} \eta x d\sigma_{S^n} = \int_{S^n} (\dot{z}_p(0) \cdot x) x d\sigma_{S^n} = |\dot{z}_p(0)| \int_{S^n} (e_p \cdot x) x d\sigma_{S^n},
\]
where \( e_p = \dot{z}_p(0)/|\dot{z}_p(0)| \). Multiplying \( e_p \) at both sides one deduces
\[ (4.10) \quad |\dot{z}_p(0)| = \left( \int_{S^n} x_1^2 d\sigma_{S^n} \right)^{-1} \int_{S^n} (x \cdot e_p) \eta d\sigma_{S^n} \leq C|\eta|_{L^1(S^n)}, \quad \text{for all } p < 1. \]

As \( B_1 \) maximises \( \tilde{J}_p \) among all \( \Omega \in \mathcal{K}_0 \{1, 2, 3, 8\} \), we have by plugging (4.9) in (4.8)
\[
0 \geq -\int_{S^n} |\nabla \eta|^2 d\sigma_{S^n} + (n + 1 - p) \int_{S^n} (\eta - \bar{\eta})^2 d\sigma_{S^n} + (p - 1) |\dot{z}_p(0)|^2 \int_{S^n} x_1^2 d\sigma_{S^n}.
\]

Sending \( p \to 1 \), we get by (4.10)
\[ (4.11) \quad \int_{S^n} (\eta - \bar{\eta})^2 d\sigma_{S^n} \leq \frac{1}{n} \int_{S^n} |\nabla \eta|^2 d\sigma_{S^n}. \]

It remains to show (4.11) is sharp. If not, then for sufficiently small \( \varepsilon > 0 \), there is an \( \eta_{\varepsilon} \neq 0, \bar{\eta}_\varepsilon = 0 \), such that
\[
\int_{S^n} |\nabla \eta_{\varepsilon}|^2 d\sigma_{S^n} = (n - \sqrt{\varepsilon}) \int_{S^n} \eta_{\varepsilon}^2 d\sigma_{S^n}.
\]

Let \( \Omega_{t}^{\eta_{\varepsilon}} = \{ z \in \mathbb{R}^{n+1} : z \cdot x \leq 1 + t \eta_{\varepsilon} \} \). Then for \( p = 1 - \varepsilon \), by (4.8) and (4.10)
\[
\frac{d^2}{dt^2}_{t=0} \tilde{J}_{1-\varepsilon}(\Omega_{t}^{\eta_{\varepsilon}}, z(t)) = (\sqrt{\varepsilon} + \varepsilon) \int_{S^n} \eta_{\varepsilon}^2 d\sigma_{S^n} - \varepsilon |\dot{z}_{1-\varepsilon}(0)|^2 \int_{S^n} x_1^2 d\sigma_{S^n}
\geq C^{-1} \sqrt{\varepsilon} |\eta_{\varepsilon}|_{L^1(S^n)} - C\varepsilon |\eta_{\varepsilon}|_{L^1(S^n)}^2

> 0,
\]
provided \( \varepsilon \) sufficiently small. This implies that \( B_1 \) is not a maximiser of \( \tilde{J}_p \), thus arriving a contradiction.

\[ \square \]
References


Centre for Mathematics and Its Applications, the Australian National University, Canberra, ACT 2601, Australia.

E-mail address: haodi.chen@anu.edu.au

School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, P.R. China.

E-mail address: chenshib@ustc.edu.cn

Centre for Mathematics and Its Applications, the Australian National University, Canberra, ACT 2601, Australia.

E-mail address: qi-rui.li@anu.edu.au