ASYMPTOTIC CONVERGENCE FOR A CLASS OF FULLY NONLINEAR CURVATURE FLOWS

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ABSTRACT. In this paper we study a class of contracting flows of closed, convex hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ with speed $r^\alpha \sigma_k$, where σ_k is the k-th elementary symmetric polynomial of the principal curvatures, $\alpha \in \mathbb{R}^1$, and r is the distance from the hypersurface to the origin. If $\alpha \geq k+1$, we prove that the flow exists for all time, preserves the convexity and converges smoothly after normalisation to a sphere centred at the origin. If $\alpha < k+1$, a counterexample is given for the above convergence. In the case $k=1$ and $\alpha \geq 2$, we also prove that the flow converges to a round point if the initial hypersurface is weakly mean-convex and star-shaped.

1. INTRODUCTION

Flows of convex hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ by functions of the principal curvatures have been extensively studied in the past four decades. Well-known examples include the mean curvature flow [25], and the Gauss curvature flow [12, 16]. Given a smooth and uniformly convex initial hypersurface, Huisken [25] proved that the mean curvature flow contracts to a point in finite time, and after a time-dependent rescaling, the flow converges smoothly to a sphere. This property was extended to other geometric flows where the speed is a homogeneous of degree one function of the principal curvatures [3, 13, 14]. For the Gauss curvature flow, Andrews [2] proved that the flow deforms a uniformly convex hypersurface into a round point when $n=2$. In higher dimensions, this result was obtained recently by combining a soliton convergence result [21] and a uniqueness result for the soliton [8].

There is a growing interest in the asymptotic behaviour of geometric flows in which the speed is a more general curvature function, in particular the cases when the speed is a homogeneous curvature function of degree not equal to 1. For examples, in [4, 6, 8] the authors studied curvature flows where the speed is a power of the Gauss curvature; and in [1, 5, 7, 11, 15, 17, 18, 32, 33, 36] the authors studied flows by different curvature functions. In many of the above mentioned paper it was proved that the flow converges to a round point in the limit, which is the most interesting property of curvature flows of closed convex hypersurfaces.

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Let $\mathcal{M}_0$ be a smooth, closed and convex hypersurface in $\mathbb{R}^{n+1}$ which encloses the origin, $n \geq 1$. In this paper we study the following geometric flow,

$$\begin{cases} \dfrac{\partial X}{\partial t}(x,t) = -r^{\alpha}\sigma_k(x,t)\nu, \\ X(x,0) = X_0(x), \end{cases} \tag{1.1}$$

where σ_k is the k-curvature, given by

$$\sigma_k(\cdot,t) = \sum_{i_1<\cdots<i_k} \kappa_{i_1}\cdots\kappa_{i_k},$$

and $\kappa_i = \kappa_i(\cdot,t)$ are the principal curvatures of the hypersurface $\mathcal{M}_t$, parametrized by $X(\cdot,t) : \mathbb{S}^n \to \mathbb{R}^{n+1}$, and $\nu(\cdot,t)$ is the unit outer normal of $\mathcal{M}_t$ at $X(\cdot,t)$. We denote by $r = |X(x,t)|$ the distance from the point $X(x,t)$ to the origin. We shall call it the radial function of $\mathcal{M}_t$ in this paper.

Flow with speed depending not only on the curvatures has recently begun to be considered. For example, flows that deform hypersurfaces by their curvature and support function were studied in [9, 26]. In this paper, our flow is driven by curvature and radial function. The study of (1.1) is also motivated by its background in convex geometry. One of the main problems in convex geometry is to characterise various geometric measures, such as the area measures S_k introduced by Fenchel-Jessen and Aleksandrov and the curvature measures C_k introduced by Federer [31]. Let Ω be a convex body containing the origin in its interior. Assume $\partial\Omega$ is uniformly convex, namely $\partial\Omega$ is C^2 with positive principal curvatures. There is an interesting relation $C_k(\Omega,\cdot) = S_k(\Omega, \mathscr{A}_\Omega(\cdot))$, where $\mathscr{A}_\Omega$ is the radial Gauss mapping of Ω [24]. More precisely we have for any $k = 1,\ldots,n$,

$$C_k(\Omega,\omega) = \int_\omega \sigma_k(\lambda_\Omega)\,dC_0(\Omega,\cdot), \quad \forall \text{ Borel set } \omega \subset \mathbb{S}^n, \tag{1.2}$$

where $\lambda_\Omega = (\lambda_{\Omega,1},\ldots,\lambda_{\Omega,n})$ are the principal radii of curvature of Ω. In [22, 23], the L_p Christoffel-Minkowski problem was studied, which is to prescribe the k-th p-area measure given by

$$\begin{aligned} S_{k,p}(\Omega,\omega) &= \int_\omega u_\Omega^{1-p}\,dS_k(\Omega,\cdot) \\ &= \int_\omega \sigma_{n-k}(\kappa_\Omega)\,dS^{(p)}(\Omega,\cdot), \quad \forall \text{ Borel set } \omega \subset \mathbb{S}^n, \end{aligned} \tag{1.3}$$

where u_Ω denotes the support function of Ω, $\kappa_\Omega = (\kappa_{\Omega,1},\ldots,\kappa_{\Omega,n})$ are the principal curvatures of Ω, and $S^{(p)}(\Omega,\cdot)$ is the L_p area measure introduced by Lutwak [30]. Analogous to (1.2) and (1.3), it is interesting to consider the following measure

$$\mathscr{C}_{k,q}(\Omega,\omega) := \int_\omega \sigma_{n-k}^{-1}(\lambda_\Omega)\,d\widetilde{C}_q(\Omega,\cdot), \quad \forall \text{ Borel set } \omega \subset \mathbb{S}^n, \tag{1.4}$$

where $\widetilde{C}_q(\Omega,\cdot)$ is the dual curvature measure introduced by Huang et al. [24]. In smooth category, $\widetilde{C}_q(\Omega,\cdot)$ is absolutely continuous w.r.t. the standard spherical measure $\sigma_{\mathbb{S}^n}$, and $d\widetilde{C}_q(\Omega,\cdot)/d\sigma_{\mathbb{S}^n} = r^{q-n-1}u/K$, where r, u, K are respectively the radial function, support function and Gauss curvature of Ω. Given a measure $d\mu = f d\sigma_{\mathbb{S}^n}$, with positive function $f \in C^\infty(\mathbb{S}^n)$, the problem of prescribing $\mathscr{C}_{k,q}$ measure is to find a convex body Ω such that $\mathscr{C}_{k,q}(\Omega,\cdot) = \mu$. It is equivalent to solving the following equation

$$r^{n+1-q}\sigma_k(\kappa_\Omega) = \frac{u}{f}(x), \quad x \in \mathbb{S}^n. \tag{1.5}$$

In particular, the dual Minkowski problem proposed in [24] is equivalently to solve (1.5) with $k = n$. When f is a constant, equation (1.5) characterises the self-similar solutions to our flow (1.1). Indeed it is not hard to see that if Ω is a convex body satisfying (1.5) with constant f, then after a proper rescaling if necessary $\mathcal{M} = \partial\Omega$ is steady under the normalised flow (1.8) below with $\alpha = n + 1 - q$.

In our previous paper [28], we studied the associated anisotropic version of (1.1) in the case $k = n$, and proved that the flow converges to solutions to the Aleksandrov problem and the dual Minkowski problem introduced in [24]. In the case $k = n$, the flow (1.1) is a decent gradient flow of an associated functional, which implies the asymptotic convergence of the normalised flow in [28] once the a priori estimates are established. For the cases $1 \le k < n$ studied in this paper, we are unaware of the existence of the functional, and so we only consider (1.1) without an anisotropic factor f. We are mainly interested in the spherical asymptotic behaviour of the flow. We prove that, when $\alpha \ge k+1$, the solution $\mathcal{M}_t$ preserves the convexity, and converges smoothly after normalisation to a sphere.

Theorem 1.1. *Let $\mathcal{M}_0$ be a smooth, closed and uniformly convex hypersurface in $\mathbb{R}^{n+1}$ enclosing the origin. If $\alpha \ge k+1$, then the flow (1.1) has a unique smooth and uniformly convex solution $\mathcal{M}_t$ for all time $t > 0$, which converges to the origin. After a proper rescaling $X \to \phi^{-1}(t)X$, where*

$$\begin{aligned} &\phi(t) = e^{-\beta t}, && \text{if } \alpha = k+1, \\ &\phi(t) = [1 + (\alpha - k - 1)\beta t]^{\frac{1}{k+1-\alpha}}, && \text{if } \alpha \ne k+1, \end{aligned} \tag{1.6}$$

and $\beta = \sigma_k(1,\cdots,1)$, the hypersurface $\widetilde{\mathcal{M}}_t = \phi^{-1}(t)\mathcal{M}_t$ converges exponentially to a sphere centred at the origin in the C^∞ topology.

For $k = 1$, we can also prove the convergence for weakly mean-convex and star-shaped initial hypersurfaces. We say a hypersurface is weakly mean-convex if its mean curvature is non-negative everywhere.

Theorem 1.2. *Let $\mathcal{M}_0$ be a smooth, closed and weakly mean-convex hypersurface in $\mathbb{R}^{n+1}$. Suppose that $\mathcal{M}_0$ is star-shaped with respect to the origin. If $k = 1$ and $\alpha \geq 2$, then the mean curvature flow* (1.1) *has a unique smooth solution $\mathcal{M}_t$ for all time $t > 0$, and $\mathcal{M}_t$ converges to the origin. After a proper rescaling $X \to \phi^{-1}(t)X$, where ϕ is given by* (1.6)*, the hypersurface $\widetilde{\mathcal{M}}_t = \phi^{-1}(t)\mathcal{M}_t$ converges exponentially to a sphere centred at the origin in the C^∞ topology.*

The study of the asymptotic behaviour of the flow (1.1) is equivalent to the long time behaviour of the normalised flow. Let

$$\widetilde{X}(\cdot,\tau) = \phi^{-1}(t)X(\cdot,t), \tag{1.7}$$

where

$$\tau = \begin{cases} t & \text{if } \alpha = k+1, \\ \dfrac{\log[1+(\alpha-k-1)\beta t]}{(\alpha-k-1)\beta} & \text{if } \alpha \neq k+1. \end{cases}$$

Then $\widetilde{X}(\cdot,\tau)$ satisfies the following normalised flow

$$\begin{cases} \dfrac{\partial X}{\partial t}(x,t) = -r^\alpha \sigma_k(x,t)\nu + \beta X(x,t), \\ X(\cdot,0) = X_0. \end{cases} \tag{1.8}$$

For convenience we still use t instead of τ to denote the time variable and omit the "tilde" if no confusions arise. In order to prove Theorems 1.1 & 1.2, we shall establish the a priori estimates for the normalised flow (1.8), and show that if $X(\cdot,t)$ solves (1.8), then $|X|$ converges exponentially to a constant as $t \to \infty$.

When $\alpha < k+1$, we find that the hypersurface evolving by (1.1) may reach the origin in finite time, before the hypersurface shrinks to a point. Hence the flow does not converge to a round sphere centred at the origin in general.

Theorem 1.3. *Suppose $\alpha < k+1$. There exists a smooth, closed, uniformly convex hypersurface $\mathcal{M}_0$, such that under the flow* (1.1)*,*

$$\mathcal{R}(X(\cdot,t)) := \frac{\max_{\mathbb{S}^n} r(\cdot,t)}{\min_{\mathbb{S}^n} r(\cdot,t)} \to \infty \ \text{ as } \ t \to T \tag{1.9}$$

for some $T > 0$.

This paper is organised as follows. In Section 2 we collect some properties of star-shaped hypersurfaces, and show that the flow (1.8) can be reduced to a parabolic equation of the radial function, or a parabolic equation of the support function provided $\mathcal{M}_t$ is uniformly convex. We will also derive the evolution equations for various geometric quantities in Section 2. In Section 3 we establish the needed a priori estimates, which ensure the longtime existence of the normalised flow (1.8). Section 4 is devoted to the sphere convergence result. The proofs of Theorems 1.1 & 1.2 will be presented in this section. Finally in Section 5 we prove Theorem 1.3.

2. Preliminaries

Let us recall some relevant geometric quantities of a smooth and star-shaped hypersurface $\mathcal{M}$ in $\mathbb{R}^{n+1}$. Given $\xi \in \mathbb{S}^n$, the ray $\{t\xi : t \geq 0\}$ intersects with $\mathcal{M}$ at exactly one point $p(\xi)$. Hence $\mathcal{M}$ can be regarded as a spherical radial graph via the mapping

$$\vec{r} : \xi \in \mathbb{S}^n \mapsto p(\xi) \in \mathcal{M}.$$

Let $r(\xi) = |p(\xi)|$ be the radial function, which is the distance from the origin to $p(\xi)$. We now give the expressions of the induced metric, second fundamental form, Weingarten curvatures of $\mathcal{M}$ in terms of the radial function. These formulae can be found in a number of papers, for example [20, 35].

Let $e_1, \cdots, e_n$ be a smooth local orthonormal frame field on $\mathbb{S}^n$, and let $\overline{\nabla}$ be the covariant derivative on $\mathbb{S}^n$. We denote by $g_{ij}, g^{ij}, \nu, h_{ij}$ the metric, the inverse of the metric, the unit outer normal and the second fundamental form of $\mathcal{M}$, respectively. Then, in terms of r, we have

$$\begin{aligned} g_{ij} &= r^2\delta_{ij} + \overline{\nabla}_i r \overline{\nabla}_j r, \\ g^{ij} &= r^{-2}\Big(\delta_{ij} - \frac{\overline{\nabla}_i r \overline{\nabla}_j r}{r^2 + |\overline{\nabla} r|^2}\Big), \\ \nu &= \frac{r\xi - \overline{\nabla} r}{\sqrt{r^2 + |\overline{\nabla} r|^2}}, \\ h_{ij} &= \frac{1}{\sqrt{r^2 + |\overline{\nabla} r|^2}}(r^2\delta_{ij} + 2\overline{\nabla}_i r \overline{\nabla}_j r - r\overline{\nabla}^2_{ij} r). \end{aligned} \tag{2.1}$$

The principal curvatures of $\mathcal{M}$ are the eigenvalues of h_{ij} with respect to g_{ij}, namely the solutions of

$$0 = \det(g^{il}h_{lj} - \kappa\delta_{ij}) = \det(a_{ij} - \kappa\delta_{ij}),$$

where the symmetric matrix $\{a_{ij}\}$ is given by

$$a_{ij} = (g^{-\frac{1}{2}})^{il} h_{lm} (g^{-\frac{1}{2}})^{mj}. \tag{2.2}$$

Here $\{(g^{-\frac{1}{2}})^{ij}\}$ is the square root of the matrix $\{g^{ij}\}$ and is given explicitly by

$$(g^{-\frac{1}{2}})^{ij} = r^{-1}\Big[\delta_{ij} - \frac{\overline{\nabla}_i r \overline{\nabla}_j r}{\sqrt{r^2 + |\overline{\nabla} r|^2}(r + \sqrt{r^2 + |\overline{\nabla} r|^2})}\Big]. \tag{2.3}$$

We call $\{a_{ij}\}$ the Weingarten matrix of $\mathcal{M}$. Throughout this paper, we will use the Einstein summation convention for convenience.

Suppose $X(\cdot,t)$ is an embedding of a time-dependent family of smooth, closed hypersurface $\mathcal{M}_t$ which is star-shaped with respect the origin. For a suitable diffeomorphism $\xi(\cdot,t):\mathbb{S}^n\to\mathbb{S}^n$, we have

$$X(x,t)=r(\xi(x,t),t)\xi(x,t),$$

where $r(\cdot,t):\mathbb{S}^n\to\mathbb{R}_+$ is the radial function of $\mathcal{M}_t$. Then we have

$$\partial_t X=\Big(\frac{\partial r}{\partial\xi^i}\frac{\partial\xi^i}{\partial t}+\frac{\partial r}{\partial t}\Big)\xi+r\frac{\partial\xi}{\partial t}.$$

Hence by (2.1)

$$\begin{aligned}\langle\partial_t X,\nu\rangle &= (r^2+|\overline{\nabla}r|^2)^{-\frac12}\Big[\frac{\partial r}{\partial\xi^i}\frac{\partial\xi^i}{\partial t}r+r\frac{\partial r}{\partial t}-r\Big\langle\overline{\nabla}r\cdot\frac{\partial\xi}{\partial t}\Big\rangle\Big]\\ &= r(r^2+|\overline{\nabla}r|^2)^{-\frac12}\frac{\partial r}{\partial t}.\end{aligned}\tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^{n+1}$. If $X(x,t)$ satisfies the normalised flow (1.8), then by (2.1)

$$\begin{aligned}\langle\partial_t X,\nu\rangle &= -r^\alpha\sigma_k+\langle\beta X,\nu\rangle\\ &= -r^\alpha\sigma_k+\beta\Big\langle r\xi,\frac{r\xi-\overline{\nabla}r}{\sqrt{r^2+|\overline{\nabla}r|^2}}\Big\rangle\\ &= -r^\alpha\sigma_k+\beta r^2(r^2+|\overline{\nabla}r|^2)^{-\frac12}.\end{aligned}\tag{2.5}$$

By (2.4) and (2.5), we conclude that the normalised flow (1.8) can be reduced to the following scalar equation for $r(\cdot,t)$,

$$\begin{cases}\dfrac{\partial r}{\partial t}(\xi,t)=-(1+|\overline{\nabla}\log r|^2)^{\frac12}r^\alpha\sigma_k(\xi,t)+\beta r(\xi,t)\ \text{ on }\mathbb{S}^n\times[0,\infty),\\ r(\cdot,0)=r_0,\end{cases}\tag{2.6}$$

where r_0 is the radial function of $\mathcal{M}_0$, and $\sigma_k(\xi,t)$ denotes the k-curvature at $r(\xi,t)\xi\in\mathcal{M}_t$.

When $k=1$, (2.6) is a quasi-linear parabolic equation. For $k\geq 2$, the equation (2.6) is parabolic, as long as $\mathcal{M}_t$ is k-convex. Namely

$$(\kappa_1(x,t),\cdots,\kappa_n(x,t))\in\Gamma_k:=\{\kappa\in\mathbb{R}^n:\sigma_m(\kappa)>0,\ \forall\ m=1,\cdots,k\},\tag{2.7}$$

where $\kappa_i(x,t)$ are the principal curvatures of $\mathcal{M}_t$ at $X(x,t)$. In what follows we shall say that $X(\cdot,t)$ (respectively $r(\cdot,t)$) is a uniformly convex solution of the flow (1.8) (respectively the equation (2.6)), if for each t, $\mathcal{M}_t$ is uniformly convex. When $\mathcal{M}_t$ is closed, this means the principal curvatures of $\mathcal{M}_t$ are positive everywhere.

It is sometimes more convenient to study the equation for the quantity

$$\varrho(\xi, t) = \log r(\xi, t). \tag{2.8}$$

By (2.1), (2.2) and (2.3), we find that

$$a_{ij} = e^{-\varrho}(1 + |\overline{\nabla}\varrho|^2)^{-\frac{1}{2}}\widetilde{a}_{ij},$$

where

$$\widetilde{a}_{ij} = \gamma_{il}(\delta_{lm} + \overline{\nabla}_l\varrho\overline{\nabla}_m\varrho - \overline{\nabla}^2_{lm}\varrho)\gamma_{mj}, \tag{2.9}$$

and

$$\gamma_{ij} = \delta_{ij} - \frac{\overline{\nabla}_i\varrho\overline{\nabla}_j\varrho}{(1 + |\overline{\nabla}\varrho|^2)^{\frac{1}{2}}(1 + (1 + |\overline{\nabla}\varrho|^2)^{\frac{1}{2}})}. \tag{2.10}$$

It then follows from (2.6) that

$$\begin{cases} \dfrac{\partial\varrho}{\partial t}(\xi, t) = -(1 + |\overline{\nabla}\varrho|^2)^{\frac{1-k}{2}} e^{(\alpha-k-1)\varrho}\sigma_k(\widetilde{a}_{ij}) + \beta \;\text{ on } \mathbb{S}^n \times [0, \infty), \\ \quad \varrho(\cdot, 0) = \log r_0, \end{cases} \tag{2.11}$$

where

$$\sigma_k(\widetilde{a}_{ij}) = \sum_{1\le i_1<\cdots<i_k\le n} \widetilde{\kappa}_{i_1}\cdots\widetilde{\kappa}_{i_k}$$

and $\widetilde{\kappa}_i$ are the eigenvalues of the matrix $\{\widetilde{a}_{ij}\}$.

If the hypersurface $\mathcal{M}$ is furthermore uniformly convex, then the geometry of $\mathcal{M}$ can be also characterised by its support function. Let $\nu^{-1} : \mathbb{S}^n \to \mathcal{M}$ be the inverse Gauss map, namely $\nu^{-1}(x)$ is the point $p(x) \in \mathcal{M}$ such that the unit outer normal of $\mathcal{M}$ at $p(x)$ is equal to x. The support function u of $\mathcal{M}$ is function defined on the unit sphere S^n, given by

$$u(x) = \langle x, \nu^{-1}(x)\rangle. \tag{2.12}$$

For a parametrisation of $\mathcal{M}_t :\; x \to X(x, t)$, in the following we may also use $u(x, t) = \langle X(x, t), \nu(x, t)\rangle$ to denote the support function, if no confusion arises. It is easy to verify that

$$\nu^{-1}(x) = u(x)x + \overline{\nabla}u(x). \tag{2.13}$$

and the principal radii of curvature of $\mathcal{M}$ at $\nu^{-1}(x)$ are the eigenvalues of the matrix

$$b_{ij} = \overline{\nabla}^2_{ij}u(x) + u(x)\delta_{ij}, \tag{2.14}$$

where the derivatives are taken with respect to an orthonormal frame on $\mathbb{S}^n$. The formulae (2.13) and (2.14) can be found in for example [3, 36].

Let $X(\cdot,t)$ be a smooth and uniformly convex solution to the normalised flow (1.8) and let $u(\cdot,t)$ be its support function. Let $\varphi(\cdot,t):\mathbb{S}^n\to\mathbb{S}^n$ be the diffeomorphism such that the unit outer normal at $X(\varphi(x,t),t)$ is x. Then

$$u(x,t)=\langle X(\varphi(x,t),t),x\rangle.$$

It follows that

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Big\langle \frac{\partial X}{\partial \varphi^i}\frac{\partial \varphi^i}{\partial t}+\frac{\partial X}{\partial t},x\Big\rangle\\ &= \big\langle -|X(\varphi(x,t),t)|^\alpha\sigma_k x+\beta X,x\big\rangle\\ &= -|X(\varphi(x,t),t)|^\alpha\sigma_k+\beta u, \end{aligned}\tag{2.15}$$

where σ_k is the k-curvature of $\mathcal{M}_t$ at $X(\varphi(x,t),t)$. By (2.14), we have

$$\sigma_k=\sum_{1\le i_1<\cdots<i_k\le n}\kappa_{i_1}\cdots\kappa_{i_k}=\frac{\sigma_{n-k}}{\sigma_n}(\overline{\nabla}^2u+uI).$$

By (2.13) and (2.15), the normalised flow (1.8) can be described by the following scalar equation of the support function $u(\cdot,t)$,

$$\begin{cases}\dfrac{\partial u}{\partial t}(x,t)=-r^\alpha Q_{n,n-k}^{-1}(\overline{\nabla}^2_{ij}u+u\delta_{ij})(x,t)+\beta u(x,t) \ \text{ on } \mathbb{S}^n\times[0,\infty),\\ u(\cdot,0)=u_0,\end{cases}\tag{2.16}$$

where u_0 is the support function of the initial hypersurface $\mathcal{M}_0$, and

$$r=\sqrt{u^2+|\overline{\nabla}u|^2}(x,t),\quad Q_{n,n-k}=\frac{\sigma_n}{\sigma_{n-k}}.$$

We now derive some evolution equations for our normalised flow (1.8). Pick any local coordinate chart $\{x_i\}_{i=1}^n$ of $\mathbb{S}^n$. Denote $\partial_i=\frac{\partial}{\partial x_i}$, $X_i=\partial_iX$ and $X_{ij}=\partial_i\partial_jX-\Gamma_{ij}^kX_k$, where Γ_{ij}^k is the Christoffel symbol of the metric of $\mathcal{M}_t$. Recall the following identities

$$\begin{aligned} X_{ij}&=-h_{ij}\nu, &&\text{(Gauss formula)}\\ \nu_i&=h_{ij}g^{jl}X_l, &&\text{(Weingarten equation)}\\ h_{ij,l}&=h_{il,j}, &&\text{(Codazzi formula)}\\ R_{ijrs}&=h_{ir}h_{js}-h_{is}h_{jr}, &&\text{(Gauss equation)}\end{aligned}\tag{2.17}$$

where $h_{ij,l}=\nabla_lh_{ij}$, ∇ denotes the derivative with respect to the metric of $\mathcal{M}_t$, and R_{ijkl} is the Riemannian curvature tensor. Making use of the Ricci identity to interchange the order of the derivatives and employing the Codazzi formula and Gauss equation in (2.17), we also obtain

$$\nabla_s\nabla_rh_{ij} = \nabla_j\nabla_ih_{rs}+(h^2)_{ij}h_{rs}-(h^2)_{is}h_{jr}+h_{is}(h^2)_{jr}-h_{ij}(h^2)_{rs},\tag{2.18}$$

where $(h^2)_{ij}=h_{il}h^l_j$, and $h^l_j=g^{lp}h_{pj}$. For convenience, the Einstein summation convention is used.

Lemma 2.1. *Denote $\Phi = r^{\alpha} F^k$, and $F = \sigma_k^{\frac{1}{k}}$. Then under the normalised flow (1.8), we have*

$$\partial_t g_{ij} = -2\Phi h_{ij} + 2\beta g_{ij}, \tag{2.19}$$

$$\partial_t \nu = \nabla \Phi, \tag{2.20}$$

and

$$\begin{aligned} \partial_t h_{ij} &= k\Phi F^{-1} F^{rs} \nabla^2_{rs} h_{ij} - (k+1)\Phi (h^2)_{ij} + k\Phi h_{ij} F^{-1} F^{rs} (h^2)_{rs} \\ &\quad + k\Phi F^{-1} F^{rs} ((h^2)_{is} h_{jr} - h_{is}(h^2)_{jr}) + \beta h_{ij} \\ &\quad + k\Phi F^{-1} F^{rs,pq} h_{rs,i} h_{pq,j} + k(k-1)\Phi \nabla_i \log F \nabla_j \log F \\ &\quad + \alpha k \Phi (\nabla_i \log r \nabla_j \log F + \nabla_j \log r \nabla_i \log F) \\ &\quad + \alpha(\alpha-1)\Phi \nabla_i \log r \nabla_j \log r + \alpha \Phi r^{-1} \nabla^2_{ij} r, \end{aligned} \tag{2.21}$$

where $F^{rs} = \frac{\partial F}{\partial h_{rs}}$ and $F^{rs,pq} = \frac{\partial F}{\partial h_{rs} \partial h_{pq}}$.

Proof. As $\langle X_i, \nu \rangle = 0$, by the Weingarten equation in (2.17),

$$\begin{aligned} \partial_t g_{ij} &= \partial_t \langle \partial_i X, \partial_j X \rangle \\ &= \langle -\partial_i(\Phi\nu - \beta X), \partial_j X \rangle + \langle \partial_i X, -\partial_j(\Phi\nu - \beta X) \rangle \\ &= -\Phi\big(\langle \partial_i \nu, \partial_j X \rangle + \langle \partial_i X, \partial_j \nu \rangle\big) + 2\beta g_{ij} \\ &= -2\Phi h_{ij} + 2\beta g_{ij}. \end{aligned}$$

This proves (2.19).

Since ν is a unit vector filed, $\partial_t \nu$ has only tangential part. Hence

$$\begin{aligned} \partial_t \nu &= \langle \partial_t \nu, \partial_i X \rangle g^{ij} \partial_j X \\ &= -\langle \nu, \partial_i(-\Phi\nu + \beta X) \rangle g^{ij} \partial_j X \\ &= \nabla \Phi. \end{aligned}$$

This verifies the second evolution equation in the lemma.

Using the Gauss formula in (2.17), we have

$$\begin{aligned} \partial_t h_{ij} &= \partial_t \langle \partial_i \partial_j X, -\nu \rangle \\ &= \langle \partial_i \partial_j (\Phi\nu - \beta X), \nu \rangle - \langle \Gamma^k_{ij} \partial_k X - h_{ij}\nu, \partial_t \nu \rangle \\ &= \partial_i \partial_j \Phi + \Phi \langle \partial_i \partial_j \nu, \nu \rangle + \beta h_{ij} - \Gamma^k_{ij} \Phi_k \\ &= \nabla^2_{ij} \Phi + \Phi \langle \partial_i (h^k_j \partial_k X), \nu \rangle + \beta h_{ij} \\ &= \nabla^2_{ij} \Phi - \Phi h_{ik} h^k_j + \beta h_{ij}. \end{aligned} \tag{2.22}$$

By (2.18), it is readily seen that

$$\begin{aligned}\nabla^2_{ij}\Phi &= k\Phi F^{-1}(F^{rs}\nabla^2_{ij}h_{rs}+F^{rs,pq}h_{rs,i}h_{pq,j})+k(k-1)\Phi\nabla_i\log F\nabla_j\log F\\ &\quad+\alpha k\Phi(\nabla_i\log r\nabla_j\log F+\nabla_j\log r\nabla_i\log F)\\ &\quad+\alpha(\alpha-1)\Phi\nabla_i\log r\nabla_j\log r+\alpha\Phi r^{-1}\nabla^2_{ij}r\\ &= k\Phi F^{-1}F^{rs}\nabla^2_{rs}h_{ij}-k\Phi(h^2)_{ij}+k\Phi h_{ij}F^{-1}F^{rs}(h^2)_{rs}\\ &\quad+k\Phi F^{-1}F^{rs}((h^2)_{is}h_{jr}-h_{is}(h^2)_{jr})\\ &\quad+k\Phi F^{-1}F^{rs,pq}h_{rs,i}h_{pq,j}+k(k-1)\Phi\nabla_i\log F\nabla_j\log F\\ &\quad+\alpha k\Phi(\nabla_i\log r\nabla_j\log F+\nabla_j\log r\nabla_i\log F)\\ &\quad+\alpha(\alpha-1)\Phi\nabla_i\log r\nabla_j\log r+\alpha\Phi r^{-1}\nabla^2_{ij}r.\end{aligned}$$

This together with (2.22) implies (2.21).

□

3. A priori estimates

In this section we establish the priori estimates and show that the normalised flow exists for long time. We first derive the L^∞-norm estimate for the radial function.

Lemma 3.1. *Let $r(\cdot,t)$ be a positive, k-convex smooth solution to* (2.6) *on $\mathbb{S}^n\times[0,T)$. If $\alpha\geq k+1$, then there is a positive constant C depending only on $\max_{\mathbb{S}^n} r(\cdot,0)$ and $\min_{\mathbb{S}^n} r(\cdot,0)$ such that*

$$1/C\leq r(\cdot,t)\leq C\quad\forall\ t\in[0,T). \tag{3.1}$$

Proof. First we consider the case $\alpha>k+1$. For the first equality of (3.1), let $r_{\min}(t)=\min_{x\in\mathbb{S}^n} r(x,t)$. By (2.1), (2.2) and (2.3), we infer that, at the point where r attains its spatial minimum,

$$\sigma_k(a_{ij})\leq\frac{\sigma_k(1,\cdots,1)}{r^k_{\min}}=\frac{\beta}{r^k_{\min}}.$$

From (2.6) it then follows that

$$\frac{d}{dt}r_{\min}\geq-\beta(r^{\alpha-k-1}_{\min}-1)r_{\min}. \tag{3.2}$$

Since $\alpha>k+1$, we may assume that $r_{\min}(t)<1$, otherwise we are through. Hence $\frac{d}{dt}r_{\min}\geq 0$. This implies

$$r(\cdot,t)\geq\min\big\{1,\min_{\mathbb{S}^n} r(\cdot,0)\big\}.$$

We obtain the first equality of (3.1). The second inequality can be proved similarly. In fact one can verify, as above,

$$r(\cdot,t)\leq\max\big\{1,\max_{\mathbb{S}^n} r(\cdot,0)\big\}.$$

This proves (3.1) for the case $\alpha > k+1$.

If $\alpha = k+1$, then (3.2) gives $\frac{d}{dt} r_{\min} \geq 0$. Similarly we have $\frac{d}{dt} r_{\max} \leq 0$. Therefore

$$\min_{\mathbb{S}^n} r(\cdot, 0) \leq r(\cdot, t) \leq \max_{\mathbb{S}^n} r(\cdot, 0).$$

□

For convex hypersurface, the gradient estimate is a direct consequence of the L^∞-norm estimate.

Lemma 3.2. *Let $r(\cdot, t)$ be a positive, smooth, uniformly convex solution to* (2.6) *on $\mathbb{S}^n \times [0, T)$. We have the gradient estimate*

$$|\overline{\nabla} r(\cdot, t)| \leq C \quad \forall\ t < T, \tag{3.3}$$

where $C > 0$ depends only on $\min_{\mathbb{S}^n \times [0,T)} r$ and $\max_{\mathbb{S}^n \times [0,T)} r$.

Proof. This lemma is due to the convexity. Given $x \in \mathbb{S}^n$, by (2.13), we have

$$X(x, t) = u(x, t)x + \overline{\nabla} u(x, t), \tag{3.4}$$

where $X(x, t)$ is the point at where the unit outer normal of $\mathcal{M}_t$ is x. Let $\xi = X(x, t)/|X(x, t)|$. Then $X(x, t) = r(\xi, t)\xi$. Multiplying ξ to both sides of the third formula in (2.1), and noting that $\nu = x$ in our current situation, one concludes that

$$\frac{r}{\sqrt{r^2 + |\overline{\nabla} r|^2}} = x \cdot \frac{X}{r} = \frac{u}{r} \tag{3.5}$$

where (3.4) was used in the second equality. This implies

$$|\overline{\nabla} r(\xi, t)| \leq \frac{r^2}{u} \leq \frac{\max_{\mathbb{S}^n \times [0,T)} r^2}{\min_{\mathbb{S}^n \times [0,T)} r}.$$

Note that in the second inequality we have used the fact

$$\min_{\mathbb{S}^n} u(\cdot, t) = \min_{\mathbb{S}^n} r(\cdot, t), \text{ and } \max_{\mathbb{S}^n} u(\cdot, t) = \max_{\mathbb{S}^n} r(\cdot, t), \tag{3.6}$$

which can be easily derived from the formula

$$|X| = \sqrt{u^2 + |\overline{\nabla} u|^2}. \tag{3.7}$$

□

Similarly we have the estimates for the support function $u(\cdot, t)$.

Lemma 3.3. *Let $X(\cdot,t)$ be a positive, smooth, uniformly convex solution to* (2.16) *on* $\mathbb{S}^n\times[0,T)$*. Let u and r be its support function and radial function respectively. Then for all $t<T$*

$$\min_{\mathbb{S}^n\times[0,T)} r \le u(\cdot,t) \le \max_{\mathbb{S}^n\times[0,T)} r, \tag{3.8}$$

and

$$|\overline{\nabla} u(\cdot,t)| \le \max_{\mathbb{S}^n\times[0,T)} r. \tag{3.9}$$

Proof. The estimates (3.8) and (3.9) follows from (3.6) and (3.7) respectivly. □

Next we show that under the normalised flow, the hypersurface $\mathcal{M}_t$ are "uniformly star-shaped".

Lemma 3.4. *Let $X(\cdot,t)$ be a positive, smooth, uniformly convex hypersurface which solves the normalised flow* (2.16) *on* $\mathbb{S}^n\times[0,T)$*, and encloses the origin. Then for any $t\in[0,T)$ and $p\in\mathcal{M}_t$, and any unit tangential vector $e(p)\in T_p\mathcal{M}_t$, we have*

$$\Big(\big\langle e(p), \frac{p}{|p|}\big\rangle\Big)^2 \le 1-\delta_0, \tag{3.10}$$

where $\delta_0>0$ is a small constant only depending on $\alpha, \min_{\mathbb{S}^n\times[0,T)} r$ and $\max_{\mathbb{S}^n\times[0,T)} r$.

Proof. Let $r(\cdot,t)$ and $u(\cdot,t)$ be the radial function and support function of $X(\cdot,t)$ respectively. For any $t<T$ and $\xi\in\mathbb{S}^n$, let $x\in\mathbb{S}^n$ be the unit outer normal of $X(\cdot,t)$ at $r(\xi,t)\xi$. For (3.10), it suffices show

$$\langle \xi, x\rangle \ge \sqrt{\delta_0}.$$

This follows from (3.5), since

$$\langle \xi, x\rangle = \frac{u(x,t)}{r(\xi,t)} \ge \frac{\min_{\mathbb{S}^n\times[0,T)} r}{\max_{\mathbb{S}^n\times[0,T)} r}.$$

□

We next derive an upper bound for the k-curvature

$$\sigma_k=\sigma_k[\kappa]=\sum_{1\le i_1<\cdots<i_k\le n}\kappa_{i_1}\cdots\kappa_{i_k},$$

where $\kappa_i=\kappa_i(x,t)$ are the principal curvatures of $\mathcal{M}_t$ at $X(x,t)$.

Lemma 3.5. *Let $X(\cdot,t)$ be a smooth, closed, uniformly convex solution to the normalised flow* (1.8) *for $t\in[0,T)$, which encloses the origin. Then there is a positive constant C depending only on $\alpha, \mathcal{M}_0, \min_{\mathbb{S}^n\times[0,T)} r, \max_{\mathbb{S}^n\times[0,T)} r$, such that*

$$\sigma_k(\cdot,t)\le C,\ \forall\ t\in[0,T). \tag{3.11}$$

Proof. Consider the auxiliary function

$$G(x,t) = \frac{-u_t}{u-\epsilon_0} = \frac{r^\alpha Q_{n,n-k}^{-1}(\overline{\nabla}^2 u + uI) - \beta u}{u-\epsilon_0},$$

where

$$\epsilon_0 = \frac{1}{2}\min_{\mathbb{S}^n\times[0,T)} u > 0.$$

We shall apply the maximum principle to G and show that G is bounded from above.

At the point where G attains its spatial maximum, we have

$$0 = \overline{\nabla}_i G = \frac{-u_{ti}}{u-\epsilon_0} + \frac{u_t u_i}{(u-\epsilon_0)^2}, \tag{3.12}$$

and

$$\begin{aligned} 0 \geq \overline{\nabla}_{ij}^2 G &= \frac{-u_{tij}}{u-\epsilon_0} + \frac{u_{ti}u_j + u_{tj}u_i + u_t u_{ij}}{(u-\epsilon_0)^2} - \frac{2u_t u_i u_j}{(u-\epsilon_0)^3} \\ &= \frac{-u_{tij}}{u-\epsilon_0} + \frac{u_t u_{ij}}{(u-\epsilon_0)^2}, \end{aligned} \tag{3.13}$$

where (3.12) was used in the second equality above, and $\overline{\nabla}_i, \overline{\nabla}_{ij}^2$ are covariant derivatives with respect to the standard metric of $\mathbb{S}^n$.

By (3.12) and (3.13), we infer that

$$-u_{tij} \leq G u_{ij}, \tag{3.14}$$

By (3.7) and (3.12),

$$r_t = \frac{uu_t + \sum u_k u_{kt}}{r} = \frac{\epsilon_0 u - r^2}{r} G. \tag{3.15}$$

Making use of (2.16), (3.14) and (3.15), we obtain

$$\begin{aligned} \partial_t G &= \frac{-u_{tt}}{u-\epsilon_0} + G^2 \\ &= \frac{-r^\alpha Q_{n,n-k}^{-2} Q_{n,n-k}^{ij}(u_{ijt} + u_t\delta_{ij})}{u-\epsilon_0} + \alpha\frac{r^{\alpha-1} r_t \sigma_k}{u-\epsilon_0} + \beta G + G^2 \\ &\leq \frac{r^\alpha Q_{n,n-k}^{-2} Q_{n,n-k}^{ij}(G u_{ij} - u_t\delta_{ij})}{u-\epsilon_0} + \alpha\frac{r^{\alpha-1} r_t \sigma_k}{u-\epsilon_0} + \beta G + G^2 \\ &= \frac{r^\alpha \sigma_k G}{u-\epsilon_0}\Big(k - \epsilon_0 Q_{n,n-k}^{-1}\sum Q_{n,n-k}^{ii}\Big) + C(1+G^2), \end{aligned} \tag{3.16}$$

where $\sigma_k = \sigma_k[\kappa]$ in (3.16) is the k-curvature. By Newton and Maclaurin's inequalities, we have

$$
\begin{aligned}
(3.17)\qquad Q^{-1}_{n,n-k}\sum Q^{ii}_{n,n-k} &= \frac{\sum\sigma_n^{ii}}{\sigma_n}(\overline{\nabla}^2 u + uI) - \frac{\sum\sigma_{n-k}^{ii}}{\sigma_{n-k}}(\overline{\nabla}^2 u + uI)\\
&= \frac{\sigma_{n-1}}{\sigma_n}(\overline{\nabla}^2 u + uI) - (k+1)\frac{\sigma_{n-k-1}}{\sigma_{n-k}}(\overline{\nabla}^2 u + uI)\\
&\geq \frac{k}{n}\frac{\sigma_{n-1}}{\sigma_n}(\overline{\nabla}^2 u + uI) = \frac{k}{n}\sigma_1[\kappa]\\
&\geq k\binom{n}{k}^{-\frac{1}{k}}\sigma_k^{\frac{1}{k}}[\kappa].
\end{aligned}
$$

Without loss of generality we assume that $\sigma_k[\kappa] \approx G \gg 1$. Plugging (3.17) into (3.16), we obtain

$$\partial_t G \leq C_0 G^2 (C_1 - \epsilon_0 G^{\frac{1}{k}})$$

for some C_0, C_1 depending on α and the L^∞-norm of u only. We therefore conclude that $G \leq C$. Our a priori bound (3.11) follows consequently. $\square$

Next we prove that if $\mathcal{M}_0$ is uniformly convex, then along the normalised flow (1.8) the principal curvatures of $\mathcal{M}_t$ remains uniformly positive. We will need the following algebra lemma.

Lemma 3.6. *Let $F = \sigma_k^{\frac{1}{k}}(h_{ij})$, and $\{\widetilde{h}^{ij}\}$ be the inverse matrix of $\{h_{ij}\}$. Then*

$$(F^{ij,rs} + 2F^{ir}\widetilde{h}^{js})\eta_{ij}\eta_{rs} \geq 2F^{-1}(F^{ij}\eta_{ij})^2. \tag{3.18}$$

Proof. This lemma can be found in [36]. We include a proof here for reader's convenience. Let

$$\widetilde{F}(\widetilde{h}^{ij}) = \Big(\frac{\sigma_n}{\sigma_{n-k}}(\widetilde{h}^{ij})\Big)^{\frac{1}{k}}.$$

Obviously,

$$F(h_{ij}) = \frac{1}{\widetilde{F}(\widetilde{h}^{ij})}.$$

Therefore

$$F^{ij} = \widetilde{F}^{-2}\widetilde{F}^{pq}\widetilde{h}^{pi}\widetilde{h}^{qj}, \tag{3.19}$$

and

$$
\begin{aligned}
(3.20)\qquad F^{ij,rs} &= -\widetilde{F}^{-2}\widetilde{F}^{pq}(\widetilde{h}^{pr}\widetilde{h}^{is}\widetilde{h}^{qj} + \widetilde{h}^{pi}\widetilde{h}^{qr}\widetilde{h}^{js})\\
&\quad -\widetilde{F}^{-2}\widetilde{F}^{pq,lm}\widetilde{h}^{pi}\widetilde{h}^{qj}\widetilde{h}^{lr}\widetilde{h}^{ms} + 2\widetilde{F}^{-3}\widetilde{F}^{lm}\widetilde{h}^{lr}\widetilde{h}^{ms}\widetilde{F}^{pq}\widetilde{h}^{pi}\widetilde{h}^{qj}.
\end{aligned}
$$

Multiplying $\eta_{ij}\eta_{rs}$ to both sides of (3.20), and using (3.19) and the symmetry of $\widetilde{F}^{pq}$, we obtain

$$F^{ij,rs}\eta_{ij}\eta_{rs} = -2F^{ir}\widetilde{h}^{js}\eta_{ij}\eta_{rs} - \widetilde{F}^{-2}\widetilde{F}^{pq,lm}\widetilde{h}^{pi}\widetilde{h}^{qj}\widetilde{h}^{lr}\widetilde{h}^{ms}\eta_{ij}\eta_{rs} + 2F^{-1}(\widetilde{F}^{ij}\eta_{ij})^2.$$

By the concavity of $\widetilde{F}$, we get (3.18) . $\square$

Lemma 3.7. *Let $X(\cdot,t)$ be a smooth, closed and uniformly convex solution to the normalised flow (1.8) for $t \in [0,T)$, which encloses the origin. Assume $\alpha \geq k+1$. Then there is a positive constant C depending only on $\alpha, \mathcal{M}_0, \min_{\mathbb{S}^n\times[0,T)} r$ and $\max_{\mathbb{S}^n\times[0,T)} r$ such that the principal curvatures of $X(\cdot,t)$ are bounded from below*

$$\kappa_i(\cdot,t) \geq 1/C, \quad \forall\ t \in [0,T) \text{ and } i = 1,\cdots,n. \tag{3.21}$$

Proof. Let $\{\widetilde{h}^{ij}\}$ be the inverse matrix of $\{h_{ij}\}$. The principal radii of curvatures of $\mathcal{M}_t$ are the eigenvalues of $\{\widetilde{h}^{il}g_{lj}\}$. To derive a positive lower bound of principal curvatures, it suffices to prove that the eigenvalues of $\{\widetilde{h}^{il}g_{lj}\}$ are bounded from above. For this, we consider the following quantity

$$W(x,t) = \log\Lambda(x,t) - \log u(x,t),$$

where

$$\Lambda(x,t) = \max\{\widetilde{h}^{ij}(x,t)\zeta_i\zeta_j : g^{ij}(x,t)\zeta_i\zeta_j = 1\},$$

and $u = \langle X,\nu\rangle$ is the support function of $\mathcal{M}_t$.

Fix an arbitrary $T' \in (0,T)$ and assume that W attains its maximum on $\mathbb{S}^n\times[0,T']$ at $P_0 = (x_0,t_0)$ with $t_0 > 0$ (otherwise W is bounded by its initial value and we are done). We choose a local orthonormal frame $e_1,\cdots,e_n$ on $\mathcal{M}_t$ such that $\nabla_{e_i}e_j = 0$ at $X(x_0,t_0)$ for all $i,j = 1,\cdots,n$, and $\{h_{ij}\}$ is diagonal at this point. By a rotation, we may also suppose that $\Lambda(x_0,t_0) = \widetilde{h}^{ij}(x_0,t_0)\zeta_i\zeta_j$ with $\zeta = (1,0,\cdots,0)$.

Let

$$w(x,t) = \log\lambda(x,t) - \log u(x,t),$$

where $\lambda(x,t) = \widetilde{h}^{11}/g^{11}$. Then $\max_{\mathbb{S}^n\times[0,T']} W = \max_{\mathbb{S}^n\times[0,T']} w$ and so w achieves its maximum at P_0. In the following we prove an upper bound for w (independent of T'). This is sufficient for Lemma 3.7, as T' is arbitrary.

By virtue of Lemma 2.1, we have, at P_0,

$$\begin{aligned}
\partial_t\lambda &= -(\widetilde{h}^{11})^2\partial_t h_{11} + \widetilde{h}^{11}\partial_t g_{11} \\
&= -(\widetilde{h}^{11})^2\cdot k\Phi F^{-1}F^{ij}\nabla^2_{ij}h_{11} + (k-1)\Phi + \beta\widetilde{h}^{11} - k\Phi\widetilde{h}^{11}F^{-1}F^{ii}h^2_{ii} \\
&\quad -(\widetilde{h}^{11})^2\cdot k\Phi F^{-1}F^{ij,rs}h_{ij,1}h_{rs,1} - (\widetilde{h}^{11})^2\Phi\Big[k(k-1)(\nabla_1\log F)^2 \\
&\quad +2\alpha k\nabla_1\log r\nabla_1\log F + \alpha(\alpha-1)(\nabla_1\log r)^2\Big] - \alpha\Phi r^{-1}\nabla^2_{11}r\cdot(\widetilde{h}^{11})^2.
\end{aligned} \tag{3.22}$$

Note that

$$
\begin{aligned}
(3.23) \qquad \nabla_i\lambda &= -(\widetilde{h}^{11})^2 h_{11,i}, \\
\nabla^2_{ij}\lambda &= -\nabla_j(\widetilde{h}^{1p}h_{pl,i}\widetilde{h}^{1l}) = -\widetilde{h}^{1p}h_{pl,ij}\widetilde{h}^{1l} - 2\nabla_j\widetilde{h}^{1p}h_{pl,i}\widetilde{h}^{1l} \\
&= -(\widetilde{h}^{11})^2 h_{11,ij} + 2\widetilde{h}^{1m}\widetilde{h}^{pq}h_{mq,j}h_{pl,i}\widetilde{h}^{1l} \\
(3.24) \qquad &= -(\widetilde{h}^{11})^2\nabla^2_{ij}h_{11} + 2(\widetilde{h}^{11})^2\widetilde{h}^{pq}h_{ip,1}h_{qj,1}.
\end{aligned}
$$

Plugging (3.24) into (3.22), and then employing (3.18), we obtain

$$
\begin{aligned}
\partial_t\lambda &= k\Phi F^{-1}F^{ij}\nabla^2_{ij}\lambda - (\widetilde{h}^{11})^2\cdot k\Phi F^{-1}\Big(F^{ij,rs}h_{ij,1}h_{rs,1} + 2F^{ij}\widetilde{h}^{pq}h_{ip,1}h_{qj,1}\Big) \\
&\quad +(k-1)\Phi + \beta\widetilde{h}^{11} - k\Phi\widetilde{h}^{11}F^{-1}F^{ii}h^2_{ii} - \Phi(\widetilde{h}^{11})^2\Big[k(k-1)(\nabla_1\log F)^2 \\
&\quad +2\alpha k\nabla_1\log r\nabla_1\log F + \alpha(\alpha-1)(\nabla_1\log r)^2\Big] - \alpha\Phi r^{-1}\nabla^2_{11}r\cdot(\widetilde{h}^{11})^2 \\
(3.25) \le\ & k\Phi F^{-1}F^{ij}\nabla^2_{ij}\lambda + (k-1)\Phi + \beta\widetilde{h}^{11} - \alpha\Phi r^{-1}\nabla^2_{11}r\cdot(\widetilde{h}^{11})^2 \\
&\quad -\Phi(\widetilde{h}^{11})^2\Big[k(k+1)(\nabla_1\log F)^2 + 2\alpha k\nabla_1\log r\nabla_1\log F + \alpha(\alpha-1)(\nabla_1\log r)^2\Big].
\end{aligned}
$$

Clearly

$$
(3.26) \qquad -2\alpha k\nabla_1\log r\nabla_1\log F \le k(k+1)(\nabla_1\log F)^2 + \frac{\alpha^2 k}{k+1}(\nabla_1\log r)^2.
$$

Direct computation gives

$$
\nabla_1 r = r^{-1}\langle X_1, X\rangle,
$$

and

$$
\begin{aligned}
\nabla^2_{11}r &= r^{-1}(\langle X_{11}, X\rangle + g_{11} - (\nabla_1 r)^2) \\
&= r^{-1}(-uh_{11} + g_{11} - (\nabla_1 r)^2).
\end{aligned}
$$

At the point P_0, $r^{-1}X$ and X_1 are two unit vectors, and X_1 is tangential, so we infer by Lemma 3.4 that

$$
g_{11} - (\nabla_1 r)^2 = 1 - \big(\langle X_1, \frac{X}{r}\rangle\big)^2 \ge \delta_0
$$

for some $\delta_0 > 0$. Hence

$$
(3.27) \qquad \nabla^2_{11}r \ge -\frac{u}{r}h_{11} + \delta_0 r^{-1}.
$$

By (3.26) and (3.27), we can estimate (3.25) as

$$
\begin{aligned}
\partial_t\lambda &\le k\Phi F^{-1}F^{ij}\nabla^2_{ij}\lambda + (k-1)\Phi + \Big(\beta + \alpha\frac{u}{r^2}\Phi\Big)\widetilde{h}^{11} - \alpha\delta_0\frac{\Phi}{r^2}(\widetilde{h}^{11})^2 \\
&\quad -\frac{\alpha(\alpha-k-1)}{k+1}\Phi(\widetilde{h}^{11})^2(\nabla_1\log r)^2.
\end{aligned}
$$

Since $\alpha \geq k+1$, we conclude that

$$\frac{\partial_t \log\lambda}{\Phi} \leq kF^{-1}F^{ij}\nabla^2_{ij}\log\lambda + kF^{-1}F^{ii}(\nabla_i\log\lambda)^2 - \frac{\alpha\delta_0}{r^2}\widetilde{h}^{11} + \frac{\beta}{\Phi} + C. \tag{3.28}$$

By using (2.17), we find that

$$\begin{aligned} \nabla^2_{ij}u &= \nabla_j\langle X, h_i^l X_l\rangle \\ &= h_{ij,m}g^{ml}\langle X, X_l\rangle - h_i^l h_{lj}u + h_{ij}. \end{aligned} \tag{3.29}$$

This together with (2.20) implies

$$\begin{aligned} \partial_t u &= \langle X, \nabla\Phi\rangle + \langle -\Phi\nu + \beta X, \nu\rangle \\ &= k\Phi F^{-1}F^{ij}h_{ij,l}\langle X_l, X\rangle + \alpha\Phi g^{ij}\langle X_i, X/|X|\rangle\langle X_j, X/|X|\rangle - \Phi + \beta u \\ &\geq k\Phi F^{-1}F^{ij}\nabla^2_{ij}u - (k+1)\Phi + \beta u. \end{aligned}$$

Consequently

$$\frac{\partial_t \log u}{\Phi} \geq kF^{-1}F^{ij}\nabla^2_{ij}\log u + kF^{-1}F^{ii}(\nabla_i\log u)^2 - \frac{k+1}{u} + \frac{\beta}{\Phi}. \tag{3.30}$$

Note that, at P_0,

$$\nabla_i\log\lambda = \nabla_i\log u,\ \forall\ i = 1,\cdots,n,$$

and

$$0 \geq \nabla^2_{ij}(\log\lambda - \log u).$$

Combining (3.28) and (3.30), we conclude that, at P_0,

$$\begin{aligned} 0 &\leq \frac{\partial_t w}{\Phi} = \frac{\partial_t\log\lambda - \partial_t\log u}{\Phi} \\ &\leq -\frac{\alpha\delta_0}{r^2}\widetilde{h}^{11} + C. \end{aligned}$$

Hence $\widetilde{h}^{11}$ is bounded. This completes the proof. □

As a consequence of Lemma 3.5 and Lemma 3.7, we obtain the corollary below.

Corollary 3.1. *Let $X(\cdot,t)$ be a smooth, closed, uniformly convex solution to the normalised flow (1.8) for $t \in [0,T)$, which encloses the origin. Assume $\alpha \geq k+1$. Then there is a constant C depending only on $\alpha, \mathcal{M}_0, \min_{\mathbb{S}^n\times[0,T)} r$ and $\max_{\mathbb{S}^n\times[0,T)} r$ such that the principal curvatures of $X(\cdot,t)$ are bounded from above and below*

$$C^{-1} \leq \kappa_i(\cdot,t) \leq C,\ \forall\ t\in[0,T) \text{ and } i = 1,\cdots,n. \tag{3.31}$$

The estimates obtained in Lemma 3.1, Lemma 3.2 and Corollary 3.1 depend only on n, α, and the geometry of the initial data $\mathcal{M}_0$. They are independent of T. By (3.1), (3.3) and (3.31), we conclude that the equation (2.6) is uniformly parabolic. By applying the Krylov-Safonov's Harnack inequality [27] to the linearised equation satisfied by r_t, we obtain the space-time Hölder estimates for $\partial_t r$. We then apply the Evans-Krylov theorem (e.g., see [10, 19]) to the uniformly elliptic equation (as a PDE of $-r$)

$$\sigma_k^{\frac{1}{k}} = \Big(\frac{\beta r - r_t}{r^\alpha\sqrt{1+|\overline{\nabla}\log r|^2}}\Big)^{\frac{1}{k}}$$

by taking exponent $\frac{1}{k}$ to the equation (2.6), which implies a space Hölder estimate for $\overline{\nabla}^2 r(\cdot,t)$ for each t. The Hölder estimate for $\overline{\nabla}^2 r$ in t can be obtained as in [34]. Estimates for higher order derivatives then follow from the bootstrap argument using the Schauder estimates. See also [12] for the regularity theory. Hence we obtain the long time existence and C^∞-smoothness of solutions for the normalised flow (1.8). The uniqueness of smooth solutions also follows from the parabolic theory. In summary, we have proved the following theorem.

Theorem 3.1. *Let $\mathcal{M}_0$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$, $n \geq 1$, which encloses the origin. If $\alpha \geq k+1$, then the normalised flow* (1.8) *has a unique smooth, closed and uniformly convex solution $\mathcal{M}_t$ for all time $t \geq 0$. Moreover, the radial function of $\mathcal{M}_t$ satisfies the a priori estimates*

$$\|r\|_{C^{k,\beta}\left(\mathbb{S}^n\times[0,\infty)\right)} \leq C,$$

where the constant $C > 0$ depends only on n, k, β, α and the geometry of $\mathcal{M}_0$.

4. Proofs of Theorems 1.1 - 1.2

In this section we prove the asymptotical convergence of solutions to the normalised flow (1.8). By Theorem 3.1, it is known that the flow (1.8) exists for all time $t > 0$ and remains smooth and uniformly convex, provided $\mathcal{M}_0$ is smooth, uniformly convex and encloses the origin. We need the following lemma.

Lemma 4.1. *Let $r(\cdot,t)$ be a smooth and uniformly convex solution to* (2.6). *If $\alpha \geq k+1$, then there exist positive constants C and γ, depending only on n, α and the geometry of $\mathcal{M}_0$, such that*

$$\max_{\mathbb{S}^n} \frac{|\overline{\nabla} r(\cdot,t)|}{r(\cdot,t)} \leq Ce^{-\gamma t}, \quad \forall\ t > 0. \tag{4.1}$$

Proof. Consider the auxiliary function

$$G = \frac{1}{2}|\overline{\nabla}\varrho|^2,$$

where $\varrho = \log r$ as in (2.8). At the point where G attains its spatial maximum, we have

$$0 = \overline{\nabla}_i G = \sum \varrho_l \varrho_{li}, \tag{4.2}$$

and

$$0 \geq \nabla^2_{ij} G = \sum \varrho_l \varrho_{lij} + \sum \varrho_{il} \varrho_{lj}. \tag{4.3}$$

By differentiating (2.11) and using (4.2), we obtain at this point

$$\begin{aligned} \partial_t G &= \sum \varrho_l \varrho_{lt} \\ &= -(1+|\overline{\nabla}\varrho|^2)^{\frac{1-k}{2}} e^{(\alpha-k-1)\varrho} \Big[(\alpha-k-1)|\overline{\nabla}\varrho|^2 \sigma_k + \sum \sigma_k^{ij} \overline{\nabla}_l \widetilde{a}_{ij} \varrho_l \Big]. \end{aligned} \tag{4.4}$$

Note that by (4.2),

$$\sum \varrho_r \overline{\nabla}_r \widetilde{a}_{ij} = -\sum \gamma_{il} \varrho_r \overline{\nabla}_r \varrho_{lm} \gamma_{mj}.$$

By the Ricci identity, we have

$$\overline{\nabla}_r \varrho_{lm} = \overline{\nabla}_m \varrho_{lr} + \delta_{lr} \varrho_m - \delta_{lm} \varrho_r.$$

Hence

$$\begin{aligned} \sum \varrho_r \overline{\nabla}_r \widetilde{a}_{ij} &= -\sum \gamma_{il} (\varrho_r \varrho_{rlm} + \varrho_l \varrho_m - \delta_{lm} |\overline{\nabla}\varrho|^2) \gamma_{mj} \\ &\geq -\sum \gamma_{il} (-\varrho_{lr} \varrho_{rm} + \varrho_l \varrho_m - \delta_{lm} |\overline{\nabla}\varrho|^2) \gamma_{mj}, \end{aligned} \tag{4.5}$$

where we have used (4.3) in (4.5). Plugging (4.5) into (4.4), we deduce that

$$\partial_t G \leq (1+|\overline{\nabla}\varrho|^2)^{\frac{1-k}{2}} e^{(\alpha-k-1)\varrho} \Big((k+1-\alpha)|\overline{\nabla}\varrho|^2 \sigma_k + \sum A^{lm} \varrho_l \varrho_m - \mathcal{A} |\overline{\nabla}\varrho|^2 \Big), \tag{4.6}$$

where $\mathcal{A} = \sum A^{ii}$ and $\{A^{lm}\}$ is the positive definition symmetric matrix given by

$$A^{lm} = \sigma_k^{ij} \gamma_{il} \gamma_{jm}.$$

If $n \geq 2$, by Corollary 3.1, we infer that

$$\max_i A^{ii} - \mathcal{A} \leq -C.$$

It then follows by (4.6) and using the assumption $\alpha \geq k+1$ that

$$\partial_t G \leq -\gamma G, \tag{4.7}$$

for some positive constant γ. This proves (4.1).

For $n = 1$ (hence $k = 1$), when $\alpha > k+1 = 2$, we still have (4.7) by using (4.6) and Corollary 3.1 (which gives a positive lower bound for σ_k). Hence it suffices to consider the case $\alpha = k+1 = 2$. Then the equation (2.11) becomes quasi-linear

$$\varrho_t = \frac{\varrho_{xx}}{1+\varrho_x^2} \quad \text{on } \mathbb{S}^1 \times [0,\infty). \tag{4.8}$$

Let

$$\bar{\varrho} := \frac{1}{2\pi} \int_{\mathbb{S}^1} \varrho(x,t) dx$$

be the average of ϱ. By the divergence theorem,

$$\frac{d}{dt}\bar{\varrho} = \frac{1}{2\pi}\int_{\mathbb{S}^1}(\arctan(\varrho_x))_x dx = 0.$$

Hence $\bar{\varrho}$ is a constant. Then it is simple to calculate

$$\begin{aligned}\frac{d}{dt}\Big(\frac{1}{2}\int_{\mathbb{S}^1}(\varrho-\bar{\varrho})^2\Big) &= \int_{\mathbb{S}^1}(\varrho-\bar{\varrho})(\arctan\varrho_x)_x dx \\ &= -\int_{\mathbb{S}^1}\varrho_x\arctan\varrho_x dx.\end{aligned}$$

Note that, $\varrho_x \arctan\varrho_x \geq \delta_0\varrho_x^2$ for some $\delta_0 > 0$ depending only on the upper bound of $|\varrho_x|$. We deduce that, by the Poincaré inequality,

$$\frac{d}{dt}\Big(\frac{1}{2}\int_{\mathbb{S}^1}(\varrho-\bar{\varrho})^2\Big) \leq -\delta_0\int_{\mathbb{S}^1}\varrho_x^2 dx \leq -C\int_{\mathbb{S}^1}(\varrho-\bar{\varrho})^2.$$

This implies ϱ exponentially converges to a constant in L^2-norm as $t\to\infty$. The exponential decay of $|\overline{\nabla}\varrho|$ now follows from the interpolation. $\square$

Remark 4.1. *In the above argument, we have actually proved the following gradient estimate. Let $r(\cdot,t)$ be a positive, k-convex solution of* (2.6) *on $\mathbb{S}^n\times[0,T)$. If $\alpha\geq k+1$, then*

$$\max_{\mathbb{S}^n}\frac{|\overline{\nabla}r(\cdot,t)|}{r(\cdot,t)} \leq \max_{\mathbb{S}^n}\frac{|\overline{\nabla}r(\cdot,0)|}{r(\cdot,0)},\ \forall\ t<T. \tag{4.9}$$

In fact, $\max_{\mathbb{S}^n}G(\cdot,t)$ *is non-increasing in t. This can be seen from* (4.6) *and by noticing*

$$\sum A^{lm}\varrho_l\varrho_m \leq \mathcal{A}|\overline{\nabla}\varrho|^2.$$

From estimate (4.9) *and Lemma 3.1, we infer that* $\max_{\mathbb{S}^n}|\overline{\nabla}r(\cdot,t)| \leq C$ *for all $t < T$. When $r(\cdot,t)$ is a convex solution, this gradient bound follows immediately from the convexity as shown in Lemma 3.2.*

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Case (i): $\alpha > k+1$.

Let $r(\cdot,t)$ be the solution to (2.6). By making a rescaling of $\mathcal{M}_0$ if necessary, we may assume

$$a := \min_{\mathbb{S}^n} r(\cdot,0) \leq 1 \leq \max_{\mathbb{S}^n} r(\cdot,0) =: b.$$

Let us introduce two time-dependent functions

$$\begin{aligned} r_1 &= [1-(1-a^q)e^{q\beta t}]^{1/q},\\ r_2 &= [1-(1-b^q)e^{q\beta t}]^{1/q},\end{aligned}$$

where $q = k + 1 - \alpha < 0$. It is easy to check that both r_1 and r_2 satisfy equation (2.6), and the spheres of radii r_1 and r_2 are solutions of (1.8). By the comparison principle, $r_1(t) \le r(\cdot, t) \le r_2(t)$. Hence

$$(b^q - 1)e^{q\beta t} \le r^q - 1 \le (a^q - 1)e^{q\beta t}.$$

Thus r converges to 1 exponentially.

By the interpolation and the a priori estimates established in Section 3, we see that $\|r(\cdot,t) - 1\|_{C^k(\mathbb{S}^n)} \to 0$ exponentially for all integers $k \ge 1$. This shows that $\mathcal{M}_t$ converges to the unit sphere centred at the origin.

Case (ii): $\alpha = k+1$. We see from (4.1) that $|\overline{\nabla} r| \to 0$ exponentially as $t \to \infty$. Hence by the interpolation and the a priori estimates in Section 3 we conclude that r converges exponentially to a constant in the C^∞ topology as $t \to \infty$. This completes the proof. $\square$

In the rest of this section, we shall prove Theorem 1.2. We first show that the mean-convexity is preserved.

Lemma 4.2. *Let $\mathcal{M}_0$ be a smooth, closed and weakly mean-convex hypersurface in $\mathbb{R}^{n+1}$, $n \ge 1$. Suppose that $\mathcal{M}_0$ is star-shaped with respect to the origin. Let $\mathcal{M}_t$ be a solution of (1.8) on $\mathbb{S}^n \times [0, T)$. Then for all $t \in [0,T)$, the hypersurface $\mathcal{M}_t$ is weakly mean-convex.*

Proof. By Lemma 2.1, we obtain

$$\begin{aligned} \partial_t H &= g^{ij}\partial_t h_{ij} - g^{ik}g^{jl}\partial_t g_{kl} h_{ij} \\ &= \Delta\Phi + \Phi|A|^2 - nH \\ &= r^\alpha \Delta H + 2\langle \nabla r^\alpha, \nabla H\rangle + (\Delta r^\alpha + r^\alpha |A|^2 - n)H. \end{aligned}$$

This implies that $\min_{\mathbb{S}^n} H(\cdot,t) \ge 0$ for all $t < T$. $\square$

Proof of Theorem 1.2. When $k = 1$, by (2.1), equation (2.6) becomes

(4.10)
$$\begin{cases} \dfrac{\partial r}{\partial t} = r^{\alpha-2}\Big(\delta_{ij} - \dfrac{r_i r_j}{r^2 + |\overline{\nabla} r|^2}\Big)\overline{\nabla}^2_{ij} r - r^{\alpha-1}\Big[n + \dfrac{|\overline{\nabla} r|^2}{r^2 + |\overline{\nabla} r|^2}\Big] + nr \ \text{ on } \mathbb{S}^n \times [0,\infty), \\ r(\cdot, 0) = r_0, \end{cases}$$

This is a quasi-linear parabolic equation. From the proof of Lemma 3.1, we infer that

$$1/C < r(\cdot,t) < C,$$

as long as the flow exists, where C is a constant depending only on the geometry of $\mathcal{M}_0$. By Lemma 4.2, $r(\cdot,t)$ remains weakly mean-convex. We have the following estimate, if either (i) $n > 1$ & $\alpha \ge k+1 = 2$ or (ii) $n = 1$ & $\alpha = k + 1 = 2$,

$$|\overline{\nabla} r| \le C e^{-\gamma t}. \tag{4.11}$$

This can be seen from (4.6) in the proof of Lemma 4.1. Indeed, when $k = 1$ and $\alpha \geq k+1 = 2$, by the weak mean-convexity, the differential inequality (4.6) gives

$$\partial_t\Big(\frac{1}{2}\frac{|\overline{\nabla} r|^2}{r^2}\Big) \leq -(n-1)r^{\alpha-2}\frac{|\overline{\nabla} r|^2}{r^2}. \tag{4.12}$$

This gives (4.11) for $n > 1$. When $n = 1$ and $\alpha = 2$, (4.11) follows from the 1D parabolic equation (4.8), as in the proof of Lemma 4.1. Inequality (4.12) also implies, for all $n \geq 1$, $\max_{\mathbb{S}^n} |\overline{\nabla} r(\cdot,t)|/r(\cdot,t)$ is non-increasing in t, and therefore

$$|\overline{\nabla} r| \leq C_0 \tag{4.13}$$

for some $C_0 > 0$ depending on the initial data $\mathcal{M}_0$. Hence $\mathcal{M}_t$ are uniformly star-shaped as long as the flow exists. Indeed, by (3.4),

$$\frac{X}{r}\cdot x = \frac{r}{\sqrt{r^2 + |\overline{\nabla} r|^2}} \geq \delta_0, \ \forall\, t > 0,$$

where $\delta_0 > 0$ is a small constant only depending on α and the initial data $\mathcal{M}_0$.

By (4.13), we infer that (4.10) is uniformly parabolic. It follows from (3.1), (4.13), and the regularity theory of quasi-linear uniformly parabolic equations (see e.g. Theorem 12.3 in [29]) that $\overline{\nabla} r$ is uniformly Hölder continuous in space-time. Therefore the coefficients of (4.10) are uniformly Hölder continuous, and we can apply the standard Schauder estimates (see e.g. Theorem 4.9 in [29]) to conclude the $C^{2,\alpha}$-estimates of r. Higher order a priori estimates follow from the standard bootstrap argument. Hence we obtain the long time existence and C^∞ regularity for the normalised flow (1.8).

Arguing as in the proof of Theorem 1.1, we deduce that r converges exponentially to a constant in the C^∞ topology as $t \to \infty$. Note that estimate (4.11) is used for the case $\alpha = k+1 = 2$. $\square$

5. Proof of Theorem 1.3

In this section we show that if $\alpha < k+1$ then the flow (1.1) may have unbounded ratio of radii, namely

$$\mathcal{R}(X(\cdot,t)) = \frac{\max_{\mathbb{S}^n} r(\cdot,t)}{\min_{\mathbb{S}^n} r(\cdot,t)} \to \infty \ \text{ as } \ t \to T \tag{5.1}$$

for some $T > 0$. Our idea is to show that $\min_{\mathbb{S}^n} r(\cdot,t) \to 0$ in finite time while $\max_{\mathbb{S}^n} r(\cdot,t)$ remains positive. The argument below is similar to that in our previous paper [28], with some necessary modifications. In [28], the result was proved for the case $k = n$.

Let $X(\cdot,t)$ be a convex solution to (1.1). Then its radial function r satisfies the equation

$$\begin{cases} \dfrac{\partial r}{\partial t}(x,t) = -r^{\alpha}\sigma_k(\kappa[r]), \\ \quad r(\cdot,0) = r_0, \end{cases} \tag{5.2}$$

where $\kappa[r] = (\kappa_1, \cdots, \kappa_n)$, and κ_i are the principal curvatures of the hypersurface whose radial function is $r(\cdot,t)$, namely the eigenvalues to the matrix a_{ij} defined in (2.2). Given a smooth, closed, uniformly convex hypersurface $\mathcal{M}_0$, our a priori estimates in Section 3 imply the existence of a smooth, closed, uniformly convex solution to the flow (1.1) for small $t > 0$.

Definition 5.1. *A time dependent family of convex hypersurfaces $Y(\cdot,t)$ is a sub-solution to* (1.1) *if its radial function w satisfies*

$$\begin{cases} \quad \dfrac{\partial w}{\partial t} \geq -r^{\alpha}\sigma_k(\kappa[w]), \\ w(\cdot,0) = w_0. \end{cases} \tag{5.3}$$

By definition, the hypersurface $\mathcal{M}_0$ (independent of t), whose radial function is r_0, is a sub-solution to (5.2). We will use the following comparison principle.

Lemma 5.1. *Let $X(\cdot,t)$ be a solution to* (1.1) *and $Y(\cdot,t)$ a sub-solution. Suppose $X(\cdot,0)$ is contained in the interior $Y(\cdot,0)$. Then $X(\cdot,t)$ is contained in the interior $Y(\cdot,t)$ for all $t > 0$, as long as the solutions exist.*

We omit the proof of Lemma 5.1 here, as a comparison principle like this is well known in geometric analysis. Note that in Lemma 5.1, we do not require that $Y(\cdot,t)$ is shrinking. Moreover, it suffices to assume that $Y(\cdot,t)$ is a sub-solution in the viscosity sense. In particular Lemma 5.1 applies if $Y(\cdot,t)$ is $C^{1,1}$ smooth.

To prove Theorem 1.3, by the comparison principle (Lemma 5.1), it suffices to construct a sub-solution $Y(\cdot,t)$ such that $\min_{\mathbb{S}^n} w(\cdot,t) \to 0$ but $\max_{\mathbb{S}^n} r(\cdot,t)$ remains positive, as $t \to T$ for some finite time $T > 0$. By a translation of time, we show below that there is a sub-solution $Y(\cdot,t)$ for $t \in (-1,0)$ such that (5.1) holds as $t \nearrow 0$.

Lemma 5.2. *There is a sub-solution $Y(\cdot,t)$, where $t \in (-1,0)$, to*

$$\begin{cases} \quad \dfrac{\partial r}{\partial t} = -ar^{\alpha}\sigma_k(\kappa[r]), \\ r(\cdot,0) = r_0. \end{cases} \tag{5.4}$$

for a sufficiently large constant $a > 0$, such that $\min_{\mathbb{S}^n} w(\cdot,t) \to 0$ but $\max_{\mathbb{S}^n} w(\cdot,t)$ remains positive, as $t \nearrow 0$.

Proof. The sub-solution we constructed is a family of closed convex hypersurfaces $\widehat{\mathcal{M}}_t := Y(\mathbb{S}^n, t)$. First note that it suffices to prove Lemma 5.2 when

$$q = k + 1 - \alpha > 0$$

is very small. Indeed, if $Y(\mathbb{S}^n, t)$ is a sub-solution to (5.4) for some α, it is also a sub-solution to (5.4) for $\alpha' < \alpha$, provided we replace a by $a \sup\{|p|^{\alpha-\alpha'};\ p \in \widehat{\mathcal{M}}_t, t \in (-1, 0)\}$.

Let $\widehat{\mathcal{M}}_t$ be the graph of the function

$$\phi(\rho, t) = \begin{cases} -|t|^\theta + |t|^{-\theta+\sigma\theta}\rho^2, & \text{if } \rho < |t|^\theta, \\ -|t|^\theta - \dfrac{1-\sigma}{1+\sigma}|t|^{\theta(1+\sigma)} + \dfrac{2}{1+\sigma}\rho^{1+\sigma}, & \text{if } |t|^\theta \le \rho \le 1, \end{cases} \tag{5.5}$$

where $x \in \mathbb{R}^n$, $\rho = |x|$, and $\sigma = \frac{q\theta - 1}{k\theta}$ and $\theta > \frac{1}{q}$ is a constant. It is easy to verify that ϕ is strictly convex, and $\phi \in C^{1,1}(B_1(0))$.

By direct computation, we have,

(i) if $0 \le \rho \le |t|^\theta$, then

$$\begin{aligned} & r^\alpha \sigma_k \ge C|t|^{\alpha\theta}|t|^{k\theta(\sigma-1)} = C|t|^{\theta-1}, \\ & |\frac{\partial}{\partial t} Y(p, t)| \le \theta|t|^{\theta-1}. \end{aligned} \tag{5.6}$$

where $p = (x, \phi(|x|, t))$ is a point on the graph of ϕ and σ_k is the k-curvature of the graph of ϕ at p.

(ii) if $|t|^\theta \le \rho \le 1$, then

$$\begin{aligned} & r^\alpha \sigma_k \ge \rho^\alpha \sigma_k \ge C\rho^\alpha \rho^{(\sigma-1)k} = C\rho^{1-\frac{1}{\theta}} \ge C|t|^{\theta-1}, \\ & |\frac{\partial}{\partial t} Y(p, t)| \le \theta|t|^{\theta-1}. \end{aligned} \tag{5.7}$$

Hence the graph of $\phi(\cdot, t)$ is a sub-solution to (5.4), provided a is sufficiently large.

Next we extend the graph of ϕ to a closed convex hypersurface $\widehat{\mathcal{M}}_t$, such that it is $C^{1,1}$ smooth, uniformly convex, rotationally symmetric, and depends smoothly on t. Moreover we may assume that the ball $B_1(z)$ is contained in the interior of $\widehat{\mathcal{M}}_t$, for all $t \in (-1, 0)$, where $z = (0, \cdots, 0, 10)$ is a point on the x_{n+1}-axis. Then $\widehat{\mathcal{M}}_t$ is a sub-solution to (5.4), for sufficiently large a. □

We are in position to prove Theorem 1.3. For a given $\tau \in (-1, 0)$, let $\mathcal{M}_0$ be a smooth, closed, uniformly convex hypersurface inside $\widehat{\mathcal{M}}_\tau$ and enclosing the ball $B_1(z)$. Let $\mathcal{M}_t$ be the solution to the flow (5.4) with initial data $\mathcal{M}_0$. By Lemma 5.1, $\mathcal{M}_t$ touches the origin at $t = t_0$, for some $t_0 \in (\tau, 0)$. We choose τ very close to 0, so that t_0 is sufficiently small.

On the other hand, let $\tilde{X}(\cdot,t)$ be the solution to

$$\frac{\partial X}{\partial t} = -ba\tilde{r}^{\alpha}\sigma_k\nu, \tag{5.8}$$

with initial condition $\tilde{X}(\cdot,\tau) = \partial B_1(z)$, where $b = 2^{\alpha}\sup\{|p|^{\alpha} :\ p \in \mathcal{M}_t, \tau < t < t_0\}$, and $\tilde{r} = |X - z|$ is the distance from z to X. We can choose τ so small that the ball $B_{1/2}(z)$ is contained in the interior of $\tilde{X}(\cdot,t)$ for all $t \in (\tau, t_0)$. By the comparison principle (Lemma 5.1), we see that the ball $B_{1/2}(z)$ is contained in the interior of $\mathcal{M}_t$ for all $t \in (\tau, t_0)$. Hence as $t \nearrow t_0$, we have $\min r(\cdot,t) \to 0$ and $\max r(\cdot,t) > |z| = 10$. Hence (5.1) is proved for $\mathcal{M}_t$.

We have proved Theorem 1.3 when $r^{\alpha}\sigma$ is replaced by $ar^{\alpha}\sigma$, for large constant $a > 0$. Making the rescaling $\widetilde{\mathcal{M}}_t = a^{-1/q}\mathcal{M}_t$, one easily verifies that $\widetilde{\mathcal{M}}_t$ solves the flow (1.1) . Theorem 1.3 is proved.

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