NONCOMMUTATIVE LOCALIZATION IN ALGEBRAIC L-THEORY

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ABSTRACT. Given a noncommutative (Cohn) localization $A \to \sigma^{-1}A$ which is injective and stably flat we obtain a lifting theorem for induced f.g. projective $\sigma^{-1}A$-module chain complexes and localization exact sequences in algebraic L-theory, matching the algebraic K-theory localization exact sequence of Neeman and Ranicki [2], [3].

CONTENTS

INTRODUCTION

In the series of papers [2], [3], we studied the algebraic K-theory of the noncommutative (Cohn) localization $\sigma^{-1}A$ of a ring A inverting a collection σ of morphisms of f.g. projective left A-modules. By definition, $\sigma^{-1}A$ is stably flat if

$$\mathrm{Tor}_i^A(\sigma^{-1}A, \sigma^{-1}A) \;=\; 0 \ (i \geq 1) \ .$$

An (A, σ)-module is an A-module T which admits a f.g. projective A-module resolution

$$0 \longrightarrow P \xrightarrow{\ s\ } Q \longrightarrow T \longrightarrow 0$$

with $s : \sigma^{-1}P \to \sigma^{-1}Q$ an isomorphism of the induced $\sigma^{-1}A$-modules. For $A \longrightarrow \sigma^{-1}A$ which is injective and stably flat we obtained an algebraic K-theory localization exact sequence

$$\cdots \to K_n(A) \to K_n(\sigma^{-1}A) \to K_{n-1}(H(A,\sigma)) \to K_{n-1}(A) \to \ldots$$

with $H(A,\sigma) = \mathcal{E}(A,\sigma)$ the exact category of (A,σ)-modules.

Let C be a bounded $\sigma^{-1}A$-module chain complex such that each $C_i = \sigma^{-1}P_i$ is induced from a f.g. projective A-module P_i. The *chain complex lifting problem* is to decide if C is chain equivalent to $\sigma^{-1}D$ for a bounded chain complex D of f.g. projective A-modules. The problem has a trivial affirmative solution for a commutative or Ore localization, by

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the clearing of denominators, when C is actually isomorphic to $\sigma^{-1}D$. In general, it is not possible to lift chain complexes: the injective noncommutative localizations $A \to \sigma^{-1}A$ which are not stably flat constructed in Neeman, Ranicki and Schofield [4, Remark 2.13] provide examples of induced f.g. projective $\sigma^{-1}A$-module chain complexes of dimensions $\geqslant 3$ which cannot be lifted.

In §1 we solve the chain complex lifting problem in the injective stably flat case, obtaining the following results (Theorems 1.4,1.5) :

Theorem 0.1. *For a stably flat injective noncommutative localization $A \to \sigma^{-1}A$ every bounded chain complex C of induced f.g. projective $\sigma^{-1}A$-modules is chain equivalent to $\sigma^{-1}D$ for a bounded chain complex D of f.g. projective A-modules. Moreover, if C is n-dimensional*

$$C \ : \ \dots \to 0 \to C_n \to C_{n-1} \to \dots \to C_1 \to C_0 \to 0 \to \dots$$

then D can be chosen to be n-dimensional. □

In §2 we consider the algebraic L-theory of a noncommutative localization, obtaining the following results (Theorems 2.4, 2.5, 2.9) :

Theorem 0.2. *Let $A \longrightarrow \sigma^{-1}A$ be a noncommutative localization of a ring with involution A, such that σ is invariant under the involution.*
(i) *There is a localization exact sequence of quadratic L-groups*

$$\dots \longrightarrow L_n(A) \longrightarrow L^I_n(\sigma^{-1}A) \xrightarrow{\partial} L_n(A,\sigma) \longrightarrow L_{n-1}(A) \longrightarrow \dots$$

with $I = \mathrm{im}(K_0(A) \longrightarrow K_0(\sigma^{-1}A))$, and $L_n(A,\sigma)$ the cobordism group of $\sigma^{-1}A$-contractible $(n-1)$-dimensional quadratic Poincaré complexes over A.
(ii) *If $\sigma^{-1}A$ is stably flat over A there is a localization exact sequence of symmetric L-groups*

$$\dots \longrightarrow L^n(A) \longrightarrow L^n_I(\sigma^{-1}A) \xrightarrow{\partial} L^n(A,\sigma) \longrightarrow L^{n-1}(A) \longrightarrow \dots$$

with $L^n(A,\sigma)$ the cobordism group of $\sigma^{-1}A$-contractible $(n-1)$-dimensional symmetric Poincaré complexes over A.
(iii) *If $A \longrightarrow \sigma^{-1}A$ is injective then $L^n(A,\sigma)$ (resp. $L_n(A,\sigma)$) is the cobordism group of n-dimensional symmetric (resp. quadratic) Poincaré complexes of (A,σ)-modules.* □

The L-theory exact sequences of Theorem 0.2 for an injective Ore localization $A \longrightarrow \sigma^{-1}A$ (which is flat and hence stably flat) were obtained in Ranicki [5]. The quadratic L-theory exact sequence of 0.2 (i) for arbitrary injective $A \longrightarrow \sigma^{-1}A$ was obtained by Vogel [8], [9]. The symmetric L-theory exact sequence of 0.2 (ii) is new.

We refer to Ranicki [7] for some of the applications of the algebraic L-theory of noncommutative localizations to topology.

1. Lifting chain complexes

If $A \longrightarrow \sigma^{-1}A$ is a stably flat localization, we know from [2, Theorem 0.4, Proposition 4.5 and Theorem 3.7] that the functor $Ti : \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c} \longrightarrow D^{\mathrm{perf}}(\sigma^{-1}A)$ is just an idempotent completion; it is fully faithful and all objects in $D^{\mathrm{perf}}(\sigma^{-1}A)$ are, up to isomorphisms, direct summands of objects in the image of Ti. A fairly easy consequence of this is the following. Let $C \in D^{\mathrm{perf}}(\sigma^{-1}A)$ be the complex

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \cdots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0,$$

with C^i all finitely generated, projective A–modules. Then there is complex $X \in D^{\mathrm{perf}}(A)$ with $C \simeq \{\sigma^{-1}A\}^L \otimes_A X$. That is, C is homotopy equivalent to the tensor product with $\sigma^{-1}A$ of a perfect complex over the ring A. In Section 1 we prove this (Theorem 1.4), and then refine the result to show that X may be chosen to be a complex of the form

$$0 \longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \cdots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0\ .$$

(Proof in Theorem 1.5).

Remark 1.1. The proof of Theorem 1.4 relies on the following fact about triangulated categories. Suppose $\mathcal{A}$ is a full, triangulated subcategory of a triangulated category $\mathcal{B}$, and suppose all objects in $\mathcal{B}$ are direct summands of objects of $\mathcal{A}$. An object $X \in \mathcal{B}$ belongs to $\mathcal{A} \subset \mathcal{B}$ if and only if $[X] \in K_0(\mathcal{B})$ lies in the image of $K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$. This fact may be found, for example, in [1, Proposition 4.5.11], but for the reader's convenience its proof is included here in Lemma 1.2 and Proposition 1.3.

□

We begin by reminding the reader of some basic facts about Grothendieck groups. For any additive category $\mathcal{A}$ we define $K_0^{\mathrm{add}}(\mathcal{A})$ to be the Grothendieck group of the split exact category $\mathcal{A}$. This means that the short exact sequences in $\mathcal{A}$ are precisely the split sequences. It is well known that every element of $K_0^{\mathrm{add}}(\mathcal{A})$ can be expressed as

$$[X] - [Y]$$

for X and Y objects of $\mathcal{A}$. The expressions $[X]-[Y]$ and $[X']-[Y']$ are equal in $K_0^{\mathrm{add}}(\mathcal{A})$ if and only if there exists an object $P \in \mathcal{A}$ and an isomorphism

$$X \oplus Y' \oplus P \;=\; X' \oplus Y \oplus P.$$

If $\mathcal{A}$ happens to be a triangulated category, then $K_0(\mathcal{A})$ means the quotient of $K_0^{\mathrm{add}}(\mathcal{A})$ by a subgroup we will denote $T(\mathcal{A})$. The subgroup $T(\mathcal{A})$ is defined as the group generated by all

$$[X] - [Y] + [Z],$$

where there exists a distinguished triangle in $\mathcal{A}$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

We prove:

Lemma 1.2. *Suppose $\mathcal{B}$ is a triangulated category. Let $\mathcal{A}$ be a full, triangulated subcategory of $\mathcal{B}$. Assume further that every object of $\mathcal{B}$ is a direct summand of an object in $\mathcal{A} \subset \mathcal{B}$.*

Then the map $f : K_0^{\mathrm{add}}(\mathcal{A}) \longrightarrow K_0^{\mathrm{add}}(\mathcal{B})$ induces a surjection $T(\mathcal{A}) \longrightarrow T(\mathcal{B})$. In symbols: $f\big(T(\mathcal{A})\big) = T(\mathcal{B})$.

Proof. Let $[X]-[Y]+[Z]$ be a generator of $T(\mathcal{B}) \subset K_0^{\mathrm{add}}(\mathcal{B})$. We need to show it lies in the image of $T(\mathcal{A}) \subset K_0^{\mathrm{add}}(\mathcal{A})$. Suppose therefore that

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

is a distinguished triangle in $\mathcal{B}$. Because every object of $\mathcal{B}$ is a direct summand of an object in $\mathcal{A}$, we can choose objects C and D with

$$X \oplus C, \qquad Z \oplus D$$

both lying in $\mathcal{A}$. But then we have a two distinguished triangles in $\mathcal{B}$

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ C & \longrightarrow & C \oplus D & \longrightarrow & D & \xrightarrow{0} & \Sigma C \end{array}$$

and their direct sum is a distinguished triangle

$$X \oplus C \longrightarrow Y \oplus C \oplus D \longrightarrow Z \oplus D \longrightarrow \Sigma(X \oplus C).$$

Two of the objects lie in $\mathcal{A}$. Since the subcategory $\mathcal{A} \subset \mathcal{B}$ is full and triangulated, the entire distinguished triangle lies in $\mathcal{A}$. Thus

$$[X \oplus C] - [Y \oplus C \oplus D] + [Z \oplus D] \quad = \quad [X] - [Y] + [Z]$$

lies in the image of $T(\mathcal{A})$. $\square$

The next proposition is well-known; again, the proof is included for the convenience of the reader.

Proposition 1.3. *Let the hypotheses be as in Lemma 1.2. That is, suppose $\mathcal{B}$ is a triangulated category. Let $\mathcal{A}$ be a full, triangulated subcategory of $\mathcal{B}$. Assume further that every object of $\mathcal{B}$ is a direct summand of an object in $\mathcal{A} \subset \mathcal{B}$.*

If X is an object of $\mathcal{B}$ and $[X]$ lies in the image of the natural map $f : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$, then $X \in \mathcal{A}$.

Proof. If we consider $[X]$ as an element of $K_0^{\mathrm{add}}(\mathcal{B})$, then saying that its image in $K_0(\mathcal{B})$ lies in the image of $K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$ is equivalent to saying that, modulo $T(\mathcal{B})$, $[X]$ lies in the image of $K_0^{\mathrm{add}}(\mathcal{A})$. That is,

$$[X] \in T(\mathcal{B}) + f\big(K_0^{\mathrm{add}}(\mathcal{A})\big) \subset K_0^{\mathrm{add}}(\mathcal{B}).$$

By Lemma 1.2 we have that $f\big(T(\mathcal{A})\big) = T(\mathcal{B})$. Thus

$$\begin{aligned} T(\mathcal{B}) + f\big(K_0^{\mathrm{add}}(\mathcal{A})\big) &= f\big(T(\mathcal{A})\big) + f\big(K_0^{\mathrm{add}}(\mathcal{A})\big) \\ &= f\big(K_0^{\mathrm{add}}(\mathcal{A})\big). \end{aligned}$$

That means there exist objects C and D in $\mathcal{A} \subset \mathcal{B}$ and an identity in $K_0^{\mathrm{add}}(\mathcal{B})$

$$[X] = [C] - [D].$$

There must therefore be an object $P \in \mathcal{B}$ and an isomorphism

$$X \oplus D \oplus P \ \simeq \ C \oplus P.$$

But P is an object of $\mathcal{B}$, hence a direct summand of an object of $\mathcal{A}$. There is an object $P' \in \mathcal{B}$ with $P \oplus P' \in \mathcal{A}$. We have an isomorphism

$$X \oplus D \oplus P \oplus P' \ \simeq \ C \oplus P \oplus P'.$$

Putting $D' = D \oplus P \oplus P$ and $C' = C \oplus P \oplus P'$ we have objects C', D' in $\mathcal{A}$, and a (split) distinguished triangle

$$D' \longrightarrow C' \longrightarrow X \longrightarrow \Sigma D'.$$

Since $\mathcal{A} \subset \mathcal{B}$ is triangulated we conclude that $X \in \mathcal{A}$. □

The relevance of these results to our work here is

Theorem 1.4. *Let $A \longrightarrow \sigma^{-1}A$ be a stably flat localization of rings. Suppose we are given a perfect complex C over $\sigma^{-1}A$. Suppose further that $C \in D^{\mathrm{perf}}(\sigma^{-1}A)$ is of the form*

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \cdots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0$$

where each C^i is a finitely generated, projective A–module. Then C is homotopy equivalent to $\{\sigma^{-1}A\}^L\otimes_A X$, for some $X \in D^{\mathrm{perf}}(A)$.

Proof. The localization is stably flat. By [2, Theorem 0.4] the functor $T : \mathcal{T}^c \longrightarrow D^{\mathrm{perf}}(\sigma^{-1}A)$ is an equivalence of categories. By [2, Proposition 4.5 and Theorem 3.7] we also know that the functor $i : \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c} \longrightarrow \mathcal{T}^c$ is fully faithful, and that every object in $\mathcal{T}^c$ is isomorphic to a direct summand of an object in the image of i. Next we apply Proposition 1.3, with $\mathcal{B} = D^{\mathrm{perf}}(\sigma^{-1}A)$ and $\mathcal{A}$ the full subcategory containing all objects isomorphic to $Ti(x)$, for any $x \in \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c}$.

Now C is an object of $D^{\mathrm{perf}}(\sigma^{-1}A)$, and in $K_0\big(D^{\mathrm{perf}}(\sigma^{-1}A)\big)$ we have an identity

$$[C] = \sum_{\ell=-\infty}^{\infty} (-1)^\ell [\sigma^{-1}C^\ell]$$

with

$$[\sigma^{-1}C^\ell] = [\{\sigma^{-1}A\} \otimes_A C^\ell] = [TiC^\ell]$$

certainly lying in the image of the map

$$K_0(Ti) : K_0\left(\frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c}\right) \longrightarrow K_0\big(D^{\mathrm{perf}}(\sigma^{-1}A)\big).$$

Proposition 1.3 therefore tells us that C is isomorphic to an object in the image of the functor Ti. There exists a perfect complex $X \in D^{\mathrm{perf}}(A)$ and a homotopy equivalence $C \simeq \{\sigma^{-1}A\}^L\otimes_A X$. □

The problem with Theorem 1.4 is that it gives us no bound on the length of the complex X with $\{\sigma^{-1}A\}^L\otimes_A X \simeq C$. We really want to know

Theorem 1.5. *Let $A \longrightarrow \sigma^{-1}A$ be a stably flat localization of rings. Suppose $C \in D^{\mathrm{perf}}(\sigma^{-1}A)$ is the complex*

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \cdots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0.$$

Then the complex $X \in D^{\mathrm{perf}}(A)$ with $C \simeq \{\sigma^{-1}A\}^L\otimes_A X$, whose existence is guaranteed by Theorem 1.4, may be chosen to be a complex

$$0 \longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \cdots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0\ .$$

If $m = n$ this is easy. For $m < n$ we need to prove something. Our proof will appeal to the results of [2, Section 4]. We remind the reader that this was the section which dealt with the subcategories $\mathcal{K}[m,n]$ of complexes in $\mathcal{R}^c$ vanishing outside the range $[m,n]$. First we need a lemma.

Lemma 1.6. *Let M and N be any finitely generated projective A–modules. We may view M and N as objects in the derived category $D^{\mathrm{perf}}(A)$, concentrated in degree 0. Then any map in $\mathcal{T}^c(\pi M, \pi N)$ can be represented as $\pi(\alpha)^{-1}\pi(\beta)$, for some α, β morphisms in $D^{\mathrm{perf}}(A)$ as below*

$$M \xrightarrow{\ \beta\ } Y \xleftarrow{\ \alpha\ } N\ .$$

The map $\alpha : N \longrightarrow Y$ fits in a triangle

$$X \longrightarrow N \xrightarrow{\ \alpha\ } Y \longrightarrow \Sigma X$$

and X may be chosen to lie in $\mathcal{K}[0,1]$.

Proof. By [2, Proposition 4.5 and Theorem 3.7] we know that the map

$$i : \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c} \longrightarrow \mathcal{T}^c$$

is fully faithful. Therefore

$$\mathcal{T}^c(\pi M, \pi N) \quad = \quad \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c}(M, N).$$

That is, any map $\pi M \longrightarrow \pi N$ can be written as $\pi(\alpha)^{-1}\pi(\beta)$, for some α, β morphisms in $D^{\mathrm{perf}}(A)$ as below

$$M \xrightarrow{\ \beta\ } Y \xleftarrow{\ \alpha\ } N\ .$$

The map $\alpha : N \longrightarrow Y$ fits in a triangle

$$X \longrightarrow N \xrightarrow{\ \alpha\ } Y \xrightarrow{\ \beta\ } \Sigma X$$

and X may be chosen to lie in $\mathcal{R}^c$. What is not clear is that we may choose X in $\mathcal{K}[0,1] \subset \mathcal{R}^c$.

The easy observation is that we may certainly modify our choice of X to lie in $\mathcal{K} \subset \mathcal{R}^c$. This follows from [3, Lemma 4.5], which tells us that for any choice of X as above there exists an X' with $X \oplus X'$ isomorphic to an object in $\mathcal{K}$. We have a distinguished triangle

$$X \oplus X' \longrightarrow N \xrightarrow{\begin{pmatrix}\alpha\\0\end{pmatrix}} Y \oplus \Sigma X' \xrightarrow{\beta\oplus 1} \Sigma(X \oplus X')$$

and a diagram

$$M \xrightarrow{\begin{pmatrix}\beta\\0\end{pmatrix}} Y \oplus \Sigma X' \xleftarrow{\begin{pmatrix}\alpha\\0\end{pmatrix}} N\ ,$$

and replacing our original choices by these we may assume $X \in \mathcal{K}$. Now we have to shorten X.

By [3, Lemma 4.7], there exists a triangle in $\mathcal{R}^c$

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathcal{K}[1, \infty)$ and $X'' \in \mathcal{K}(-\infty, 1]$. The composite $X' \longrightarrow X \longrightarrow N$ is a map from $X' \in \mathcal{K}[1, \infty)$ to $N \in \mathcal{S}^{\leq 0}$, which must vanish. Hence we have that $X \longrightarrow N$ factors as $X \longrightarrow X'' \longrightarrow N$. We complete to a morphism of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & N \arrow[r, "\alpha"] \arrow[d, "1"'] & Y \arrow[r] \arrow[d, "\gamma"'] & \Sigma X \arrow[d] \\
X'' \arrow[r] & N \arrow[r, "\gamma\alpha"] & Y'' \arrow[r] & \Sigma X''
\end{tikzcd}$$

and another representative of our morphism is the diagram

$$M \xrightarrow{\gamma\beta} Y'' \xleftarrow{\gamma\alpha} N$$

We may, on replacing Y by Y'', assume $X \in \mathcal{K}(-\infty, 1]$.

Applying [3, Lemma 4.7] again, we have that any $X \in \mathcal{K}(-\infty, 1]$ admits a triangle

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathcal{K}[0, 1]$ and $X'' \in \mathcal{K}(-\infty, 0]$. Form the octahedron

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & N \arrow[r, "\alpha'"] \arrow[d, "1"'] & Y' \arrow[r] \arrow[d, "\gamma"'] & \Sigma X' \arrow[d] \\
X \arrow[r] & N \arrow[r, "\alpha"] & Y \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
 & & \Sigma X'' \arrow[r, "1"] & \Sigma X''
\end{tikzcd}$$

The composite $M \longrightarrow Y \longrightarrow \Sigma X''$ is a map from the projective module M, viewed as a complex concentrated in degree 0, to $\Sigma X'' \in \mathcal{K}(\infty, -1]$. This composite must vanish.

The map $\beta : M \longrightarrow Y$ therefore factors as $M \xrightarrow{\beta'} Y' \xrightarrow{\gamma} Y$, and our morphism in $\mathcal{T}^c$ has a representative

$$M \xrightarrow{\beta'} Y' \xleftarrow{\alpha'} N$$

so that in the triangle

$$X' \longrightarrow N \xrightarrow{\alpha'} Y' \longrightarrow \Sigma X'$$

X' may be chosen to lie in $\mathcal{K}[0,1]$. □

Now we are ready for

Proof of Theorem 1.5. We are given a complex $C \in D^{\mathrm{perf}}(\sigma^{-1}A)$ of the form

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \cdots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0.$$

To eliminate the trivial case, assume $m \leq n-1$. Shifting, we may assume $m = 0$ and $n \geq 1$. Theorem 1.4 guarantees that C is homotopy equivalent to $\{\sigma^{-1}A\}^L \otimes_A D$, with $D \in D^{\mathrm{perf}}(A)$. But D need not be supported on the interval $[0,n]$. We need to show how to shorten D. Assume therefore that D is supported on $[-1,n]$. We will show how to replace D by a complex supported on $[0,n]$. Shortening a complex supported on $[0,n+1]$ is dual, and we leave it to the reader.

We may suppose therefore that $D \in D^{\mathrm{perf}}(A)$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow D^{-1} \longrightarrow D^0 \longrightarrow \cdots \longrightarrow D^n \longrightarrow 0 \longrightarrow \cdots$$

and that there is a homotopy equivalence of $\sigma^{-1}D$ with a shorter complex, that is a commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[r] \arrow[d] & \sigma^{-1}D^{-1} \arrow[r, "\partial"] \arrow[d] & \sigma^{-1}D^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \sigma^{-1}D^n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \sigma^{-1}C^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \sigma^{-1}C^n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & \sigma^{-1}D^{-1} \arrow[r, "\partial"] & \sigma^{-1}D^0 \arrow[r] & \cdots \arrow[r] & \sigma^{-1}D^n \arrow[r] & 0 \arrow[r] & {}
\end{tikzcd}$$

so that the composite is homotopic to the identity. In particular, there is a map $d : \sigma^{-1}D^0 \longrightarrow \sigma^{-1}D^{-1}$ so that $d\partial : \sigma^{-1}D^{-1} \longrightarrow \sigma^{-1}D^{-1}$ is the identity.

By [3, Proposition 3.1] the map $d : \sigma^{-1}D^0 \longrightarrow \sigma^{-1}D^{-1}$ lifts uniquely to a map $d' : \pi D^0 \longrightarrow \pi D^{-1}$. By Lemma 1.6 the map d' can be represented as $\pi(\alpha)^{-1}\pi(\beta)$, where α and β are, respectively, the chain maps

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & D^{-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & X \arrow[r, "r"] & Y \arrow[r] & 0 \arrow[r] & {}
\end{tikzcd}$$

and

$$\begin{array}{ccccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & D^0 & \longrightarrow & 0 & \longrightarrow \\ & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle g} & & \big\downarrow & \\ \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{r} & Y & \longrightarrow & 0 & \longrightarrow \end{array}$$

The fact that $\sigma^{-1}\alpha$ is an equivalence tells us that the map $\sigma^{-1}r : \sigma^{-1}X \longrightarrow \sigma^{-1}Y$ is injective, with cokernel $\sigma^{-1}D^{-1}$. The fact that $\alpha^{-1}\beta$ agrees with d' means that the composite

$$\sigma^{-1}D^0 \xrightarrow{\sigma^{-1}g} \sigma^{-1}Y \longrightarrow \mathrm{Coker}(\sigma^{-1}r)$$

is just the map $d : \sigma^{-1}D^0 \longrightarrow \sigma^{-1}D^{-1}$. Let X be the chain complex

$$\longrightarrow 0 \longrightarrow D^0 \oplus X \xrightarrow{\begin{pmatrix} \partial & 0 \\ g & r \end{pmatrix}} D^1 \oplus Y \longrightarrow \cdots \longrightarrow D^n \longrightarrow 0 \longrightarrow$$

Let $f : X \longrightarrow D$ be the natural map of chain complexes

$$\begin{array}{ccccccccccccc} \longrightarrow 0 & \longrightarrow & 0 & \longrightarrow & D^0 \oplus X & \xrightarrow{\begin{pmatrix} \partial & 0 \\ g & r \end{pmatrix}} & D^1 \oplus Y & \longrightarrow & \cdots & \longrightarrow & D^n & \longrightarrow 0 \longrightarrow \\ & & \big\downarrow & & {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} & & & & \big\downarrow & \\ \longrightarrow 0 & \longrightarrow & D^{-1} & \longrightarrow & D^0 & \xrightarrow[\partial]{} & D^1 & \longrightarrow & \cdots & \longrightarrow & D^n & \longrightarrow 0 \longrightarrow \end{array}$$

where the vertical maps labelled π_1 are the projections to the first factor of the direct sum. The map $\sigma^{-1}f$ is easily seen to be homotopy equivalence. Thus $\sigma^{-1}X$ is homotopy equivalent to $\sigma^{-1}D \cong C$. □

2. Algebraic L-theory

An *involution* on a ring A is an anti-automorphism

$$A \longrightarrow A \; ; \; r \mapsto \overline{r} \; .$$

The involution is used to regard a left A-module M as a right A-module by

$$M \times A \longrightarrow M \; ; \; (x, r) \mapsto \overline{r}x \; .$$

The *dual* of a (left) A-module M is the A-module

$$M^* \;=\; \mathrm{Hom}_A(M, A) \; , \; A \times M^* \longrightarrow M^* \; ; \; (r, f) \mapsto (x \mapsto f(x)\overline{r}) \; .$$

The *dual* of an A-module morphism $s : P \longrightarrow Q$ is the A-module morphism

$$s^* \; : \; Q^* \longrightarrow P^* \; ; \; f \mapsto (x \mapsto f(s(x))) \; .$$

If M is f.g. projective then so is M^*, and

$$M \longrightarrow M^{**} \; ; \; x \mapsto (f \mapsto \overline{f(x)})$$

is an isomorphism which is used to identify $M^{**} = M$.

Hypothesis 2.1. *In this section, we assume that*

(i) *A is a ring with involution,*
(ii) *the duals of morphisms $s : P \longrightarrow Q$ in σ are morphisms $s^* : Q^* \longrightarrow P^*$ in σ,*
(iii) *$\epsilon \in A$ is a central unit such that $\overline{\epsilon} = \epsilon^{-1}$ (e.g. $\epsilon = \pm 1$).*

The noncommutative localization $\sigma^{-1}A$ is then also a ring with involution, with $\epsilon \in \sigma^{-1}A$ a central unit such that $\overline{\epsilon} = \epsilon^{-1}$. □

We review briefly the chain complex construction of the f.g. projective ϵ-quadratic L-groups $L_*(A,\epsilon)$ and the ϵ-symmetric L-groups $L^*(A,\epsilon)$. Given an A-module chain complex C let the generator $T \in \mathbb{Z}_2$ act on the $\mathbb{Z}$-module chain complex $C \otimes_A C$ by the ϵ-transposition duality

$$T_\epsilon \ : \ C_p \otimes_A C_q \longrightarrow C_q \otimes_A C_p \ : \ x \otimes y \mapsto (-1)^{pq}\epsilon y \otimes x \ .$$

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$W \ : \ \dots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \ .$$

The *ϵ-symmetric* (resp. *ϵ-quadratic*) *Q-groups* of C are the $\mathbb{Z}_2$-hypercohomology (resp. $\mathbb{Z}_2$-hyperhomology) groups of $C \otimes_A C$

$$\begin{aligned} Q^n(C,\epsilon) \ &= \ H^n(\mathbb{Z}_2; C \otimes_A C) \ = \ H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_A C)) \ , \\ Q_n(C,\epsilon) \ &= \ H_n(\mathbb{Z}_2; C \otimes_A C) \ = \ H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_A C)) \ . \end{aligned}$$

The Q-groups are chain homotopy invariants of C. There are defined forgetful maps

$$\begin{aligned} &1 + T_\epsilon \ : \ Q_n(C,\epsilon) \longrightarrow Q^n(C,\epsilon) \ ; \ \psi \mapsto (1+T_\epsilon)\psi \ , \\ &Q^n(C,\epsilon) \longrightarrow H_n(C \otimes_A C) \ ; \ \phi \mapsto \phi_0 \ . \end{aligned}$$

For f.g. projective C the function

$$C \otimes_A C \longrightarrow \mathrm{Hom}_A(C^*, C) \ ; \ x \otimes y \mapsto (f \mapsto \overline{f(x)}y)$$

is an isomorphism of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes, with $T \in \mathbb{Z}_2$ acting on $\mathrm{Hom}_A(C^*, C)$ by $\theta \mapsto \epsilon\theta^*$. The element $\phi_0 \in H_n(C \otimes_A C) = H_n(\mathrm{Hom}_A(C^*, C))$ is a chain homotopy class of A-module chain maps $\phi_0 : C^{n-*} \longrightarrow C$.

An *n-dimensional ϵ-symmetric complex over A* (C,ϕ) is a bounded f.g. projective A-module chain complex C together with an element $\phi \in Q^n(C,\epsilon)$. The complex (C,ϕ) is *Poincaré* if the A-module chain map $\phi_0 : C^{n-*} \longrightarrow C$ is a chain equivalence.

Example 2.2. A 0-dimensional ϵ-symmetric Poincaré complex (C,ϕ) over A is essentially the same as a nonsingular ϵ-symmetric form (M,λ) over (A,σ), with $M = (C_0)^*$ a f.g. projective A-module and

$$\lambda \ = \ \phi_0 \ : \ M \times M \longrightarrow A$$

a sesquilinear pairing such that the adjoint

$$M \longrightarrow M^* \ ; \ x \mapsto (y \mapsto \lambda(x,y))$$

is an A-module isomorphism.

$\square$

See pp. 210–211 of [6] for the notion of an ϵ-*symmetric (Poincaré) pair.* The *boundary* of an n-dimensional ϵ-symmetric complex (C,ϕ) is the $(n-1)$-dimensional ϵ-symmetric Poincaré complex

$$\partial(C,\phi) \ = \ (\partial C,\partial\phi)$$

with $\partial C = C(\phi_0 : C^{n-*} \longrightarrow C)_{*+1}$ and $\partial\phi$ as defined on p. 218 of [6]. The *n-dimensional ϵ-symmetric L-group* $L^n(A,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric Poincaré complexes (C,ϕ) over A with C n-dimensional. In particular, $L^0(A,\epsilon)$ is the Witt group of nonsingular ϵ-symmetric forms over A.

An n-dimensional ϵ-symmetric complex (C,ϕ) over A is *$\sigma^{-1}A$-Poincaré* if the $\sigma^{-1}A$-module chain map $\sigma^{-1}\phi_0 : \sigma^{-1}C^{n-*} \longrightarrow \sigma^{-1}C$ is a chain equivalence, in which case $\sigma^{-1}(C,\phi)$ is an n-dimensional ϵ-symmetric Poincaré complex over $\sigma^{-1}A$.

The *n-dimensional ϵ-symmetric Γ-group* $\Gamma^n(A \longrightarrow \sigma^{-1}A,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric $\sigma^{-1}A$-Poincaré complexes (C,ϕ) over A such that $\sigma^{-1}C$ is chain equivalent to an n-dimensional induced f.g. projective $\sigma^{-1}A$-module chain complex. The *n-dimensional ϵ-symmetric L-group* $L^n(A,\sigma,\epsilon)$ is the cobordism group of $(n-1)$-dimensional ϵ-symmetric Poincaré complexes over A (C,ϕ) such that C is $\sigma^{-1}A$-contractible, i.e. $\sigma^{-1}C \simeq 0$.

Similarly in the ϵ-quadratic case, with groups $L_n(A,\epsilon)$, $\Gamma_n(A \longrightarrow \sigma^{-1}A,\epsilon)$, $L_n(A,\sigma,\epsilon)$. The ϵ-quadratic L- and Γ-groups are 4-periodic

$$\begin{aligned}
&L_n(A,\epsilon) \ = \ L_{n+2}(A,-\epsilon) \ = \ L_{n+4}(A,\epsilon) \ , \\
&\Gamma_n(A \longrightarrow \sigma^{-1}A,\epsilon) \ = \ \Gamma_{n+2}(A \longrightarrow \sigma^{-1}A,-\epsilon) \ = \ \Gamma_{n+4}(A \longrightarrow \sigma^{-1}A,\epsilon) \ , \\
&L_n(A,\sigma,\epsilon) \ = \ L_{n+2}(A,\sigma,-\epsilon) \ = \ L_{n+4}(A,\sigma,\epsilon) \ .
\end{aligned}$$

Proposition 2.3. *For any ring with involution A and noncommutative localization $\sigma^{-1}A$ there is defined a localization exact sequence of ϵ-symmetric L-groups*

$$\cdots \longrightarrow L^n(A,\epsilon) \longrightarrow \Gamma^n(A \longrightarrow \sigma^{-1}A,\epsilon) \xrightarrow{\partial} L^n(A,\sigma,\epsilon) \longrightarrow L^{n-1}(A,\epsilon) \longrightarrow \cdots \ .$$

Similarly in the ϵ-quadratic case, with an exact sequence

$$\cdots \longrightarrow L_n(A,\epsilon) \longrightarrow \Gamma_n(A \longrightarrow \sigma^{-1}A,\epsilon) \xrightarrow{\partial} L_n(A,\sigma,\epsilon) \longrightarrow L_{n-1}(A,\epsilon) \longrightarrow \cdots \ .$$

Proof. The relative group of $L^n(A,\epsilon) \longrightarrow \Gamma^n(A \longrightarrow \sigma^{-1}A,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric $\sigma^{-1}A$-Poincaré pairs over A $(f : C \longrightarrow D,(\delta\phi,\phi))$ with (C,ϕ) Poincaré. The effect of algebraic surgery on (C,ϕ) using this pair is a cobordant $(n-1)$-dimensional ϵ-symmetric Poincaré complex (C',ϕ') with C' $\sigma^{-1}A$-contractible. The function $(f : C \longrightarrow D,(\delta\phi,\phi)) \mapsto (C',\phi')$ defines an isomorphism between the relative group and $L^n(A,\sigma,\epsilon)$. $\square$

Define

$$I \;=\; \mathrm{im}(K_0(A) \longrightarrow K_0(\sigma^{-1}A)) ,$$

the subgroup of $K_0(\sigma^{-1}A)$ consisting of the projective classes of the f.g. projective $\sigma^{-1}A$-modules induced from f.g. projective A-modules. By definition, $L^n_I(\sigma^{-1}A,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric Poincaré complexes over $\sigma^{-1}A$ (B,θ) such that $[B]\in I$. There are evident morphisms of Γ- and L-groups

$$\begin{aligned}
\sigma^{-1}\Gamma^* &: \Gamma^n(A\longrightarrow \sigma^{-1}A,\epsilon) \longrightarrow L^n_I(\sigma^{-1}A,\epsilon) \;;\; (C,\phi)\mapsto \sigma^{-1}(C,\phi) ,\\
\sigma^{-1}\Gamma_* &: \Gamma_n(A\longrightarrow \sigma^{-1}A,\epsilon) \longrightarrow L_n^I(\sigma^{-1}A,\epsilon) \;;\; (C,\psi)\mapsto \sigma^{-1}(C,\psi) .
\end{aligned}$$

In general, the morphisms $\sigma^{-1}\Gamma^*,\sigma^{-1}\Gamma_*$ need not be isomorphisms, since a bounded f.g. projective $\sigma^{-1}A$-module chain complex D with $[D]\in I$ need not be chain equivalent to $\sigma^{-1}C$ for a bounded f.g. projective A-module chain complex C.

It was proved in Chapter 3 of Ranicki [5] that if $A\longrightarrow \sigma^{-1}A$ is an injective Ore localization then the morphisms $\sigma^{-1}Q^*,\sigma^{-1}Q_*,\ \sigma^{-1}\Gamma^*,\sigma^{-1}\Gamma_*$ are isomorphisms, so that there are defined localization exact sequences for both the ϵ-symmetric and the ϵ-quadratic L-groups

$$\begin{gathered}
\cdots \longrightarrow L^n(A,\epsilon) \longrightarrow L^n_I(\sigma^{-1}A,\epsilon) \xrightarrow{\partial} L^n(A,\sigma,\epsilon) \longrightarrow L^{n-1}(A,\epsilon) \longrightarrow \cdots ,\\
\cdots \longrightarrow L_n(A,\epsilon) \longrightarrow L_n^I(\sigma^{-1}A,\epsilon) \xrightarrow{\partial} L_n(A,\sigma,\epsilon) \longrightarrow L_{n-1}(A,\epsilon) \longrightarrow \cdots .
\end{gathered}$$

Special cases of these sequences were obtained by Milnor-Husemoller, Karoubi, Pardon, Smith, Carlsson-Milgram.

Let $G\pi : D(A)\to D(A)$ be the functor of Proposition 6.1 of [2], with $D(A)$ the derived category of A. It follows from the work of Vogel [9] that for any bounded f.g. projective A-module chain complex C

$$G\pi(C) \;=\; \varinjlim_{(B,\beta)} B$$

with the direct limits taken over all the bounded f.g. projective A-module chain complexes B with a chain map $\beta : C\longrightarrow B$ such that $\sigma^{-1}\beta:\sigma^{-1}C\longrightarrow \sigma^{-1}B$ is a $\sigma^{-1}A$-module chain equivalence. The natural A-module chain map

$$G\pi(C) \;=\; \varinjlim_{(B,\beta)} B \longrightarrow \sigma^{-1}C$$

induces morphisms

$$\begin{aligned}
\sigma^{-1}Q^* &: Q^n(G\pi(C),\epsilon) \;=\; \varinjlim_{(B,\beta)} Q^n(B,\epsilon) \longrightarrow Q^n(\sigma^{-1}C,\epsilon) ,\\
\sigma^{-1}Q_* &: Q_n(G\pi(C),\epsilon) \;=\; \varinjlim_{(B,\beta)} Q_n(B,\epsilon) \longrightarrow Q_n(\sigma^{-1}C,\epsilon) .
\end{aligned}$$

The natural projection $D\otimes_A D\longrightarrow D\otimes_{\sigma^{-1}A} D$ is an isomorphism for any bounded f.g. projective $\sigma^{-1}A$-module chain complex D (since this is already the case for $D=\sigma^{-1}A$), so the Q-groups of $\sigma^{-1}C$ are the same whether $\sigma^{-1}C$ is regarded as an A-module or $\sigma^{-1}A$-module chain complex.

Theorem 2.4. (Vogel [9], Theorem 8.4) *For any ring with involution A and noncommutative localization $\sigma^{-1}A$ the morphisms*

$$\sigma^{-1}\Gamma_* \ : \ \Gamma_n(A \longrightarrow \sigma^{-1}A, \epsilon) \longrightarrow L^I_n(\sigma^{-1}A, \epsilon) \ ; \ (C, \psi) \mapsto \sigma^{-1}(C, \psi)$$

are isomorphisms, and there is a localization exact sequence of ϵ-quadratic L-groups

$$\cdots \longrightarrow L_n(A, \epsilon) \longrightarrow L^I_n(\sigma^{-1}A, \epsilon) \xrightarrow{\partial} L_n(A, \sigma, \epsilon) \longrightarrow L_{n-1}(A, \epsilon) \longrightarrow \cdots .$$

Proof. By algebraic surgery below the middle dimension it suffices to consider only the special cases $n = 0, 1$. In effect, it was proved in [9] that $\sigma^{-1}Q_*$ is an isomorphism for 0- and 1-dimensional C. □

It was claimed in Proposition 25.4 of Ranicki [6] that $\sigma^{-1}\Gamma^*$ is also an isomorphism, assuming (incorrectly) that the chain complex lifting problem can always be solved. However, we do have :

Theorem 2.5. *If $\sigma^{-1}A$ is a noncommutative localization of a ring with involution A which is stably flat over A, there is a localization exact sequence of ϵ-symmetric L-groups*

$$\cdots \longrightarrow L^n(A, \epsilon) \longrightarrow L^n_I(\sigma^{-1}A, \epsilon) \xrightarrow{\partial} L^n(A, \sigma, \epsilon) \longrightarrow L^{n-1}(A, \epsilon) \longrightarrow \cdots .$$

Proof. For any bounded f.g. projective A-module chain complex C the natural A-module chain map $G\pi(C) \longrightarrow \sigma^{-1}C$ induces isomorphisms in homology

$$H_*(G\pi(C)) \ \cong \ H_*(\sigma^{-1}C) \ .$$

Thus the natural $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$G\pi(C) \otimes_A G\pi(C) \longrightarrow \sigma^{-1}C \otimes_A \sigma^{-1}C \ = \ \sigma^{-1}C \otimes_{\sigma^{-1}A} \sigma^{-1}C$$

induces isomorphisms of ϵ-symmetric Q-groups

$$\sigma^{-1}Q^* \ : \ \varinjlim_{(B,\beta)} Q^n(B, \epsilon) \longrightarrow Q^n(\sigma^{-1}C, \epsilon)$$

(and also isomorphisms $\sigma^{-1}Q_*$ of ϵ-quadratic Q-groups). By Theorem 0.1 every n-dimensional induced f.g. projective $\sigma^{-1}A$-module chain complex D is chain equivalent to $\sigma^{-1}C$ for an n-dimensional f.g. projective A-module chain complex C, with

$$Q^n(D, \epsilon) \ = \ Q^n(\sigma^{-1}C, \epsilon) \ = \ \varinjlim_{(B,\beta)} Q^n(B, \epsilon) \ .$$

It follows that the morphisms of ϵ-symmetric Γ- and L-groups

$$\sigma^{-1}\Gamma^* \ : \ \Gamma^n(A \longrightarrow \sigma^{-1}A, \epsilon) \longrightarrow L^n_I(\sigma^{-1}A, \epsilon) \ ; \ (C, \phi) \mapsto \sigma^{-1}(C, \phi)$$

are also isomorphisms, and the localization exact sequence is given by Proposition 2.3. □

Hypothesis 2.6. *For the remainder of this section, we assume Hypothesis 2.1 and also that $A \longrightarrow \sigma^{-1}A$ is an injection.* $\square$

As in Proposition 2.2 of [3] it follows that all the morphisms in σ are injections.

We shall now generalize the results of Ranicki [5] and Vogel [8] to prove that under Hypotheses 2.1,2.6 the relative L-groups $L^*(A,\sigma,\epsilon)$, $L_*(A,\sigma,\epsilon)$ in the L-theory localization exact sequences are the L-groups of $H(A,\sigma)=\mathcal{E}(A,\sigma)$ with respect to the following duality involution.

Define the *torsion dual* of an (A,σ)-module M to be the (A,σ)-module

$$M\widehat{\ } \;=\; \mathrm{Ext}^1_A(M,A)\ ,$$

using the involution on A to define the left A-module structure. If M has f.g. projective A-module resolution

$$0 \longrightarrow P_1 \xrightarrow{s} P_0 \longrightarrow M \longrightarrow 0$$

with $s\in\sigma$ the torsion dual $M\widehat{\ }$ has the dual f.g. projective A-module resolution

$$0 \longrightarrow P_0^* \xrightarrow{s^*} P_1^* \longrightarrow M\widehat{\ } \longrightarrow 0$$

with $s^*\in\sigma$.

Proposition 2.7. *Let* $M=\mathrm{coker}(s:P_1\longrightarrow P_0)$, $N=\mathrm{coker}(t:Q_1\longrightarrow Q_0)$ *be* (A,σ)-*modules.*
(i) *The adjoint of the pairing*

$$M\times M\widehat{\ } \longrightarrow \sigma^{-1}A/A\ ;\ (g\in P_0, f\in P_1^*)\mapsto fs^{-1}g$$

defines a natural A-module isomorphism

$$M\widehat{\ } \longrightarrow \mathrm{Hom}_A(M,\sigma^{-1}A/A)\ ;\ f\mapsto (g\mapsto fs^{-1}g)\ .$$

(ii) *The natural A-module morphism*

$$M \longrightarrow M\widehat{\ }\widehat{\ }\ ;\ x\mapsto (f\mapsto \overline{f(x)})$$

is an isomorphism.
(iii) *There are natural identifications*

$$\begin{aligned} M\otimes_A N \;&=\; \mathrm{Tor}_0^A(M,N) \;=\; \mathrm{Ext}^1_A(M\widehat{\ },N) \;=\; H_0(P\otimes_A Q)\ ,\\ \mathrm{Hom}_A(M\widehat{\ },N) \;&=\; \mathrm{Tor}_1^A(M,N) \;=\; \mathrm{Ext}^0_A(M\widehat{\ },N) \;=\; H_1(P\otimes_A Q)\ .\end{aligned}$$

The functions

$$\begin{aligned} M\otimes_A N &\longrightarrow N\otimes_A M\ ;\ x\otimes y\mapsto y\otimes x\ ,\\ \mathrm{Hom}_A(M\widehat{\ },N) &\longrightarrow \mathrm{Hom}_A(N\widehat{\ },M)\ ;\ f\mapsto f\widehat{\ }\end{aligned}$$

determine transposition isomorphisms

$$T\ :\ \mathrm{Tor}_i^A(M,N) \longrightarrow \mathrm{Tor}_i^A(N,M)\ (i=0,1)\ .$$

(iv) *For any finite subset* $V = \{v_1, v_2, \dots, v_k\} \subset M \otimes_A N$ *there exists an exact sequence of* (A,σ)*-modules*

$$0 \longrightarrow N \longrightarrow L \longrightarrow \oplus_k M\widehat{\ } \longrightarrow 0$$

such that $V \subset \ker(M \otimes_A N \longrightarrow M \otimes_A L)$.

Proof. (i) Apply the snake lemma to the morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(P_0, A) & \longrightarrow & \mathrm{Hom}_A(P_0, \sigma^{-1}A) & \longrightarrow & \mathrm{Hom}_A(P_0, \sigma^{-1}A/A) & \longrightarrow & 0 \\
 & & \Big\downarrow{s^*} & & \Big\downarrow{s_1^*} & & \Big\downarrow{s_2^*} & & \\
0 & \longrightarrow & \mathrm{Hom}_A(P_1, A) & \longrightarrow & \mathrm{Hom}_A(P_1, \sigma^{-1}A) & \longrightarrow & \mathrm{Hom}_A(P_1, \sigma^{-1}A/A) & \longrightarrow & 0
\end{array}$$

with s^* injective, s_1^* an isomorphism and s_2^* surjective, to verify that the A-module morphism

$$M\widehat{\ } = \mathrm{coker}(s^*) \longrightarrow \mathrm{Hom}_A(M, \sigma^{-1}A/A) = \ker(s_2^*)$$

is an isomorphism.
(ii) Immediate from the identification

$$s^{**} = s : (P_0)^{**} = P_0 \longrightarrow (P_1)^{**} = P_1 .$$

(iii) Exercise for the reader.
(iv) Lift each $v_i \in M \otimes_A N$ to an element

$$v_i \in P_0 \otimes_A Q_0 = \mathrm{Hom}_A(P_0^*, Q_0) \ (1 \le i \le k) .$$

The A-module morphism defined by

$$u = \begin{pmatrix} s^* & 0 & 0 & \dots & 0 \\ 0 & s^* & 0 & \dots & 0 \\ 0 & 0 & s^* & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_1 & v_2 & v_3 & \dots & t \end{pmatrix} : U_1 = (\oplus_k P_0^*) \oplus Q_1 \longrightarrow U_0 = (\oplus_k P_1^*) \oplus Q_0$$

is in σ, so that $L = \mathrm{coker}(u)$ is an (A,σ)-module with a f.g. projective A-module resolution

$$0 \longrightarrow U_1 \xrightarrow{u} U_0 \longrightarrow L \longrightarrow 0 .$$

The short exact sequence of 1-dimensional f.g. projective A-module chain complexes

$$0 \longrightarrow Q \longrightarrow U \longrightarrow \oplus_k P^{1-*} \longrightarrow 0$$

is a resolution of a short exact sequence of (A,σ)-modules

$$0 \longrightarrow N \longrightarrow L \longrightarrow \oplus_k M\widehat{\ } \longrightarrow 0 .$$

The first morphism in the exact sequence

$$\mathrm{Tor}_1^A(M, \oplus_k M\widehat{\ }) \longrightarrow M \otimes_A N \longrightarrow M \otimes_A L \longrightarrow M \otimes_A (\oplus_k M\widehat{\ }) \longrightarrow 0$$

sends $1_i \in \mathrm{Tor}_1^A(M, \oplus_k M\widehat{\ }) = \oplus_k \mathrm{Hom}_A(M\widehat{\ }, M\widehat{\ })$ to $v_i \in \ker(M \otimes_A N \longrightarrow M \otimes_A L)$. $\square$

Given an (A,σ)-module chain complex C define the *ϵ-symmetric* (resp. *ϵ-quadratic*) *torsion Q-groups* of C to be the $\mathbb{Z}_2$-hypercohomology (resp. $\mathbb{Z}_2$-hyperhomology) groups of the ϵ-transposition involution $T_\epsilon = \epsilon T$ on the $\mathbb{Z}$-module chain complex $\mathrm{Tor}_1^A(C,C) = \mathrm{Hom}_A(C^\frown, C)$

$$\begin{aligned} Q^n_{\mathrm{tor}}(C,\epsilon) &= H^n(\mathbb{Z}_2;\mathrm{Tor}_1^A(C,C)) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W,\mathrm{Tor}_1^A(C,C)))\ , \\ Q_n^{\mathrm{tor}}(C,\epsilon) &= H_n(\mathbb{Z}_2;\mathrm{Tor}_1^A(C,C)) = H_n(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(\mathrm{Tor}_1^A(C,C)))\ . \end{aligned}$$

There are defined forgetful maps

$$\begin{aligned} &1+T_\epsilon\ :\ Q_n^{\mathrm{tor}}(C,\epsilon)\longrightarrow Q^n_{\mathrm{tor}}(C,\epsilon)\ ;\ \psi\mapsto(1+T_\epsilon)\psi\ , \\ &Q^n_{\mathrm{tor}}(C,\epsilon)\longrightarrow H_n(\mathrm{Tor}_1^A(C,C))\ ;\ \phi\mapsto\phi_0\ . \end{aligned}$$

The element $\phi_0\in H_n(\mathrm{Tor}_1^A(C,C))$ is a chain homotopy class of A-module chain maps $\phi_0 : C^{n-\frown}\longrightarrow C$.

An *n-dimensional ϵ-symmetric complex over* (A,σ) (C,ϕ) is a bounded (A,σ)-module chain complex C together with an element $\phi\in Q^n_{\mathrm{tor}}(C,\epsilon)$. The complex (C,ϕ) is *Poincaré* if the A-module chain maps $\phi_0 : C^{n-\frown}\longrightarrow C$ are chain equivalences.

Example 2.8. A 0-dimensional ϵ-symmetric Poincaré complex (C,ϕ) over (A,σ) is essentially the same as a nonsingular ϵ-symmetric linking form (M,λ) over (A,σ), with $M=(C_0)^\frown$ an (A,σ)-module and

$$\lambda\ =\ \phi_0\ :\ M\times M\longrightarrow\sigma^{-1}A/A$$

a sesquilinear pairing such that the adjoint

$$M\longrightarrow M^\frown\ ;\ x\mapsto(y\mapsto\lambda(x,y))$$

is an A-module isomorphism.

□

The *n-dimensional torsion ϵ-symmetric L-group* $L^n_{\mathrm{tor}}(A,\sigma,\epsilon)$ is the cobordism group of n-dimensional ϵ-symmetric Poincaré complexes (C,ϕ) over (A,σ), with C n-dimensional. In particular, $L^0_{\mathrm{tor}}(A,\sigma,\epsilon)$ is the Witt group of nonsingular ϵ-symmetric linking forms over (A,σ).

Similarly in the ϵ-quadratic case, with torsion L-groups $L_n^{\mathrm{tor}}(A,\sigma,\epsilon)$. The ϵ-quadratic torsion L-groups are 4-periodic

$$L_n^{\mathrm{tor}}(A,\sigma,\epsilon)\ =\ L_{n+2}^{\mathrm{tor}}(A,\sigma,-\epsilon)\ =\ L_{n+4}^{\mathrm{tor}}(A,\sigma,\epsilon)\ .$$

Theorem 2.9. *If $A\longrightarrow\sigma^{-1}A$ is injective the relative L-groups in the localization exact sequences of Proposition 2.3*

$$\cdots\longrightarrow L^n(A,\epsilon)\longrightarrow\Gamma^n(A\longrightarrow\sigma^{-1}A,\epsilon)\xrightarrow{\partial}L^n(A,\sigma,\epsilon)\longrightarrow L^{n-1}(A,\epsilon)\longrightarrow\cdots$$

$$\cdots\longrightarrow L_n(A,\epsilon)\longrightarrow\Gamma_n(A\longrightarrow\sigma^{-1}A,\epsilon)\xrightarrow{\partial}L_n(A,\sigma,\epsilon)\longrightarrow L_{n-1}(A,\epsilon)\longrightarrow\cdots$$

are the torsion L-groups

$$\begin{aligned} L^*(A,\sigma,\epsilon) &= L^*_{\text{tor}}(A,\sigma,\epsilon) \ , \\ L_*(A,\sigma,\epsilon) &= L_*^{\text{tor}}(A,\sigma,\epsilon) \ . \end{aligned}$$

Proof. For any bounded (A,σ)-module chain complex T there exists a bounded f.g. projective A-module chain complex C with a homology equivalence $C \longrightarrow T$. Working as in [8] there is defined a distinguished triangle of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$\Sigma \mathrm{Tor}_1^A(T,T) \longrightarrow C \otimes_A C \longrightarrow T \otimes_A T \longrightarrow \Sigma^2 \mathrm{Tor}_1^A(T,T)$$

with $\mathbb{Z}_2$ acting by the ϵ-transposition T_ϵ on the $\mathbb{Z}$-module chain complex $\mathrm{Tor}_1^A(T,T)$ and by the $(-\epsilon)$-transpositions $T_{-\epsilon}$ on $C \otimes_A C$ and $T \otimes_A T$, inducing long exact sequences

$$\cdots \longrightarrow Q^n_{\text{tor}}(T,\epsilon) \longrightarrow Q^{n+1}(C,-\epsilon) \longrightarrow Q^{n+1}(T,-\epsilon) \longrightarrow Q^{n-1}_{\text{tor}}(T,\epsilon) \longrightarrow \cdots$$

$$\cdots \longrightarrow Q_n^{\text{tor}}(T,\epsilon) \longrightarrow Q_{n+1}(C,-\epsilon) \longrightarrow Q_{n+1}(T,-\epsilon) \longrightarrow Q_{n-1}^{\text{tor}}(T,\epsilon) \longrightarrow \cdots \ .$$

Passing to the direct limits over all the bounded (A,σ)-module chain complexes U with a homology equivalence $\beta : T \longrightarrow U$ use Proposition 2.7 (iv) to obtain

$$\begin{aligned} \varinjlim_{(U,\beta)} Q^{n+1}(U,-\epsilon) &= 0 \ , \\ \varinjlim_{(U,\beta)} Q_{n+1}(U,-\epsilon) &= 0 \end{aligned}$$

and hence

$$\begin{aligned} \varinjlim_{(U,\beta)} Q^n_{\text{tor}}(U,\epsilon) &= Q^{n+1}(C,-\epsilon) \ , \\ \varinjlim_{(U,\beta)} Q_n^{\text{tor}}(U,\epsilon) &= Q_{n+1}(C,-\epsilon) \ . \end{aligned}$$

□

Remark 2.10. The identification $L_*(A,\sigma,\epsilon) = L_*^{\text{tor}}(A,\sigma,\epsilon)$ for noncommutative $\sigma^{-1}A$ was first obtained by Vogel [8].

□

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