Crystals

Combinatorial Algorithms and Tensor Categories

Lecture 0 Categorical prerequisites

Categories are (for us) a useful language for keeping track of a large amount of mathematical data.

We will happily ignore all set theoretic issues!

Key example A category collects together a type of mathematical object and its properties. The main example to have in mind is the category of vector spaces over a field $\mathbb{F}$

Def A category $\mathcal{C}$ consists of

- A collection of objects
- For any two objects $A, B \in \mathcal{C}$, a collection $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of morphisms from A to B
- A rule for composing morphisms
$$\mathrm{Hom}_{\mathcal{C}}(A\,B) \times \mathrm{Hom}_{\mathcal{C}}(B\,C) \to \mathrm{Hom}_{\mathcal{C}}(A\,C)$$
$$f: A \to B\,,\ g: B \to C \mapsto g \circ f: A \to C$$
- For each object $A \in \mathcal{C}$ a morphism
$$\mathrm{id}_A \in \mathrm{End}_{\mathcal{C}}(A) := \mathrm{Hom}_{\mathcal{C}}(A, A)$$

such that

i) (Associativity) For morphisms

$f: A \to B$, $g: B \to C$, $h: C \to D$
we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

ii) (identity) for any morphism $f: A \to B$

$$id_B \circ f = f = f \circ id_A$$

Example $\mathrm{Vect}_{\mathbb{F}}$ is the category of vector spaces over the field $\mathbb{F}$:

objects: vector spaces $/\mathbb{F}$
morphisms: $\mathrm{Hom}(V, W) = \{\text{linear maps } V \to W\}$.

with composition and identity defined as usual

Exercise Think of as many examples of categories as you can. At least 10!

- Can you think of two categories that have the same objects but different morphisms?
- Can you think of two categories with different objects that "feel" the same (ie they should be equivalent in some sense, in the same way that $\mathbb{Z}$ and the infinite cyclic group feel the same before you learn about group homomorphisms).

Functors

Def A functor $F: \mathcal{C} \to \mathcal{D}$ between two categories is an assignment of

- an object $F(A) \in \mathcal{D}$ for every object $A \in \mathcal{C}$
- a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

such that

i) $F(id_A) = id_{F(A)}$

ii) $F(g \circ f) = F(g) \circ F(f)$

Example Let Set be the category with objects sets, and morphisms functions.

$$F: \mathrm{Vect}_{\mathbb{F}} \longrightarrow \mathrm{Set}$$

is the functor that assigns to a vectorspace the set of its elements, and to a linear map, the same map considered as an ordinary function.

Example: Any category has the identity functor $id: \mathcal{C} \to \mathcal{C}$

Exercise Let Ab be the category of abelian groups. Fix any abelian group M and define

$$H: Ab \longrightarrow Ab; \quad H(A) := \mathrm{Hom}(M, A)$$

$$T: Ab \longrightarrow Ab; \quad T(A) := A \otimes_{\mathbb{Z}} M$$

prove these are functors

Exercise Try and think of as many examples of functors as you can.

Example Let Cat be the category whose objects are categories and whose morphisms are functors. This is a category!

Maps

In mathematics we like to study objects up to some kind of isomorphism (or homeomorphism, homotopy, isotopy, or other version of sameness.)

Def In a category $\mathcal{C}$, a morphism $f \in \mathrm{Hom}(A,B)$ is an isomorphism if there exists a $g \in \mathrm{Hom}(B,A)$ s.t.

$$g \circ f = \mathrm{id}_A \quad \text{and} \quad f \circ g = \mathrm{id}_B$$

g is called the inverse of f and is denoted f^{-1}

Exercise Show that f^{-1} (if it exists) is unique.

Def We call a morphism $f \in \mathrm{Hom}(A,B)$ a monomorphism if for any $g, h \in \mathrm{Hom}(X,A)$

$$f \circ g = f \circ h$$

(ie left cancellation holds)

Similarly an epimorphism is on where right cancellation holds.

Remark It is true that injections are always monomorphisms, but the converse is not true!

Similary, all surjections are epimorphisms, but the converse is not true.

Exercise Let Ring be the category of rings. Consider the inclusion

$$\varphi : \mathbb{Z} \hookrightarrow \mathbb{Q}$$

Is it a monomorphism? Is it an epimorphism? Is it an isomorphism?

Def A subobject of an object $A \in \mathcal{C}$ is simply a monomorphism $i : U \to A$.

The category of subobjects of A, $\mathrm{sub}(A)$, has objects monomorphisms $i : U \to A$ an morphisms

$$\mathrm{Hom}(i : U \to A, j : V \to A) = \{ f \in \mathrm{Hom}(U,V) \mid i = j \circ f \}.$$

ie morphisms making the diagram
$$\begin{tikzcd} U \arrow[dr, "i"] \arrow[d, "f"'] & \\ V \arrow[r, "j"'] & A \end{tikzcd}$$
commute.

Exercise Check that in the category Set, isomorphism classes of subobjects are simply subsets.
Check the above is actually a category and define

quotient objects in analogy with this (use epimorphisms) What are the quotient objects in the category of sets?

Remark The notion of sub and quotient object above is maybe a little too weak and gives some strange results (eg $\mathbb{Q}$ is a quotient of $\mathbb{Z}$ in Ring) We wont worry about it here as it wont effect us but it is something to be aware of.

Direct sums

The categorical notion of a direct sum is

Def If $A, B \in \mathcal{C}$ are objects of a category, a **coproduct** of A and B is an object $A \sqcup B \in \mathcal{C}$ along with two morphisms

$$A \xrightarrow{\iota_A} A \sqcup B \xleftarrow{\iota_B} B$$

such that, for any object $C \in \mathcal{C}$ and morphisms

$$A \xrightarrow{f_A} C \xleftarrow{f_B} B$$

there is a unique morphism $\varphi : A \sqcup B \longrightarrow C$ such that

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f_A"] \arrow[r, "\iota_A"'] & A \sqcup B \arrow[u, "\varphi"] & B \arrow[ul, "f_B"'] \arrow[l, "\iota_B"] \end{tikzcd}$$

commutes.

We can think of $A \sqcup B$ as the universal object that "contains" both A and B.

Exercise Show that disjoint union in Set and direct sum in Vect give coproducts.

Proposition If $A \sqcup B$ exists, then it is unique up to unique isomorphism.

proof: Suppose

$$A \xrightarrow{i_A} X \xleftarrow{i_B} B$$

and

$$A \xrightarrow{j_A} Y \xleftarrow{j_B} B$$

are two coproducts. Then by the universal property, there exist morphisms $\varphi : X \to Y$, $\psi : Y \to X$ such that

$$\begin{tikzcd}
 & Y \arrow[d, bend right, "\psi"'] & \\
A \arrow[ur, "j_A"] \arrow[dr, "i_A"'] & & B \arrow[ul, "j_B"'] \arrow[dl, "i_B"] \\
 & X \arrow[u, bend right, "\varphi"'] &
\end{tikzcd}$$

commutes. Thus we get a diagram

$$\begin{tikzcd}
 & X & \\
A \arrow[ur, "i_A = \psi \circ j_A"] \arrow[r, "i_A"'] & X \arrow[u, "\psi \circ \varphi"] & B \arrow[ul, "\psi \circ j_B = i_B"'] \arrow[l, "i_B"]
\end{tikzcd}$$

which commutes. But by uniqueness, we must have $\psi \circ \varphi = \mathrm{id}_X$. Similary we show $\varphi \circ \psi = \mathrm{id}_Y$ $\square$.

The construction of a coproduct can be extended to arbitrary families. If $\{X_i\}$, $i \in I$ is a family of objects in $\mathcal{C}$, a coproduct of $\{X_i\}$ is an object $\coprod_{i\in I} X_i$ and a family of maps $\rho_i : X_i \to \coprod_i X_i$ such that for any other object Y and maps $f_i : X_i \to Y$, there is a unique map $\varphi : \coprod_i X_i \to Y$ such that

$$\begin{tikzcd} & Y \\ X_i \arrow[ur, "f_i"] \arrow[r, "\rho_i"'] & \coprod_{i\in I} X_i \arrow[u, "\varphi"'] \end{tikzcd}$$

commute for all $i \in I$.

Exercise Suppose X_1, X_2, X_3 are objects in a category $\mathcal{C}$. Assume that the coproducts

$$X_1 \sqcup X_2,\ (X_1 \sqcup X_2) \sqcup X_3 \text{ and } \coprod_{i=1,2,3} X_i$$

exist. Show that there is a unique isomorphism

$$(X_1 \sqcup X_2) \sqcup X_3 \simeq \coprod_{i=1,2,3} X_i$$

making the diagram

$$\begin{tikzcd} & (X_1 \sqcup X_2) \sqcup X_3 \arrow[d, leftrightarrow, "\simeq"] \\ X_i \arrow[ur] \arrow[r] & \coprod_{i=1,2,3} X_i \end{tikzcd}$$

commute.

Exercise Let P be a poset regarded as a category by setting

$$\mathrm{Hom}(a,b) = \begin{cases} \{*\} & \text{if } a \leq b \\ \phi & \text{otherwise} \end{cases}$$

(ie an arrow $a \to b$ whenever $a \leq b$).
Interpret the notion of a coproduct in the language of posets.

Proposition There is a natural bijection

$$\mathrm{Hom}(A \sqcup B, C) \simeq \mathrm{Hom}(A,C) \times \mathrm{Hom}(B,C)$$

proof: Suppose we have a pair of maps

$$A \xrightarrow{f} C \xleftarrow{g} B$$

then by the universal property, there is a unique map $\varphi : A \sqcup B \to C$. We denote $f \sqcup g := \varphi$.
Now we prove that

$$\begin{aligned} \mathrm{Hom}(A,C) \times \mathrm{Hom}(B,C) &\longrightarrow \mathrm{Hom}(A \sqcup B, C) \\ (f,g) &\longmapsto f \sqcup g \end{aligned}$$

is a bijection.
First injectivity. If $f_1 \sqcup g_1 = f_2 \sqcup g_2$, then we have, by commutativity of

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f_1"] \arrow[r, "\iota_A"'] & A \sqcup B \arrow[u, shift left, "f_1 \sqcup g_1"'] \arrow[u, shift right, "f_2 \sqcup g_2"] & A \arrow[ul, "f_2"'] \arrow[l, "\iota_A"] \end{tikzcd}$$

that

$$f_1 = f_1 \sqcup g_1 \circ \iota_A = f_2 \sqcup g_2 \circ \iota_A = f_2$$

and similarly $g_1 = g_2$.

Now surjectivity. Suppose $\varphi \in \mathrm{Hom}(A \sqcup B, C)$, then

$$\varphi = \varphi \circ \iota_A \sqcup \varphi \circ \iota_B . \qquad \square.$$

Lecture 1: Natural transformations.

Suppose we have two constructions we can apply to a vector space

$$V \longmapsto \hat{V}$$
$$V \longmapsto \mathring{V}$$

(eg we could have $\hat{V} = V^{\oplus 2}$ and $\mathring{V} = \mathrm{Hom}(k^2, V)$)

a natural map $\hat{V} \to \mathring{V}$ is a linear map whose definition does <u>not</u> depend on the specific vector space in question, but can be applied to all vector spaces uniformally.

For example, we have a natural map

$$\varphi_V : V^{\oplus 2} \longrightarrow \mathrm{Hom}(k^2, V)$$
$$(u,v) \longmapsto \left(\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \mapsto \lambda u + \mu v \right)$$

notice we didn't have to choose a basis or anything else specific to V.

We can interpret this categorically. A "construction" is a functor. A map between constructions is a morphism between images of the functors.

Suppose $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ are functors, then

a natural morphism should be a collection of morphisms
$$\alpha_A : F(A) \longrightarrow G(A),$$
one for each object $A \in \mathcal{C}$. But not just this! What does it mean for this family to not depend on choices?

Lets go back to the example of vector spaces. Essentially we want to avoid picking any specific representation of V. E.g. picking a basis.

Choosing two different bases gives an isomorphism
$$f : V \longrightarrow V.$$
Checking that our map φ_V is the same for these two different bases is the same as checking that

$$\begin{array}{ccc} V^{\oplus 2} & \xrightarrow{(f,f)} & V^{\oplus 2} \\ \downarrow{\scriptstyle \varphi_V} & & \downarrow{\scriptstyle \varphi_V} \\ \mathrm{Hom}(k^2, V) & \xrightarrow{\hat{f}} & \mathrm{Hom}(k^2, V) \\ & m \longmapsto \left(\begin{pmatrix} a \\ b \end{pmatrix} \mapsto f \circ m \right) & \end{array}$$

commutes. We can generalise this notion.

Def Suppose $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ are two functors. A natural transformation $\alpha : F \Longrightarrow G$ is a collection of morphisms (in $\mathcal{D}$)

$$\{\alpha_A : F(A) \longrightarrow G(A)\}_{A \in \mathcal{C}}$$

for each object in $\mathcal{C}$, such that, for each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ in $\mathcal{C}$,

$$\begin{tikzcd} F(A) \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "Gf"] & G(B) \end{tikzcd}$$

commutes.

We say α is a natural isomorphism if each α_A is an isomorphism.

Exercise Define two functors $\mathcal{P}, 2^{-} : \mathrm{Set} \longrightarrow \mathrm{Set}$

$\mathcal{P}(X)$ = the power set

$2^X = \mathrm{Fun}(X, \{0\ 1\})$ functions $X \to \{0\ 1\}$.

Complete the definition of these functors and show they are naturally isomorphic.

Exercise Let $\mathrm{Nat}(F,G)$ be all natural transformations. Show that the set of objects functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ with morphisms natural transformations is a category.

Monoidal Categories

Loosely, a monoidal category is a category that has a notion of tensor product, modelled on the properties of the tensor product of vector spaces. The properties we will axiomatise are

- $U \otimes V$ is functorial. I.e. linear maps $f : U \to U'$ and $g : V \to V'$ give a map $f \otimes g : U \otimes V \longrightarrow U' \otimes V'$.
- $(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$ in a natural way
- $\Bbbk \otimes V \simeq V \simeq V \otimes \Bbbk$, in a natural way.

Def A monoidal category is a tuple
$$(\mathcal{C}, \otimes, a, \mathbb{1}, \ell, r)$$

where

- $\mathcal{C}$ is a category
- $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a functor (the tensor product)
- a is a natural isomorphism
$$\otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \Longrightarrow \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)$$
of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ (the associator)
- $\mathbb{1}$ is an object of $\mathcal{C}$ (the identity)
- ℓ and r are natural isomorphisms
$$\mathbb{1} \otimes - \overset{\ell}{\Longrightarrow} \mathrm{id}_{\mathcal{C}} \qquad - \otimes \mathbb{1} \overset{r}{\Longrightarrow} \mathrm{id}_{\mathcal{C}}$$

such that

(i) the following diagram commutes for all objects $A, B, C, D \in \mathcal{C}$:

(pentagon axiom)

$$
\begin{tikzcd}
 & ((A\otimes B)\otimes C)\otimes D \arrow[dl, "a_{A,B,C}\otimes \mathrm{id}_D"'] \arrow[dr, "a_{A\otimes B,C,D}"] & \\
(A\otimes(B\otimes C))\otimes D \arrow[d, "a_{A,B\otimes C,D}"] & & (A\otimes B)\otimes(C\otimes D) \arrow[d, "a_{A,B,C\otimes D}"] \\
A\otimes((B\otimes C)\otimes D) \arrow[rr, "\mathrm{id}_A\otimes a_{B,C,D}"'] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd}
$$

(ii) the following diagram commute for all objects $A, B \in \mathcal{C}$:

(triangle axiom)

$$
\begin{tikzcd}
(A\otimes \mathbb{1})\otimes B \arrow[rr, "a_{A,\mathbb{1},B}"] \arrow[dr, "r_A\otimes \mathrm{id}_B"'] & & A\otimes(\mathbb{1}\otimes B) \arrow[dl, "\mathrm{id}_A\otimes l_B"] \\
 & A\otimes B &
\end{tikzcd}
$$

Lecture 2 Examples

Ex 1 The prototypical example is of course $\mathrm{Vect}_{\mathbb{k}}$

- $U \otimes V$ is the usual tensor product.
- $a_{U,V,W}((u \otimes v) \otimes w) := u \otimes (v \otimes w)$ extended linearly.
- $\mathbb{1} := \mathbb{k}$
- $\ell_V(\lambda \otimes v) = \lambda v \quad r_V(v \otimes \lambda) = \lambda v.$

Ex 2 In fact $\mathrm{Vect}_{\mathbb{k}}$ has another monoidal structure: $(\mathrm{Vect}_{\mathbb{k}}, \oplus, \mathrm{id}, \{0\}, \ell, r)$.

Ex 3 Set has a monoidal structure given by disjoint union.

- associator is the identity
- $\mathbb{1} = \phi$
- ℓ, r are the identity.

Exercise Determine another monoidal structure on Set.

Ex 4 Let G be a finite group. Define $\mathrm{Vect}_{k}(G)$ to be the category of G-graded vector spaces. I.e.:

objects: vector spaces with a decomposition $V = \bigoplus_{g \in G} V_g$

morphisms: linear maps preserving the grading (ie $\varphi(V_g) \subseteq W_g$ if $\varphi: V \longrightarrow W$).

We can define a G-grading on the usual tensor product $V \otimes W$ by

$$(V \otimes W)_g := \bigoplus_{\substack{x,y \in G \\ xy = g}} V_x \otimes W_y .$$

This is functorial because for any morphisms

$$\varphi : V \to V' \ , \ \psi : W \to W'$$

the linear map $\varphi \otimes \psi$ preserves the grading, i.e.

$$\varphi \otimes \psi (V_x \otimes W_y) \subseteq V'_x \otimes W'_y$$

and so

$$\varphi \otimes \psi ((V \otimes W)_g) \subseteq (V' \otimes W')_g .$$

The associator from $\mathrm{Vect}_{\Bbbk}$ also works in $\mathrm{Vect}_{\Bbbk}(G)$ because

$$a_{U,V,W}((U_x \otimes V_y) \otimes W_z) \subseteq U_x \otimes (V_y \otimes W_z)$$

and so

$$a_{U,V,W}(((U \otimes V) \otimes W)_{(xy)z}) \subseteq (U \otimes (V \otimes W))_{x(yz)}$$

$$= (U \otimes (V \otimes W))_{xyz}$$

The identity object (as a v.sp) should clearly be $\mathbb{1}=\mathbb{k}$. But what should the grading be? We must have

$$\mathbb{1}_g = \begin{cases} \mathbb{k} & \text{for } g=p \\ 0 & \text{otherwise} \end{cases}$$

for some fixed $p \in G$. Thus

$$V_g \simeq (\mathbb{1} \otimes V)_g = \bigoplus_{xy=g} \mathbb{1}_x \otimes V_y$$

$$= \mathbb{1}_p \otimes V_{p^{-1}g}$$

$$= V_{p^{-1}g}$$

So we must have $g = p^{-1}g$ ie $p=1$. Now we can use the same l, r as for $\mathrm{Vect}_{\mathbb{k}}$.

Ex 5 Let R be a ring. An (R,R)-bimodule is an abelian group M, with the structure of both a left and right R-module such that

$$(a \cdot m) \cdot b = a \cdot (m \cdot b) \quad \text{for } m \in M,\ a, b \in R.$$

Tensoring over R gives a monoidal structure.

Ex 6 The category of endofunctors $\mathrm{End}(\mathcal{C})$ is monoidal with composition.

Ex 7 Define SVect to be the category of super-vector spaces, that is, vector spaces with a decomposition

$$V = V_0 \oplus V_1$$

we say V_0 is the even part, V_1 is the odd part. Morphisms are grading preserving linear maps.

Tensor product is defined as follows:

$V \otimes W$ is the usual tensor product

$$(V \otimes W)_0 := (V_0 \otimes V_0) \oplus (V_1 \otimes V_1)$$

$$(V \otimes W)_1 := (V_0 \otimes V_1) \oplus (V_1 \otimes V_0)$$

ie v is homogeneous.

If $v \in V_i$ we use the notation $|v| := i$. We can choose the following associator:

$$\alpha_{UVW} : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$$

$$a_{UVW}((u \otimes v) \otimes w) := (-1)^{|u|+|w|}\, u \otimes (v \otimes w)$$

for u, v, w homogeneous.

important: preserves the grading!

$$
\begin{tikzcd}[column sep=small]
& ((uv)w)x \arrow[dl, "a_{uvw}\otimes id"'] \arrow[dr, "a_{u\otimes v, w, x}"] & \\
(-1)^{|u|+|w|}(u(vw))x \arrow[d, "a_{u\, v\otimes w\, x}"'] & & (-1)^{|u|+|v|+|x|}(uv)(wx) \arrow[d, "a_{u\, v\, w\otimes x}"] \\
(-1)^{|u|+|w|}(-1)^{|u|+|x|}u((vw)x) \arrow[dr, "id_u\otimes a_{vwx}"'] & & (-1)^{|u|+|v|+|x|}(-1)^{|u|+|w|+|x|}u(v(wx)) \arrow[dl, phantom, "\shortparallel"] \\
& (-1)^{|u|+|w|}(-1)^{|u|+|x|}(-1)^{|v|+|w|}u(v(wx)) &
\end{tikzcd}
$$

What should the unit be? 1-dim obviously, so either $\mathbb{1} = k_0$ or $\mathbb{1} = k_1$. Note that

$$k_1 \otimes k_1 = k_0 \quad \text{, so we must have } \mathbb{1} = k_0$$

Now we need to determine the morphisms l, r!
We need maps l_V, r_V such that

$$
\begin{tikzcd}
(U\otimes \mathbb{1})\otimes V \arrow[rr, "a_{U\mathbb{1}V}"] \arrow[dr, "r_U\otimes id_V"'] & & U\otimes(\mathbb{1}\otimes V) \arrow[dl, "id_U\otimes l_V"] \\
& U\otimes V &
\end{tikzcd}
$$

Translating: if $u \in U$, $v \in V$ homogeneous,

$$r_U(u \otimes \lambda) \otimes v = (-1)^{|u|+|v|} u \otimes l_V(\lambda \otimes v)$$

So it looks like

$$r_U(u \otimes \lambda) = (-1)^{|u|} \lambda u \qquad l_V(\lambda \otimes v) = (-1)^{|v|} \lambda v$$

would work! Again, it is important these are actually morphisms in $\mathrm{SVect}_{\mathbb{k}}$.

Braided monoidal categories

We have ignored a key property of $\otimes$ in $\mathrm{Vect}_{\Bbbk}$. Namely that $v \otimes w \mapsto w \otimes v$ gives a natural isomorphism

$$V \otimes W \simeq W \otimes V.$$

Remark Such a structure does not exist in all cases. Take for example $\mathrm{Vect}_{\Bbbk}(G)$ and consider

$$\Bbbk_g \otimes \Bbbk_h \simeq \Bbbk_{gh} \qquad \Bbbk_h \otimes \Bbbk_g \simeq \Bbbk_{hg}$$

If $gh \neq hg$ in G then there are no grading preserving linear maps between them!

Again, we will not simply require that $A \otimes B \simeq B \otimes A$ but instead specify how they are isomorphic.

Def A braided monoidal category consists of a monoidal category $\mathcal{C}$ and a natural isomorphism

$$c : \otimes \Rightarrow \otimes \circ \tau : \mathcal{C} \times \mathcal{C} \Rightarrow \mathcal{C}$$

Here $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the "flip" functor.

$$\tau(A,B) = (B,A), \quad \tau(f,g) = (g,f).$$

The braiding maps c_{AB} are required to satisfy the following two commutative diagrams:

(hexagon axiom)

$$
\begin{tikzcd}
 & (B\otimes A)\otimes C \arrow[r, "a_{B,A,C}"] & B\otimes(A\otimes C) \arrow[dr, "id_B\otimes c_{A,C}"] & \\
(A\otimes B)\otimes C \arrow[ur, "c_{A,B}\otimes id_C"] \arrow[dr, "a_{A,B,C}"'] & & & B\otimes(C\otimes A) \\
 & A\otimes(B\otimes C) \arrow[r, "c_{A,B\otimes C}"'] & (B\otimes C)\otimes A \arrow[ur, "a_{B,C,A}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[r, "c_{A\otimes B,C}"] & C\otimes(A\otimes B) \arrow[dr, "a^{-1}_{C,A,B}"] & \\
A\otimes(B\otimes C) \arrow[ur, "a^{-1}_{A,B,C}"] \arrow[dr, "id_A\otimes c_{B,C}"'] & & & (C\otimes A)\otimes B \\
 & A\otimes(C\otimes B) \arrow[r, "a^{-1}_{A,C,B}"'] & (A\otimes C)\otimes B \arrow[ur, "c_{A,C}\otimes id_B"'] &
\end{tikzcd}
$$

Example The category Vect_k is braided with braiding given by $\mathrm{flip}: V \otimes W \longrightarrow W \otimes V$

Similarly Set with the monoidal structure given by Cartesian product is braided. These two examples are special, in the sense that doing the braiding twice is the identity

Def If $\mathcal{C}$ is a braided monoidal category with braiding c, we say $\mathcal{C}$ is symmetric if

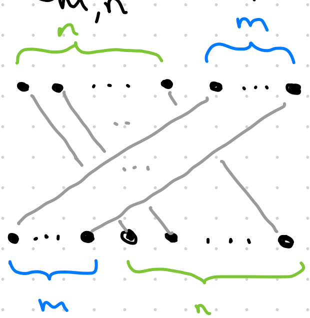
$$c_{XY}^{-1} = c_{YX} \quad \text{or equivalently if} \quad c_{YX} \circ c_{XY} = \mathrm{id}_{X \otimes Y}$$

for any two objects $X, Y \in \mathcal{C}$.

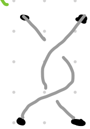
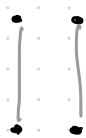
Not all braided categories are symmetric!

Example The category of tangles, Tang, is braided by using the braiding

$$c_{m,n} : m+n \longrightarrow n+m$$



(exercise: check this gives a natural isomorphism)

clearly $c_{1,1} \circ c_{1,1} = c_{1,1}^2 =$  $\neq$  $= \mathrm{id}_{1+1}$

Why consider braided monoidal categories instead of simply symmetric ones?

Yang-Baxter equation

The Yang-Baxter equation is an equation that asks for an operator

$$R \in \mathrm{End}(V \otimes V) \quad , \quad V \in \mathrm{Vect}.$$

to satisfy the equation

$$R_{23} R_{13} R_{12} = R_{12} R_{13} R_{23}$$

where $R_{ij} : V \otimes V \otimes V \longrightarrow V \otimes V \otimes V$ is the operator obtained from R by acting on the i^{th} and j^{th} tensor factors.
i.e. $R_{23} = \mathrm{id}_V \otimes R$. $R_{12} = R \otimes \mathrm{id}_V$, $R_{13} = \tau_{12} \circ R_{23} \circ \tau_{12}$ ← flip

In fact, we can try and make sense of this in any monoidal category by noticing the YBE has an equivalent form:

Let $Q = \tau \circ R$. Then the YBE is equivalent to

$$(Q \otimes \mathrm{id}) \circ (\mathrm{id} \otimes Q) \circ (Q \otimes \mathrm{id}) = (\mathrm{id} \otimes Q) \circ (Q \otimes \mathrm{id}) \circ (\mathrm{id} \otimes Q)$$

The YBE originated in statistical physics. Operators of the form R appear in statistical mechanical systems and if they satisfy the YBE then the system is integrable, Hence there was a large amount of interest in finding solutions to the YBE.

In fact, braided monoidal categories give a procedure for producing a large number of solutions:

Proposition If $\mathcal{C}$ is a braided monoidal cat, with braiding c (and trivial associator) then the braiding c_{XX} satisfies the XBE, or more generally, for any 3 objects $X, Y, Z \in \mathcal{C}$:

$$(c_{YZ} \otimes id_X) \circ (id_Y \otimes c_{XZ}) \circ (c_{XY} \otimes id_Z)$$

$$= (id_Z \otimes c_{XY}) \circ (c_{XZ} \otimes id_Y) \circ (id_X \otimes c_{YZ})$$

proof: We want to show the following hexagon commutes

$$\begin{tikzcd}
 & Y \otimes X \otimes Z \arrow[rr, "id_Y \otimes c_{XZ}"] \arrow[drrr, "c_{Y \otimes X, Z}"'] \arrow[dd, phantom, "3"] & & Y \otimes Z \otimes X \arrow[dr, "c_{YZ} \otimes id_X"] \arrow[d, phantom, "2"] & \\
X \otimes Y \otimes Z \arrow[ur, "c_{XY} \otimes id_Z"] \arrow[dr, "id_X \otimes c_{YZ}"'] \arrow[drrr, "c_{X \otimes Y, Z}"] & & & {} & Z \otimes Y \otimes X \\
 & X \otimes Z \otimes Y \arrow[rr, "c_{XZ} \otimes id_Y"'] \arrow[ur, phantom, "1"] & & Z \otimes X \otimes Y \arrow[ur, "id_Z \otimes c_{XY}"'] &
\end{tikzcd}$$

Triangles 1 and 2 commute as these are exactly the hexagon axioms once we forget about associators.

Square 3 commutes since $c : \otimes \Rightarrow \otimes \circ \underset{\text{flip functor}}{\tau}$ is a natural transformation and so the square

$$\begin{tikzcd} A\otimes B \arrow[r, "f\otimes g"] \arrow[d, "c_{A,B}"'] & A'\otimes B' \arrow[d, "c_{A',B'}"] \\ B\otimes A \arrow[r, "g\otimes f"] & B'\otimes A' \end{tikzcd}$$

commutes, in particular, for

$A = X\otimes Y$

$B = Y\otimes X$

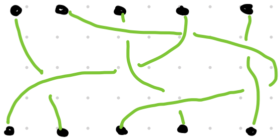
$f = c_{XY}$

$g = id_Z$ $\square$.

Apart from showing that braided monoidal categories give us solutions to the YBE, it also generates actions of the braid group:

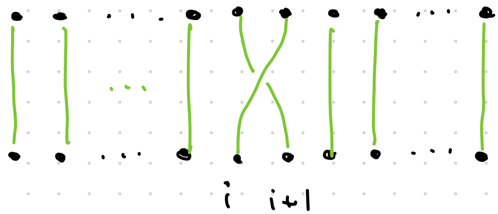
Braid groups

Def The braid group on n strands, B_n, is the group whose elements are isotopy classes of braid diagrams:

eg: 

with multiplication given by vertical stacking.

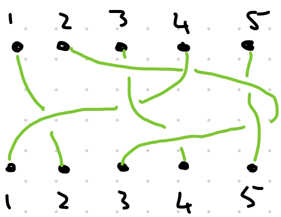
More formally, a braid diagram is one that can be isotoped to a vertical stacking of the diagrams

$\sigma_i =$ 

Theorem B_n has presentation

$$B_n = \left\langle \sigma_i \, , i=1\dots n-1 \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \end{array}\right\rangle$$

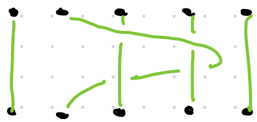
Notice that there is a natural map $\pi: B_n \longrightarrow S_n$ that is given by reading off the permutation given by the strands

eg 

$\longmapsto (1432)$

ie by sending $\sigma_i \mapsto s_i = (i, i+1) \in S_n$. This is a surjective map and if we define the pure braid group as

$$PB_n = \ker \pi = \left\{ \begin{array}{l} \text{diagrams whose strands match up} \\ \text{dots in order} \end{array} \right\}.$$

e.g.  $\in PB_5$

then there is an exact sequence

$$1 \longrightarrow PB_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 1.$$

Remark If you know some topology this comes from the following fact:
Let $\mathbb{C}^n_{reg} := \{ z \in \mathbb{C}^n \mid z_i \neq z_j \}$ then $S_n \curvearrowright \mathbb{C}^n_{reg}$ freely and

$$PB_n \simeq \pi_1(\mathbb{C}^n_{reg})$$

$$B_n \simeq \pi_1(\mathbb{C}^n_{reg}/S_n)$$

In general, if a finite group acts freely on a top space X, there is an exact sequence

$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/G) \longrightarrow G \longrightarrow 1$$

Lecture 5 Motivating examples

We will first consider the category $\tilde{\mathcal{D}}_1$ consisting of

objects: (V, e, f, h) V a finite dimensional vector space$/\mathbb{C}$
$e, h, f \in \mathrm{End}(V)$ s.t.
$$[h\, e] = 2e, \quad [h\, f] = -2f \quad [e\, f] = h.$$

morphisms: $\mathrm{Hom}_{\tilde{\mathcal{D}}_1}(V, W) =$ linear map $\varphi : V \longrightarrow W$
such that
$$\varphi \circ e_V = e_W \circ \varphi$$
$$\varphi \circ f_V = e_W \circ \varphi$$
$$\varphi \circ h_V = e_W \circ \varphi$$

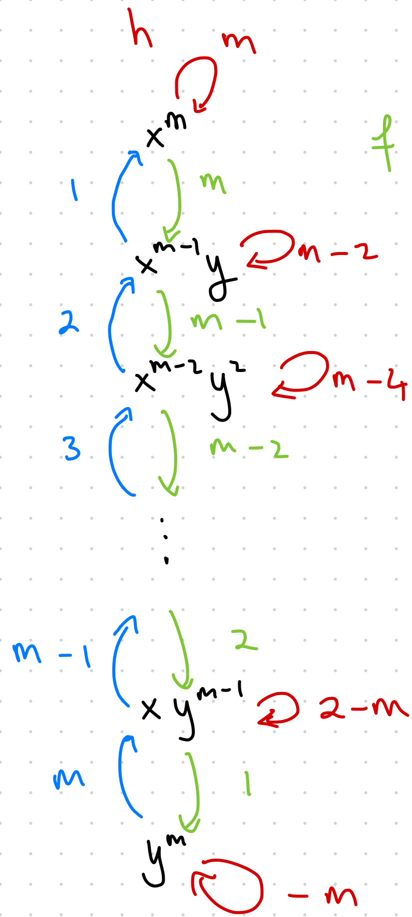
Remark This is the category of sl_2-representations where sl_2 is the Lie algebra of 2×2 traceless matrices. The following shows that it also has something to do with differential operators on $\mathbb{P}^1$.

Example Consider the (infinite dim) vector space $\mathbb{C}[x\, y]$ of polynomials in two variables and the operators
$$e := x\partial_y \quad f := y\partial_x \quad h := x\partial_x - y\partial_y$$
These operators satisfy the necessary relations! (check).
Let
$$V(m) := \mathbb{C}[x\, y]_m := \mathrm{span}\{x^n, x^{n-1}y, \dots, y^n\}$$
ie the subspace of homogeneous degree n polynomials.
The operators e, f, h preserve each $V(m)$ so

we get that $(V(m), e, f, h) \in \tilde{\mathcal{D}}_1$.

Lemma The operators e, f are nilpotent and h is diagonalisable on $V(m)$

proof: The operators e, f, h can be depicted as



ie $e^{m+1} = f^{m+1} = 0$ and $\{x^a y^b\}$ is an eigenbasis for h.

For two objects $V, W \in \tilde{\mathcal{D}}$, we can form the direct sum $V \oplus W$ with $e_{V\oplus W} = \left(\begin{array}{c|c} e_V & 0 \\ \hline 0 & e_W \end{array}\right)$.
and so on.

Def $\mathcal{D}_1 \subseteq \tilde{\mathcal{D}}_1$ is the subcategory of objects isomorphic to a direct sum of the objects $V(m)$ ie $V \in \mathcal{D}_1$ if there exist integers a_m such that

$$V \simeq \bigoplus_m V(m)^{\oplus a_m}$$

Remark In fact $\mathcal{D}_1 = \tilde{\mathcal{D}}_1$. This isn't a particularly difficult fact but it would take us more time than I'd like to spend on the issue, so I define our way out of the problem.

Prop $\mathrm{Hom}(V(m), V(n)) = \begin{cases} \mathbb{C}\mathrm{id} & \text{if } m=n \\ 0 & \text{o/w}. \end{cases}$

proof: Let $\varphi : V(m) \longrightarrow V(n)$ be a morphism and set $v := \varphi(x^m)$. Note

$$e(v) = e \circ \varphi(x^m) = \varphi \circ e(x^m) = \varphi(0) = 0$$

Thus $v = \lambda x^n$ for some $\lambda \in \mathbb{C}$. Now note

$$h(v) = h \circ \varphi(x^m) = \varphi \circ h(x^m) = \varphi(m x^m) = mv$$

But $h(\lambda x^n) = \lambda n x^n$ so we must have either

$$\lambda = 0 \text{ or } n = m.$$

If $\lambda = 0$ then

$$\varphi(x^{m-a}y^a) = \varphi\left(\frac{(m-a)!}{m!} f^a(x^m)\right)$$

$$= \frac{(m-a)!}{m!} \varphi \circ f^a(x^m)$$

$$= \frac{(m-a)!}{m!} f^a \circ \varphi(x^m) = 0$$

so $\varphi = 0$.

On the other hand, if $m = n$ then by a similar argument as above,

$$\varphi(x^{m-a}y^a) = \varphi\left(\frac{(m-a)!}{m!} f^a(x^m)\right)$$

$$= \frac{(m-a)!}{m!} f^a \circ \varphi(x^m)$$

$$= \frac{(m-a)!}{m!} f^a(\lambda x^m)$$

$$= \lambda x^{m-a} y^a$$

so $\varphi = \lambda \cdot \mathrm{id}$. $\square$.

We can put a monoidal structure on $\mathcal{D}_1$:

$V \otimes W$ = usual tensor of vector spaces.

$$e_{V\otimes W} = e_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes e_W$$

$$f_{V\otimes W} = f_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes f_W$$

$$h_{V\otimes W} = h_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes h_W$$

with $V(0)$ as the identity object and trivial associator

Example Consider $V(1)\otimes V(1) =: W$. This has a basis

$$\begin{tikzcd}[row sep=large]
x\otimes x \arrow[loop right, "2"] \arrow[d, bend right, "e:\ 1"'] \\
y\otimes x + x\otimes y \arrow[u, bend right, "f:\ 2"'] \arrow[loop right, "0"] \arrow[d, bend right, "2"'] \\
y\otimes y \arrow[loop right, "-2"] \arrow[u, bend right, "1"']
\end{tikzcd}
\qquad\qquad
y\otimes x - x\otimes y \ \circlearrowleft 0$$

So $V(1)\otimes V(1) \simeq V(2)\oplus V(0)$

Remark It is clear that $V\otimes W \in \tilde{\mathcal{D}}_1$ (once you check the relations) but it is a priori not clear why $V\otimes W \in \mathcal{D}_1$. You will check this on the problem set.

Prop The natural transformation $c_{VW} = \text{flip}$ defines a braiding for $\mathcal{D}_1$.

proof: What do we need to prove?
- that it is a natural transformation (square commute)
- hexagon axioms commute (immediate since we know it for Vect).

Recall c_{VW} must be morphisms in $\mathcal{D}_1$!

$$c_{VW} \circ e_{V \otimes W}(v \otimes w) = c_{VW}(e.v \otimes w + v \otimes e.w)$$

$$= w \otimes e.v + e.w \otimes v$$

$$= (id_W \otimes e_V + e_W \otimes id_V)(w \otimes v)$$

$$= e_{W \otimes V} \circ c_{VW}(v \otimes w) \qquad \square$$

Cor In fact $\mathcal{D}_1$ is a symmetric monoidal category. $B_n \curvearrowright V^{\otimes n}$ is simply permutation of tensor factors.

Remark One way to get a more interesting braiding is to define the associator differently (using the monodromy of KZ equation - a differential equation on

$$\mathbb{C}^n_{reg} := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\}.$$

in fact $B_n \cong \pi_1(\mathbb{C}^n_{reg}/S_n)$

A powerful philosophy in modern algebra:
"Deform/quantise to reveal hidden structure"

Lets define $\widetilde{D}_q$ (q is formal variable)

objects: (V, e, f, K) where

rational functions

- V is a fin. dim. v.sp over $\mathbb{C}(q)$
- $e, f, K \in \mathrm{End}_{\mathbb{C}(q)}(V)$ such that

(think $K = q^h$)

K is invertible

$$KeK^{-1} = q^2 e \qquad KfK^{-1} = q^{-2} f$$

$$[e,f] = \frac{K - K^{-1}}{q - q^{-1}} \text{ "} = \frac{q^h - q^{-h}}{q - q^{-1}} = [h]_q \text{"}$$

Def The q-analogues of integers are defined as the rational functions:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{q^{-n}}{q^{-1}}\left(\frac{q^{2n} - 1}{q^2 - 1}\right) = q^{1-n}\left(q^{2n-2} + q^{2n-4} + \dots + 1\right)$$

$$= \underbrace{q^{n-1} + q^{n-3} + \dots + q^{1-n}}_{n}$$

so $\lim_{q \to 1} [n]_q = n$.

we also define the factorials

$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q$$

Remark $\hat{\mathcal{D}}_q$ is in fact representations of $U_q(sl_2)$ the quantum group for sl_2.

Example Consider the vector space $\mathbb{C}(q)[x,y]$, with operators

no q-derivative

$$e := x\partial_y^q \qquad f := y\partial_x^q \qquad K := q^{x\partial_x - y\partial_y}$$

where $\partial_x^q(x^n y^m) = [n]_q x^{n-1} y^m$

$$\partial_y^q(x^n y^m) = [m]_q x^n y^{m-1}$$

How do we interpret K?

$$K(x^n y^m) = q^{x\partial_x - y\partial_y}(x^n y^m) := q^{n-m} x^n y^m$$

We can check the relations:

$$KeK^{-1}(x^a y^b) = q^{b-a} Ke(x^a y^b)$$

$$= q^{b-a}[b]_q K(x^{a+1} y^{b-1})$$

$$= q^{b-a}[b]_q q^{a-b+2} x^{a+1} y^{b-1}$$

$$= q^2 e(x^a y^b)$$

similarly for the other two.

Observation e, f, K can be restricted to

$$V_q(m) := \mathbb{C}(q)[x\, y]_m = \text{homogeneous of deg. } m \text{ poly's.}$$

Def $\mathcal{D}_q$ is the subcategory of $\tilde{\mathcal{D}}_q$ consisting of all objects W, isomorphic to a direct sum of the $V_q(m)$.

Remark Actually $\mathcal{D}_q$ is almost all of $\tilde{\mathcal{D}}_q$. It is the so called type 1 representations.

Exercises Repeat the structure analysis of $V_q(m)$ as for $V(m)$, calculate

$$\mathrm{Hom}(V_q(m), V_q(n)).$$

We can define a monoidal structure on $\mathcal{D}_q$:

$V \otimes W$ = the usual tensor product of v.sp's.

$$e_{V \otimes W} = e_V \otimes id_W + K_V \otimes e_W$$

$$f_{V \otimes W} = f_V \otimes K_W + id_V \otimes f_W$$

$$K_{V \otimes W} = K_V \otimes K_W$$

<u>Note</u>: We've broken a symmetry with respect to flipping tensor products.

Solution to 7a:

We would like to show that in a braided monoidal category $\mathcal{C}$, the following diagram commutes

$$\begin{tikzcd}
A1 \arrow[rr, "c_{A,1}"] \arrow[dr, "r_A"'] & & 1A \arrow[dl, "l_A"] \\
& A &
\end{tikzcd} \qquad (i)$$

We will suppress the tensor symbol completely, and sometimes the subscript on natural transformations. Consider the diagram:

$$\begin{tikzcd}
& A(11) \arrow[rr, "c"] \arrow[d, "il"] \arrow[drr, phantom, "2"] & & (11)A \arrow[dl, "li"'] \arrow[dr, "a"] \arrow[d, phantom, "4"] & \\
(A1)1 \arrow[ur, "a"] \arrow[r, "ri"] \arrow[dr, "ci"'] \arrow[ur, phantom, "1"'] \arrow[r, phantom, "*"', shift right=4] & A1 \arrow[r, "c"] & 1A \arrow[d, phantom, "3"] & & 1(1A) \arrow[ll, "l"'] \\
& (1A)1 \arrow[u, "li"] \arrow[rr, "a"'] \arrow[r, phantom, "4"] & & 1(A1) \arrow[ull, "l"] \arrow[ur, "ic"'] &
\end{tikzcd} \qquad (ii)$$

We want to prove $*$ commutes as this is simply (i) tensored by $\mathbb{1}$ on the right. So if $*$ commutes so does (i) by the following argument:

$$
\begin{tikzcd}
A1 \arrow[rrrr, "c"] \arrow[dddrr, "r"'] & & & & 1A \arrow[dddll, "\ell"] \\
& (A1)1 \arrow[ul, "r"] \arrow[rr, "ci"] \arrow[dr, "ri"] & & (1A)1 \arrow[ur, "r"'] \arrow[dl, "\ell i"'] & \\
& & A1 \arrow[d, "r"] & & \\
& & A & &
\end{tikzcd}
$$

The inner diagram is what we assumed commutes, and the outer squares commute by naturality of r. ie take $X = A1$ and $f = r_A$, ℓ_A or $c_{A,1}$ in:

$$
\begin{tikzcd}
X1 \arrow[r, "fi"] \arrow[d, "r"'] & X1 \arrow[d, "r"] \\
X \arrow[r, "f"] & X
\end{tikzcd}
$$

Thus the outer triangle commutes.

Now we come back to (ii). To show $*$ commutes, note that the outer hexagon axiom commutes by the hex axiom. 1 commutes by the triangle axiom. 2 commutes by the naturality of c: ie take $X = A$, $Y = 11$, $S = A$, $T = 1$ and $f = i$ and $g = \ell_1$ in the following.

$$\begin{tikzcd} XY \arrow[r, "fg"] \arrow[d, "c"'] & ST \arrow[d, "c"] \\ YX \arrow[r, "gf"] & TS \end{tikzcd}$$

③ commutes by the naturality of ℓ. ie take $X = A1$, $Y = 1A$, $f = c_{A,1}$ in

$$\begin{tikzcd} 1X \arrow[r, "if"] \arrow[d, "\ell"'] & 1Y \arrow[d, "\ell"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

We are left with only having to show the commutivity of the two triangles ④. These follow from a more general triangle:

$$\begin{tikzcd} (1X)Y \arrow[rr, "a"] \arrow[dr, "\ell i"'] & & 1(XY) \arrow[dl, "\ell"] \\ & XY & \end{tikzcd}$$

(iii)

One can show the commutativity of this triangle using the pentagon axiom (which is what you should expect given it only involves a, ℓ).

$$
\begin{tikzcd}
 & & ((11)X)Y \arrow[dll, "ai"'] \arrow[dl, "(ri)i"] \arrow[drr, "a"] & & \\
(1(1X))Y \arrow[r, "(i\ell)i"'] \arrow[ddr, "a"'] & (1X)Y \arrow[drr, "a"] & & & (11)(XY) \arrow[dl, "ri"'] \arrow[dd, "a"] \\
 & & & 1(XY) & \\
 & 1((1X)Y) \arrow[uurr, "i(\ell i)"] \arrow[rrr, "ia"'] & & & 1(1(XY)) \arrow[ul, "i\ell"']
\end{tikzcd}
$$

1 commute by the triangle axiom, while 2 commute by naturality of a. This forces ☆ to commute which is (iii) tensored by 1. By a similar argument to the above, (iii) must commute and the proof is complete.

Lecture 7 A braided and a not so braided category.

Observation The map $c = \text{flip}$ does not give a braiding on $\mathcal{D}_q$. Indeed on $V_q(1) \otimes V_q(1)$:

$$e \circ \text{flip}(x \otimes y) = e(y \otimes x)$$
$$= e \cdot y \otimes x + K \cdot y \otimes e \cdot x$$
$$= x \otimes x.$$

$$\text{flip} \circ e(x \otimes y) = \text{flip}(e \cdot x \otimes y + K \cdot x \otimes e \cdot y)$$
$$= \text{flip}(q x \otimes x)$$
$$= q x \otimes x.$$

We need to "deform" our flip map. Consider the operator

ie $q^{\frac{1}{2}h\otimes h}(v \otimes w) = q^{\frac{ab}{2}}$ if $Kv = q^a$, $Kw = q^b$.

$$R_{V,W} := q^{\frac{1}{2}h \otimes h} \circ \sum_{i=0}^{\infty} q^{\binom{i}{2}} \frac{(q-q^{-1})^i}{[i]_q!} e_V^i \otimes f_W^i$$

Note: the operators e and f are nilpotent so the sum is finite! Here $q^{\frac{1}{2}h\otimes h}(u \otimes v) = q^{\frac{1}{2}mn} u \otimes v$ if $Ku = q^m u$ and $Kv = q^n v$.

Example Let $V = V_q(1)$. $R_{VV} = q^{\frac{1}{2}h\otimes h} \circ \left(1 + (q-q^{-1}) e \otimes f\right)$

$R_{VV}(x \otimes x) = q^{\frac{1}{2}} \cdot x \otimes x$ $\qquad$ $R_{VV}(y \otimes x) = q^{-\frac{1}{2}}\left(y \otimes x + (q-q^{-1}) x \otimes y\right)$

$R_{VV}(x \otimes y) = q^{-\frac{1}{2}} x \otimes y$ $\qquad$ $R_{VV}(y \otimes y) = q^{\frac{1}{2}} y \otimes y$

$$R_{VV} = q^{-\frac{1}{2}} \begin{pmatrix} q & & & \\ & 1 & q-q^{-1} & \\ & & 1 & \\ & & & q \end{pmatrix}.$$

Exercise: Check that is commutes with e, f, K. Check this satisfies the hexagon axiom on $V \otimes V \otimes V$.

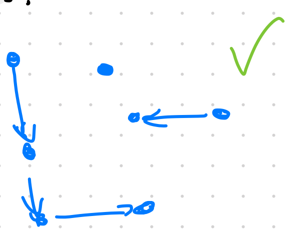
Thm The map $c_{VW} := R_{WV} \circ \text{flip}$ defines a braiding on $\mathcal{D}_q$ that is not symmetric.

proof: We wont give a proof here but it is entirely possible to varify the hexagon axioms explicitly. It should be clear that $c \neq \text{flip}$ unless $R = \text{id}$.

Remark This is a deformation of $\mathcal{D}_1$ in the sense that as $q \to 1$ (interpreted appropriately) we get $R_{VW} \to \text{id}$ so $c_{VW} \to \text{flip}$.

One more monoidal category

We will consider directed graphs, with no loops, and with every vertex having at most one incoming and at most one outgoing edge. We will call these **line graphs**.



If $v \in L$ is a vertex of a line graph we write fv for the neighbouring vertex in positive direction and $f^{-1}v$ for the neighbouring vertex in negative direction

$$\begin{tikzcd} \underset{f^{-1}v}{\bullet} \arrow[r] & \underset{v}{\bullet} \arrow[r] & \underset{fv}{\bullet} \end{tikzcd}$$

We let $fv = 0$ or $f^{-1}v = 0$ if they don't exist.

Define $\quad \varphi(v) := \max \{ a \mid f^a v \neq 0 \}$.

$\quad \varepsilon(v) := \max \{ a \mid f^{-a} v \neq 0 \}$.

Example

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \underset{v}{\bullet} \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

$\varphi(v) = 2 \quad \varepsilon(v) = 4$

Prop For any vertex v, $\varphi(v) + \varepsilon(v) + 1 = \#$ of vertices in the connected component of v.

Let $\mathcal{D}_0$ be the category with:

objects: line graphs with finitely many vertices

morphisms: maps of directed graps $\alpha : L \to M$ such that $\varphi(\alpha v) = \varphi(v)$

Example Let $B(m)$ be the unique connected line graph with $m+1$ vertices

$$\begin{tikzcd} \underset{b_m}{\bullet} \arrow[r] & \underset{fb_m}{\bullet} \arrow[r] & \underset{f^2 b_m}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{f^{m-1} b_m}{\bullet} \arrow[r] & \underset{f^m b_m}{\bullet} \end{tikzcd}$$

Prop $\mathrm{Hom}(B(m), B(n)) = \begin{cases} \{id\} & \text{if } m=n \\ \phi & \text{o/w} \end{cases}$

proof: In either situation if $\alpha \in \mathrm{Hom}(B(m), B(n))$ then $\alpha(b_m) = b_n$ (unique vertex with single out arrow).

But $\varphi(b_m) = m$ and $\varphi(b_n) = n$ so if $b_m \neq b_n$ then α cannot exist. If $m=n$ then since α is a map of graphs we must have $\alpha = id$. $\square$.

A tensor product

We want to define a line graph $L \otimes M$

vertices: $\{a \otimes b \mid a \in L, b \in M\}$.

edges: Determined by the following rule

$$f(a\otimes b) = \begin{cases} fa \otimes b & \text{if } \varepsilon(a) \geq \varphi(b) \\ a \otimes fa & \text{if } \varepsilon(a) < \varphi(b) \end{cases}$$

It is clear this defines a line graph.

Example $B(1)\otimes B(2)$

$$\begin{tikzcd}
 & & b_2 \arrow[r] & fb_2 \arrow[r] & f^2b_2 \\
 & & \varphi: 2 & 1 & 0 \\
\varepsilon: 0 & b_1 \arrow[dd] & b_1\otimes b_2 \arrow[r] & b_1\otimes fb_2 \arrow[r] & b_1\otimes f^2b_2 \arrow[dd] \\
 & & & & \\
1 & fb_1 & fb_1\otimes b_2 \arrow[r] & fb_1\otimes fb_2 & fb_1\otimes f^2b_2
\end{tikzcd}$$

Exercise Prove that
$$\varphi(a\otimes b) = \varphi(a) + \max\{0, \varphi(b) - \varepsilon(a)\}.$$
$$\varepsilon(a\otimes b) = \varepsilon(b) + \max\{0, \varepsilon(a) - \varphi(b)\}.$$

The associator is defined to be the identity map.

Thm This gives the structure of a monoidal category.

proof: We need to show $\otimes$ is a functor
ie if we have maps $\alpha: L \to L'$, $\beta: M \to M'$
then $\alpha\otimes\beta : L\otimes M \to L'\otimes M'$; $\alpha\otimes\beta(u\otimes v) := \alpha(u)\otimes\beta(v)$
is a morphism, ie
$$\varphi(\alpha\otimes\beta(u\otimes v)) = \varphi(u\otimes v) \qquad \text{(use above exercise)}$$
it respects composition (immediate).

We also need to check that $\alpha = \mathrm{id}$ is a morphism in the category, after which naturality and the pentagon axiom are immediate. (again, use above) $\square$.

What about a braiding?

Lets Look at $B(1)\otimes B(1)$:

$$\begin{tikzcd} & b_1 \arrow[r] & fb_1 \\ b_1 \arrow[d] & b_1\otimes b_1 \arrow[r] & b_1\otimes fb_1 \arrow[d] \\ fb_1 & fb_1\otimes b_1 & fb_1\otimes fb_1 \end{tikzcd}$$

The flip doesn't work! In fact:

Thm The monoidal category $\mathcal{D}_0$ cannot be made into a braided monoidal category.

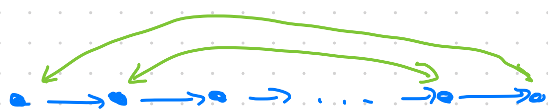
proof: see problem set.

We can however still find a nice natural iso:

$$c \quad : L\otimes M \longrightarrow M\otimes L$$

Define $\xi_L : L \longrightarrow L$ to be the "upside down" map that turns each connected component upside down:

eg $\xi_{B(m)} : B(m) \longrightarrow B(m) \qquad \xi(f^a b_m) := f^{m-a} b_m$



Warning: ξ is only a map of sets! It is not a morphism in $\mathcal{D}_0$. It is still natural in the sense that is is a natural isomorphism

$$\xi : U \Longrightarrow U$$

where $U : \mathcal{D}_0 \longrightarrow \mathrm{Set}$ is the forgetful functor.

Def Define $c_{L,M} : L \otimes M \longrightarrow M \otimes L$ by

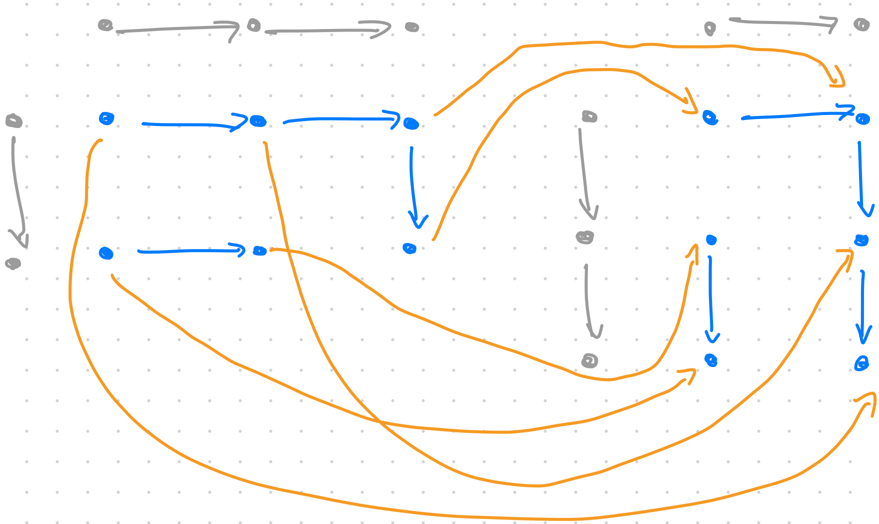
$$c_{L,M}(u \otimes v) := \xi_{M \otimes L}\left(\xi_M(v) \otimes \xi_L(u)\right)$$

ie $\quad c_{LM} = \xi_{M \otimes L} \circ (\xi_M \otimes \xi_L) \circ \mathrm{flip}$

Prop c_{LM} is a morphism in $\mathcal{D}_0$

proof: Thursday.

Example $\quad c_{B(1),B(2)} \qquad \xi_{B(2)} \otimes \xi_{B(1)}$



So if c is not a braiding, what is it?

Def A cobaundary monoidal category is a monoidal category $\mathcal{C}$ with a natural isomorphism

$$c : \otimes \Longrightarrow \otimes \circ \mathrm{flip}$$

satisfying:

- $c_{M,L} \circ c_{L,M} = \mathrm{id}_{L \otimes M}$
- the following commutes

$$\begin{tikzcd}
& (L\otimes M)\otimes N \arrow[r, "c_{L,M}\otimes \mathrm{id}_N"] \arrow[dl, "a_{L,M,N}"'] & (M\otimes L)\otimes N \arrow[dr, "c_{M\otimes L, N}"] & \\
L\otimes(M\otimes N) \arrow[dr, "\mathrm{id}_L\otimes c_{M,N}"'] & & & N\otimes(M\otimes L) \arrow[dl, "a^{-1}_{N,M,L}"] \\
& L\otimes(N\otimes M) \arrow[r, "c_{L,N\otimes M}"'] & (N\otimes M)\otimes L &
\end{tikzcd}$$

Thm The category $\mathcal{D}_0$ is coboundary monoidal

Remark There is a precise sense in which
$$\mathcal{D}_0 = \lim_{q\to 0} \mathcal{D}_q .$$
As monoidal categories. Notice that $\lim_{q\to 0} R_{V(1)V(1)}$ does not exist. If we take
$$R_{VW}\left(R_{WV}\circ R_{VW}\right)^{-\frac{1}{2}} \circ \text{flip}$$
this defines a coboundary structure on $\mathcal{D}_q$ and we can say the limit is true a coboundary cats.

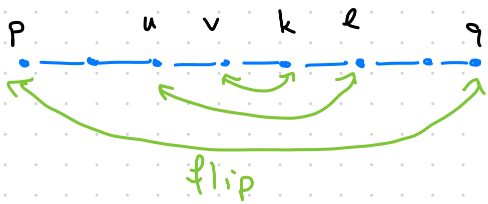
What is the analogue of the braid group?

Previously, we had $B_n \curvearrowright V^{\otimes n}$. Let $\mathcal{C}$ be a coboundary category and $L \in \mathcal{C}$. Can we find groups G_n such that $G_n \curvearrowright L^{\otimes n}$?

Def The cactus group C_n is the group generated by symbols s_{pq}, $1 \leq p < q \leq n$ and relations:

interval

- $s_{pq}^2 = 1$
- $s_{pq}s_{kl} = s_{kl}s_{pq}$ if $[p,q] \cap [k,l] = \emptyset$.
- $s_{pq}s_{kl} = s_{uv}s_{pq}$ if $[k,l] \subseteq [p,q]$



ie $u = q-(k-p) = p+q-k$

$v = q-(\ell-p) = p+q-\ell$

Prop There is a surjection $C_n \to S_n$ given by $s_{pq} \mapsto (p,q)(p+1,q-1)\cdots$ (ie flip the interval $[p,q]$). The kernel PC_n is called the pure cactus group.

proof: It isn't difficult to check this is a group hom and clearly the image of $s_{12}\, s_{23} \ldots s_{n-1n}$ generate S_n. $\square$.

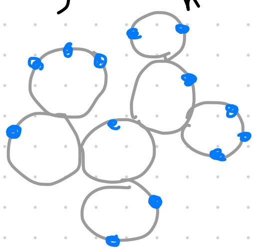
Remark The exact sequence

$$1 \to PC_n \to C_n \to S_n \to 1$$

also comes from topology. Namely

$$PC_n = \pi_1(\overline{M}_{0n+1}(\mathbb{R})) \qquad C_n = \pi_1([\overline{M}_{0n+1}(\mathbb{R})/S_n])$$

moduli space of stable rational curves with n+1 marked points



Lecture 7

Thm If $\mathcal{C}$ is a coboundary category, for every object $L \in \mathcal{C}$, there is a map
$$C_n \longrightarrow \mathrm{Aut}(L^{\otimes n}).$$

proof: We want to think about the image of $s_{1,n}$ as "flipping" the tensor product $L \otimes L \otimes \dots \otimes L$.

For any objects $L_1, L_2, \dots, L_r$ define
$$\sigma_{L_1, L_2 \dots L_r} := c_{L_{r-1}, L_r} \otimes \mathrm{id}^{\otimes r-2} \circ c_{L_{r-2}, L_{r-1} \otimes L_r} \otimes \mathrm{id}^{\otimes r-3}$$
$$\circ \dots \circ c_{L_2, L_3 \otimes \dots \otimes L_r} \otimes \mathrm{id} \circ c_{L_1, L_2 \otimes \dots \otimes L_r}$$
$$: L_1 \otimes L_2 \otimes \dots \otimes L_r \longrightarrow L_r \otimes L_{r-1} \otimes \dots \otimes L_1$$

So we have isomorphisms
$$\sigma_{pq} := \mathrm{id}^{\otimes p-1} \otimes \sigma_{L_p, L_{p+1}, \dots, L_q} \otimes \mathrm{id}^{\otimes r-q}$$

We can define the map $C_n \longrightarrow \mathrm{Aut}(L^{\otimes n})$ by $s_{pq} \mapsto \sigma_{pq}$. We just need to show that the σ_{pq} obey the relations in the cactus group.

First we check that $\sigma_{pq}^2 = 1$. We concentrate on $p=1$, $q=n$, the other cases being similar.

Note $\sigma_{1n} = \sigma_{1\,n-1} \otimes id \circ c_{L, L^{\otimes n-1}}$

so

$$\sigma_{1n}^2 = \sigma_{1\,n-1} \otimes id \circ c_{L, L^{\otimes n-1}} \circ \underbrace{\sigma_{1\,n-1} \otimes id \circ c_{L, L^{\otimes n-1}}}_{=}$$

$$= \sigma_{1\,n-1} \otimes id \circ \underbrace{c_{L, L^{\otimes n-1}} \circ c_{L, L^{\otimes n-1}}}_{=1} \circ id \otimes \sigma_{1\,n-1}$$

$$= \sigma_{1\,n-1}^2 = 1 \quad \text{by induction}$$

by $c_{BA} \circ c_{AB} = id$ axiom.

by repeated application of the coboundary axiom.

The relation $s_{pq} s_{k\ell} = s_{k\ell} s_{pq}$ when $[k\ell] \cap [pq] \neq \phi$ is clear since s_{pq} and $s_{k\ell}$ act on different tensor factors.

The final relation for $[k\ell] \subseteq [pq]$ follows from a similar but more involved calculation to the $s_{pq}^2 = id$ case $\square$.

Aim of the course Construct interesting coboundary categories and calculate the associated cactus grp actions to discover some nice combinatorics!

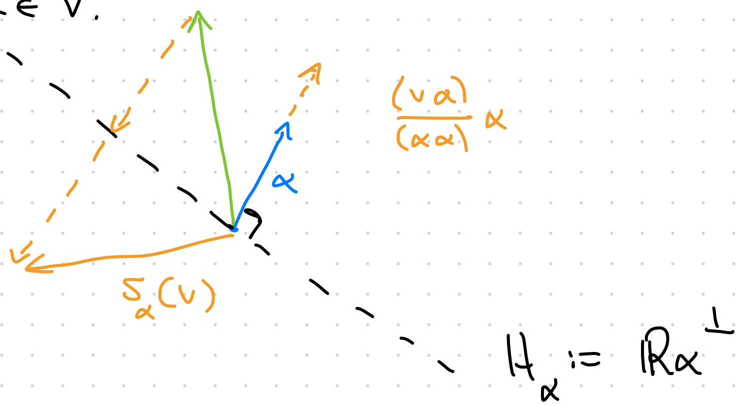
Reflection groups

Let V be a finite dimensional innerproduct space over k $(=\mathbb{C}, \mathbb{R}, \mathbb{Q})$. A reflection $s \in GL(V)$ is a finite order operator of the form

$$s(v) = s_\alpha(v) := v - 2\frac{(v\,\alpha)}{(\alpha\,\alpha)}\,\alpha$$

If $k=\mathbb{C}$ replace with $(\xi - 1)$ for a root of unity ξ.

for some $\alpha \in V$.



Note that $H_\alpha := \ker(s_\alpha - id) = \mathbb{R}\alpha^\perp$ so s_α fixes a $\dim V - 1$ dimensional subspace and $s_\alpha(\alpha) = -\alpha$ so

$$s_\alpha = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & -1 \end{pmatrix}$$

or ξ.

in some basis, ie it is diagonalisable.

Def A reflection group over k is a pair (W, V) of an inner product space V, and a finite group $W \subseteq GL(V)$

generated by reflections.

The rank of W is $\dim V$, and we say W is essential if

$$V^W := \{v \in V \mid gv = v \text{ for all } g \in W\} = \{0\}$$

Remark A rational reflection group is sometimes called a crystallographic group or a Weyl group.

Remark Any rational reflection group can be made into a real (or complex) reflection group by considering the induced transformations of $V \otimes_{\mathbb{Q}} \mathbb{R}$ (i.e. extending scalars).

Note that $\sigma_{1n} = \sigma_{1,n-1} \otimes id \circ c_{L,L^{\otimes n-1}}$

$$= id \otimes \sigma_{1,n-1} \circ c_{L^{\otimes n-1},L}$$

by induction on n and the coboundary axiom.

In addition, the following commutes by naturality

$$\begin{tikzcd}
L^{\otimes n-1} \otimes L \arrow[r, "\sigma_{1n-1}\otimes id"] \arrow[d, "c_{L^{n-1},L}"'] & L^{\otimes n-1} \otimes L \arrow[d, "c_{L^{n-1},L}"] \\
L \otimes L^{\otimes n-1} \arrow[r, "id \otimes \sigma_{1n-1}"] & L \otimes L^{\otimes n-1}
\end{tikzcd}$$

So

$$\sigma_{1n}^2 = \overbrace{\sigma_{1n-1} \otimes id \circ c_{L,L^{\otimes n-1}}}^{=\sigma_{1n} \text{ by def}} \circ \overbrace{id \otimes \underbrace{\sigma_{1,n-1} \circ c_{L^{\otimes n-1},L}}_{= \text{ by}}}^{=\sigma_{1n} \text{ by}}$$

$$= \sigma_{1n-1} \otimes id \circ \underbrace{c_{L,L^{\otimes n-1}} \circ c_{L^{\otimes n-1},L}}_{=id \text{ by symmetry axiom}} \circ \sigma_{1n-1} \otimes id$$

$$= \sigma_{1n-1} \otimes id \circ \sigma_{1n-1} \otimes id$$

$$= \sigma_{1n-1}^2 \otimes id$$

$= id$ by induction on n.

Lecture 10

Examples

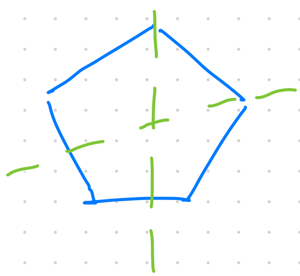
(i) The only reflection on $\mathbb{Q}$ is $s_1 : \lambda \mapsto -\lambda$ so the only rank 1 reflection group is

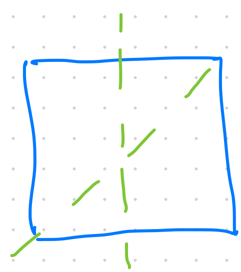
$$\mathbb{Z}/_2 \simeq \langle \lambda \mapsto -\lambda \rangle.$$

(ii) Embed a regular n-gon into $\mathbb{R}^2$ centered at the origin. We can take

$$W = \langle s_\alpha \mid \alpha \text{ any vertex or midpoint of a side} \rangle.$$

Then $W \simeq D_n$ the dihedral group of order $2n$. D_n is a rational reflection group if and only if $n = 3, 4, 5$





(iii) Let ε_i be a basis of $\mathbb{R}^n$. $S_n \subseteq GL_n(\mathbb{R})$ where $\sigma \in S_n$ is given by the appropriate permutation matrix. We have that

$$(a,b) = s_{\varepsilon_a - \varepsilon_b}$$

so S_n is generated by (rational reflections).

Note: It is not essential! $(\mathbb{R}^n)^{S_n} = \mathbb{R}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n)$

But for this reason we can see $S_n \subseteq GL\left(\mathbb{R}^n / \mathbb{R}(\varepsilon_1 + \dots + \varepsilon_n)\right)$

This is essential.

Proposition If (W, V) is a reflection group then there is an injective map

$$W \hookrightarrow GL\left(V/V^W\right)$$

and $(W, V/V^W)$ is an essential reflection group.

proof: The map is well defined since V^W is W-invariant, and if $g \in W$ acts as the identity on V/V^W, that means, for any $v \in V$

$$gv + V^W = V^W$$

ie $gv \in V^W$ by that means $v \in V^W$ so $gv = v$. Hence the map is injective.

Now we must calculate $(V/V^W)^W$. Suppose $v + V^W \in (V/V^W)^W$. Then

$$gv + V^W = v + V^W$$

for all $g \in W$. ie $gv - v \in V^W$. W is generated by reflections s, so

$$s(sv - v) = sv - v$$

$$v - sv = sv - v$$

$$sv = v$$

so $v \in V^W$ and hence $(V/V^W)^W = \{0\}$.

Remark We could alternatively take $(W, (V^W)^\perp)$ to be an equivalent essential reduction.

Goal Classify real and rational reflection groups.

The above says it is enough to only consider essential reflection groups. We will make one more reduction:

If (W_1, V_1) and $(W_2 V_2)$ are two reflection groups then define

$$(W_1, V_1) \oplus (W_2 V_2) = (W_1 \times W_2, V_1 \oplus V_2)$$

Def A reflection group (W, V) is irreducible if it does not allow a nontrivial decomposition as above.

Prop If $w \in W$ is an element of a reflection group, and $s_\alpha \in W$ then $s_{w\alpha} \in W$.

proof: This follows from $w s_\alpha w^{-1} = s_{w\alpha}$. To see this:

1. $w s_\alpha w^{-1}(w\alpha) = -w\alpha$
2. $w s_\alpha w^{-1}$ fixes $H_{w\alpha}$ pointwise.

1. $w s_\alpha w^{-1}(w\alpha) = w s_\alpha(\alpha) = -w\alpha$
2. We need to show that $w s_\alpha w^{-1}(\lambda) = \lambda$ for any $\lambda \in H_{w\alpha}$.

Note that $(\lambda, w\alpha) = (w^{-1}\lambda, \alpha)$ since w is an orthogonal operator. I.e. $\lambda \in H_{w\alpha}$ if and only if $w^{-1}\lambda \in H_\alpha$.

Thus $w s_\alpha w^{-1}(\lambda) = w(w^{-1}\lambda) = \lambda$. $\square$.

Lecture 11 Root systems.

Let W be a real reflection group. We saw on the problem set that W acts faithfully on
$$\hat{R} = \{\mathbb{R}_{>0}\alpha \mid s_\alpha \text{ a reflection in } W\}$$
In fact, the combinatorics of $\hat{R}$ will allow us to understand W and classify refl. groups.

We define a modified verson of $\hat{R}$:

Def A root system $\Phi \subseteq V$ in a Euclidean v.sp. is a finite set of vectors satisfying

(R1) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ if $\alpha \in \Phi$

(R2) $s_\alpha \cdot \Phi = \Phi$ for $\alpha \in \Phi$.

Example If $V = \mathbb{R}$, then $\{\pm 1\}$ is a root system.

If $V = \mathbb{R}^2$ some possible root systems are

(i) $\{(\pm 1, 0), (0, \pm 1)\}$

(ii) $\{\pm(2,0), \pm(1,\sqrt{3}), \pm(-1,\sqrt{3})\}$.

Let $W(\Phi)$ be the group generated by s_α, $\alpha \in \Phi$. Then

Prop $W(\Phi)$ is a finite reflection group and every reflection group occurs in this way.

proof: (exercise).

We will use root systems to understand the structure and classify reflection groups. First job is to come up with a nice minimal generating set and relations.

A total order on the real vector space V is an ordering such that

(i) if $\lambda, \mu \in V$ then either $\lambda < \mu$, $\lambda = \mu$, $\mu > \lambda$

(ii) if $\eta \in V$ and $\lambda < \mu$ then $\lambda + \eta < \mu + \eta$

(iii) if $\lambda < \mu$ and $c \in \mathbb{R}$ then

$$c\lambda < c\mu \quad \text{if } c > 0$$
$$c\lambda > c\mu \quad \text{if } c < 0$$

Given any ordered basis $\{\lambda_1 \ldots \lambda_n\}$ of V the lexiographic ordering is a total order (i.e. where

$$2\lambda_1 + \lambda_2 > \lambda_1 + \lambda_2$$

and

$$\lambda_1 - 3\lambda_2 + \lambda_3 > \lambda_1 - 3\lambda_2 - \lambda_3$$

Def A subset $\Pi \subseteq \Phi$ is called a positive system if
$$\Pi = \{\alpha \in \Phi \mid \alpha > 0\}$$
for some total order on V

A subset $\Delta \subseteq \Phi$ is called a simple system if Δ is a basis for span Φ and for any $\alpha \in \Phi$ we have
$$\alpha \in \pm \mathbb{R}_{>0} \Delta$$
(ie any root is either a positive or negative linear combination of Δ).

Lemma If $D \subseteq \Pi$ is minimal wrt every $\beta \in \Pi$ is a positive linear combination of D, then $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta \in D$.

proof: Suppose $(\alpha, \beta) > 0$, then
$$s_\alpha \beta = \beta - \frac{2(\alpha\,\beta)}{\underbrace{(\alpha\,\alpha)}_{>0}} \alpha = \sum_{\gamma \in D}' c_\gamma \gamma$$
either all coefficients are +ve or -ve. Suppose $c_\gamma \geq 0$.

Two cases

① $c_\beta < 1$; then
$$(1 - c_\beta)\beta = \frac{2(\alpha\,\beta)}{(\alpha\,\alpha)} \alpha + \sum_{\gamma \in D - \{\beta\}}' c_\gamma \gamma$$
So D is not minimal (β in positive span of $D - \{\beta\}$)

② $c_\beta \geq 0$, so $\quad 0 = (c_\beta - 1) + \frac{2(\alpha\beta)}{(\alpha\alpha)}\alpha + \sum_{\gamma \neq \beta}' c_\gamma \gamma$

impossible since $c_\beta - 1 \geq 0 \quad \frac{2(\alpha\beta)}{(\alpha\alpha)} > 0$, $c_\gamma \geq 0$

Similarly, we rule out the possibility that $s_\alpha \beta < 0$. $\square$.

Lemma D as above is a simple system.

proof: We just need to show that D is linearly independent. Suppose

$$\sum_{\gamma \in D}' c_\gamma \gamma = 0$$

We rewrite as

$$\sum_{c_\gamma > 0}' c_\gamma \gamma = \sum_{c_\gamma < 0}' -c_\gamma \gamma = \eta$$

Calculate $0 \leq (\eta\, \eta) = \left(\sum_{c_\gamma > 0} c_\gamma \gamma \, , \sum_{c_\gamma < 0}' -c_\gamma \gamma \right) \leq 0$

So $\eta = 0$ and so $c_\gamma = 0$ for all γ. $\square$

Theorem (i) If $\Delta \subseteq \Phi$ is a simple system then there is a unique positive system Π st.

$$\Delta \subseteq \Pi \subseteq \Phi$$

(ii) If Π is a positive system then there is a unique simple system Δ st

$$\Delta \subseteq \Pi \subseteq \Phi.$$

proof: (i) Firstly if Δ is a simple system, choose an ordering of Δ and take Π to be the resulting positive system.

Suppose Π' is another positive system containing Δ. If $\sum_{\alpha\in\Delta}' c_\alpha \alpha = \gamma$ is such that $c_\alpha > 0$ then since $\alpha > 0$ for all $\alpha \in \Delta$, we have $\gamma > 0$, so $\gamma \in \Pi'$. But this shows

$$\Pi' = \{\gamma \in \Phi \mid \gamma \text{ is a +ve sum of } \Delta\} = \Pi.$$

(ii) If Π is a positive system, we take $\Delta \subseteq \Pi$ to be the minimal subset such that Π is positive linear sums of Φ.

If Δ' is another simple system in Π. Clearly

$$\Delta' \supseteq \left\{\alpha \in \Pi \;\middle|\; \begin{array}{l}\alpha \text{ cannot be expressed} \\ \text{as a nontrivial pos. sum} \\ \text{in } \Pi\end{array}\right\} \subseteq \Delta$$

But since Δ and Δ' are linearly independent these must be equalities $\square$.

Now we want to show that all positive/simple systems are alike.

Prop If $\Delta \subset \Pi$ are simple/pos. systems and $w \in W$ then so are $w\Delta \subset w\Pi$.

We will show that $\{w\Delta \subseteq w\Pi\}_{w\in W}$ is a complete collection of simple/pos. systems for Φ.

First we analyse $s_\alpha \Pi$.

Prop If $\alpha \in \Delta \subseteq \Pi$ then $s_\alpha \Pi = (\Pi - \{\alpha\}) \cup \{-\alpha\}$

proof: Let $\sum_{\gamma\in\Delta} c_\gamma \gamma \in \Pi - \{\alpha\}$.

$$s_\alpha\Big(\sum{}' \underset{\geq 0}{c_\gamma} \gamma\Big) = \underbrace{\sum{}' c_\gamma \gamma \;\; - c\alpha}_{\text{still has at least one pos. coef}} \in \Pi$$

Note $s_\alpha\left(\sum' c_\gamma \gamma\right) \neq \alpha$ as otherwise

$$s_\alpha^2(\text{——}) = s_\alpha \alpha = -\alpha \neq \sum{}' c_\gamma \gamma. \qquad \square.$$

Thm If $\Delta \subseteq \Pi$ and $\Delta' \subseteq \Pi'$ are simple/pos. systems, there exists $w \in W$ st

$$w\Delta = \Delta' \quad \text{and} \quad w\Pi = \Pi'$$

proof: Induction on $n = \#(\Pi \cup \Pi' \setminus \Pi')$. Clearly if $n = 0$ then $\Pi = \Pi'$ so we take $w = 1$.

If $n > 0$, choose $\alpha \in \Delta$ s.t. $\alpha \notin \Pi'$.

$$\#(s_\alpha \Pi \cup \Pi' \setminus \Pi') = n - 1$$

so we can find $w' \in W$ st $w' s_\alpha \Pi = \Pi'$ and $w' s_\alpha \Delta = \Delta'$
let $w = w' s_\alpha$. $\square$

Some notation: If $\beta \in \Phi$ and $\beta \in \sum_\Delta' c_\alpha \alpha$ then

$$ht(\beta) := \sum' c_\alpha$$

so $\Pi = \{ \beta \in \Phi \mid ht(\beta) > 0 \}$.

called simple reflections

Thm W is generated by s_α, $\alpha \in \Delta$.

proof: Let $W' = \langle s_\alpha \mid \alpha \in \Delta \rangle$. Our strategy is to show that $s_\beta \in W'$ for every $\beta \in \Pi$. Since

$$W = \langle s_\beta \mid \beta \in \Pi \rangle$$

this implies $W' = W$.

First observe that if $\beta \in \Pi \setminus \Delta$ then $\exists\, \alpha \in \Delta$ s.t.

$$ht(s_\alpha \beta) < ht(\beta) \quad \text{and} \quad s_\alpha \beta \in \Pi.$$

indeed $s_\alpha \beta = \beta - \frac{2(\alpha\, \beta)}{(\alpha\, \alpha)} \alpha$ so if this is not true then $(\alpha\, \beta) \leq 0$ for all $\alpha \in \Delta$, Hence

$$0 < (\beta\, \beta) = \sum_{\gamma \in \Delta}' \underbrace{c_\alpha}_{\geq 0} \underbrace{(\beta\, \alpha)}_{\leq 0} \leq 0 \implies \beta = 0.$$

This means we can find a sequence of simple

roots $\alpha_1 \alpha_2 \ldots \alpha_r$ s.t. $ht(s_{\alpha_1} \cdots s_{\alpha_r} \beta)$ is minimal
Suppose $\gamma = s_{\alpha_1} \cdots s_{\alpha_r} \beta$ is not simple, then we can choose $\alpha \in \Delta$ s.t. $s_\alpha \gamma \in \Pi$ and $ht(s_\alpha \gamma) < ht(\gamma)$
Contradiction! So $\gamma \in \Delta$.

This tells us, if $w = s_{\alpha_1} \cdots s_{\alpha_r} \in W'$ that

$$w s_\beta w^{-1} = s_\gamma \in W'$$

ie $$s_\beta = w s_\gamma w^{-1} \in W' \qquad \square.$$

Lecture 13 Relations in $W(\Phi)$

We saw that if we fix a simple system $\Delta \subseteq \Phi$ then $W = W(\Phi)$ is generated by s_α, $\alpha \in \Delta$.

What relations are there? There are some obvious ones:

$$s_\alpha^2 = 1, \quad (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1$$

Turns out these are the only ones!

First we need so notation, for $w \in W$

$$\ell(w) = \min\{r \mid s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r} = w\} \quad \text{length}$$

$$n(w) = \#\{\beta \in \Pi \mid w\beta < 0\} \quad \text{number of flips}$$

Note: $\ell(s_\alpha) = 1$, and $n(s_\alpha) = 1$ for $\alpha \in \Delta$

If $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{\ell(w)}}$ we say this is a reduced expression for w.

Lemma If $\alpha \in \Delta$, $w \in W$ then

(i) if $w\alpha > 0$ then $n(ws_\alpha) = n(w) + 1$

(ii) if $w\alpha < 0$ then $n(ws_\alpha) = n(w) - 1$

proof: We must analyse, if $w\alpha > 0$

$$\{\beta > 0 \mid ws_\alpha \beta < 0\} = s_\alpha\{\beta > 0 \mid w\beta < 0\} \sqcup \{\alpha\}.$$

Since $\alpha \notin \{\beta > 0 \mid w\beta < 0\}$ and if $\beta \in$ then $ws_\alpha(s_\alpha\beta) < 0$

so $n(ws_\alpha) = n(w) + 1$

If $w\alpha < 0$ then

$$\{\beta > 0 \mid ws_\alpha\beta < 0\} \sqcup \{-\alpha\} = s_\alpha \underbrace{\{\beta > 0 \mid w\beta < 0\}}_{\alpha \in}$$

So $n(ws_\alpha) = n(w) - 1$. $\square$

Cor for any $w \in W$, $n(w) \leq \ell(w)$

proof: If $w = s_1 s_2 \dots s_r$ (with $s_i = s_{\alpha_i}$ for some $\alpha_i \in \Delta$) we can build w from the left

1	$n(1) = 0$
s_1	$n(s_1) = 1$
$s_1 s_2$	$n(s_1 s_2) = 2$ or 0
$s_1 s_2 s_3$	$n(s_1 s_2 s_3) = 3$ or 1
$\vdots$	$\vdots$
w	$n(w) \leq r$

thus if $r = \ell(w)$ we see $n(w) \leq \ell(w)$. $\square$

Now we come to the most important property of W:

Thm If $w = s_1 s_2 \dots s_r$ (with $s_i = s_{\alpha_i}$ $\alpha_i \in \Delta$) is any expression and $r > n(w)$ then there exist $1 \leq i < j \leq r$ s.t.

$$s_{i+1}s_{i+2}\cdots s_j = s_i s_{i+1}\cdots s_{j-1} \qquad (1)$$

and thus

delete $s_i\ s_j$.

$$w = s_1\cdots s_{i-1}\, s_{i+1}\cdots s_{j-1}\, s_{j+1}\cdots s_r \qquad (2)$$

(we can reduce expressions by deleting in pairs)

proof: Assume (1). We will show (2)

$$w = s_1\cdots s_i \underbrace{s_{i+1}\cdots s_j}_{\text{replace}} s_{j+1}\cdots s_r$$

$$= s_1\cdots s_i s_i s_{i+1}\cdots s_{j-1} s_{j+1}\cdots s_r$$

$$= s_1\cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$$

Now we show (1): We have that $n(w) < r$. Consider

$$\begin{array}{ll} s_1\alpha_2 > 0 & \\ s_1 s_2 \alpha_3 > 0 & \\ s_1 s_2 s_3 \alpha_4 > 0 & \\ \vdots & \\ s_1\cdots s_{j-1}\alpha_j > 0 & \\ \phantom{s_1\cdots s_{j-1}\alpha_j} < 0 & \\ \vdots & \end{array}$$

if this is true, then Lemma above $\Rightarrow n(w) \geq r$

So, for some $1 \leq j \leq r$ we must have

I.e. for some $1 \leq j \leq r$ we have $s_1\cdots s_{j-1}\alpha_j < 0$

Thus we must have

$$\alpha_j > 0$$

$$s_{j-1}\alpha_j > 0$$

$$\vdots$$

$$s_{i+1}\cdots s_{j-1}\alpha_j > 0$$

first time it swaps from pos to neg.

$$s_i s_{i+1}\cdots s_{j-1}\alpha_j < 0$$

But the only positive root swapped to negative by s_i is α_i so

$$s_{i+1}\cdots s_{j-1}\alpha_j = \alpha_i$$

This means, if $w' = s_{i+1}\cdots s_{j-1}$ that

$$w s_j w^{-1} = s_i$$

$$w s_j = s_i w$$

$$s_{i+1}\cdots s_{j-1} s_j = s_i s_{i+1}\cdots s_{j-1} \qquad \square.$$

Cor $n(w) = \ell(w)$

proof: We know $n(w) \leq \ell(w)$. If $\ell(w) > n(w)$ then we can take an expr.

$$w = s_1 \cdots s_{\ell(w)}$$

and find two elements to delete, contradicting minimality. $\square$

Thm $W \simeq \langle s_\alpha \mid s_\alpha^2 = (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1 \rangle$

ie, these are the only relations in W.

proof: We will show that any relation

$$s_1 s_2 \cdots s_r = 1 \qquad (1)$$

Is a consequence of the relations above.

Note $r = 2q$ is even. Indeed

$$\det(s_i) = -1, \text{ and } \det(1) = 1$$

We will induct on q. The case $q=1$ is immediate.

Assume $q > 1$. Rewrite as:

$$\underbrace{s_1 s_2 \cdots s_{q+1}} = s_r s_{r-1} \cdots s_{q+2}$$

This cannot be a reduced expr. So the deletion condition applies.

Thus, there exists $1 \le i < j \le q+1$ s.t.

$$s_{i+1} \cdots s_j = s_i \cdots s_{j-1}$$

or alternatively

$$1 = s_i s_{i+1} \cdots s_{j-1} s_j s_{j-1} \cdots s_{i+1}$$

This involves $2(j-i)$ simple reflections, so we can apply induction (!!! unless $2(j-i) = r$)

This must be a result of the stated relations

$$1 = s_1 \dots s_r = s_1 \dots s_i (s_i \dots s_{j-1}) s_{j+1} \dots s_r$$
$$= s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_r$$

which again by induction is the result of the stated relations

Now if $2(j-i)=r$ (ie $i=1$, $j=q+1$) we get

$$s_2 s_3 \dots s_{q+1} = s_1 s_2 \dots s_q \qquad \mathrm{I}$$

We could rewrite ① as

$$s_2 s_3 \dots s_r s_1 = 1 \qquad ②$$

repeating the same steps:

$$s_2 s_3 \dots s_{q+2} = s_1 s_r s_{r-1} \dots s_{q+3}$$

so the LHS must not be reduced and we can again find subwords that imply the theorem unless

$$s_3 s_4 \dots s_{q+2} = s_2 s_3 \dots s_{q+1}$$

which implies

$$s_3 s_2 s_3 \dots s_{q+1} s_{q+2} s_{q+1} \dots s_4 = 1$$

We now try the original trick again:

$$s_3 s_2 s_3 \dots s_{q+1} = s_4 s_5 \dots s_{q+2}$$

The LHS is not reduced so we can repeat the argument which will imply (2) and therefor (1) follow from the relations unless

$$s_2 s_3 \dots s_{q+1} = s_3 s_2 s_3 \dots s_q \qquad \mathrm{II}$$

But I and II imply that $s_1 = s_3$.

Repeating with

$$s_3 s_4 \dots s_r s_1 s_2 = 1$$

will imply $s_2 = s_4$ and repeating with further cyclic permutations will eventually give

$$s := s_1 = s_3 = \dots = s_{r-1} \quad \text{and} \quad s_2 = s_4 = \dots = s_r =: t$$

But this reduces (1) to $(st)^q = 1$ which is a known relation $\square$.

Lecture 14

If W is a finite reflection group $W \subseteq GL(V)$, we fix a root system, with a simple system

$$W \circlearrowright \Phi \supseteq \Delta$$

$$W \cong \langle s_\alpha,\ \alpha \in \Delta \mid s_\alpha^2 = (s_\alpha s_\beta)^{m(\alpha\beta)} = 1 \rangle$$

Def A Coxeter group is a pair (W, S) where $S \subseteq W$ is a finite generating set and

$$W \cong \langle s \in S \mid s^2 = (st)^{m(s,t)} = 1 \rangle$$

for some $m(s,t) \in \mathbb{Z}_{>1} \cup \{\infty\}$

Example (i) D_∞ the infinite dihedral group is

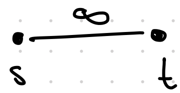
$$\langle s,t \mid s^2 = t^2 = 1 \rangle$$

here $S = \{s,t\}$ and $m(s,t) = \infty$

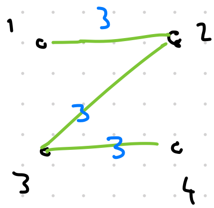
(ii) $$S_n \cong \langle s_1 \dots s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = \underset{|i-j|>1}{(s_i s_j)^2} = 1 \rangle$$

Def To a Coxeter group (W, S) we associate a Coxeter graph,

- vertices: S
- edge labelled $s \overset{m(s,t)}{\text{———}} t$ whenever $m(s,t) > 2$

Example (i) D_∞ 

(ii) S_5

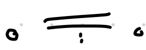
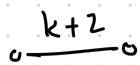


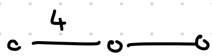
S_n 

Remarks (i) $(st)^2 = 1$ means $stst = 1$

ie $st = t^{-1}s^{-1} = ts$ so $(st)^2 = 1$ means s, t commute.

(ii) 3 occurs a lot, so we often leave it off

(iii) k-multiple edges  means 

so  is the same as 

Def Let $W \leq GL(V)$ and $W' \leq GL(V')$ be reflection groups. We say W and W' are isomorphic as reflection groups if there exists an surjective isometry

$$\phi : V \longrightarrow V'$$

s.t. the induced map $\phi : GL(V) \longrightarrow GL(V')$ identifies W and W', ie $\phi(W) = W'$

Exercise Find an example of two reflection groups that are the same as abstract groups but not isomorphic as reflection groups.

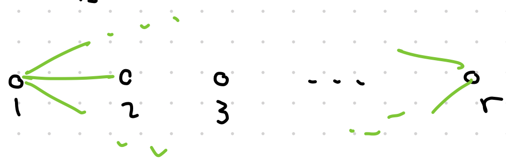
Thm Let $W \leq GL(V)$ and $W' \leq GL(V')$ be essential reflection groups. $W \simeq W'$ as reflection groups if and only if their Coxeter graphs are isomorphic.

proof: If $W \simeq W'$, then clearly their Coxeter graphs are isomorphic.

Now suppose the Coxeter graphs agree. Fix root systems, and simple systems

$$\Delta = \{\alpha_1, \dots \alpha_r\} \subset \Phi \subset V \quad \text{and} \quad \Delta' = \{\alpha_1', \dots \alpha_r'\} \subseteq \Phi' \subseteq V'$$

consisting of unit vectors, so that the common Coxeter graph is



Now consider the linear map

$$\phi : V \longrightarrow V', \quad \alpha_i \longmapsto \alpha_i'.$$

This is an isomorphism since Δ and Δ' are bases for $\operatorname{span} \Phi = V$ and $\operatorname{span} \Phi' = V'$.

Furthermore, the value of (α_i', α_j') is determined by $m(\alpha_i', \alpha_j') = m(\alpha_i\, \alpha_j)$ (using the fact that the Coxeter graphs coincide). So

$$(\alpha_i'\, \alpha_j') = (\alpha_i\, \alpha_j)$$

and hence ϕ is an isometry.

ϕ identifies W and W' since it identifies Φ and Φ'. $\square$.

We have learnt that

$$\left\{\begin{array}{c}\text{Irreducible}\\ \text{finite reflection}\\ \text{groups}\end{array}\right\} \Big/ \text{iso} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{Connected}\\ \text{Coxeter graphs}\\ \text{coming from finite}\\ \text{Coxeter groups}\end{array}\right\} \Big/ \text{iso}$$

Def If (W,S) is a Coxeter group the subgroup

$$W_I = \langle s \mid s \in I \rangle \qquad I \subset S$$

is called a parabolic subgroup

Exercise (W_I, I) is a Coxeter group and $W_{I \cap J} = W_I \cap W_J$.

Thm If $W \subset GL(V)$ is a refl. group, fix $\Delta \subset \Phi$. If $\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \dots \sqcup \Gamma_r$ is the Coxeter graph decomposed into connected components, then

$$W \simeq W_{S_1} \times W_{S_2} \times \dots \times W_{S_r}$$

where S_i are the generators corresponding to the vertices in Γ_i.

More over if Γ is connected, then W is irreducible.

proof: First we show W is irreducible if Γ is connected. Suppose Γ is connected and $W = W_1 \times W_2$. This means $V = V_1 \oplus V_2$ and so Δ must be partitioned into $\Delta \cap V_1$ and $\Delta \cap V_2$. We also must have

$$s_\alpha s_\beta = s_\beta s_\alpha$$

for $\alpha \in \Delta \cap V_1$, $\beta \in \Delta \cap V_2$. But this would imply Γ is disconnected.

What remains to prove is that if $S = I \sqcup J$ and Γ_I and Γ_J are not connected by a path then

$$W \simeq W_I \times W_J.$$

Clearly W_I and W_J are subgroups that commute with each other and $W = W_I W_J$.

Furthermore $W_I \cap W_J = W_{I \cap J} = W_\phi = \{id\}$
so
$$W \simeq W_I \times W_J.$$
We just need to find a compatible decomposition of V, which is given by
$$\Delta = \Delta_I \sqcup \Delta_J \quad , \quad \Delta_I = \{\alpha \in \Delta \mid s_\alpha \in S_I\}$$
and $V = \operatorname{span} \Delta_I \oplus \operatorname{span} \Delta_J$. $\square$.

Lecture 15

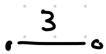
Recall we have reduced our problem to classifying Coxeter graphs that come from finite Coxeter groups

$$\left\{\begin{array}{c}\text{irreducible}\\ \text{finite reflection}\\ \text{group}\end{array}\right\}\Big/\text{iso} \quad \overset{1:1}{\longleftrightarrow} \quad \left\{\begin{array}{c}\text{Connected Coxeter}\\ \text{graphs whose Coxeter}\\ \text{group is finite}\end{array}\right\}\Big/\text{iso}$$

When is a Coxeter group finite?

Def For a Coxeter group (W,S) or a Coxeter graph Γ (vertices S), the Coxeter matrix is

$$A = \left(-2\cos\frac{\pi}{m(s,t)}\right)_{s,t\in S} \qquad \left(\text{set } m(s,s)=1\right)$$

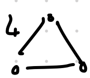
Example (i) S_3 

$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ det $=3$, evals $=3,1$

(ii) S_n 
$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$ det $=n$, evals >0

(iii) D_∞ 

$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ det $=0$, evals $=4,0$

(iv) 

$A = \begin{pmatrix} 2 & -1 & \sqrt{2} \\ -1 & 2 & -1 \\ -\sqrt{2} & -1 & 2 \end{pmatrix}$ det $=-2\sqrt{2}$

evals $= 2+\sqrt{2}, \frac{1}{2}\left(4-\sqrt{2}\pm\sqrt{10}\right)$

Let $V_{(W,S)} := \mathrm{span}\{e_s\}_{s\in S}$. A defines a bilinear form:

$$(x,y) := \tfrac{1}{2} x^t A y$$

If $W \subseteq GL(V)$ is a reflection group and S comes from a choice of simple roots $\Delta \subset \Phi$ then $S = \{s_\alpha\}_{\alpha\in\Delta}$

$$(e_{s_\alpha}, e_{s_\beta}) = -\cos\frac{\pi}{m(\alpha\beta)}$$

in $V_{(W,S)}$, but this is exactly the inner product $(\alpha\ \beta)$ in V. So

$$V \longrightarrow V_{(W,S)};\ \alpha \mapsto e_{s_\alpha}$$

is an isometry. In this case $(-\ -)$ is an inner product, hence it is positive definite, ie

$$(x, x) > 0 \quad \text{for } x \neq 0$$

So, if W is finite then $(-,-)$ is positive definite

Fact The following are equivalent:

(i) $(x,y) := \frac{1}{2}x^t A y$ is positive definite (positive semi-def.)

(ii) A has only positive eigenvalues (nonnegative evals)

(iii) The principal minors of A are positive (nonnegative)

positive semidefinite means

$$(x\ x) \geq 0 \quad \text{for all } x.$$

Thus

$$\left\{\begin{array}{l}\text{Connected Coxeter}\\ \text{graphs coming}\\ \text{from finite Coxeter}\\ \text{groups}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Connected coxeter}\\ \text{groups with}\\ A \text{ positive definite}\end{array}\right\}$$

Our strategy

(1) Find all of these

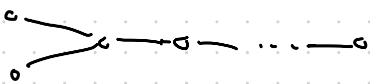
(2) Show all of them actually have a finite coxeter group (ie the above is an equality).

Prop The following are positive definite

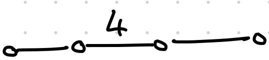
A_n 

(n vertices)

B_n/C_n 

D_n 

E_n  $n = 6, 7, 8$

F_4 

H_3 

H_4 

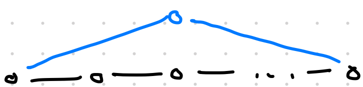
$I_2(m)$ 

proof: For any graph on the list, deleting one vertex gives another graph on the list. Make this vertex index the final row/col in A. All the proper principal minors are therefore >0. So we only need to check that $\det A > 0$. Compute! $\square$.

Prop The following are all positive semidefinite but not positive definite

(n+1 vertices)

$\tilde{A}_1$ 

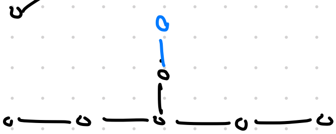
$\tilde{A}_n$ 

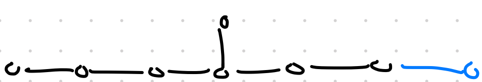
$\tilde{B}_2/\tilde{C}_2$ 

$\tilde{B}_n$ 

$\tilde{C}_n$ 

$\tilde{D}_n$ 

$\tilde{E}_6$ 

$\tilde{E}_7$ 

$\tilde{E}_8$ 

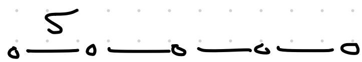
$\tilde{F}_4$ 

$\tilde{G}_2$ 

proof: Note that the removal of the blue vertex on each graph gives a positive definite one. Thus we only need to check that $\det A = 0$. Compute! $\square$.

Lemma The following are not positive definite

Z_4 

Z_5 

We will eventually prove that any positive definite graph must appear on the list above. To do this we will rule out the types of subgraphs that can apper.

Def Let Γ and Γ' be Coxeter graphs, we say Γ' is a subgraph of Γ if Γ' is obtained from Γ by deleting vertices and/or lowering weights of edges.

We sa a matrix A is indecomposable if no simultaneous permutation of rows and columns gives a block diagonal matrix. It is clear that

Lemma Γ is connected, if and only if A is indecomposable.

Prop If A is any real symmetric positive semi-definite matrix which is indecomposable and with all off diagonal entries ≤ 0, then:

(a) $\ker A = \{x \in \mathbb{R}^n \mid x^t A x = 0\}$ and has $\dim \leq 1$

(b) the smallest eigenvalue has multiplicity 1 and has an eigenvector with all entries positive

proof: Frobenious-Perron theory.

Cor If Γ is a connected Coxeter graph that is positive semidefinite then every proper subgraph is positive definite.

proof: Suppose Γ' is a subgraph of a connected graph Γ. Let A and A' be the respective matrices. Suppose A is $n \times n$, and A' is $k \times k$. We have

$$a'_{ij} = -2\cos\frac{\pi}{m'(ij)} \geq -2\cos\frac{\pi}{m(ij)} = a_{ij}$$

since $m(ij) \geq m'(ij)$. Suppose for contradiction that A' is not positive definite, ie, there exists $0 \neq x \in \mathbb{R}^k$ st

$$x^t A' x \leq 0$$

Now consider the vector

$$\hat{x} = \begin{pmatrix} |x_1| \\ \vdots \\ |x_k| \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$$

$$0 \leq \hat{x}^t A \hat{x} = \sum_{1 \leq i,j \leq k}' a_{ij} |x_i||x_j| \leq \sum_{1 \leq i,j \leq k}' a'_{ij} |x_i||x_j|$$

$$\le \sum_{1 \le i,j \le k}' a'_{ij} x_i x_j = x^t A' x \le 0$$

Thus $\hat{x}^t A \hat{x} = 0$ and so b the Prop $A\hat{x} = 0$ and so $\ker A \ne 0$. By part (b) of the prop $\hat{x}$ has strictly positive entries, so $n = k$.

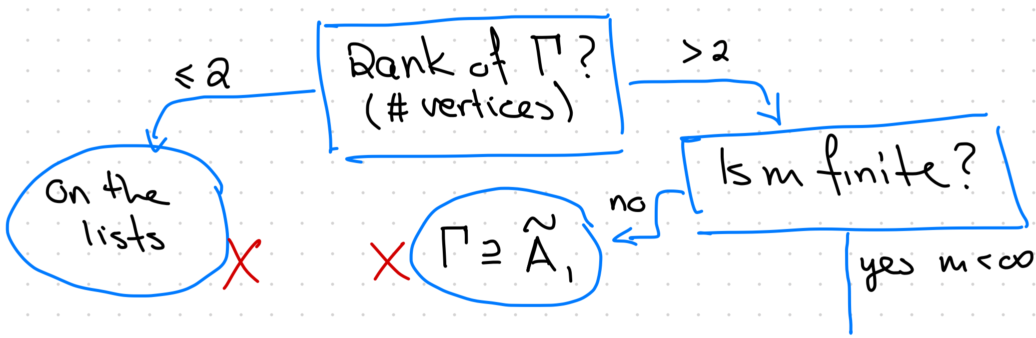
But then since

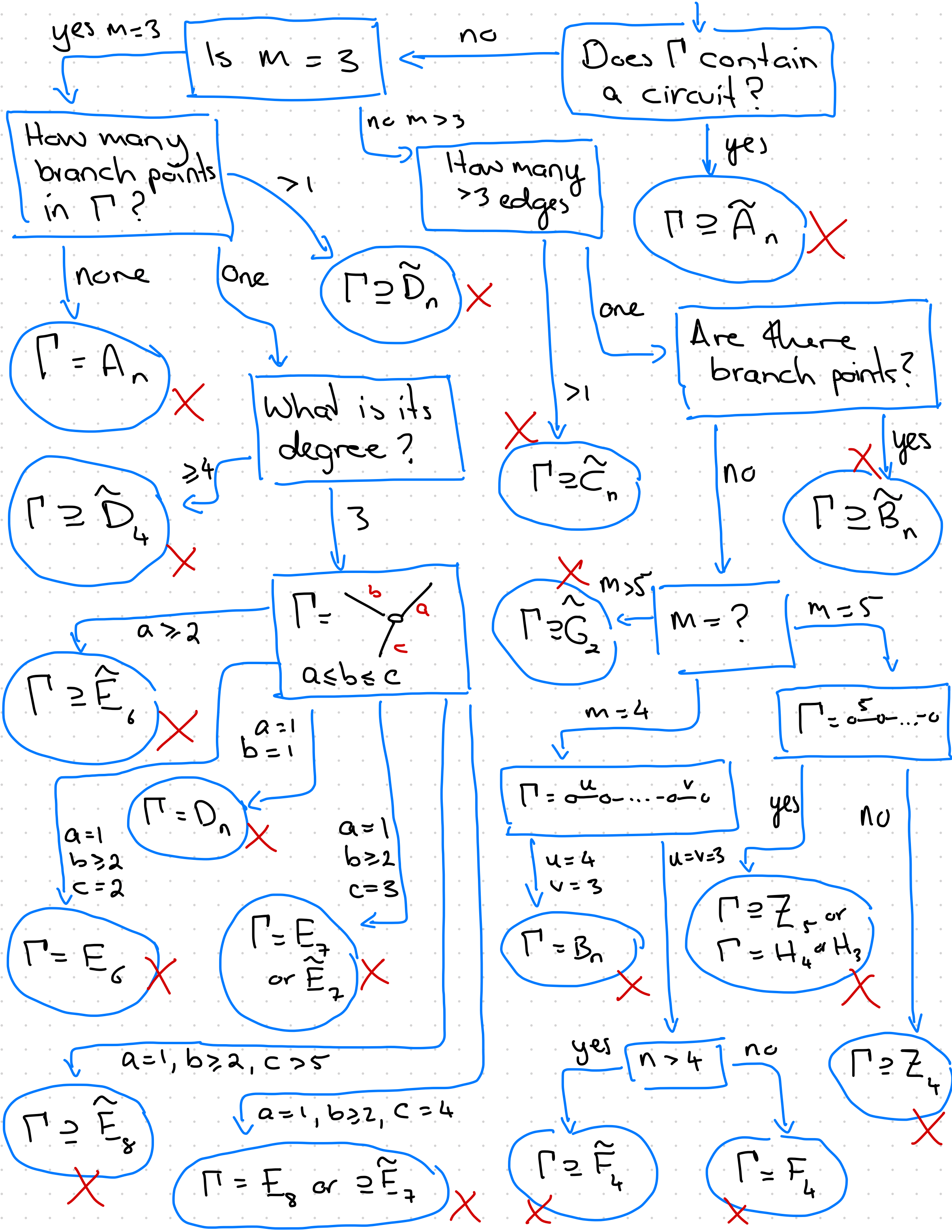
$$\sum_{1 \le i,j \le n}' a_{ij} |x_i||x_j| = \sum_{1 \le i,j \le n}' a'_{ij} |x_i||x_j|$$

and $a'_{ij} \ge a_{ij}$, we must have $a_{ij} = a'_{ij}$. Contradiction! $\square$.

Theorem The lists above are the only connected Coxeter graphs that are positive (semi)-definite

proof: By flow chart: Suppose Γ is a positive semi-definite graph not on the lists above. Let m = maximum weight, n = # vertices.




Does Γ contain a circuit?
no
yes
Is m = 3
yes m=3
no m>3
$\Gamma \supseteq \tilde{A}_n$
How many branch points in Γ?
none
One
>1
$\Gamma \supseteq \tilde{D}_n$
$\Gamma = A_n$
How many >3 edges
one
>1
$\Gamma \supseteq \tilde{C}_n$
Are there branch points?
yes
no
$\Gamma \supseteq \tilde{B}_n$
What is its degree?
≥4
3
$\Gamma \supseteq \tilde{D}_4$
Γ=
b
a
c
a≤b≤c
a≥2
$\Gamma \supseteq \tilde{E}_6$
a=1 b=1
$\Gamma = D_n$
a=1 b≥2 c=2
$\Gamma = E_6$
a=1 b≥2 c=3
$\Gamma = E_7$ or $\tilde{E}_7$
a=1, b≥2, c>5
$\Gamma \supseteq \tilde{E}_8$
a=1, b≥2, c=4
$\Gamma = E_8$ or $\supseteq \tilde{E}_7$
m = ?
m>5
$\Gamma \supseteq \tilde{G}_2$
m=5
m=4
Γ = o–5–o–o–…–o
Γ = o–u–o–…–o–v–o
yes
no
u=4 v=3
u=v=3
$\Gamma = B_n$
$\Gamma \supseteq Z_5$ or $\Gamma = H_4$ or H_3
$\Gamma \supseteq Z_4$
n > 4
yes
no
$\Gamma \supseteq \tilde{F}_4$
$\Gamma = F_4$

Lecture 17 Rational reflection groups.

Recall if $W \subseteq GL(V_{\mathbb{Q}})$ for $V_{\mathbb{Q}}$ a $\mathbb{Q}$-vector space, then we can look at the corresponding group

$$W \subseteq GL(V_{\mathbb{R}}) \qquad V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$$

and this is a real reflection group.

If V is a real vector space, a $\mathbb{Q}$-lattice, is a $\mathbb{Q}$-subspace $U \subseteq V$ s.t. the natural map $U \otimes_{\mathbb{Q}} \mathbb{R} \to V$ is an isomorphism. I.e. U must be the $\mathbb{Q}$-span of a $\mathbb{R}$-basis of V.

Example $V = \mathbb{R}^2$

Here are two, 2-dim'l $\mathbb{Q}$-subspaces

- $U = \mathrm{span}_{\mathbb{Q}}\left\{\begin{pmatrix}1\\0\end{pmatrix}\ \begin{pmatrix}0\\1\end{pmatrix}\right\}$. this is a $\mathbb{Q}$-lattice

- $U' = \mathrm{span}_{\mathbb{Q}}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\sqrt{2}\\0\end{pmatrix}\right\}$ this is not a $\mathbb{Q}$-lattice.

$W \subseteq GL(V)$

Remark A real reflection group is (or comes from) a rational reflection group, if there exists some $\mathbb{Q}$-lattice $U \subseteq V$ st W preserves U.

We can also think about integral reflection groups. These are called "Crystallographic".

A $\mathbb{Z}$-lattice (or just lattice) in an $\mathbb{R}$-vector space V is a $\mathbb{Z}$-submodule $M \subset V$ st. the natural map $M \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow V$ is an isomorphism (ie $\mathbb{Z}$-span of a basis).

Def If $W \subseteq GL(V)$ is a reflection group for V a real vector space, we say W is Crystallographic if it preserves a $\mathbb{Z}$-lattice in V.

Prop If $W \subseteq GL(V)$ is a Crystallographic reflection group, then if $\Phi \supseteq \Delta$ is a root system, with simple roots, then for any $\alpha \neq \beta \in \Delta$

$$m(\alpha, \beta) = 2, 3, 4 \text{ or } 6$$

proof: Since W is Crystallographic, it can be represented by integer matrices, and thus have integer trace.

Consider $s_\alpha s_\beta$ is a rotation in the plane spanned by α, β, with angle $2\pi / m(\alpha, \beta)$.

So $\operatorname{tr}(s_\alpha s_\beta) = \dim V - 2 + 2\cos \frac{2\pi}{m(\alpha \beta)} \in \mathbb{Z}$.

So $\cos \frac{2\pi}{m(\alpha \beta)} \in \frac{1}{2}\mathbb{Z}$. ie $m(\alpha \beta) = 2, 3, 4$ or 6. $\square$.

Def Let Φ be a root system, Φ is called Crystallographic if, for any $\alpha, \beta \in \Phi$

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha \alpha)} \in \mathbb{Z}.$$

Note: $s_\alpha \beta = \beta - \langle \beta \alpha \rangle \alpha$

Lemma If Φ is a Crystallographic root system, then $W(\Phi)$ is Crystallographic (ie it preserves a lattice).

proof: $W(\Phi)$ preserves $\mathbb{Z}\Phi$ $\square$

Aim:
- Show that Crystallographic refl. grps are the same as rational refl grps.
- Classify Crystallographic root systems.

Lecture 17

$$
\begin{tikzcd}
\left\{\begin{matrix}\text{irreducible}\\ \text{real refl.}\\ \text{groups}\end{matrix}\right\}\Big/_{\text{iso}} \arrow[r, phantom, "="] \arrow[d, phantom, "\cup\!\!|"] & \left\{\begin{matrix}\text{connected}\\ \text{pos. def.}\\ \text{Coxeter gphs}\end{matrix}\right\}\Big/_{\text{iso}} & \left\{\text{root systems}\right\}\Big/_{\text{iso}} \arrow[l, two heads] \\
\left\{\begin{matrix}\text{irr. rational}\\ \text{refl. grps}\end{matrix}\right\}\Big/_{\text{iso}} \arrow[d, phantom, "\cup\!\!|"] & & \\
\left\{\begin{matrix}\text{irr. crystallographic}\\ \text{refl. grps}\end{matrix}\right\}\Big/_{\text{iso}} & & \left\{\begin{matrix}\text{crystallographic}\\ \text{root systems}\end{matrix}\right\}\Big/_{\text{iso}} \arrow[ll, two heads] \arrow[uu, hook]
\end{tikzcd}
$$

① show this is an equality.

② classify this

③ understand this

Lattices

Let k be a field and $R \subseteq k$ a ring, and V a k vector space. An R-lattice $M \subset V$ is an R-submodule such that the natural map

$$M \otimes_R k \longrightarrow V; \quad m \otimes \lambda \mapsto \lambda m.$$

is an isomorphism (ie it is the R-span of a basis)

Remark If $B \subseteq V$ is a k-basis, then $R \cdot B =: M$ is a lattice in V.

Thm Suppose G is a finite group, R a PID and k its field of fractions. If V is a fin.dim. G-module $/k$, then there exists a free R-submodule $M \subseteq V$, that is G-invariant and is an R-lattice.

proof: We will find a k-basis T s.t. $\overset{M:=}{R \cdot T}$ is a free R-module, and invariant under G.

Fix any basis B of V and define $\tilde{B} := \bigcup_{g \in G} gB$ and let

$$M = R \cdot \tilde{B}$$

M is a finitely generated free R-module since V is a free module over R which is a PID.

Let T be a free R-basis for M, we will prove that T is a basis for V. To see that T spans V, consider

$$V \ni x = \sum_{b \in \tilde{B}}' \lambda_b b = \sum_{b \in \tilde{B}}' \frac{p_b}{q_b} b \qquad \text{for some } p_b, q_b \in R$$

$\lambda_b \in k$

Let $c = \prod_b q_b$, thus $cx \in M$ so

$$cx = \sum_{t \in T}' r_t t \qquad \text{since } T \text{ is a basis}$$

and so $$x = \sum_{t \in T}' \frac{r_t}{c} t \in \operatorname{span}_k T.$$

To see T is lin ind, suppose

$$\sum_{t\in T}' \frac{p_t}{q_t} t = 0 \qquad \text{for some } p_t\, q_t \in R$$

and let $c = \prod_{t\in T} q_t$, then

$$\mathrm{span}_R T \ni \sum_{t\in T}' c \frac{p_t}{q_t} t = 0$$

and so $c\frac{p_t}{q_t} = 0$ for all t, since T is an R-basis for M.

Since R is an integral domain, $p_t/q_t = 0$ and so T is lin. ind. $\square$.

Cor If W is a rational reflection group, then W is a crystallographic refl. grp.

proof: $W \subseteq GL(V_{\mathbb{Q}})$ for a fin. dim. $\mathbb{Q}$-vector sp. $V_{\mathbb{Q}}$. By the above theorem, there exists a $\mathbb{Z}$-lattice $M \subset V_{\mathbb{Q}}$, preserved by W. $\square$.

Recall:

Def A root system Φ is crystallographic if

$$\langle \beta\, \alpha \rangle := \frac{2(\alpha\ \beta)}{(\alpha\ \alpha)} \in \mathbb{Z} \quad \text{for } \alpha, \beta \in \Phi.$$

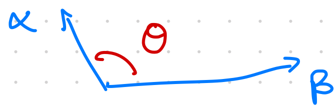
and we say Φ and Φ' are isomorphic if there is an isomorphism of vector spaces

$$\begin{array}{ccc} f: V & \longrightarrow & V' \\ \cup\!| & & \cup\!| \\ \Phi & & \Phi' \end{array}$$

such that $f(\Phi) = \Phi'$ and $\langle f(\beta), f(\alpha) \rangle = \langle \beta, \alpha \rangle$ for all $\alpha\ \beta \in \Phi$

Remark Any root system Φ is isomorphic to $\lambda\Phi$ for any $\lambda \in \mathbb{R} - \{0\}$.

Consider two simple roots $\alpha, \beta \in \Delta \subseteq \Phi$



$$\langle \beta\, \alpha \rangle = \frac{2(\alpha\ \beta)}{(\alpha\ \alpha)}$$

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \frac{4(\alpha, \beta)^2}{\|\alpha\|^2 \|\beta\|^2} = 4 \cos^2 \theta$$

If Φ is Crystallographic, this must be an integer! So $\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ and the corresponding values of $\langle \alpha\, \beta \rangle \langle \beta\, \alpha \rangle$ are

$$\langle \alpha\, \beta \rangle \langle \beta\, \alpha \rangle = 0,\ 1,\ 2,\ 3$$

But $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ must themselves be negative integers (since the angle must be obtuse).

We can summarise this in a table

$4\cos^2\theta$	$\langle \alpha\ \beta \rangle$	$\langle \beta\ \alpha \rangle$	θ	$\|\beta\|^2 / \|\alpha\|^2$
0	0	0	$\frac{\pi}{2}$	*
1	-1	-1	$\frac{2\pi}{3}$	1
2	-1 or -2	-2 or -1	$\frac{3\pi}{4}$	2 or $1/2$
3	-1 or -3	-3 or -1	$\frac{5\pi}{6}$	3 or $1/3$

The last column is calculated using $\langle \beta, \alpha \rangle = 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta$ so

$$\langle \beta\ \alpha \rangle^2 = 4\cos^2\theta \frac{\|\beta\|^2}{\|\alpha\|^2}$$

This leads to an amazing fact:

(i) vertices in the Coxeter graph of Φ connected by an edge $\circ\!\!-\!\!\!-\!\!\circ$ weight 3, must be equal length roots.

(ii) vertices connected by $\circ\overset{4}{-\!\!-}\circ$ wt 4, must be roots with square lengths in ratio 2.

iii) vertices connected by $\circ\overset{6}{-\!\!-}\circ$ wt 6 must be roots with square length in ratio 3.

It is clear that any isometry gives an isomorphism of a root system, thus since the Coxeter graph determines all the angles between all the roots, we only need to know the lengths of each root to completely determine Φ.

Example (i) There is a single (up to iso) root system with Coxeter graph

$$\circ - \circ - \circ - \cdots - \circ$$

This is because, whatever the length of the first simple root, the fact above demonstrates all other roots must have equal length.

(ii) There are precisely two root systems with Coxeter graph

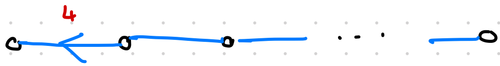
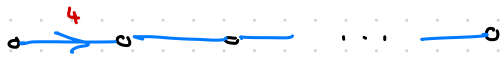
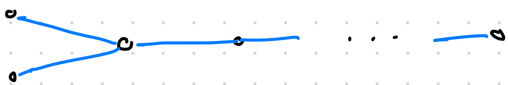
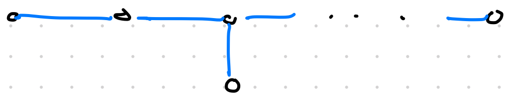
$$\circ \overset{4}{-\!-} \circ - \circ - \circ \cdots - \circ$$

If we fix the square length of the first root, the square length of all the others must be either twice, or half this length.

These examples demonstrate

Prop A crystallographic root system is determined by its Coxeter graph decorated with an arrow on each $\circ \overset{4}{-\!-} \circ$ or $\circ \overset{6}{-\!-} \circ$ edge, pointing to the longest root.
This is called its Dynkin diagram

Thm The crystallographic root systems such that span $\Phi \cong V$ are

Type	Dynkin diagram	
A_n		$n \geq 1$
B_n		$n \geq 2$
C_n		$n \geq 3$
D_n		$n \geq 4$
$E_{n=6,7,8}$		$n = 6, 7, 8$
F_4		
G_2		

proof: All that one needs to do is check that a crystallographic root system exists which we have done (except in type E).

Rmk This means each of the corresponding reflection groups is rational. Note that the reflection groups in type B and C are isomorphic but their root systems are not!

$$\langle \beta, \alpha \rangle = \frac{2(\beta\,\alpha)}{(\alpha\,\alpha)}$$

Lecture 18

Some lattices. Let Φ be a crystallographic root system with simple roots Δ.

Def (i) The lattice $\Lambda_{root} := \mathbb{Z}\Phi$ is called the root lattice

(ii) A lattice $\Lambda \subseteq V$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for every $\alpha \in \Phi$ is called a choice of weight lattice (eg Λ_{root} is a weight lattice)

Example If $\operatorname{span}\Phi = V$ (call this semisimple.) then define

$$\Lambda_{sc} = \{ \lambda \in V \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}, \alpha \in \Phi \}.$$

This is a weight lattice, and every weight lattice Λ has the property $\Lambda_{root} \subseteq \Lambda \subseteq \Lambda_{sc}$.

It is desirable to find a basis for Λ:

Def If $\alpha \in \Phi$, $\alpha^\vee := \frac{2}{(\alpha\alpha)}\alpha$ and if $\Delta = \{\alpha_i \mid i \in I\}$. A choice of fundamental weights ϖ_i $i \in I$ are vectors $\varpi_i \in V$ such that

$$(\varpi_i, \alpha_j^\vee) = \delta_{ij}$$

Example (i) Take $V = \mathbb{R}^2$ and $\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}_{i,j=1,2}$

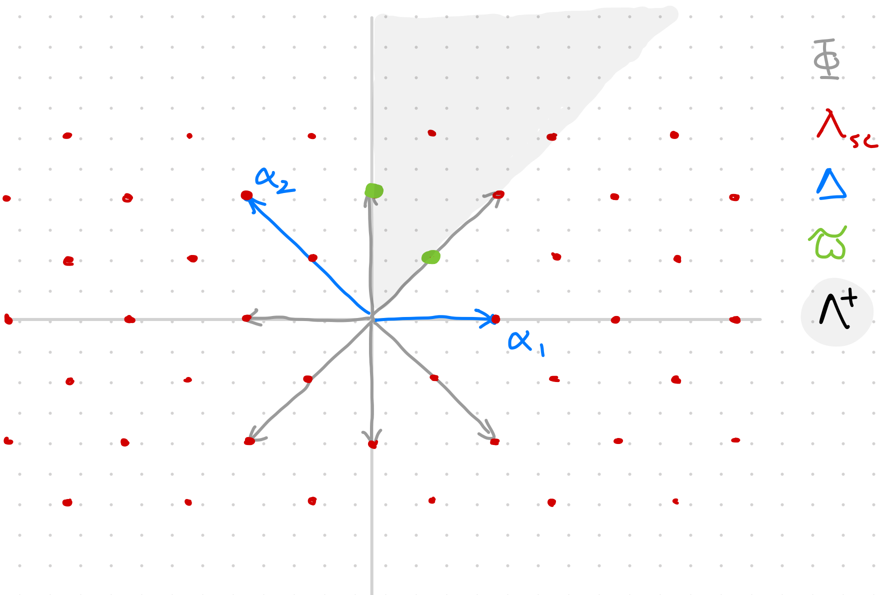
$\Delta = \{ \alpha_1 = \varepsilon_1, \alpha_2 = \varepsilon_2 - \varepsilon_1 \}$ $\quad \alpha_i^\vee, \varpi_i, \Lambda$

Then $\alpha_1^\vee = 2\varepsilon_1$, $\alpha_2^\vee = \varepsilon_2 - \varepsilon_1$ so

$\lambda = a\varepsilon_1 + b\varepsilon_2 \in \Lambda$ if

$$2a = (\lambda, \alpha_1^\vee) \in \mathbb{Z} \quad \text{ie} \quad a \in \tfrac{1}{2}\mathbb{Z}$$

$$b - a = (\lambda, \alpha_2^\vee) \in \mathbb{Z}.$$



If $\varpi_1 = a\varepsilon_1 + b\varepsilon_1$ then $2a = 1$, $b - a = 0$ $\quad \varpi_1 = \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2$

If $\varpi_2 = a\varepsilon_1 + b\varepsilon_2$ then $2a = 0$, $b - a = 1$ $\quad \varpi_2 = \varepsilon_2$

Def $\Lambda^+ = \{ \lambda \in \Lambda \mid \overset{\langle \lambda, \alpha_i \rangle}{\overset{\parallel}{(\lambda, \alpha_i^\vee)}} \geq 0 \text{ for all } i \in I \}$

Is called the dominant cone/chamber and its elements are dominat weights

We will call a root system semisimple if $\operatorname{span}\Phi = V$

Example $\Phi = \{ \alpha_{ij} := \pm\varepsilon_i \mp \varepsilon_j \mid i \neq j \} \subseteq \mathbb{R}^n$ (GL_n)

Take $\Delta = \{\alpha_i = \alpha_{i,i+1} \mid i = 1 \dots n-1\}$. $\alpha_i^\vee = \alpha_i$

$\lambda = \sum a_i \varepsilon_i \in \Lambda$ if $(\lambda, \alpha_i^\vee) = a_i - a_{i+1} \in \mathbb{Z}$.

This is not semisimple!!! This means

$$\Lambda_{sc} = \text{a lattice} + (\operatorname{span}\Phi)^\perp$$
$$\cong \mathbb{Z}^{n-1} + \mathbb{R}$$

We need to make a choice. Lets set

$$\Lambda = \mathbb{Z}^n$$

This certainly satisfies $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$.

Now we can set: $\varpi_i = \sum a_i \varepsilon_i$ $a_j - a_{j+1} = 0$ for $j \neq i$ and $a_i - a_{i+1} = 1$, thus

$$\varpi_i = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i \qquad 1 \leq i \leq n$$

This is the only non semisimple root system we work with.

Lecture 19 Crystals

We fix

- Φ a crystallographic root system
- $\Delta = \{\alpha_i \mid i \in I\}$ a set of simple roots indexed by a set I.
- Λ a weight lattice (usually Λ_{sc})
- ϖ_i, $i \in I$, a set of fundamental weights

Def A Kashiwara crystal (or just crystal) is a set B, with functions

$$e_i, f_i : B \longrightarrow B \sqcup \{0\}$$
$$\varepsilon_i \ \varphi_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\} \qquad i \in I$$
$$wt : B \longrightarrow \Lambda$$

satisfying

(A1) If $x, y \in B$, then $e_i(x) = y \iff x = f_i(y)$

(A1.5) If $e_i(x) = y$ then

$$wt(y) = wt(x) + \alpha_i, \quad \varepsilon_i(y) = \varepsilon_i(x) - 1 \quad \text{and}$$
$$\varphi_i(y) = \varphi_i(x) + 1$$

(A2) $\varphi_i(x) = \langle wt(x), \alpha_i^\vee \rangle + \varepsilon_i(x)$.

If $\varphi_i(x) = -\infty$ then we must have $e_i(x) = f_i(x) = 0$.

Def A crystal $\mathcal{B}$ is

- finite type if $\varphi_i(x), \varepsilon_i(x) \neq -\infty \quad \forall x \in B$
- seminormal if

$$\varphi_i(x) = \max\{k \in \mathbb{N} \mid f_i^k(x) \neq 0\}$$

$$\varepsilon_i(x) = \max\{k \in \mathbb{N} \mid e_i^k(x) \neq 0\}$$

The crystal graph of $\mathcal{B}$ is the coloured graph with vertices B and edges $x \xrightarrow{i} y$ if $f_i(x) = y$.

If B is seminormal then B is determined by its graph plus the data of $wt(x)$ for every maximal element x.

We can use graph theoretic language to describe B: connected, connected components, paths, etc.

Rmk For $\lambda, \mu \in \Lambda$ we say $\lambda \preceq \mu$ if

$$\lambda - \mu \in \mathbb{N}\Delta$$

(e.g. if $\beta \in \Pi$ then $\beta \succ 0$)

Def If $b \in B$, we say b is a highest weight element (hwe) if $e_i(b) = 0$ for all $i \in I$.

Proposition Let B be a seminormal crystal and $b \in B$ a hwe. Then $wt(b) \in \Lambda^+$ (ie it is a dominant weight)

proof: Since $e_i(b) = 0$, we must have $\varepsilon_i(b) = 0$, so

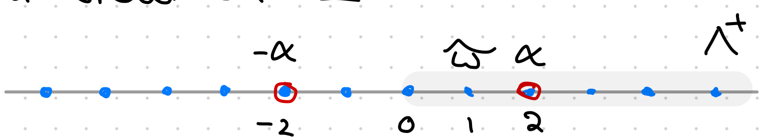
$$0 < \varphi_i(b) = \varphi_i(b) - \varepsilon_i(b) = \langle wt(b), \alpha_i^\vee \rangle$$

for all $i \in I$, so $wt(b)$ is dominant. $\square$

Example A_1 root system $\Phi = \{\pm\alpha = \pm 2\} \subseteq \mathbb{R} = V$. with $\langle \lambda \mu \rangle = \frac{1}{2}\lambda\mu$, $\Delta = \{2\}$ and $\alpha^\vee = 2$

$$\Lambda = \Lambda_{sc} = \{\lambda \in \mathbb{R} \mid \langle \lambda, \alpha^\vee \rangle = \lambda \in \mathbb{Z}\} = \mathbb{Z}$$

Then $\langle \varpi, \alpha^\vee \rangle = \varpi = 1$, $\Lambda^+ = \mathbb{N}$ and $\preceq$ is the normal order on $\mathbb{Z}$.



Let B be a connected seminormal crystal with a single highest weight element b, $wt(b) = k$

$$\varphi(b) = \varphi(b) - \varepsilon(b) = \langle wt(b), \alpha_i^\vee \rangle = \frac{1}{2}k \cdot 2 = k$$

So our crystal is completely determined!

$$\underset{k}{b} \longrightarrow \underset{k-2}{fb} \longrightarrow \underset{k-4}{f^2b} \longrightarrow \cdots \longrightarrow \underset{-k}{f^k b}$$

wt: k $\quad k-2 \quad k-4 \quad -k$

These are the objects of $\mathcal{D}_0$!

Def A morphism of crystals $f: B \longrightarrow C$ is a map $\pi: B \longrightarrow C$ such that

(1) $wt(\pi(b)) = wt(b)$

(2) $\varepsilon_i(\pi(b)) = \varepsilon_i(b)$ and $\varphi_i(\pi(b)) = \varphi_i(b)$

(3) $f_i\pi(b) = \pi(f_i b)$ and $e_i\pi(b) = \pi(e_i b)$.

This make the collection of crystals for (Φ, Δ, Λ) a category. Isomorphisms are bijections.

Def If B and C are crystals, we define a crystal $B \otimes C$ with elements $b \otimes c$, $b \in B, c \in C$

$$wt(b \otimes c) = wt(b) + wt(c)$$

$$\varphi_i(b \otimes c) = \max\{\varphi_i(b), \varphi_i(c) + \langle wt(b), \alpha_i^\vee \rangle\}$$

$$\varepsilon_i(b \otimes c) = \max\{\varepsilon_i(c), \varepsilon_i(b) - \langle wt(c), \alpha_i^\vee \rangle\}$$

$$f_i(b \otimes c) = \begin{cases} f_i b \otimes c & \text{if } \varphi_i(c) \leq \varepsilon_i(b) \\ b \otimes f_i c & \text{if } \varphi_i(c) > \varepsilon_i(b) \end{cases}$$

$$e_i(b\otimes c)=\begin{cases} e_i b\otimes c & \text{if } \varphi_i(c)<\varepsilon_i(b) \\ b\otimes e_i c & \text{if } \varphi_i(c)\geq\varepsilon_i(b)\end{cases}$$

Prop $B\otimes C$ is a crystal.

proof: We must check the crystal axioms

A1 ($e_i x=y \iff x=f_i y \dots$)

Suppose $e_i(b\otimes c)=b'\otimes c'$.

Case 1: $\varphi_i(c)<\varepsilon_i(b)$, then $b'\otimes c'=e_i b\otimes c$, so

$\varphi_i(c')=\varphi_i(c)\leq\varepsilon_i(b)-1=\varepsilon_i(e_i b)=\varepsilon_i(b')$ so

$$f_i(b'\otimes c')=f_i b'\otimes c'=f_i e_i b\otimes c=b\otimes c$$

Case 2 and $f_i(b\otimes c)=b'\otimes c'$ similar.

$$wt(e_i(b\otimes c))=wt(b')+wt(c')=wt(b\otimes c)+\alpha_i$$

Similar for φ_i and ε_i.

A2 ($\varphi_i(x)=\langle wt(x),\alpha_i^\vee\rangle+\varepsilon_i(x)$):

We calculate $\varphi_i(b\otimes c)$

Case 1 $\varphi_i(b\otimes c)=\varphi_i(b)$ then

$$\varepsilon_i(b)+\langle wt(b),\alpha_i^\vee\rangle=\varphi_i(b)\geq\varphi_i(c)+\langle wt(b),\alpha_i^\vee\rangle$$
$$=\varepsilon_i(c)+\langle wt(b),\alpha_i^\vee\rangle+\langle wt(c),\alpha_i^\vee\rangle.$$

So $\varepsilon_i(b)-\langle wt(c),\alpha_i^\vee\rangle\geq\varepsilon_i(c)$

and thus

$$\varepsilon_i(b\otimes c) = \varepsilon_i(b) - \langle wt(c), \alpha_i^\vee\rangle$$

but $wt(b\otimes c) = wt(b) + wt(c)$ so

$$\varepsilon_i(b\otimes c) + \langle wt(b\otimes c), \alpha_i^\vee\rangle = \varepsilon_i(b) - \langle wt(c), \alpha_i^\vee\rangle + \langle wt(b), \alpha_i^\vee\rangle + \langle wt(c), \alpha_i^\vee\rangle$$

$$= \varepsilon_i(b) + \langle wt(b), \alpha_i^\vee\rangle$$

$$= \varphi_i(b) = \varphi_i(b\otimes c)$$

<u>Case 2</u>: similar $\square$.

Prop If B and C are seminormal so is $B\otimes C$

proof: Similar arguments.

Lecture 19

We fix

- Φ a crystallographic root system
- $\Delta = \{\alpha_i \mid i \in I\}$ simple roots
- Λ a weight lattice
- ϖ_i, $i \in I$ fundamental weights

(A root datum)

Note $\Phi \subseteq V$ a real inner product space with inner product $\langle -,- \rangle$ (to match conventions in Bump-Schilling).

Note if $\alpha \in \Phi$, $\alpha^\vee := \frac{2}{\langle \alpha, \alpha \rangle} \alpha$ and so

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

Def A Kashiwara crystals (or just crystal) is a set B with functions

- $e_i, f_i : B \longrightarrow B \sqcup \{0\}$ ← crystal operators.
- $\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}$
- $\mathrm{wt} : B \longrightarrow \Lambda$

$i \in I$

one for each simple root.

satisfying

(A1) if $x, y \in B$ then $e_i x = y \iff x = f_i y$

(A1.5) If $e_i x = y$ then

$\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i$, $\varepsilon_i(y) = \varepsilon_i(x) - 1$, $\varphi_i(y) = \varphi_i(x) + 1$

(A2) $\varphi_i(x) = \varepsilon_i(x) + \langle \mathrm{wt}(x), \alpha_i^\vee \rangle$

and if $\varphi_i(x) = -\infty$ then $e_i(x) = f_i(x) = 0$

Def A crystal B is

- seminormal if

$$\varphi_i(x) = \max\{k \in \mathbb{N} \mid f_i^k x \neq 0\}$$

$$\varepsilon_i(x) = \max\{k \in \mathbb{N} \mid e_i^k x \neq 0\}.$$

- finite type if $\varphi_i(x), \varepsilon_i(x) \neq -\infty \quad \forall x \in B$.

Exercise seminormal $\Rightarrow$ finite type.

The crystal graph of B is the directed coloured graph with vertices B and an arrow $x \xrightarrow{i} y$ whenever $f_i x = y$.

We say B is connected whenever its graph is.

Rmk If $\lambda, \mu \in \Lambda$, we say $\lambda \succeq \mu$ if

$$\lambda - \mu \in \mathbb{N}\Delta$$ ← non-negative span of the simple roots.

Def If $b \in B$, we say b is highest weight if

$$e_i b = 0 \quad \forall i \in I$$

(ie no incoming arrows to b).

Prop Let B be a seminormal crystal an b a highest weight element, then

$$wt(b) \in \Lambda^+ \quad \text{(is a dominant weight)}$$

proof: recall $\Lambda^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0\}$. Since $e_i b = 0$, we must have $\varepsilon_i(b) = 0$ so

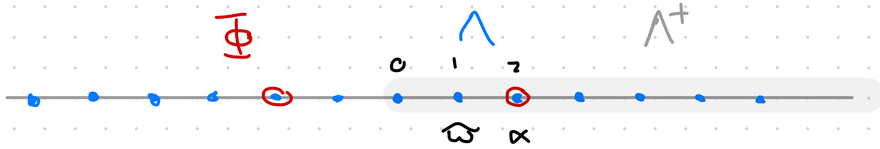
$$0 \leq \varphi_i(b) = \varphi_i(b) - \varepsilon_i(b) = \langle wt(b), \alpha_i^\vee \rangle \qquad \square.$$

Example (A_1) $\Phi = \{\pm\alpha = \pm 2\} \subseteq \mathbb{R} = V$ with $\langle \lambda, \mu \rangle := \frac{1}{2}\lambda\mu$.

Then $\Delta = \{2\}$ and $\alpha = 2$, $\alpha^\vee = 2$ and $\langle \lambda, \alpha^\vee \rangle = \frac{1}{2}\lambda \cdot 2 = \lambda$

$$\Lambda = \Lambda_{sc} = \{\lambda \in \mathbb{R} \mid \langle \lambda, \alpha^\vee \rangle = \lambda \in \mathbb{Z}\} = \mathbb{Z}.$$

Then $\langle \varpi, \alpha^\vee \rangle = \varpi = 1$, $\Lambda^+ = \mathbb{N}$.



Let B be a connected seminormal crystal. B has either no, or a single highest weight element.

Suppose B has a single highest weight element $b \in B$, with $wt(b) = k \in \mathbb{N}$.

$$\varphi(b) = \varphi(b) - \underset{\substack{\| \\ 0}}{\varepsilon(b)} = \langle wt(b), \alpha^\vee \rangle = wt(b) = k$$

So the crystal graph is

$$\begin{array}{ccccccccc} & b & \longrightarrow & fb & \longrightarrow & f^2 b & \longrightarrow \cdots \longrightarrow & f^k b \\ wt & k & & k-2 & & k-4 & & -k \end{array}$$

These are the objects of D_0!

1. Let $W = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \sum_i a_i = 0 \right\} \subseteq \mathbb{R}^n$.

find a basis for W.

1) Make a list of hints.

- Give me an example of $x \in W$,
- How do you find a basis / what is a basis

  find a spanning set, make it lin. ind.
- Find a vector lin. ind w/ x.
- Repeat it?
- Is it spanning.

2. Find the nullity of $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y+z \\ y+z \\ 2z-x \end{pmatrix}$.

- Can you find v, st $T(v)=0$
- What is the nullity $\rightsquigarrow$ dim of null(T).
- Matrix of T.
- find a basis of null space...

What if no-one is speaking?
- easier and easier questions
- stay silent.
- VERY LAST RESORT: give a partial answer

One person dominates
- Directing questions
- What do others think?
- Ask dominant person to explain to the others.
- Have a scribe

Sidetracked conversation
- Maybe not a bad thing.
- Steer the conversation back.

No one understands whats going on
- Go back and explain first principles
- Change question to something simpler

Specifics to SDU
- Style of doing mathematics
- English language ⇝ alot of interaction will be via chat.
  Speak + type
- Some majors are much weaker.

- Students will be very good at computational tasks.
- Can struggle with conceptual tasks.
- Students will be in a classroom w/ personal device.

Lecture 20

Def A (strict) morphism of crystals $\pi : B \longrightarrow C$ is a function such that

(1) $wt(\pi(b)) = wt(b)$

(2) $\varphi_i(\pi(b)) = \varphi(b)$ and $\varepsilon_i(\pi(b)) = \varepsilon_i(b)$

(3) $f_i \pi(b) = \pi(f_i b)$ and $e_i(\pi(b)) = \pi(e_i b)$.

(where $\pi(0) = 0$).

Rmk (1) B and C should be crystals for the same root datum.

(2) This is a restrictive definition

(3) This makes the collection of crystals for $(\Phi, \Delta, \Lambda, \varpi_c)$ a category.

(4) The isomorphisms are bijections.

Def If B and C are crystals, we define a crystal $B \otimes C$ with elements $b \otimes c$, $b \in B$, $c \in C$

$$wt(b \otimes c) = wt(b) + wt(c)$$

$$\varphi_i(b \otimes c) = \max\{\varphi_i(b), \varphi_i(c) + \langle wt(b), \alpha_i^\vee \rangle\}$$

$$\varepsilon_i(b \otimes c) = \max\{\varepsilon_i(c), \varepsilon_i(b) - \langle wt(c), \alpha_i^\vee \rangle\}$$

$$f_i(b \otimes c) = \begin{cases} f_i b \otimes c & \text{if } \varphi_i(c) \leq \varepsilon_i(b) \\ b \otimes f_i c & \text{if } \varphi_i(c) > \varepsilon_i(b) \end{cases}$$

$$e_i(b\otimes c)=\begin{cases} e_i b\otimes c & \text{if } \varphi_i(c)<\varepsilon_i(b)\\ b\otimes e_i c & \text{if } \varphi_i(c)\geq\varepsilon_i(b)\end{cases}$$

Example (GL_n) $\Phi=\{\alpha_{ij}=d_i-d_j\}\subseteq\mathbb{R}^n$, $\Delta=\{\alpha_i=\alpha_{i,i+1}\}$

$\Lambda=\mathbb{Z}^n$, $\varpi_i=d_1+d_2+\dots+d_i$ $\quad i=1\dots n-1$

$$\begin{array}{l} B_{\varpi_1} \\ \parallel \\ B_{(1)} \end{array}\qquad
\begin{tikzcd}
\boxed{1} \arrow[r, "f\ 1"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[r, "3"] & \cdots \arrow[r, "n-1"] & \boxed{n} \\
\varpi_1 = d_1 & d_2 & d_3 & & d_n
\end{tikzcd}$$

Example $B_{(1)}\otimes B_{(1)}$ for (GL_3)

$$\begin{tikzcd}
 & \overset{\varphi_2:\ 1\,0}{\underset{\varepsilon_2:\ 0\,0}{\boxed{1}}} \arrow[r] & \overset{0\,1}{\underset{1\,0}{\boxed{2}}} \arrow[r] & \overset{0\,0}{\underset{0\,1}{\boxed{3}}} \\
\overset{\varphi_2}{1\ 0}\ \boxed{1}\ \overset{\varepsilon_2}{0\ 0} \arrow[d] & \boxed{1}\cdot\boxed{1} \arrow[r] & \boxed{1}\cdot\boxed{2} \arrow[r] \arrow[d] & \boxed{1}\cdot\boxed{3} \arrow[d] \\
0\ 1\ \boxed{2}\ 1\ 0 \arrow[d] & \boxed{2}\cdot\boxed{1} \arrow[d] & \boxed{2}\cdot\boxed{2} \arrow[r] & \boxed{2}\cdot\boxed{3} \arrow[d] \\
0\ 0\ \boxed{3}\ 0\ 1 & \boxed{3}\cdot\boxed{1} \arrow[r] & \boxed{3}\cdot\boxed{2} & \boxed{3}\cdot\boxed{3}
\end{tikzcd}$$

Question For (GL_n) can we find a combinatorial rule for the arrows in $B_{(1)}^{\otimes n}$. Can we identify the connected components?

Prop $B \otimes C$ is a crystal.

proof. We check the crystal axioms.

(A1: $e_i x = y \Leftrightarrow x = f_i y$).

Suppose $e_i(b \otimes c) = b' \otimes c'$.

<u>Case 1</u> $b' = e_i b$ and $c' = c$, ie $\varphi_i(c) < \varepsilon_i(b)$.

We will calculate $f_i(b' \otimes c')$.

$\varphi_i(c') = \varphi_i(c) \leq \varepsilon_i(b) - 1$. $=$ $\varepsilon_i(e_i b) = \varepsilon_i(b')$

by crystal axioms for B.

So $f_i(b' \otimes c') = f_i b' \otimes c' = b \otimes c$.

<u>Case 2</u> and $f_i(b \otimes c) = b' \otimes c'$ follow by similar arguments.

(A1.5: $wt(e_i x)$, $\varepsilon_i(e_i x)$, $\varphi_i(e_i x)$)

$$wt(e_i(b \otimes c)) = \begin{cases} wt(e_i b \otimes c) = wt(e_i b) + wt(c) \\ wt(b \otimes e_i c) = wt(b) + wt(e_i c) \end{cases} = wt(b \otimes c) + \alpha_i$$

similar for ε_i, φ_i.

(A2: $\varphi_i(x) = \varepsilon_i(x) + \langle wt(x), \alpha_i^\vee \rangle$).

We need to calculate $\varphi_i(b \otimes c)$

$$\varphi_i(b \otimes c) = \max\{\varphi_i(b), \varphi_i(c) + \langle wt(b), \alpha_i^\vee \rangle\}$$

$$\varepsilon_i(b \otimes c) = \max\{\varepsilon_i(c), \varepsilon_i(b) - \langle wt(c), \alpha_i^\vee \rangle\}$$

<u>Case 1</u>: $\varphi_i(b \otimes c) = \varphi_i(b)$, then

$$\varepsilon_i(b) + \langle wt(b), \alpha_i^\vee \rangle = \varphi_i(b) \geq \varphi_i(c) + \langle wt(b), \alpha_i^\vee \rangle$$
$$= \varepsilon_i(c) + \langle wt(c), \alpha_i^\vee \rangle + \langle wt(b), \alpha_i^\vee \rangle$$

so $\varepsilon_i(b) - \langle wt(c), \alpha_i^\vee \rangle \geq \varepsilon_i(c)$

so $\varepsilon_i(b \otimes c) = \varepsilon_i(b) - \langle wt(c), \alpha_i^\vee \rangle$.

$$\varepsilon_i(b \otimes c) + \langle wt(b \otimes c), \alpha_i^\vee \rangle = \varepsilon_i(b) - \langle wt(c), \alpha_i^\vee \rangle + \langle wt(b), \alpha_i^\vee \rangle + \langle wt(c), \alpha_i^\vee \rangle$$
$$= \varepsilon_i(b) + \langle wt(b), \alpha_i^\vee \rangle$$
$$= \varphi_i(b) = \varphi_i(b \otimes c)$$

<u>Case 2</u> Similar $\square$.

Prop If B and C are seminormal, so is $B \otimes C$.

proof: similar arguments.

$(GL_n) \rightsquigarrow \{\alpha_{ij}\}$, $\Lambda = \mathbb{Z}^n$, $\varpi_i = \underline{e}_1 + \underline{e}_2 + \dots + \underline{e}_i$

$$\Lambda^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0\}$$

$$\langle \lambda, \alpha_i \rangle = \langle \lambda, \underline{e}_i - \underline{e}_{i+1} \rangle = \lambda_i - \lambda_{i+1}$$

$$= \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$$

$$= \mathcal{P}_{n-1} \times \mathbb{Z}.$$

$\mathcal{P}_n$ = partions with at most n parts

$\hookrightarrow \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$

For $1 \leq k \leq n$ $\quad (k), (\underbrace{1, 1 \dots 1}_{k}) = (1^k) \in \mathcal{P}_n$

Exemple $B_{(k)}$: elements $\underbrace{\boxed{i_1 \mid i_2 \mid i_3 \mid \dots \mid i_k}}_{k}$

s.t. $1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n$

$\varphi_i(\boxed{i_1 \mid i_2 \mid \dots}) = \#$ of i's

$\varepsilon_i(\boxed{i_1 \mid i_2 \mid \dots}) = \#$ of $i{+}1$'s

$\mathrm{wt}(\boxed{i_1 \mid i_2 \mid \dots}) = \sum_i \varphi_i(-)\,\underline{e}_i$

$$f_i(\boxed{i_1 \mid i_2 \mid \dots}) = \begin{cases} \text{change the rightmost } i \text{ to } i{+}1 \\ 0 \quad \text{if no } i\text{'s.} \end{cases}$$

e.g. GL_3

$\boxed{1 \mid 1 \mid 2}$

$\varphi_1(-) = 2$

$\mathrm{wt}(-) = (2,1,0)$

$f_1(-) = \boxed{1 \mid 2 \mid 2}$

$f_2(-) = \boxed{1 \mid 1 \mid 3}$

Exercise: draw the graph of $B_{(3)}$ for (GL_3)
label with wts.

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \otimes \boxed{1}$

$$\boxed{2} \otimes \boxed{1\ 1} \cong \overset{112}{\begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix}} \quad (2\ 1\ 0)$$

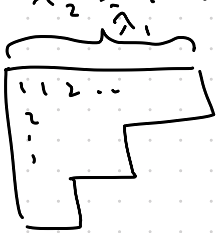
$$\begin{tikzcd}
 & \begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix} \ (2\ 1\ 0) \arrow[dl, "1"'] \arrow[dr] & \\
(1\ 2\ 0)\ \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix} \arrow[d, dotted] & & \begin{smallmatrix} 1 & 1 \\ 3 & \end{smallmatrix}\ (2\ 0\ 1) \arrow[d, dotted] \\
(1\ 1\ 1)\ \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} \arrow[d, "2"'] & & \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}\ (1\ 1\ 1) \arrow[d] \\
(1\ 0\ 2)\ \begin{smallmatrix} 1 & 3 \\ 3 & \end{smallmatrix} \arrow[dr] & & \begin{smallmatrix} 2 & 2 \\ 3 & \end{smallmatrix}\ (0\ 2\ 1) \arrow[dl] \\
 & \begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix}\ (0\ 1\ 2) &
\end{tikzcd}$$

$B_{(1)}^{\otimes 3}$

$B_{(1)} \otimes B_{(2)}$

$B_{(1^k)} \otimes B_{(1)}$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$$



Lecture 21

We want to understand the tensor product rule better. Consider three crystals: A, B, C.

If $a \in A$, $b \in B$, $c \in C$. What is

$$f_i((a \otimes b) \otimes c) \; ? \qquad (\text{in } (A \otimes B) \otimes C).$$

Recall

$$f_i(x \otimes y) = \begin{cases} f_i x \otimes y & \varphi_i(y) \leq \varepsilon_i(x) = \varphi_i(x) - \langle wt(x), \alpha_i^\vee \rangle \\ & \varphi_i(y) + \langle wt(x), \alpha_i^\vee \rangle \leq \varphi_i(x) \\ x \otimes f_i y & \varphi_i(y) > \varepsilon_i(x) \\ & \varphi_i(y) + \langle wt(x), \alpha_i^\vee \rangle > \varphi_i(x) \end{cases}$$

Reinterpreting:

$\varphi_i(x)$	$f_i x \otimes y$
$\varphi_i(y) + \langle wt(x), \alpha_i^\vee \rangle$	$x \otimes f_i y$

we look at the max of these

For $(A \otimes B) \otimes C$:

$\varphi_i(a)$	$f_i a \otimes b \otimes c$
$\varphi_i(b) + \langle wt(a), \alpha_i^\vee \rangle$	$a \otimes f_i b \otimes c$
$\varphi_i(c) + \langle wt(a), \alpha_i^\vee \rangle + \langle wt(b), \alpha_i^\vee \rangle$	$a \otimes b \otimes f_i c$

Act with f_i in accordance with the first row where the maximum value occurs.

Check We get same table for $A \otimes (B \otimes C)$!

Prop The map $(a \otimes b) \otimes c \longmapsto a \otimes (b \otimes c)$ is an isomorphism of crystals

$$(A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

Thus the category of all crystals (for a fixed root datum) is a monoidal category

proof: The above shows that the bijection commutes with f_i (and thus with e_i). It follows from the definitions that φ_i, ε_i, and wt are preserved. $\square$.

Generalising the above picture to a tensor product of crystals:

$$B_1 \otimes B_2 \otimes \dots \otimes B_m$$

Prop If $x_1 \otimes x_2 \otimes \cdots \otimes x_m \in B_1 \otimes B_2 \otimes \cdots \otimes B_m$ then

$$\varphi_i(x_1 \otimes \cdots \otimes x_m) = \max\left\{ \varphi_i(x_j) + \sum_{k=1}^{j-1} \langle wt(x_k), \alpha_i^\vee \rangle \;\middle|\; j = 1 \dots m \right\}$$

and

$$f_i(x_1 \otimes \cdots \otimes x_m) = x_1 \otimes \cdots \otimes f_i x_r \otimes \cdots \otimes x_m$$

where $1 \leq r \leq m$ is the minimal value where the expression above acheives its maximum.

proof: by induction $\square$.

Exercise Determine a similar statement for ε_i and e_i.

Recall the (GL_n) crystal $B_{(1)}$

$$\begin{array}{lccccccccc}
f & \boxed{1} & \xrightarrow{1} & \boxed{2} & \xrightarrow{2} & \boxed{3} & \xrightarrow{3} & \cdots & \xrightarrow{n-1} & \boxed{n} \\
wt: & \underline{e}_1 & & \underline{e}_2 & & \underline{e}_3 & & & & \underline{e}_n
\end{array}$$

We will interpret the above rule for $B_{(1)}^{\otimes m}$.
What is

$$f_i\left(\boxed{x_1} \otimes \boxed{x_2} \otimes \cdots \otimes \boxed{x_m}\right) \;?$$

Note: $\varphi_i(\boxed{x}) = \delta_{xi}$ and

$$\langle wt(\boxed{x}), \alpha_i^\vee \rangle = \langle \underline{e}_x, \underline{e}_i - \underline{e}_{i+1} \rangle = \begin{cases} 0 & \text{if } x \neq i, i+1 \\ 1 & \text{if } x = i \\ -1 & \text{if } x = i+1 \end{cases}$$

Example

$$\varphi_2\left(\boxed{3} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{1} \right)$$

$0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 2 \quad 1$

$$= \boxed{3} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes f_2\boxed{2} \otimes \boxed{3} \otimes \boxed{1}$$

$$= \boxed{3} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{3} \otimes \boxed{1}$$

Case 1: If all i's appear to the left of all $i+1$'s.

The maximum first occurs at the rightmost i

So f_i changes the rightmost i to an $i+1$.

Case 2: If case 1 doesn't apply.

There exists a sequence

$$\xi = \boxed{x_a} \otimes \boxed{x_{a+1}} \otimes \cdots \otimes \boxed{x_b}$$

$0 \quad -1 \quad -1 \cdots \quad 0$

where $x_a = i+1$, $x_b = i$, and no i's or $i+1$'s occur between. Then

$$f_i\left(\boxed{x_1} \otimes \boxed{x_2} \otimes \cdots \otimes \boxed{x_n}\right) = f_i\left(\boxed{x_1} \otimes \cdots \otimes \boxed{x_{a-1}} \otimes \xi \otimes \boxed{x_{b+1}} \otimes \cdots \otimes \boxed{x_m}\right)$$

Note $\varphi_i(\xi) = 0$ and $f_i(\xi) = 0$

Now we repeat by induction.

Algorithm for calculating $f_i\left(\boxed{x_1} \otimes \dots \otimes \boxed{x_m}\right)$

1. Under each box put a $+$ for an $i+1$

   $-$ for an i

   nothing for other

2. Cancel pairs of $(+,-)$

3. Repeat 2 until no longer able to

4. Apply f_i to the rightmost i labbelled by a non-cancelled $-$

5. Or, if no $-$'s, $f_i(-) = 0$

Example

$$f_2\left(\boxed{1} \otimes \underset{+}{\boxed{3}} \otimes \underset{+}{\boxed{3}} \otimes \underset{-}{\boxed{2}} \otimes \underset{-}{\boxed{2}} \otimes \boxed{1} \otimes \underset{-}{\boxed{2}} \otimes \underset{+}{\boxed{3}}\right)$$

$$= \boxed{1} \otimes \boxed{3} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{3}$$

Example (GL_4) $\quad B_{(1)}^{\otimes 12}$

$\varphi_3(3)=1 \quad \varepsilon_3(3)=0$

$$f_3\big(\underset{+}{4} \cdot 1 \cdot \underset{-}{3} \cdot \underset{-}{3} \cdot \underset{+}{4} \cdot 2 \cdot 2 \cdot \underset{+}{4} \cdot \underset{-}{3} \cdot \underset{-}{3} \cdot \underset{+}{4} \cdot 2 \big)$$

$\varepsilon_3(4)=1$

$$= 4 \cdot 1 \cdot 3 \cdot \underline{4} \cdot 4 \cdot 2 \cdot 2 \cdot 4 \cdot 3 \cdot 3 \cdot 4 \cdot 2$$

Lecture 22

Thm (signature rule). If $x_1 \otimes x_2 \otimes \cdots \otimes x_m \in B_1 \otimes \cdots \otimes B_m$ then

$$f_i(x_1 \otimes \_ \otimes x_m) = x_1 \otimes \_ \otimes f_i x_k \otimes \_ \otimes x_m$$

where k is found using the following process.

1. Decorate each x_j with $\varphi_i(x_j)$ $-$'s, and $\varepsilon_i(x_j)$ $+$'s

$$x_j: \quad \underbrace{- - \cdots -}_{\varphi_i(x_j)}\ \underbrace{+ + \cdots +}_{\varepsilon_i(x_j)}$$

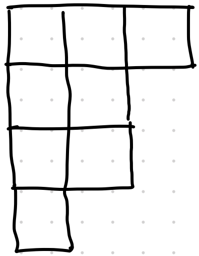
2. Inductively cancel $+-$ pairs until we are left with a sequence $- - \cdots - + + \cdots +$
3. k is the index associated with the rightmost $-$.

Tableaux

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a weakly decreasing sequence of positive integers.

The diagram of a partition is the left justified arrangement of boxes, with λ_i boxes in the i^{th} row

Example $(3\,2\,2\,1)$ is a partition of 8 with diagram



A semistandard tableau of shape λ is an arrangement of positive integers in the diagram of λ, such that

(i) rows are weakly increasing.
(ii) columns are strictly increasing.

A standard tableau is a semistandard tableau with strictly increasing rows.

Example Tableaux of shape $(3\,2\,2\,1)$

1	2	5
3	4	
6	8	
7		

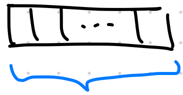
standard

1	1	2
2	3	
4	4	
5		

semistandard.

Special cases: $(GL_n) \curvearrowright \mathbb{C}^n$

$\lambda = (k)$

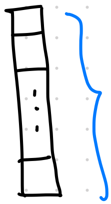


k boxes

$S^k\mathbb{C}^n \rightsquigarrow B_{(k)} = \left\{ \begin{array}{l} \text{semistandard tableaux} \\ \text{shape } \lambda = (k) \text{ with} \\ \text{entries } 1, 2, \dots, n \end{array} \right\}$

crystals

$\lambda = (1^k) = (1, 1, \dots, 1)$



k

$B_{(1^k)} = \left\{ \begin{array}{l} \text{semistandard tableaux} \\ \text{shape } \lambda = (1^k) \text{ with} \\ \text{entries } 1, 2, \dots, n \end{array} \right\} \rightsquigarrow \Lambda^k\mathbb{C}^n$

Goal Understand $B_{(1)}^{\otimes m}$

Prop The map $RR : B_{(k)} \longrightarrow B_{(1)}^{\otimes k}$ given by

$$RR\left(\boxed{x_1 \,|\, x_2 \,|\, \cdots \,|\, x_k}\right) = \boxed{x_1} \otimes \cdots \otimes \boxed{x_k}$$

is a morphism of crystals.

proof: $wt\left(\boxed{x_1 \,|\, \cdots \,|\, x_k}\right) = (\mu_1 \dots \mu_n)$ where

$$\mu_i = \#\{ j \mid x_j = i \}.$$

On the other hand

$$\mathrm{wt}\left(\boxed{x_1}\otimes\cdots\otimes\boxed{x_k}\right) = \sum_{j=1}^{k} \mathrm{wt}(\boxed{x_j})$$

$$= \sum_{j=1}^{k} \underline{e}_{x_j}$$

$$= (\mu_1 \cdots \mu_n)$$

So RR preserves wt. Note, if we can show RR commutes with e_i, f_i then it also preserves ε_i and φ_i, since both $B_{(k)}$ and $B_{(1)}^{\otimes k}$ are seminormal.

We have to show, if $T, T' \in B_{(k)}$ the

$$f_i T = T' \quad \text{if and only if} \quad f_i RR(T) = RR(T')$$

f_i changes the rightmost i of T to an $i+1$.

f_i acts on $RR(T)$ via the signature rule, but all i's occur before $i+1$'s in $RR(T)$, so f_i increases the rightmost i to an $i+1$.

e_i is handled similarly. $\square$.

For the (GL_n) root datum, and for a partition define

$$B_\lambda := \left\{ \begin{array}{l} \text{semistandard tableaux} \\ \text{shape } \lambda, \text{ with entries} \\ 1 \dots n. \end{array} \right\}.$$

Rmk $B_\lambda = \phi$ if λ has more than n parts.

Let $\lambda_1 + \lambda_2 + \dots + \lambda_r = m$, be the number of boxes.
Define a map

$$RR : B_\lambda \longrightarrow B_{(1)}^{\otimes m}$$

by letting $RR(T)$ be the row reading word of T:

Example If $T = \begin{array}{llll} 1 & 1 & 3 & 4 \\ 2 & 3 & 4 & \\ 3 & 4 & & \end{array}$ (so $\lambda = (4\ 3\ 2)$).

then $RR(T) = 3 \otimes 4 \otimes 2 \otimes 3 \otimes 4 \otimes 1 \otimes 1 \otimes 3 \otimes 4$
(read left to right, bottom to top).

We define the Yamanouchi tableau u_λ of shape λ, to be the tableau with all 1's the first row, 2's in the second row and so on.

Thm The image $RR(B_\lambda) \subseteq B_{(1)}^{\otimes m}$ is a connected component, with unique highest weight element u_λ.

proof: delayed.

A way to create a new tableau from old:

T = a semistandard tableau

x = a positive integer

$T \leftarrow x$ = the insertion tableau

Example $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} \end{array} \quad \leftarrow \quad x = 1$

$$\begin{array}{cccc} 1 & 1 & 1 & 4 \\ 2 & 3 & 3 & \\ 4 & 4 & & \end{array} \quad \begin{array}{c} \\ \leftarrow \; 2 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 1 & 4 \\ 2 & 2 & 3 & \\ 4 & 4 & & \end{array} \quad \begin{array}{c} \\ \\ \leftarrow \; 3 \end{array}$$

1 1 1 4

2 2 3

3 4

$\longleftarrow 4$

$$T \leftarrow 1 := \begin{ytableau} 1 & 1 & 1 & 4 \\ 2 & 2 & 3 \\ 3 & 4 \\ 4 \end{ytableau}$$

Insertion algorithm

$$T = \begin{array}{ccccc} 1 & 1 & 3 & 5 & 5 \\ 2 & 3 & 4 & 4 & \\ 3 & 4 & & & \end{array} \quad \longleftarrow 2$$

$$\begin{array}{ccccc} 1 & 1 & 2 & 5 & 5 \\ 2 & 3 & 4 & 4 & \\ 3 & 4 & & & \end{array} \quad \longleftarrow 3$$

$$\begin{array}{ccccc} 1 & 1 & 2 & 5 & 5 \\ 2 & 3 & 3 & 4 & \\ 3 & 4 & & & \end{array} \quad \longleftarrow 4$$

$$\begin{array}{ccccc} 1 & 1 & 2 & 5 & 5 \\ 2 & 3 & 3 & 4 & \\ 3 & 4 & 4 & & \end{array} \quad =: T \leftarrow 4.$$

ie an element of $B_{(1)}^{\otimes M}$

Def Let w be a word in $\{1, 2, \dots, n\}$. The P-symbol of w is: if $w = w_1 w_2 \cdots w_r$

$$P(w) := (\cdots((\phi \leftarrow w_1) \leftarrow w_2) \leftarrow \cdots \leftarrow w_r)$$

empty tableau

Example If $w = 132$

$$\phi \leftarrow 1 = \boxed{1}$$

$$\boxed{1} \leftarrow 3 = \boxed{1\,|\,3}$$

$$\boxed{1\,|\,3} \leftarrow 2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} = P(w)$$

ϕ	$1\ 3\ 2$
$\boxed{1}$	$3\ 2$
$\boxed{1\,|\,3}$,	2

Question Experiment with $P \circ RR$

- Do you have a guess for what this is?

  $P \circ RR = \mathrm{id}$

- Can you prove it in some special cases?

  Columns, rows $\rightsquigarrow$ almost immediate.

- What about $RR \circ P$?

  not the identity, eg $w = 132$

  $$P(132) = \begin{matrix} 1 & 2 \\ 3 & \end{matrix}$$

  $$RR\left(\begin{matrix} 1 & 2 \\ 3 & \end{matrix}\right) = 312.$$

  $$P(312) = \begin{matrix} 1 & 2 \\ 3 & \end{matrix}.$$

We have a map

$$\coprod_{\lambda \vdash m} B_\lambda \underset{P}{\overset{RR}{\rightleftarrows}} B_{(1)}^{\otimes m} \text{ " } = \text{words}(1 \ldots n) \text{ length } m \text{ "}$$

λ is a partition on m, $\lambda_1 + \dots + \lambda_r = m$

$\emptyset \qquad 312$

$\begin{array}{|c|}\hline 3 \\ \hline\end{array} \qquad 12$

$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} \qquad 2$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

$w = w_1 w_2 \dots w_r$

Def Let $P(w)$ be the P-symbol (or insertion tableau). Define Q-symbol (or recording tableau) to be the standard tableau defined by

$$sh(P(w_1)) \subseteq sh(P(w_1 w_2)) \subseteq \dots \subseteq sh(P(w_1 w_2 \dots w_r))$$

Example $w = 132$

$P(1) = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$

$P(13) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}$

$P(132) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

sh
$\begin{array}{|c|}\hline \; \\ \hline\end{array}$
$\cap$
$\begin{array}{|c|c|}\hline \; & \; \\ \hline\end{array}$
$\cap$
$\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array}$

$Q(132) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

$w = 312$

$P(3) = \begin{array}{|c|}\hline 3 \\ \hline\end{array}$

$P(31) = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}$ $\qquad Q(312) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$

$P(312) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

<u>Question</u>

- Apply P, Q to $w = 4213$
- Reverse the process.

RSK

words $\qquad$ semistd $\qquad$ std. tableaux

$$B_{(1)}^{\otimes m} \underset{RSK^{-1}}{\overset{RSK}{\rightleftarrows}} \bigsqcup_{\lambda \vdash m} B_\lambda \times ST(\lambda)$$

$RSK(w) = (P(w), Q(w))$. $\qquad RSK(w^{-1}) = (Q(w), P(w))$

this is a bijection, and a crystal morphism.

<u>Cor</u> $\qquad n^m = \sum_{\lambda \vdash m}' \# B_\lambda \# ST(\lambda)$

$$n! = \sum_{\lambda \vdash m}' \# ST(\lambda)^2 \qquad (m = n)$$

Lecture 22

RSK correspondence (Robinson-Schensted-Knuth)

$$B_{(1)}^{\otimes m} \longrightarrow \bigsqcup_{\lambda \vdash m} B_\lambda \times ST(\lambda)$$

$\shortparallel$ words of length m

$w \longmapsto (P(w), Q(w))$

semistd std

partitions of m.

Thm (RSK) This is a bijection

Thm This is a morphism (and therefore an isomorphism) of crystals).

proof sketch: We set up an induction

$$\begin{tikzcd} B_{(1)}^{\otimes m} \arrow[r, "RSK_m"] \arrow[d, "="'] & \bigsqcup_{\lambda \vdash m} B_\lambda \times ST(\lambda) \\ B_{(1)}^{\otimes m-1} \otimes B_{(1)} \arrow[r, "RSK_{m-1} \otimes id"] & \bigsqcup_{\mu \vdash m-1} B_\mu \otimes B_{(1)} \times ST(\mu) \arrow[u, "\text{decompose } B_\mu \otimes B_{(1)}"'] \end{tikzcd}$$

is crystal morphism by induction

Fact: $B_\mu \otimes B_{(1)} \simeq \bigsqcup B_\lambda$ where λ ranges over shapes obtained from μ by adding a single box.

Example for GL_n $n \geq 3$

$$B_{(3,1)} \otimes B_{(1)} \simeq B_{(4,1)} \sqcup B_{(3,2)} \sqcup B_{(3,1,1)}$$

for $n=2$ $B_{(3,1,1)} = \phi$. $\square$.

What about $B_\lambda \otimes B_\mu$? The answer is given by the Littlewood-Richardson rule.

Thm $B_\lambda \otimes B_\mu \simeq \bigsqcup_\nu B_\nu^{\oplus c^\nu_{\lambda\mu}}$ where $c^\nu_{\lambda\mu}$ is the "Littlewood-Richardson coefficient".

This means the set $\{B_\lambda\}$ is closed under tensor product.

How would we compute $c^\nu_{\lambda\mu}$? $\lambda \vdash p$, $\mu \vdash q$

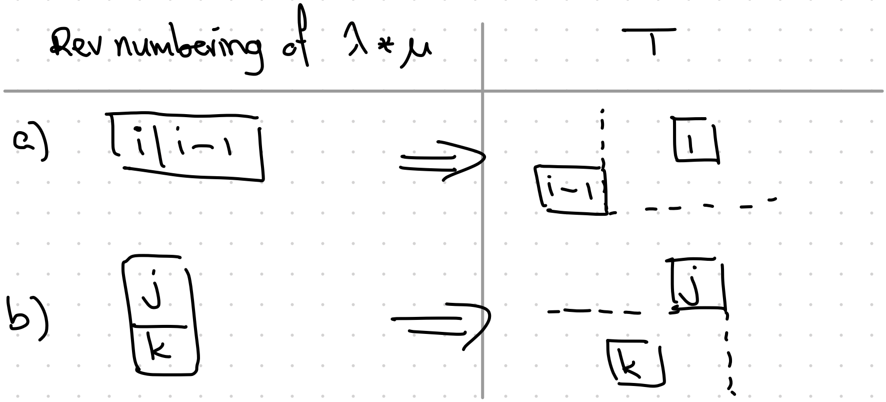
$$B_\lambda \otimes B_\mu \xrightarrow{RR \otimes RR} B_{(1)}^{\otimes p} \otimes B_{(1)}^{\otimes q} \xrightarrow{RSK} \bigsqcup_{\nu \vdash p+q} B_\nu \times ST(\nu)$$

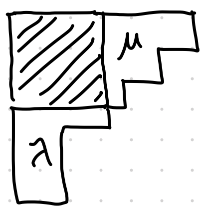
what is the image of this map?

$c^\nu_{\lambda\mu} = \# T \in ST(\nu)$ that are in the image.

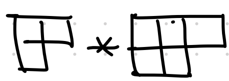
Answer:

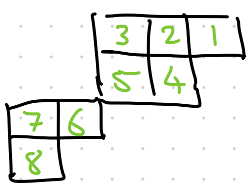
$$c^{\nu}_{\lambda\mu} = \#\{T \in ST(\nu) \mid \text{two properties hold:}\}$$



$\lambda * \mu =$ 

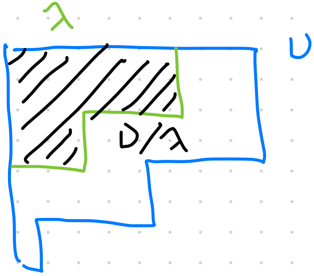
number boxes from right to left and top to bottom.

eg rev numbering of  is



Alternatively:

Thm $c^{\nu}_{\lambda\mu} = \#\{$LR tableaux of shape ν/λ and weight $\mu\}$

ie. 

- semistandard
- $\#1\text{'s} = \mu_1$, $\#2\text{'s} = \mu_2$, ...
- reading rows right to left and top to bottom produces a lattice word.

Let $\mathrm{crys}(GL_n)$ be the category of crystals whose connected components are all isomorphic to B_λ for some λ.

What is the coboundary structure on $\mathrm{crys}(GL_n)$?

There is a unique map of sets

$$\xi_\lambda : B_\lambda \longrightarrow B_\lambda$$

such that

- $\mathrm{wt}(\xi_\lambda b) = w_0 \cdot \mathrm{wt}(b)$ $\qquad w_0 = (1,n)(2,n-1)\dots$
- $\xi_\lambda(e_i b) = f_{n-i}(\xi_\lambda b)$
- $\xi_\lambda(f_i b) = e_{n-i}(\xi_\lambda b)$

We can extend to a map $\xi_B : B \longrightarrow B$ by applying ξ_λ to the connected components

Thm (Kamnitzer-Henriques) The map

$$C_{A,B} : A \otimes B \longrightarrow B \otimes A$$
$$a \otimes b \longmapsto \xi_{B\otimes A}\big(\xi_B(b) \otimes \xi_A(a)\big)$$

gives $\mathrm{crys}(GL_n)$ a coboundary structure.

Questions

(1) What is $\xi_\lambda(T)$ for T a semistandard tableau?

(2) If $w \in B_{(1)}^{\otimes m}$, what is the cactus group action?