ANU: Crystals　　　　　　Problem set 1

Homework should be handed in by Monday 16 August. For students in the 3000/4000 level you must complete 20 points woth of questions. For those at the 6000 level, you must hand in 25 points worth of questions.

1. (1 point) Let $\mathcal{P}$ be a poset. We can regard $\mathcal{P}$ as a category with objects the elements of $\mathcal{P}$ and a single morphism $a \longrightarrow b$ for each relation $a \leq b$. Interpret the coproduct of two elements in the language of posets.

2. (4 points) Sometimes (and especially in geometry) functorial constructions swap the directions of arrows. E.g. to every topological space M, we can associate the ring of continuous functions from M to $\mathbb{R}$, denoted $\mathcal{O}(M)$. To every continuous map $f : M \longrightarrow N$, we get a map $\mathcal{O}(f) : \mathcal{O}(N) \longrightarrow \mathcal{O}(M)$ given by $\varphi \mapsto \varphi \circ f$. Notice this swaps the direction of the arrow φ! People usually call this a contravariant functor. Here is the language that deals with this:
   (a) For any category $\mathcal{C}$ define the category $\mathcal{C}^{\mathrm{op}}$ which has the same objects as $\mathcal{C}$ but reversed morphisms, i.e. $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) := \mathrm{Hom}(B, A)$. Confirm this is indeed a category. *Hint: there isn't much to do.*
   (b) Check that $\mathcal{O}$ above defines a functor from $\mathbf{Top}^{\mathrm{op}}$ to **Ring**, where **Top** is the category of topological spaces with continuous maps and **Ring** is the category of rings with ring homomorphisms.

3. (4 points) Let **Set** be the category of sets with morphisms $\mathrm{Hom}(A, B)$ being all functions. Define $\mathcal{P}(X)$ to be the power set of X (the set of all subsets) and $2^X = \mathrm{Hom}(X, \{0, 1\})$.
   (a) Extend the definitions of $\mathcal{P}$ and 2^- to functors $\mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. *Hint: think about inverse images!*
   (b) Show that these two functors are naturally isomorphic.

4. (2 points) Let G be a group and consider it as a category with a single object and morphisms given by G (with composition determined by multiplication). What is a functor $F : G \longrightarrow \mathbf{Set}$ in more normal language? What is a natural transformation between two of these functors?

5. (4 points) For any category $\mathcal{C}$, the monoid $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$ of natural endomorphisms of the identity functor is called the *centre* of the category and is denoted $Z(\mathcal{C})$.
   (a) Determine the centre of **Set**. *Hint: the object* $\{*\}$ might come in handy!
   (b) For a unital ring R, determine the centre of $R - \mathbf{Mod}$, the category of left R-modules. *Hint: the name is suggestive and the useful thing about* $\{*\}$ *above was that the morphisms* $\{*\} \to X$ *are the same as elements of set.*
   (c) Determine the centre of **Grp**, the category of all groups.

6. (10 points) Let G be a finite group and k a field. Define the category of G-graded vector spaces $\mathbf{Vect}_k(G)$ as having objects vector spaces over k with a decomposition

$$V = \bigoplus_{g \in G} V_g$$

   and morphisms being linear maps that preserve the grading, i.e. a morphism $\phi : U \longrightarrow V$ is a linear map such that $\phi(U_g) \subseteq V_g$. We can define a tensor product by letting $V \otimes W$ be the usual tensor product of vector spaces and defining the grading by

$$(V \otimes W)_g = \bigoplus_{p \in G} V_p \otimes W_{p^{-1}g}$$

   (a) Show that this indeed defines a functor.
   (b) In order to make $\mathbf{Vect}_k(G)$ into a monoidal category, we need a identity object. There is only one possibility. What should it be?
   (c) If we want to define an associator, a, show that it must act on subspaces of the form $U_p \otimes V_q \otimes W_r$ by multiplying by a nonzero scalar. *Hint: consider the one dimensional objects.*

(d) These scalars should determine a map $\sigma : G^{\times 3} \longrightarrow k^{\times}$. What property does this map need to have in order for it to determine an associator?

(e) Determine furthermore the natural isomorphisms l and r needed for the monoidal structure. Does this place any further restrictions on σ?

7. (4 points) In a braided monoidal category $\mathcal{C}$ with braiding c, check the following identities are automatically satisfied.

   (a) $l_A \circ c_{A,\mathbf{1}} = r_A$

   (b) $r_A \circ c_{\mathbf{1},A} = l_A$

   (c) $c_{A,\mathbf{1}}^{-1} = c_{\mathbf{1},A}$

8. (5 points) In class, we defined a monoidal structure on the category $\mathbf{sVect}_k$ of superspaces using the associator $u \otimes v \otimes w \mapsto (-1)^{|u|+|w|} u \otimes v \otimes w$.

   (a) Find a braiding on $\mathbf{sVect}_k$ that is compatible with this monoidal structure.

   (b) Consider $\mathbf{sVect}_k$ instead with the monoidal structure given by a trivial associator. Find a braiding that is compatible with this.

9. (8 points) We saw in class, that the category **Tang** of tangles has two "obvious" braidings, the under and over braidings.

   (a) Experiment and see if you can find other braidings (I don't know if they exist!)

   (b) Without defining precisely what is meant by "generators", find objects and morphisms, that generate all other objects and morphisms in the category **Tang** under the operations $\otimes$ and composition. *You don't need to prove anything here, just having the right vibe is fine.*

   (c) Let **Braid** be the subcategory of **Tang** with the same objects but we only consider morphisms each of whose strings start on the bottom and finish on the top of the diagram, with no circles floating around (so there are no morphisms between unequal numbers). What is $\mathrm{End}(n)$ in this category and what is the resulting map $B_n \longrightarrow \mathrm{End}(1^{\otimes n})$. If you feel like this is interesting, what is the map $B_n \longrightarrow \mathrm{End}(m^{\otimes n})$?

10. (6 points) We saw in class that for a general finite group G the category $\mathbf{Vect}_k(G)$ cannot be braided because $k_g \otimes k_h = k_{gh}$ is only isomorphic to $k_h \otimes k_g = k_{hg}$ when $gh = hg$. This means, the only chance for a braiding to exist is when G is abelian. Assume this is the case. Assume that we have chosen an appropriate function $\omega : G^{\times 3} \longrightarrow k^{\times}$ that defines an associator on our category. *Hint: if this is tricky try the case when $\omega(g, h, k) = 1$.*

    (a) Show that any possible braiding $c_{UV} : U \otimes V \longrightarrow V \otimes U$ on $\mathbf{Vect}_k(G)$ must act on subspaces of the form $U_g \otimes V_h$ by the flip multiplied by a scalar.

    (b) Interpret the braiding as a function $G \times G \longrightarrow k^{\times}$ and write down the condition it must satisfy to indeed define a braiding.

    (c) Show that a braiding does indeed exist.

11. (1 point) (or more... speak to me) If you feel like doing some background reading you can make sense of the following statement: A monoidal category is the same as a 2-category with a single object. In fact, more is true, we have "higher levels" of this correspondence but the identifications are muuuuch harder (you need to understand the Eckmann-Hilton argument)

    (a) A braided monoidal category is the same as a 3-category with a single object and single 1-morphism.

    (b) A symmetric monoidal category is the same as a 4-category with a single object, single 1-morphism and single 2-morphism.

    Another statment that you might like to read about: a braided monoidal category is the same as an algebra over the little 2-cubes operad

12. When should two categories be equivalent? Clearly we want some notion of "bijectivity" for a functor. Let's let category theory guide us!
    (a) We can consider the category **Cat** of all categories (set theoretic issues are ignored!). The morphisms are functors. An isomorphism between objects in a category is a function that has a two-sided inverse. This would imply that an isomoprhism of categories is a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, such that there exists a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ so that $G \circ F = \mathrm{id}_{\mathbf{C}}$ and $F \circ G = \mathrm{id}_{\mathbf{D}}$.. No exercise here, just digest this!
    (b) Let $\mathbf{Mat}_k$ be the category whose objects are natural numbers and $\mathrm{Hom}(n, m)$ is the set of $m \times n$ matrices with entries in k (composition is matrix multiplication). Is $\mathbf{Mat}_k$ isomorphic to $\mathbf{Vect}_k^{fd}$, the category of finite dimensional vector spaces?
    (c) Clearly the two categories above should be equivalent. So we should define a slightly looser notion of equivalence than isomorphism. The problem stems from the fact that in category theory it is usually a bad idea to declare things are equal when they could instead be isomorphic. I.e. we should ask that $G \circ F \cong \mathrm{id}_{\mathbf{C}}$ and $F \circ G \cong \mathrm{id}_{\mathbf{D}}$. We say $\mathcal{C}$ and $\mathcal{D}$ are equivalent if functors as above exist.
    (d) Are $\mathbf{Mat}_k$ and $\mathbf{Vect}_k^{fd}$ equivalent categories?
    (e) A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called *full* if every map $\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FA, FB)$ given by $f \mapsto Ff$ is surjective. The functor is called *faithfull* if the above maps are injective. The functor is called *essentially surjective* if every object $X \in \mathcal{D}$ is isomporphic to FA for some $A \in \mathcal{C}$. Show that $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories if and only if a full, faithfull and essentially surjective functor between them exists.

13. What does it mean for a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between monoidal categories to preserve the monoidal sturcture? Obviously, we want $F(\mathbf{1}_{\mathcal{C}}) \cong F(\mathbf{1}_{\mathcal{D}})$ and $F(A \otimes_{\mathcal{C}} B) \cong F(A) \otimes_{\mathcal{D}} F(B)$. But as we discussed in class, we usually want to specify *how* these objects are isomorphic.
    (a) A *monoidal functor* between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a triple (F, J, ϕ) where $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, $J : F \circ \otimes_{\mathcal{C}} \Rightarrow \otimes_{\mathcal{D}} \circ F \times F$ is a natural isomophism, and $\phi \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}_{\mathcal{D}}, F\mathbf{1}_{\mathcal{C}})$ is an isomoprhism. This data must satisfy some coherence relations with respect to the associators and unit maps. Try and come up with them, and if need be look them up!
    (b) Put a monoidal structure on the category $\mathbf{Mat}_k$ where $m \otimes n = mn$. Find a monoidal functor from $\mathbf{Vect}_k^{fd}$ to $\mathbf{Vect}_k^{fd}$.
    (c) Can you come up with the notion of a natural transformation of monoidal functors? *Hint: a monoidal functor is a structure on a functor, i.e. there may be more than one way for a functor to be monoidal. being a natural transformation of monoidal functors is a property of the natural transformation.*

14. If you like the categories $\mathbf{Vect}_k(G)$, read the definition of group cohomology $H^n(G, k^\times)$ (e.g Wikipedia) and then read section 2.6 of Etingof et al *Tensor categories.* Your answer to question 6 essentially said that a choice of associator on this category is same as a 3-cocycle, and this reading will show you exactly when two of these associators give monoidally equivalent categories. (Warning: Etingof et al actually talk about slightly different categories $\mathcal{C}_G^\omega$ - see Example 2.3.6 - but the story is basically the same - it is a good exercise to write out the details for $\mathbf{Vect}_k(G)$.)

15. If you *really* like these categories and want to know even more, then you will have to work even harder. The obvious question is what structure classifies the braidings on $\mathbf{Vect}_k(G)$? This is answers in section 8.4 of Etingof et al's book, but there they work out the story for slightly more general categories.

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