ANU: Crystals Problem set 2

Homework should be handed in by Monday 30 August. For students in the 3000/4000 level you must complete 20 points woth of questions. For those at the 6000 level, you must hand in 25 points worth of questions.

1. (5 points) Prove that a monoidal category is symmetric if and only if it is both braided and coboundary (using the same commutator for both structures).

2. (4 points) Let V be an object of $\tilde{\mathcal{D}}_1$ and $v \in V$ a vector such that $ev = 0$ and $hv = mv$.
   (a) Consider the subspace $U = \text{span}\{v, fv, f^2v, \dots\}$. Show that there is a $k \geq 0$ such that $f^k v \neq 0$ and $f^{k+1}v = 0$.
   (b) Show that the set $\{v, fv, f^2v, \dots, f^kv\}$ is linearly independent and thus a basis of U. *Hint: consider the h eigenvalues.*
   (c) Show that $k = m$ and thus $\dim U = m + 1$. *Hint: consider $ef^{k+1}v$, compute it in two different ways.*
   (d) Show $U \cong V(m)$.

3. (4 points) Let V be an object of $\tilde{\mathcal{D}}_1$ and $u, v \in V$ linearly independent, such that $eu = ev = 0$, $hu = mu$ and $hv = nv$.
   (a) Suppose $m \geq n$ and $m - 2a = n$. Prove that $f^a u$ and v are linearly independent.
   (b) Show that $\{f^a u, f^b v\}_{\substack{0 \leq a \leq m \\ 0 \leq b \leq n}}$ is linearly independent. *Hint: remember, vectors in different h-eigenspaces are linearly independent. You will also need to generalise the above slightly.*
   (c) Deduce that $\text{span}\{f^a u, f^b v\}_{a,b \geq 0} \cong V(m) \oplus V(n)$.

4. (10 points) Let us consider the categories $\mathcal{D}_1, \mathcal{D}_q$ that were introduced in class. Let $V(n)$ be the $n + 1$ dimensional objects used to define the cateory in each case.
   (a) Show that $V(n)$ is the span of $f^k(x^n)$ for $k \geq 0$ and use this to deduce that $V(n)$ cannot be expressed as the direct sum of other objects, i.e. $V(n)$ is indecomposable.
   (b) Explicitly determine the tensor product rule in each case. That is, given that $V(m) \otimes V(n) = \bigoplus_{k \geq 0} V(k)^{\oplus a_k}$, determine the integers a_k.
   (c) Do the same for $\mathcal{D}_0$.

5. (2 points) Find all the one dimensional objects in $\tilde{\mathcal{D}}_q$. For each one dimensional object S, give an explicit description of the operators e, f and K on $S \otimes V_q(m)$.

6. (5 points) Let $V = V_q(1)$. Compute explicitly the homomoprhisms $\text{Hom}(V \otimes V, V \otimes V)$. Which ones are isomoprhisms?

7. (3 points) Are the categories $\mathcal{D}_1$, $\mathcal{D}_q$, and $\mathcal{D}_0$ equivalent? What about as monoidal categories? As braided monoidal categories? No proofs necessary, a good discussion suffices.

8. (2 points) Show that the cactus group C_3 is isomorphic to the infinite dihedral group, i.e. the group with presentation $D_\infty = \langle r, s \mid s^2 = 1, srs = r^{-1} \rangle$.

9. (6 points) This question will guide you through a proof showing the monoidal category $\mathcal{D}_0$ *cannot* be made into a braided monoidal category. The proof is due to A. Savage. The strategy is to show that any natural isomophism $c : \otimes \Rightarrow \otimes \circ \text{flip}$ cannot possibly satisfy the hexagon axiom.
   (a) Show that $c_{B(1),B(1)}$ must the identity map. Thus what is the map $(\text{id}_{B(1)} \otimes c_{B(1),B(1)}) \circ (c_{B(1),B(1)} \otimes \text{id}_{B(1)})$?
   (b) Now calculate $c_{B(2),B(1)}(fb_2 \otimes b_1)$.
   (c) Use this to calculate $c_{B(1) \otimes B(1),B(1)}(b_1 \otimes fb_1 \otimes b_1)$. *Hint: you will need to use naturality here using a map $B(2) \to B(1) \otimes B(1)$.*

(d) Deduce that c cannot be a braiding.

10. (5 points) (for any 50% of this) This is a question to guide you through the computation of group cohomology for those that are interested. We will concentrate on the case when our group is $G = \mathbb{Z}/2$. For any group G and a G-module A (i.e. an abelian group with a G-action, or equivalently a $\mathbb{Z}G$-module) we first define $C^n(G, A)$ to be the set of functions $G^{\times n} \longrightarrow A$. We define maps

$$C^0(G,A) \xrightarrow{d^0} C^1(G,A) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} C^n(G,A) \xrightarrow{d^n} C^{n+1}(G,A) \xrightarrow{d^{n+1}} \cdots$$

$$d^n(f)(g_0, g_1, \ldots, g_n) := g_0 \cdot f(g_1, \ldots, g_n) + \sum_{i=1}^{n} (-1)^i f(g_0, \ldots, g_{i-2}, g_{i-1}g_i, g_{i+1}, \ldots, g_n) + (-1)^{n+1} f(g_0, \ldots, g_{n-1})$$

Note we are using + for the group operation in A.

(a) Check that $d^n \circ d^{n-1} = 0$, i.e that $\operatorname{im} d^{n-1} \subseteq \ker d^n$. We define the n^{th} cohomology group of G with coefficients in A as the abelian group $H^n(G, A) := \ker d^n / \operatorname{im} d^{n-1}$.

(b) Give a simple interpretation of $H^0(G, A)$ (set $C^{-1}(G, A) = 0$ and $d^{-1} = 0$ by convention).

(c) Suppose that G acts trivially on A. What is $H^1(G, A)$? *Hint: the image of* d^0 *is trivial.*

(d) Recall your condition for the associator on $\mathbf{Vect}_k(G)$ to be given by a function $\omega \in C^3(G, k^\times)$. What is this condition in the language above? (*Warning: above you are using additive notation for* A*, but if* $A = k^\times$ *then it makes more sense to use multiplicative notation.*)

(e) Now consider the case when $G = \mathbb{Z}/2$ and $A = k^\times$. Show that the kernel of d^1 has two elements (or does it? What role does the characteristic play?). This should give you $H^1(\mathbb{Z}/2, k^\times)$. Does this mesh with your answer to b?

(f) Now try calculating $H^2(\mathbb{Z}/2, k^\times)$. *Hint: it is quite a small group.* Does your answer depend on the characteristic?

(g) If you want a long slog of a calculation, try $H^3(\mathbb{Z}/2, k^\times)$. *Hint: it should also be quite small.*

(h) Pick two different elements of $\ker d^3$ that are cohomologous (i.e. the same mod $\operatorname{im} d^2$). These should give two monoidal stuctures on $\mathbf{sVect}_k$. Find a monoidal functor giving an equivalence between them.

(i) If you want even more calculations to do, you could figure out how the calculations change when you let $\mathbb{Z}/2$ act non trivially on $k^\times$ by $\lambda \mapsto \lambda^{-1}$.