ANU: Crystals Problem set 3

Homework should be handed in by Monday 20 September. For students in the 3000/4000 level you must complete 20 points woth of questions. For those at the 6000 level, you must hand in 25 points worth of questions.

1. (3 points) Let L and M be finite line graphs (i.e. objects in $\mathcal{D}_0$). For the vertex $u \otimes v \in L \otimes M$, show that

$$\phi(u \otimes v) = \phi(u) + \max\{0, \phi(v) - \epsilon(u)\}, \text{ and}$$
$$\epsilon(u \otimes v) = \epsilon(v) + \max\{0, \epsilon(u) - \phi(v)\}.$$

2. (5 points) Show that the categtory $\mathcal{D}_0$ with the natural isomorphism $c_{LM} = \xi_{M\otimes L} \circ \text{flip} \circ (\xi_L \otimes \xi_M)$ is a coboundary monoidal category. *Hint: in this category, disjoint union is a direct sum, thus you can check the axioms when L and M are both connected line graphs.*

3. (3 points) (for part a) Let $B = B(1)$. If we identify b_1 with the number 1 and fb_1 with 2, we can think of an element of $B^{\otimes n}$ as a string of 1's and 2's of length n. E.g. $212211212 \in B^{\otimes 9}$.
   (a) In this language, can you come up with a rule that gives the arrows?
   (b) What about the cactus group action? Can you come up with a rule that tells us what $s_{1n} \cdot w$ is for any $w \in B^{\otimes n}$? *This is probably not an easy question, so I don't expect you to actually come up with an answer without really serious effort, instead just have a play around and note that there is something nontrivial to answer here. We will come up with an answer later in the course that (I think) is very beautiful.*

4. (7 points) Let k be a field and V a finite dimensional vector space over k. A *psuedo-reflection* $s \in \text{GL}(V)$ is an operator of finite order whose fixed points are codimension 1, that is

$$\dim \ker(s - \text{id}) = \dim V - 1.$$

   A *psuedo-reflection group*, is a finite subgroup $G \subseteq \text{GL}(V)$ that is generated by psuedo-reflections.
   (a) What are the possibilities of the Jordan block decomposition of s? What are the possible eigenvalues. Your answer should say precisely when these options can arise depending on the field. *Hint: over a field of characteristic zero, only one of these options actually arises.*
   (b) Give an example of a psuedo-reflection group over $\mathbb{C}$ that is not a psuedo-reflection group over $\mathbb{R}$.
   (c) Show that $\text{SL}_2(\mathbb{F}_p)$ is a psuedo-reflection group for any prime p.
   (d) Let $k = \mathbb{R}$ or $\mathbb{C}$ and suppose G is a finite subgroup of $\text{GL}(V)$. Choose any inner product $\langle .,. \rangle_0$ on V and consider the bilinear function $\langle u, v \rangle = \sum_{g\in G} \langle gu, gv \rangle_0$. Check this is an inner product. What is $\langle gu, gv \rangle$?
   (e) Let $k = \mathbb{R}$ or $\mathbb{C}$ and show that if s is a psuedo-reflection, then you can define a suiable inner product so that is a reflection with respect to that inner product.
   (f) Let $k = \mathbb{R}$ or $\mathbb{C}$. Show that a subgroup $G \subset \text{GL}(V)$ is a reflection group for some suitable inner product, if and only if it is a psuedo-reflection group.

5. (7 points) This question is about the *hyperoctahedral group*. By definition HO_n is the group of permutation σ of the set $\{-n, -n+1, \ldots, -1, 1, 2, \ldots, n\}$ such that $\sigma(-i) = -\sigma(i)$. This clearly contains S_n as a subgroup.
   (a) Let $V = \mathbb{R}^n$ and $\{\epsilon_i\}$ be the standard basis. Consider the reflections $t_i = s_{\epsilon_i}$ and $s_{ij} = s_{\epsilon_i - \epsilon_j}$. Show that the reflection group generated by the t_i and s_{ij} is essential.
   (b) In class we saw the s_{ij} generate a group isomorphic to S_n. What is the group generated by the t_i?
   (c) Show that $W \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \cong HO_n$. This is the type B_n reflection group (it is also the type C_n reflection group so we will often call it type BC_n).
   (d) Show that $u_{ij} := s_{\epsilon_i+\epsilon_j} \in W$.

(e) Let K be the subgroup of W generated by u_{ij} and s_{ij}. Show that K is an essential reflection group and that it again splits as a semidirect product. Show that it has index 2 in $W \cong HO_n$. This is the reflection group of type D_n.

6. (10 points) Let V be a real vector space and fix a subgroup $W \subset \mathrm{GL}(V)$ and let $R \subset W$ be the set of all reflections in W. Suppose W is generated by R. We do not assume that W or R are finite. *Note: R is the set of all reflections, not simply a generating set. This is important, otherwise the punchline of the exercise is not true!*

(a) If $s \in R$ and $w \in W$, show that $wsw^{-1} \in R$.

(b) Find an example of irreducible, essential reflection groups W, where the action on R has a single orbit, and an example that has two orbits.

(c) Let $\mathrm{Sym}(R)$ be the symmetric group on R. Is the induced map $W \longrightarrow \mathrm{Sym}(R)$ injective?

(d) Show that $W/Z(W)$ is finite if R is finite. *(here $Z(W)$ is the centre of W)*

(e) The above suggests that we might be able to upgrade the result to showing "W is finite if and only if R is finite" if we can find an appropriate set on which W acts faithfully. Note that the fundamental problem with R is that for $w \in Z(W)$, we have $ws_\alpha w^{-1} = s_\alpha$ even though it might not be true that $w\alpha = \alpha$. Show that for each $w \in Z(W)$, there exists at least one $s_\alpha \in R$ such that $w\alpha = -\alpha$.

(f) Let $\hat{R} := \{\mathbb{R}_{>0}\alpha \mid \alpha \in V \text{ such that } s_\alpha \in R\}$ (i.e. the set of rays that define all the reflections - note you get a positive and negative ray for each reflection - so $\hat{R}$ is some sort of double cover of R). Show that W acts on $\hat{R}$ and that this action is faithful (i.e. the kernal of the natural map to $\mathrm{Sym}(\hat{R})$ is injective). Use this to conclude W is finite if and only if R is finite.