ANU: Crystals **Problem set 4**

Homework should be handed in by Monday 4 October. For students in the 3000/4000 level you must complete 20 points woth of questions. For those at the 6000 level, you must hand in 25 points worth of questions.

1. (3 points) Prove the following: If Φ is a root system and $W = W(\Phi)$ is group generated by s_α for $\alpha \in \Phi$ then W is a finite reflection group and moreover every finite reflection group arises in this way. *Hint: this shouldn't be hard, you have proved all the ingredients already, you just need to reinterpret and synthesise.*

2. (3 points) If $\Delta \subset \Pi \subset \Phi$ are simple and positive systems respectively in a root system Φ, show that $w\Delta \subset w\Pi$ are also simple and positive systems for any $w \in W(\Phi)$.

3. (5 points) Let Φ be a root system and $H \subset V$ a hyperplane such that $H \cap \Phi = \emptyset$. Choose a vector $\lambda \in V$ such that $H = \ker(-, \lambda)$ (i.e. a side of H that we call the positive direction). Let $\Pi = \{\alpha \in \Phi \mid (\alpha, \lambda) > 0\}$. Show that Π is a positive system and that all positive systems arise in this way.

4. (6 points) Find a root system in $\mathbb{R}^{n-1}$ such that the corresponding reflection group is S_n. Find a positive and simple system. This is called the type A_{n-1} root system.

5. (10 points) Find root systems for the hyperoctahedral group and it's index two subgroup discussed in the last problem set. These are called the type B_n and D_n root systems respectively.
   (a) A root system is called *crystallographic* if $2(\alpha, \beta)/(\alpha, \alpha)$ is an integer for every pair of roots $\alpha, \beta \in \Phi$. Can you adjust your root systems so they are crystallographic?
   (b) Find simple and positive systems.

6. (5 points) Can you find a crystallographic root system for the order $2n$ dihedral group?

7. (2 points) Let $\alpha, \beta \in \Phi$ be two roots. Since the reflection group W associated to Φ is finite, the order of $s_\alpha s_\beta$ is finite. I.e. there is an integer $m(\alpha, \beta)$ such that $(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$. Characterise the number $m(\alpha, \beta)$ in terms of the geometry of $\alpha, \beta \in \Phi$.

8. (5 points) Let $\ell(w)$ and $n(w)$ be the statistics defined in class (length and number of "flips").
   (a) Show that $\det w = (-1)^{\ell(w)} = (-1)^{n(w)}$
   (b) For the root system type A_{n-1} root system, take as simple system Δ such that the corresponding simple reflections are the transpositions $(i, i+1)$. Show that $\ell(w) = n(w) = \#\{(i,j) \mid i < j \text{ and } w(i) > w(j)\}$.

9. (5 points) Write a computer program that takes as input a Coxeter graph and outputs whether it is positive definite, positive semidefinite or neither.

10. (4 points) Show presidely which of the Coxeter graphs of type $I_2(m)$ can be extended to a positive semidefinite graph (without using the classification results).

11. (6 points) Let $V = \mathbb{R}^n$ with the standard inner product. Consider the set Φ of vectors in $\mathbb{Z}^n$ that have length 2 or $2\sqrt{2}$.
    (a) Determine Φ explicitely.
    (b) Show that Φ is a root system and find a simple system.
    (c) Determine the Coxeter graph
    (d) Is Φ crystallographic?

12. (4 points) Let $W \subset \mathrm{GL}(V)$ be an irreducible (and essential) reflection group. Show that V is an irreducible representation of W. *Hint: consider a direct sum decomposition, where do the root vectors lie?*