ANU: Crystals **Problem set 5**

Homework should be handed in by Monday 1 November. For students in the 3000/4000 level you must complete 20 points woth of questions. For those at the 6000 level, you must hand in 25 points worth of questions.

1. (2 points) Let $\Phi \subset V$ be a crystallographic root system and let $\alpha^\vee := \frac{2}{(\alpha,\alpha)}\alpha$.
   (a) Show that $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ is also a crystallographic root system. This is called the dual root system.
   (b) Show furthermore that if $\Delta \subset \Phi$ is a simple system, then $\Delta^\vee$ is a simple system for $\Phi^\vee$.
   (c) For each irreducible root system in the list from class, determine the type of the dual system. *Hint: there isn't as much to do here as it seems. The angles in the dual system do not change, only the lengths.*

2. (4 points) Consider the GL_n root system we defined in class, $\Phi = \{\alpha_{ij} = \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\} \subset \mathbb{R}^n$ with simple system $\Delta = \{\alpha_i = \alpha_{i,i+1} \mid 1 \leq i < n\}$.
   (a) If we make the choice $\Lambda = \mathbb{Z}^n$, what are the possible choices for the fundamental weights $\varpi_1, \varpi_2, \ldots, \varpi_{n-1}$ so that they lie in Λ?
   (b) Give a simple description of the dominant weights.

3. (6 points) Calculate the simply connected weight lattice and the fundamental weights for the semisimple root system of type A: $\Phi = \{\alpha_{ij} = \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\} \subset V$ where V is the subspace of $\mathbb{R}^n$ consisting of vectors whose coordinates add to zero. We can choose $\Delta = \{\alpha_i = \alpha_{i,i+1} \mid 1 \leq i < i\}$ as our simple roots.

4. (4 points) The dual crystal. If B is a crystal, we can form another crystal $B^\vee$ with elements $x^\vee$ for every $x \in B$ (so as sets they are the same). We set $\mathrm{wt}(x^\vee) = -\mathrm{wt}(x)$ and we swap the roles of the e_i's and f_i's, specifically, we set $\varepsilon_i(x^\vee) = \varphi_i(x)$, $\varphi_i(x^\vee) = \varepsilon_i(x)$ and $e_i x^\vee = f_i x$ and $f_i x^\vee = e_i x$. Show that $B^\vee$ is a crystal, that it is seminormal if B is, and compute it for some examples you have seen in class.

5. (5 points) For each of B_n, C_n, D_n find a connected crystal with highest weight ϖ_1 (here we assume that we have fixed the simply connected weight lattice).

6. (4 points) Construct a crystal for GL_4 with highest weight ϖ_2. It should have six elements.

7. (6 points) Construct some examples of the tensor products $B_{(k)} \otimes B_{(1)}$ and $B_{(1^k)} \otimes B_{(1)}$. Can you make a guess at a general rule for how many connected components there will be and what they might be isomorphic to?

8. (4 points) If B is a crystal and $\theta \in \Lambda$ such that $\langle \theta, \Phi \rangle = 0$ (i.e. it is orthogonal to all the roots) then we can define a *twisted* crystal B^θ which coincides with B in every way except we define a new weight function $\mathrm{wt}^\theta(b) := \mathrm{wt}(b) + \theta$.
   (a) Show that B^θ is a crystal.
   (b) For the (GL_n) root datum, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition, and a an integer, when is $B_{\lambda+(a^k)}$ a twist of B_λ?

9. (3 points) If $\emptyset$ is the empty partition, then $B_\emptyset$ is the (GL_n) crystal with a single element of weight 0.
   (a) Show $B_\emptyset$ is the identity object for the monoidal structure.
   (b) Confirm this by calculating the Littlewood-Richardson coefficients $c^\nu_{\lambda\emptyset}$ (i.e. calculate the number of standard tableaux of shape ν meeting the conditions given in lecture for the reverse numbering of $\lambda\emptyset$).