Combinatorics of Pavings and Paths

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Contents

Abstract

The principle theme of this thesis is the enumeration of directed lattice paths. An ancillary theme is the development of a notion of pavings due to Viennot. We then utilize pavings in the solution of path problems.

Pavings

We build on Viennot's conception of 'pavings' on a path graph, along with associated 'paving polynomials'; and Viennot's bijection between these pavings and cycles on the digraphs associated with Ballot and Motzkin paths.

- We show that a very natural generalization extends Viennot's pavings and cycle bijection to d-up and Lukasiewicz paths; but *not* to Jump-step paths.
- We further generalize the mapping between cycles and pavings to enable the calculation of Jump-step paving polynomials.
- We define a new kind of path graph paving called a 'Laurent paving', which has some complexity/combinatorial advances over the original pavings; and prove the combinatorial relationship between the two.
- We use the combinatorics of pavings and Laurent pavings to find closed-form expressions for paving polynomials and Laurent paving polynomials associated with weighted Ballot, Motzkin and other more general paths.

Paths

- We derive the generating function for n-banded directed ballot paths in the half plane explicitly for all n, thus showing that it is always a quadratic function. In the case of bi-banded directed ballot paths in the half plane, we also find the weight function explicitly, and in so doing proved a new combinatorial interpretation of the Narayana numbers.

- We give a new combinatorial derivation of the known single variable constant coefficient recurrence relations on path weight polynomials. This defines a new combinatorial derivation of generating functions for directed paths in a strip in terms of pavings.

- We define a new form of 'Constant Term' method applicable to weighted Ballot-like and Motzkin-like paths with finite numbers of decorations. We use the method to solve an open enumeration problem from the 1970's and generalizations thereof.

Declaration

This is to certify that

1. *the thesis comprises only my original work except where indicated in the preface,*

2. *due acknowledgement has been made in the text to all other material used,*

3. *the thesis is less than* 100,000 *words in length, exclusive of tables, maps, bibliographies and appendices.*

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Preface

This thesis was written under the supervision of

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Chapter 2

- Largely review; work done by Judy-anne Osborn (JO).

Chapter 3

- Work done by JO, with thanks to Richard Brak (RB) for discussion of n-banded generating functions, which led to the finding of Theorem 30.

Chapters 4, 5 and 6

- Work done by JO, with thanks to RB for the original idea of calculating paving polynomials via breaking and recombining pavings.

Chapter 7

- Work done by JO.

Chapter 8

- The work in Section 8.1 was done equally by JO and RB. The work in Sections 8.2 – 8.4 was done by JO.
- A paper based on the first two sections of this chapter is in preparation by JO and RB [24].

Chapter 9

- Work done by JO.

Chapter 10

- Work in this Chapter was done equally by JO and RB. In particular, the proofs of Theorem 78, Lemma 79, Theorem 80 were done equally by JO and RB. Theorems 81 and 82 are the work of RB. Theorems 83 and 84 are the work of JO.
- Some of the work in this chapter has appeared in a paper [15].
- A further paper based on this chapter is in preparation by JO and RB [25].

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For interest, pride and occasional provision of quiet kitchen tables at which to work, thanks to my extensive family.

List of Figures

Notation

$\mathbb{N}$ = the set of natural numbers $\{1, 2, 3, \ldots\}$

$\mathbb{Z}$ = the set of integers $\{\ldots - 2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Z}_{\geq 0}$ = the set of non-negative integers $\{0, 1, 2, 3, \ldots\}$

$\lceil x \rceil$ = the least integer which is greater than or equal to x

$\lfloor y \rfloor$ = the greater integer which is less than or equal to y

The binomial coefficient $\binom{a}{b} := \begin{cases} \frac{a!}{(a-b)!b!} & 0 \leq b \leq a \text{ and } a, b \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$

Path definitions

Following are the basic definitions which pertain to paths. These are fairly standard graph theoretical terminology. At the beginning of Chapter 4 another collation of definitions is given providing the foundational terminology required for pavings, in the sense required for this thesis. In addition to these two general collections, individual definitions are provided throughout the text as needed.

A **graph** is a pair $G = (V, E)$, where V is a countable set of elements termed **vertices**, and E is a set of pairs of vertices, called **edges**. An edge $\{u, v\}$ may be abbreviated either uv or vu. A pair of vertices are termed **adjacent** if they are part of the same edge. A graph may be visualized as a collection of dots representing vertices, and lines connecting the dots, which represent edges.

A **directed graph**, or **digraph**, is a pair $G = (V, A)$ where V is a countable set of elements termed **vertices**, and A is a set of ordered pairs of vertices, called **arcs**. The arc (u, v) is abbreviated uv. If the arc uv is present we say that u is **adjacent to** v. If both arcs uv and vu are present then we say that u is **adjacent with** v; [62]. A digraph may be visually represented in the same way as a graph, but with the addition of arrows on the lines indicating the direction of the arcs.

A **walk** on a (di)graph is a sequence of vertices and edges/arcs

$$w = v_0 e_1 v_1 e_2 v_2 ... e_n v_n, \tag{1}$$

where we require that the edge/arc e_j comprises the vertices v_{j-1} and v_j, in that order for a digraph. We often abbreviate by listing the edges only as:

$$w = e_1 e_2 ... e_n; \tag{2}$$

or the vertices only as

$$w = v_0 v_1 v_2 ... v_n. \tag{3}$$

The **length** of a walk is the number of edges traversed, counting multiple visits multiple times. Thus

$$l(w) \equiv l(v_0e_1v_1e_2v_2...e_nv_n) \equiv l(e_1e_2...e_n) = n. \tag{4}$$

We visually indicate a walk on a (di)graph by thickening or colouring the lines representing the edges/arcs included in the walk.

A **path** is a special case of a walk, where all vertices are distinct. A **path graph** is a graph with vertices labeled $0, 1, 2, ..., n$ and edges of the form $\{i, i + 1\}$ for $0 \leq i \leq n - 1$.

A **cycle** is a walk $v_0v_1v_2...v_{n-1}v_n$ such that $v_0 = v_n$ and $v_0v_1v_2...v_{n-1}$ is a path. The **cycle graph**, C_n, is a graph with vertices $0, 1, 2, ..., n$. It contains n edges, $n - 1$ of which are of the form $\{i, i + 1\}$ for $0 \leq i \leq n - 1$, in addition to the edge $\{n, 0\}$.

A **lattice** is a regular array of points contained in $\mathbb{Z}^n$, for some n. Our use of the term 'lattice' corresponds to what is also sometimes called a **point lattice** in the wider literature. We also use the term 'lattice' to refer to a 'lattice digraph', to be formally defined below, which is a digraph with a lattice as its underlying vertex set.

The one-dimensional lattices used most commonly throughout this work this work are the set $\mathbb{Z}$, which we term the **(number) line**; the set $\mathbb{Z}_{\geq 0}$, which we term the **half (number) line** and $\{0, 1, 2, ..., L\}$, which we term a **(number) line segment**. We also use $\{[0], [1], ..., [L - 1]\}$, where each $[i]$ is the equivalence class of i modulo L. We call this set a **(number) circle of circumference** c.

The two dimensional lattices used most commonly throughout this work are $\mathbb{Z}^2$, which we term the **plane**; $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$, which we term the **half plane** and $\mathbb{Z} \times \{0, 1, 2, ..., L\}$, which we term a **strip** in the half plane. We also use $\mathbb{Z} \times \{[0], [1], ..., [L-1]\}$, which we term a **(number) cylinder** of **circumference** L.

A lattice may be considered a special case of a digraph by regarding the points as **vertices** and defining an appropriate set of arcs. The arcs we shall define may each be visualized as straight line segments connecting pairs of vertices, with an arrow to indicate the direction. Each arc $e = uv$ is a vector anchored at the point u.

We define an arc $e = uv$ to be **of the form *w*** provided

$$v - u = w. \tag{5}$$

An **allowed step set** of a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$ is a set, S, of vectors

$$S = \{w_1, w_2, \cdots\} \subseteq \{v - u | u, v \in \mathcal{L}\}. \tag{6}$$

We define a **lattice digraph** $\mathcal{L}_S$ to be a digraph with vertex set $\mathcal{L}$ and arc set A, where A is the set of all possible arcs uv of the form w for $w \in S$, with the vertices u, v in $\mathcal{L}$ and S being some allowed step set on $\mathcal{L}$. The object we are terming a 'lattice digraph' is commonly known as simply a 'lattice', and we also use this short terminology when the step set is understood from the context.

Examples of lattice digraphs are the 'square lattice' and 'rotated square lattice'.

The **square lattice** is the lattice $\mathbb{Z}^2$ with allowed step set

$$\{N, E\} \tag{7}$$

where

$$\begin{aligned} N &= (1,0) \text{ ... a } \textbf{North Step} & (8)\\ E &= (0,1) \text{ ... an } \textbf{East Step} & (9)\end{aligned}$$

The **(45 degrees) rotated square lattice** is the lattice $\mathbb{Z}^2$ with allowed step set

$$\{U, D\} \tag{10}$$

where

$$\begin{aligned} U &= (1,1) \text{ ... an } \textbf{Up Step} & (11)\\ D &= (1,-1) \text{ ... a } \textbf{Down Step} & (12)\end{aligned}$$

A **binomial box** is a rectangular portion of either the square lattice or the rotated square lattice, which inherits the relevant step set. The size of the box is determined by the number of steps from boundary to boundary. In the square lattice, an $m \times n$ binomial box requires m steps of type E to pass from the leftmost to rightmost boundary, and n steps of type N to pass from the lower to the upper boundary. In the rotated square lattice, an $m \times n$ binomial box requires m steps of type D to pass from the upper left to lower right boundary, and n steps of type U to pass from the lower left to upper right boundary.

A lattice digraph is **Ballot-like** if it has allowed step set $\{U, D\}$ with U and D defined as in Equations (11) and (12). A path is **Ballot-like** if it is defined on a Ballot-like lattice. A lattice digraph is **Motzkin-like** if it has allowed step set

$$\{U, A, D\} \tag{13}$$

where U and D are defined as in Equations (11) and (12) and

$$A = (1, 0) \text{ ... an } \textbf{Across Step}. \tag{14}$$

A path is **Motzkin-like** if it is defined on a Motzkin-like digraph.

A vertex $v = (t, y)$ has **height** y. In a walk $w = v_0e_1v_1e_2...e_nv_n$, the height of vertex v_0 is termed the **initial height** of the walk; and the height of vertex v_n is termed the **final height** of the walk.

A **binomial path** is a path in the square lattice. A binomial path of length t always falls within a binomial box of size $m \times (t - m)$, for some $m \geq 0$. A **Ballot path** is a Ballot-like path in the half plane with initial height 0. A **rigged Ballot path** is a Ballot-like path in the half plane with initial height ≥ 0. A **Dyck path** is a Ballot-like path in the half plane with initial and final heights both 0. A **Motzkin path** is a Motzkin-like path in the half plane with initial and final heights both 0.

Ballot paths get their name from the **Ballot problem**, which asks for the number of ways in which votes may be cast in an election between two candidates A and B such that candidate A is always tied with or ahead of candidate B in a cumulative count of the votes. Solving the Ballot problem is equivalent to counting Ballot paths, since a vote for A may be equated with an Up Step, and a vote for B with a Down Step. Then the requirement that A is always ahead of or tied with B is equivalent to the restriction that the path is confined to the upper half plane. The origin of the name 'rigged Ballot path' also follows from this interpretation, since paths starting at some positive height correspond to an election in which candidate A has already been accorded some positive number of votes prior to the commencement of voting.

We also need the concept of a weighting. A **weight function** is a function which maps a set of combinatorial objects into a field. The field is usually $\mathbb{C}$ or $\mathbb{R}$; or an extension of $\mathbb{C}$ or $\mathbb{R}$. Examples of combinatorial objects to which we assign weights are: vertices, edges, arcs, walks and paths. We denote a weight function with the letter w so that, for example, $w(v)$, $w(e)$, $w(a)$, $w(w)$ and $w(p)$ are the weights of a vertex v, edge e, arc a, walk w and path p respectively.

A **weighted (di)graph** is a (di)graph with weights assigned to its vertices and edges/arcs. An **unweighted (di)graph** is a (di)graph without weights assigned to its vertices or edges/arcs. Our convention with regard to both weighted and unweighted digraphs is that any vertex or edge without a weight explicitly assigned to it may for convenience be regarded as weighted, with weight '1'. Thus we may refer to all the arcs of an unweighted digraph as having the 'same weight', that weight being 1.

A **uniformly weighted digraph** or **background weighted digraph** is a digraph for which both of the following conditions hold:
(i) All vertices of the digraph carry the same weight, and
(ii) All arcs of the same form in the digraph carry the same weight.
Note 1: Recall from the definition above that arcs of the same form are translates of each other. Thus, for example, in the rotated square lattice defined above, all Up Steps are of the same form and would thus carry the same weight in a uniformly weighted digraph.
Note 2: The class of uniformly weighted digraphs includes the class of unweighted digraphs.

A **decorated digraph** is a digraph with weighted arcs and unweighted vertices, for which arc weights vary across arcs of the same form. A **decoration** is a weight assigned to an arc which differs from the weights assigned to other arcs of the same form. An **undecorated digraph** is a uniformly weighted digraph.

Weight functions of composite objects are defined in terms of the weights of the component parts.

In particular, let G be a weighted graph or digraph. Then the **weight of a path** on G is defined as the product of the weights of the vertices and the weights of the edges/arcs in the path, i.e.

$$w(p) = \prod_{v,e \in p} w(v)w(e). \tag{15}$$

for p the path on G, each v a vertex and each e an edge (or an arc if G is a digraph). When vertices are unweighted, i.e. implicitly taken to have weight 1, Equation 15 simplifies to

$$w(p) = \prod_{e \in p} w(e). \tag{16}$$

A **(path) weight polynomial** is a polynomial in the weight variables which is the sum of the weights of all paths on a lattice, $\mathcal{L}$, of length n, and subject to some specifications such as initial and final height.

$$P_n(\text{weight variables}) = \sum_{\{p | p \in \mathcal{P}_n\}} w(p), \tag{17}$$

where $\mathcal{P}_n$ is the specified set of paths of length n on the lattice $\mathcal{L}$. In the case of unweighted paths we may still use the term ‘polynomial’ although the polynomials are reduced to integer constants in this situation.

A **counting sequence of (path) weight polynomials** is a sequence $\{P_n\}_{n\in\mathbb{N}_{\geq 0}}$ of (path) weight polynomials indexed by the lengths of the paths. For unweighted paths, where $\{P_n\}_{n\in\mathbb{N}_{\geq 0}}$ is a sequence of integers, we may still use the term 'counting sequence of (path) weight polynomials' but more often simply refer to the **(path) counting sequence**.

The **enumeration problem**, for a given class of paths, is to find an expression for the nth weight polynomial. The coefficients of this multivariate polynomial count the number of paths carrying each possible combination of weights. For unweighted paths in a given class, the weight polynomial reduces to a pure number which enumerates paths in the class.

"Begin at the beginning," the King said gravely, "and go on till you come to the end: then stop."

Lewis Carroll, 'Alice's Adventures in Wonderland'

Part I

Paths

Chapter 1

Introduction

This thesis has as its main focus the enumeration of directed lattice paths, with the development of a theory of pavings as a supporting auxilliary theme.

1.1 Why count lattice paths?

Lattice path enumeration problems are mathematically interesting because

- We can draw pictures; and apply geometric intuition – for a pair of paradigmatic historical examples see the discussion of the Reflection Principle and Many-weighted Paths, beginning on pages 12 and 28 respectively; for a recent example, the power of paving representations of recurrences throughout the whole of Part II relies upon being able to see the whole recurrence as a picture, then break and recombine it at an arbitrary point.
- Their solution often involves bringing together techniques from distinct areas of mathematics: combinatorics, algebra, analysis, computation. For example the 'Constant Term' method we develop in Chapter 10 closely entwines analysis and combinatorics in its derivation.
- The problems are easy to state, and some very simple to solve, yet seemingly slight variations can turn easy problems into much harder problems. For instance the DiMarzio-Rubin problem [37] which has been open since 1971 and which we solve in Chapter 10 is just a perturbation by addition of two decorations to the classic and much easier Ballot Path in a Strip problem which is discussed in Chapter 2.

These problems are physically interesting because

- In Chemistry, lattice paths model long-chain polymers in solution, [100]. For example, in investigation of steric stablization and sensitized flocculation paths are used represent polymers respectively holding apart and drawing together larger colloidal particles, [27]. Collapse transitions for polymers in dilute solutions are modeled, for example in [22], [23], [59], [83], [79], [97], [6]. Two-coloured lattice paths are used to model copolymers, for instance in [67], [68], [82], [60], [111], [110]. Adsorption of polymers on surfaces [38], [39] as well as pulling polymers off surfaces is studied, for example in [75], [74], [84], [80].

- In Medicine, lattice paths model the behaviour of proteins [32], [64], [114], [95], [86], [42], [94], [92], [28], [81], [111].

- In Statistical Mechanics, lattice paths arise in a plethora of models [100], [5], [108], [93]. Lattice paths relate to alternating sign matrices which are also of importance in Statistical Mechanics [13], [26].

- Path enumerations also unexpectedly solve the stationary state of a stochastic traffic model [41], [19], [9], [8], [36], [35], [18].

Because lattice path enumeration draws upon such a broad range of areas and techniques, solutions to the problems can have additional benefits. These include

- Advances in other aspects of Combinatorics, via
  - bijections to other combinatorial objects [99], [105], [29], [72], [73], [51], [48], [13], [31], [30], [46]
  - the development of new combinatorial objects as auxiliary tools [103], [24]

- Advances in computational algorithms to deal with exponential growth problems [76], [34], [61], [85], [78], [115], [21]

- Advances in techniques to manipulate
  - matrices, especially for finding high powers thereof, [98] (See Chapter 4 Section 4.7.), [46], [25], [45]
  - partial difference equations with boundary conditions; [77], [17]

  since both are used extensively in lattice path enumeration.

- Furthermore, [3] lists examples of applications of lattice paths to: probability theory, statistics, analysis of algorithms such as 'merge-sort' and 'shellsort', formal semantics, automata, continued fractions, birth-and-death processes as well as queueing theory.

1.2 Which paths?

The most tractable of the lattice path enumeration problems are those of *directed paths* [112], [110], which are the focus of this thesis. Directed paths are those for which most exact enumeration results have been obtained, and upon which our broadest range of techniques may be brought to bear. They stand in contrast to SAWs (Self Avoiding Walks), about which exact results are rare [102], [53], [54], [55] and which are commonly approached via analysis of series [112], [52], [56], [21], [6] obtained through computer enumeration. Under certain conditions, directed problems provide insight into the more difficult SAWs [101], which is an additional motivation to the study of the former.

Furthermore, [3] lists examples of applications of lattice paths to: probability theory, statistics, the study of permutations, the study of other lattice objects such as lattice animals, analysis of algorithms such as 'mergesort' and 'shellsort', formal semantics, automata, continued fractions, birth-and-death processes and queueing theory.

In general, lattice paths are characterized by

- The underlying lattice, and dimension thereof
- The allowed step set (which determines amongst other things whether paths are directed)
- Rules governing the interaction of paths with boundaries, with other paths (in the case of multiple paths), or with themselves (in the case of SAWs)
- Weights assigned to paths, either as decorations (which depend on position of edges and vertices) or as functions of the shape of the path

Given a space of paths defined by the above considerations, we enumerate them according to

- Length
- Starting position

- Ending position

The most widely useful and well-studied class of directed lattice paths are Ballot-like lattice paths (of which Dyck paths are a special case), closely followed by Motzkin-like lattice paths. Both are defined on a two-dimensional lattice. The former, Ballot-like lattice paths, owe their inclusive range of applicability to their identification with various classes of binary sequences, via the Ballot-like step set comprising just the two possibilities of U, for 'Up', and D, for Down. Hence Ballot-like lattice paths have potential application in contexts where there is some sequence of 0/1, Yes/No, True/False, On/Off, $P/\neg P$ choices. The Motzkin-like step set adds the natural third possibility of a neutral step, A, for 'Across'.

Much of this thesis is devoted to enumerating Ballot-like and Motzkin-like paths in the plane, half plane and strip, according to various weighting schemes, with a finite or infinite number of weights. We consider both the situation of weights determined by position (of edges in the path) and also by shape (the number of turns made by a path). Interestingly, in Chapter 3, we find a bijection between two such differently assigned weightings.

At this point one may generalize in any of at least three ways, to

1. Many paths (where one counts configurations of paths), for example [16], [70], [10], [33] [71], [57], [58], [72], [73].

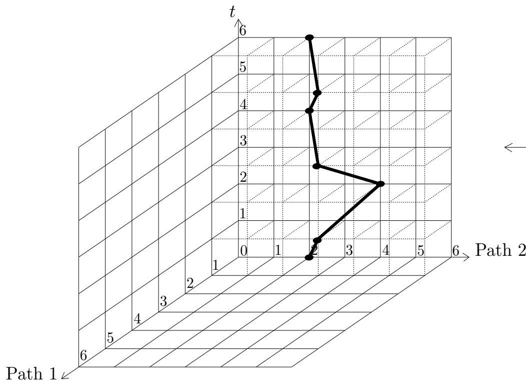
2. Many dimensions, for example [40], [49], [90].

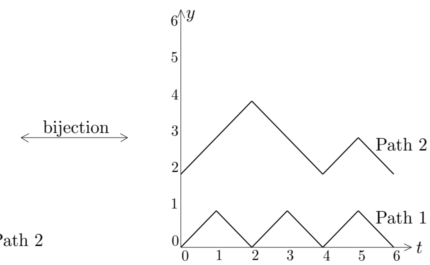
3. Increased allowed step set, for example [19], [3], [96] – (sequence numbers A091156, A001003, A011117, AO59231, A010683, A071945, A071946, A059435, A071943, ... and many others), [11], [4], [65], [66].

We focus on the third possibility, that of increasing the allowed step set. We also make some exploration of the second, increasing the number of dimensions, in Chapter 7. We note that this focus on increased step set does not preclude applicability of the results to Items 1 and 2, as the three generalizations are linked.

- Many paths $\longleftrightarrow$ Many dimensions

  Figure 1.1 shows a single path in three dimensions bijecting to a pair of paths in two dimensions. This illustrates the general point that a single path in $n+1$ dimensions bijects with n paths in two dimensions, given appropriate boundary conditions on the single paths. It is worth





t	0	1	2	3	4	5	6
Path 1	0	1	0	1	0	1	0
Path 2	2	3	4	3	2	3	2

Figure 1.1: A single path in $n+1$ dimensions bijects with n paths in two dimensions.

noting as an aside that Figure 1.1 shows multiple (in this case two) *non-intersecting paths*. For these paths which are not allowed to touch each other, the enumeration problem is satisfactorily solved in terms of two-dimensional single path results by the famous Gessel-Viennot determinant formula [50], [48]. The difficult many-path problems are those where paths are allowed to touch (osculating paths), or share edges (n-friendly paths). The easy many path enumeration is where paths can cross arbitrarily, and is given by the product of the single path results, since in this case paths 'don't see each other', and we may combine the independent results.

- Many dimensions $\longleftrightarrow$ Increased step set

The in-principle connection between many dimensional problems and two dimensional problems with an increased step set relies on the famous observation that the product of any finite number of countable sets is countable. In other words, given a 2 or 3 or 4 or n dimensional lattice, we can find a listing of the vertices of the lattice that lie on a straight (number) line in bijection with the natural numbers. Hence allowing steps up or down of varying distance mimics walking in many dimensions. See Figure 1.2 for a basic example. Figure 1.2 illustrates both the utility and the drawback of conceiving of paths in many

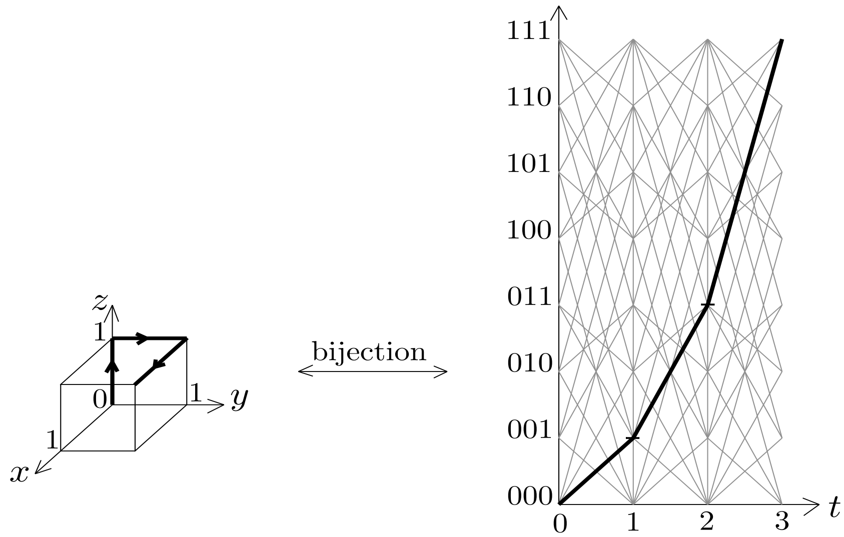


Figure 1.2: A walk on a small piece of cubic lattice corresponds to a walk in two dimensions with varying sized steps allowed.

dimensions as paths in two dimensions with many allowable steps. The utility is that the resulting two-dimensional path is easy to draw and is amenable to the techniques established for paths in the plane. The drawback is that the resulting mixed step set problems (which will differ depending on the choice of labeling) look formidable. Whether this bijection is useful as a practical tool for calculations is explored further in Chapter 7.

An immediately employable bonus of focusing on increased step sets is the bijection between 'Jump Any-step Paths' and 'Partially Directed Paths', an instance of which is illustrated in Figure 1.3. The partially directed path shown in the left hand side of the Figure has steps in both directions on the North-South axis, but only Easterly steps on the East-West axis. The path to which it bijects contains steps up, down and across which span varying heights. As we will see in Section 5.9, paths whose steps may jump up or down by *any* distance are particularly amenable to calculate with. Consequently, partially directed path enumerations are also within our practical capabilities.

As with Motzkin and Ballot like paths, we consider the various increased step set paths subject to a variety of weightings.

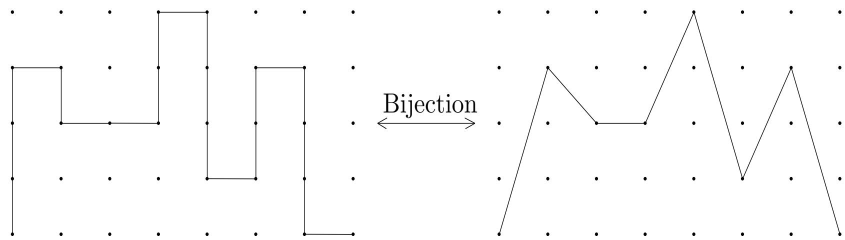


Figure 1.3: A 'Partially Directed path' bijects with a 'Jump-Any path'.

The next two sections describe methods. Generic approaches to path enumeration problems are reviewed in Section 1.3, with examples all drawn from the class of unweighted Ballot-like paths. Then a 'many-weighted' Ballot-like enumeration by combinatorial means is reviewed in Section 1.4.

1.3 A Cook's tour of existing approaches

Lattice path enumeration problems are typically approached from one of three main directions. These may be broadly classified as

1. Combinatorial
2. Transfer Matrix / Difference Equations
3. Generating Functions

'Combinatorial' is a name to collect together a variety of approaches which have in common a direct utilization of the geometry of the problem. Combinatorial solutions are often 'one off' insights into the structure of a particular geometry. These solutions may be particularly satisfying as they allow one to 'see in a picture' why an answer is correct. Bijections, involutions and inclusion/exclusion are the stock-in-trade of combinatorial methods.

Transfer matrices provide a systematic way to express the enumeration problem in terms of matrix equations. Obtaining closed form solutions for the weight polynomials depends on finding practical ways to evaluate high powers of matrices whose form is fixed (for instance, for Ballot-like paths the form will always be tridiagonal) but whose order may not be.

Difference equations provide a systematic way of expressing the weight polynomial as the solution of a recurrence. Solving the enumeration problem this way requires solving a partial difference equation with boundary conditions. The recurrences arising in this approach are closely related to recurrences on the entries of the eigenvectors of the transfer matrix; hence the intimate connection between transfer matrix and difference equation formulations.

Generating functions are, in Wilf's picturesque phrase, 'a clothesline to hang a sequence on' [113]. Using generating functions to solve the enumeration problem requires

- first, find the generating function, and

- secondly, extract the coefficients.

The first task may be accomplished by using combinatorial methods or transfer matrices. Then there are several ways both classical and new (see Chapters 1 and 10) of extracting the coefficients.

Generating functions are also of interest in their own right. Though asymptotics is not the subject of this work, we comment that generating functions are particularly useful in investigating the 'large n' behaviour of the weight polynomial sequence. Even without explicitly extracting the weight polynomials, an analysis of the closest singularity to the origin of the generating function gives asymptotic information about the coefficient sequence. Thus a natural extension of the work in this thesis would be do asymptotics on some of the generating functions we find.

In this section we apply an instance of each of the three approaches to the problem of enumerating unweighted Ballot paths. This classic problem was solved in the 19th century and is usually attributed to André [2], but there is some question about this [91]. The purpose of its re-working here is to give a concrete example of each generic type of approach.

1.3.1 Combinatorial

The method used in the late 1800's to solve the Ballot problem is called the **André Principle** or **Reflection Principle** when referred to in the combinatorics literature. A similar and possibly more general idea arose independently in the physics literature [43], [88] in a slightly different formulation, where it is known as the **Method of Images**. The two ideas are the same as applied to directed lattice paths, and we use the various names interchangeably.

In this subsection we describe the solution to the Ballot problem using the Reflection Principle, and show how this principle is a particular case of an involution, which latter is a ubiquitously useful structure in the solution of combinatorial problems.

The Reflection Principle, or Method of Images, for Ballot paths requires the solution to the binomial path problem first. This counting problem is easy once one has may a bijection between binomial paths and **binomial words**, the latter being finite sequences of letters drawn from an alphabet of two letters. See Figure 1.4. Depicted is a binomial box, which we recall is a sub-digraph of the square lattice consisting of all vertices of the form (i, j) where $x \leq i \leq x+m$ *and* $y \leq j \leq y+n$; along with the connecting arcs induced by these vertices.

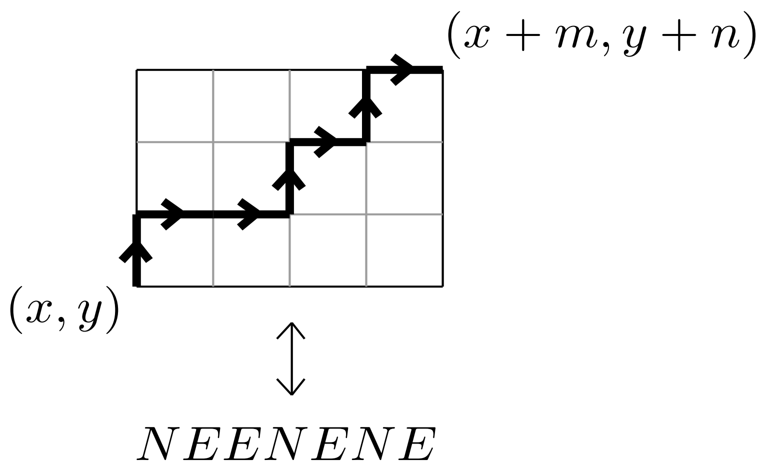


Figure 1.4: Binomial paths biject to binomial words.

We observe that a path from the lower left hand corner to the upper right hand corner of an $m \times n$ binomial box must consist of m steps East and n steps North; a total of $m+n$ steps. Hence the corresponding binomial word consists of $m+n$ letters, of which m of them are the letter E and the rest are the letter N. There are $\binom{m+n}{m}$ ways to arrange m copies of E in $m+n$ possible places, thus there are $\binom{m+n}{m}$ binomial words of length $m+n$. Since any binomial word corresponds to a valid binomial path and vice versa, we have classic result

Lemma 1. *The number of binomial paths with initial vertex* (x, y) *and final vertex* $(x+m, y+n)$ *is equal to*

$$Bin(m, n) = \binom{m+n}{m}. \tag{1.1}$$

We now use binomial paths to count Ballot paths.

The idea is to first count a set which is too big: Ballot paths plus some extra, 'bad paths'. Then we count the 'bad paths'. By subtracting the number of 'bad paths' away from the first number we obtained, we are left with the correct number of Ballot paths.

1. Consider Figure 1.5. It shows a binomial box of size $(t+y)/2 \times (t-y)/2$ which has been rotated 45 degrees clockwise about the origin. Any path which is incident with the line $y = -1$ is *not* a Ballot path and is termed a **bad path**. The rest of the paths within the box constitute the set of Ballot paths ending at vertex (t, y).

2. Figure 1.6 shows the bad path after reflection, where that portion of it after it hits the line $y = -1$ has been reflected in the line $y = -1$.

3. After reflection, all bad paths end up in their own binomial box, indicated in Figure 1.7. Furthermore, all binomial paths in the new box shown are reflections of bad paths. Hence there is a bijection between bad paths and paths in the binomial box of size $(t + y + 2)/2 \times (t - y - 2)/2$.

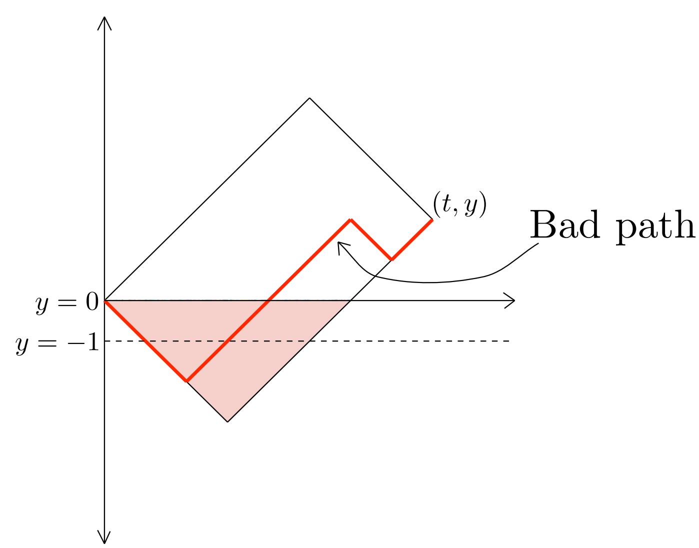


Figure 1.5: All Ballot paths that end at vertex (t, y) occur within the unshaded part of the binomial box.

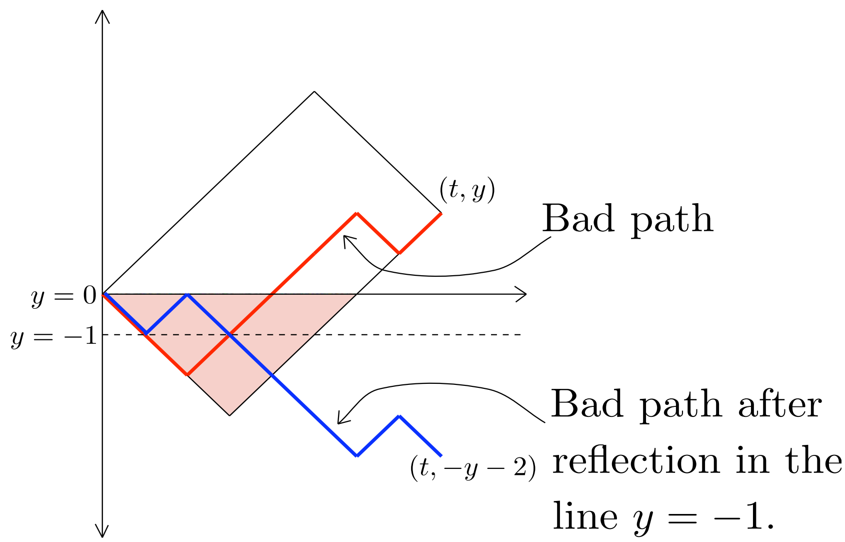


Figure 1.6: Those paths which intersect the line $y = -1$ have the portion after their first intersection reflected in the line $y = -1$.

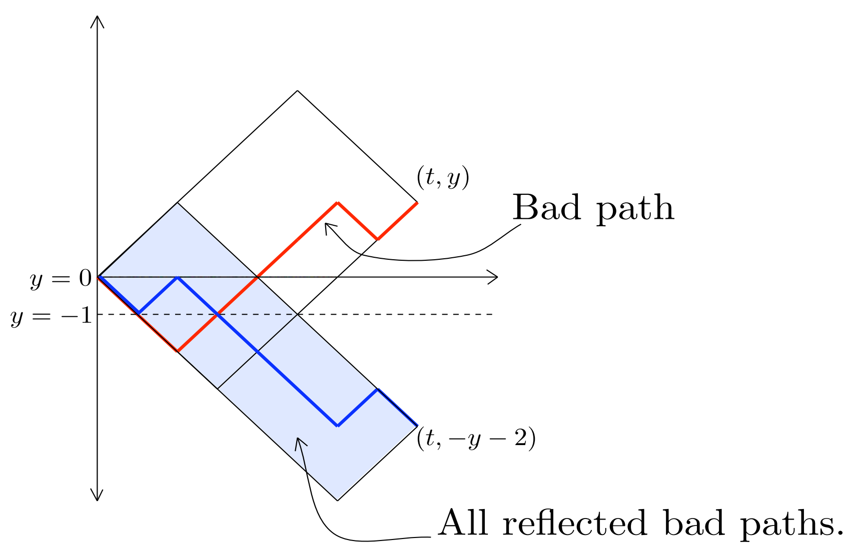


Figure 1.7: The reflected paths occupy their own binomial box.

Putting figures 1.5–1.7 together with Lemma 1 gives classic theorem counting Ballot paths:

Theorem 2. *The number of Ballot paths of length t ending at height y is*

$$B_{t,y} = \binom{t}{(t+y)/2} - \binom{t}{(t+y+2)/2}. \tag{1.2}$$

The Reflection Principle is a particular instance of a 'Sign Reversing Involution'.

Definition 1. *An* **Involution**, *ν, is a mapping $\nu : \Omega \mapsto \Omega$ such that*

$$\nu^2 = I. \tag{1.3}$$

In other words, an involution is a function which is its own inverse. The **fixed point set**, *$\mathcal{F} \subseteq \Omega$, of an involution is the set of all $p \in \Omega$ such that*

$$\nu(p) = p. \tag{1.4}$$

A **sign reversing involution**, *$\nu : \Omega \mapsto \Omega$, is an involution such that there exists a partitioning of Ω into a* ***positive set***, *Ω_+, and a* ***negative set*** *Ω_- such that*

$$\Omega := \Omega_+ \cup \Omega_- \quad \textit{and} \quad \Omega_+ \cap \Omega_- = \emptyset \tag{1.5}$$

and so that the fixed point set of the involution lies entirely within Ω_+, i.e.

$$p \in \Omega_- \quad \Rightarrow \quad \nu(p) \in \Omega_+, \ \textit{and} \tag{1.6}$$

$$p \in \Omega_+ \quad \Rightarrow \quad \textit{either } \nu(p) \in \Omega_- \textit{ or } \nu(p) = p. \tag{1.7}$$

Sign-reversing involutions are a classic way to count any set for which we can construct a sign-reversing involution for which the desired set is the fixed point set [47].

Lemma 3 (Involution Principle)**.** *Let mapping $\nu: \Omega \mapsto \Omega$ be a sign-reversing involution with notation as in Definition 1. Then the size of the fixed point set is*

$$|\mathcal{F}| = |\Omega_+| - |\Omega_-|. \tag{1.8}$$

Figure 1.8 illustrates the Involution Principle of Lemma 3 applied to the Reflection Principle.

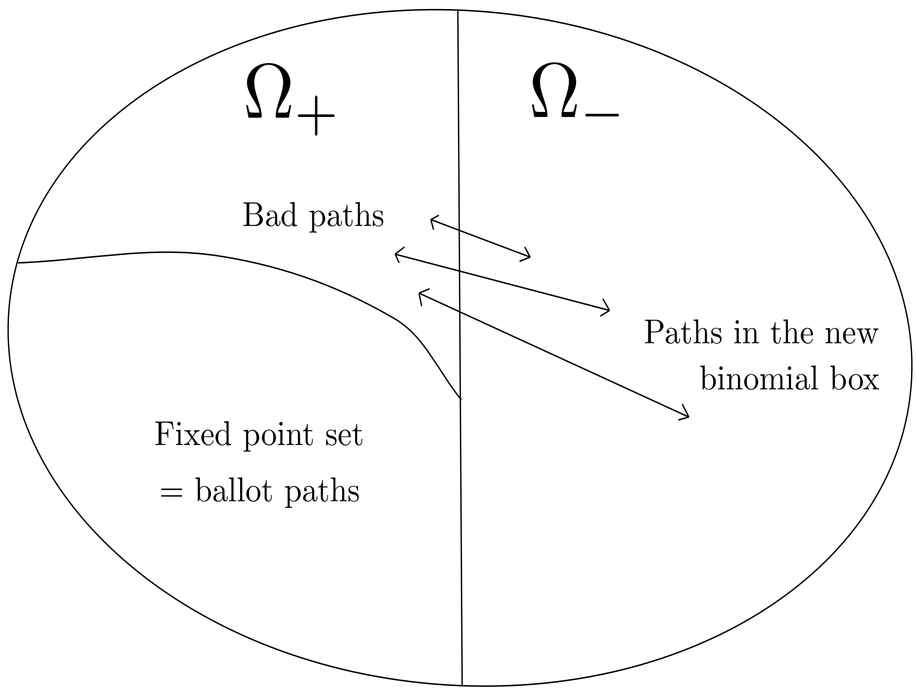


Figure 1.8: The Method of Images is an involution for which Ω_+ is the set of all paths in the original binomial box illustrated in Figure 1.5, and Ω_- is the set of paths in the lower binomial box shown in Figure 1.7.

1.3.2 Transfer Matrices / Difference Equations

Transfer matrices and partial difference equations both allow systematic expressions of the Ballot enumeration problem. We present first a transfer matrix method and show it's relation to an ordinary difference equation. We then describe the partial difference equation formalism and show the relationship between this expression and the ordinary difference equations that arise in the transfer matrix method.

Transfer Matrices

We begin by defining state vectors and transfer matrices.

Definition 2. *A* **state vector** *is a vector,* $\underline{B}^{(t)} = (B_0^{(t)}, B_1^{(t)}, B_2^{(t)}, ...)^T$*, which records the state of a system at 'time'* t*. For lattice paths in a* (t, y) *coordinate system, the* y^{th} *entry of the* t^{th} *state vector specifies the number of paths (or weight polynomial) at height* y *and horizontal coordinate* t*.*

Definition 3. *A* **transfer matrix** *is a matrix,* $T = (T_{ij})$*, which, when multiplied by the* t^{th} *state vector, gives the* $(t+1)^{th}$ *state vector; i.e.*

$$\underline{B}^{(t+1)} = T\underline{B}^{(t)}. \tag{1.9}$$

The $(i, j)^{th}$ *entry of the matrix (for indices* $i, j = 0, 1, 2, ...$*) gives the number of ways of passing from height* i *to height* j *in a single step. In terms of weights,*

$$T_{ij} = w(\text{arc from vertex } (t, i) \text{ to vertex } (t+1, j)), \tag{1.10}$$

and $T_{ij} = 0$ *when there is no arc from vertex* (t, i) *to vertex* $(t+1, j)$*.*

Iterating the definition of a transfer matrix gives classic result

Lemma 4. *The state vector for Ballot paths is given by matrix equation*

$$\underline{B}^{(t)} = T^t \underline{B}^{(0)}, \tag{1.11}$$

with initial state vector $\underline{B}^{(0)} = (1, 0, 0, ...)^T$ *and tridiagonal transfer matrix:*

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & \ddots \\ 1 & 0 & 1 & 0 & \ddots \\ 0 & 1 & 0 & 1 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.12}$$

Thus the problem of counting Ballot paths becomes one of evaluating high powers of a matrix. For a matrix of fixed finite order, this is a standard problem in linear algebra and, for a diagonalizable matrix, is achieved by diagonalizing T, so that we would have

$$T^t = PD^tP^{-1}, \tag{1.13}$$

for some diagonal matrix of eigenvalues D and column matrix of eigenvectors P.

But for Ballot paths, which occupy the entire upper half plane, the transfer matrix is infinite and the commonly used diagonalization procedure does not terminate. We have two alternatives. They are

1. (a) Work directly with the infinite matrix.
   (b) Use a recurrence to calculate infinite eigenvectors.
   (c) The final step of multiplying out expression (1.13), which for finite matrices would be a sum over entries of P, D^t and P^{-1}, becomes an integral.
2. (a) Truncate T to a finite matrix T_L, which counts Ballot paths in a strip of height L.
   (b) Use a recurrence to calculate eigenvectors.
   (c) Either
      i. Evaluate the sum implicit in Equation (1.13) as a function of L and then take a limit as $L \to \infty$, or
      ii. Evaluate the sum implicit in Equation (1.13) subject to the assumption that $L > t$, so that paths never get long enough to be affected by the upper wall of the strip.

We illustrate Option 2.

Lemma 5. *The transfer matrix for Ballot paths in a strip of height L is*

$$T_L = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \\ L-1 \\ L \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & & & L-1 & L \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & & & & \\ 1 & 0 & 1 & 0 & & & & \\ 0 & 1 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & & \ddots & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix} \end{array}. \tag{1.14}$$

Powers of T_L are expressible in the form

$$T_L^t = Q^T D^t Q \tag{1.15}$$

where D is a diagonal matrix with diagonal entries of the form

$$\rho + \rho^{-1} \tag{1.16}$$

such that $\rho \in \mathcal{R}$ with

$$\mathcal{R} = \{e^{\frac{i\pi}{L+2}}, e^{\frac{i2\pi}{L+2}}, ..., e^{\frac{i(L+1)\pi}{L+2}}\} \tag{1.17}$$

and Q is an orthogonal matrix with column vectors

$$\underline{B}(\rho) = (B_0(\rho), B_1(\rho), ..., B_L(\rho))^T, \tag{1.18}$$

indexed by ρ. The entries of the vectors take the form

$$B_y(\rho) = \alpha(\rho^y - \rho^{-y}), \tag{1.19}$$

where $\alpha \in \mathbb{C}$ is a normalization constant such that

$$\alpha^2 = \frac{-1}{2(L+2)}. \tag{1.20}$$

Comments on Proof. The proof is straightforward calculation. It relies on the observation that the entries of an eigenvector $\underline{B} = (B_0, ..., B_L)$ of T_L satisfy constant coefficient recurrence relation

$$B_{y-1} + B_{y+1} = \mu B_y \tag{1.21}$$

where μ is an eigenvalue; and with boundary conditions

$$B_1 = \mu B_0 \tag{1.22}$$
$$B_{L+1} = 0. \tag{1.23}$$

We note that recurrence relation (1.21) is a rearrangement of the standard three-term recurrence for Chebyshev's '$S(n, x)$' Polynomials [96][1]

$$B_{y+1}(\mu) = \mu B_y(\mu) - B_{y-1}(\mu) \tag{1.24}$$

We also note that the parametrization of the eigenvalues of T_L in the form

$$\mu = \rho + \rho^{-1} \tag{1.25}$$

occurs naturally when solving the recurrence relation. Finally, we note that $\mathcal{R}$ is the set of solutions to:

$$B_{L+1}(\mu(\rho)) = 0, \tag{1.26}$$

where $\mu(\rho)$ is given by Equation (1.25) and $B_{L+1}(\mu)$ is the $(L+1)^{\text{th}}$ Chebyshev polynomial. It is the simple form of the parametrized polynomial which allows explicit determination of its roots. □

Corollary 6. *The number of Ballot paths of length t ending at height y and confined to a strip of height L is given by the sum*

$$B_{t,y}(L) = \frac{-1}{2(L+2)} \sum_{\rho\in\mathcal{R}} (\rho+\rho^{-1})^t(\rho-\rho^{-1})(\rho^{y+1}-\rho^{-(y+1)}), \tag{1.27}$$

where $\mathcal{R} = \{e^{\frac{i\pi}{L+2}}, e^{\frac{i2\pi}{L+2}}, ..., e^{\frac{i(L+1)\pi}{L+2}}\}$. Evaluating this sum for general L gives

$$B_{t,y}(L) = \sum_{j=-\infty}^{\infty} \left(\binom{t}{\frac{t+y+(2L+4)j}{2}} - \binom{t}{\frac{t+y+2+(2L+4)j}{2}} \right). \tag{1.28}$$

Comments on Proof. Equation (1.27) follows directly from Equation (1.15). Evaluating Equation (1.27) to get Equation (1.28) is a long and straightforward but tedius calculation that relies both on the fact that we know the members of $\mathcal{R}$ explicitly and that these are evenly distributed around the unit circle so that we may employ geometric series. □

In the half plane, Corollary 6 immediately specializes to

Corollary 7. *The number of Ballot paths of length t ending at height y is*

$$B_{t,y} = \binom{t}{\frac{t+y}{2}} - \binom{t}{\frac{t+y+2}{2}}. \tag{1.29}$$

Difference Equations

Another way to pose the Ballot enumeration problem is directly in terms of partial difference equations. Let $B_{t,y}$ be the number of Ballot paths with length t that end at height y. Then

$$B_{t,y} = B_{t-1,y-1} + B_{t-1,y+1} \tag{1.30}$$

$$B_{t,0} = B_{t-1,1} \tag{1.31}$$

$$B_{0,y} = \delta_{0,y}. \tag{1.32}$$

where (1.30) applies for $y, t \geq 1$, (1.31) applies for $t \geq 1$ and $\delta_{0,y}$ is the delta function whose value is 1 when its arguments coincide, and 0 otherwise.

A three step method to solve partial difference equation (1.30) with boundary conditions was developed by Brak et al [20] in 1998

The first two steps are standard. The third step is something new, and was the genesis of the theory presented in Chapter 10.

Step One: Ansatz

$$B(t, y) = \mu^t \rho^y \tag{1.33}$$

substituted into bulk equation (1.30) gives

$$\mu = \rho + \rho^{-1} \tag{1.34}$$

and symmetry $\mu(\rho) = \mu(-p)$ implies bulk equation (1.30) is also satisfied by

$$B_{t,y} = \mu^t (C\rho^y + D\rho^{-y}) \tag{1.35}$$

for unknown constants C and D.

Step Two: Solve for D in terms of C using boundary equation (1.31) and substitute back into trial solution (1.35) to get

$$B_{t,y} = \mu^t C(\rho^y - \rho^{-y-2}). \tag{1.36}$$

Step Three: Choose for C the expression obtained by taking the bracketed component of Equation (1.36), replacing ρ with ρ^{-1} and replacing y with the initial height of the path - for Ballot paths this is zero. Then take half the constant term of the resulting expression as a Laurent series expanded in ρ about 0, to obtain the number of Ballot paths. We get

$$B_{t,y} = \frac{1}{2} \left[\mu^t (1 - \rho^2)(\rho^y - \rho^{-y-2}) \right] \tag{1.37}$$

Rearranging Equation (1.37) and recalling Equation (1.34) we have result due to Brak, Essam and Owczarek [20]

Lemma 8. *The solution to partial difference equation (1.30) with boundary Equations (1.31) and (1.32) is*

$$B_{t,y} = \frac{-1}{2} \operatorname{CT} \left[(\rho + \rho^{-1})^t (\rho - \rho^{-1})(\rho^{y+1} - \rho^{-(y+1)}) \right], \tag{1.38}$$

where

$$\operatorname{CT}[\cdots a_{-1}\rho^{-1} + a_0 + a_1\rho + \cdots] = a_0 \tag{1.39}$$

and the argument of the constant term is a Laurent Series in ρ expanded about 0.

Proof. The proof is in the pudding. Direct substitution shows the right hand side of Equation (1.38) to satisfy Equations (1.30) – (1.32). □

1.3.3 Generating functions

Definition 4. *The* **(ordinary) generating function** *for the sequence* $\{c_t\}_{t\geq 0}$ *is the series*

$$G(x) = c_0 + c_1 x + c_2 x^2 + ... \tag{1.40}$$

Finding the generating function

Two common approaches to finding generating functions are

1. When the desired sequence is given by powers of a transfer matrix T, the generating function

   $$G(x)_{y',y} = I + (T)_{y',y} x + (T^2)_{y',y} x^2 + ... \tag{1.41}$$

   is a geometric series, which may be written concisely as

   $$G(x)_{y',y} = (I - xT)^{-1}\big|_{y',y}. \tag{1.42}$$

   The matrix $(I - xT)^{-1}$ is called the **resolvent** of T. Sometimes, we are able to use what we know about T to find an explicit expression for the resolvent of T, hence the generating function. We will pursue this approach in detail in the latter part of this thesis for a variety of classes of weighted paths.

2. (a) Use combinatorics to find a functional relationship on G
   (b) Solve the relationship to obtain a formula for $G(x)$

We choose to pursue the second approach in this section. We specialize to the famous case of Dyck paths, which we recall are Ballot paths which end as well as begin at height 0.

Let $\{c_t\}_{t\geq 0}$ be the counting sequence for Dyck paths, and let $G(x)$ be its ordinary generating function. The first few terms in the series are

$$G(x) = 1 + x^2 + 2x^4 + 5x^6 + ..., \tag{1.43}$$

as illustrated by Figure 1.9. We find a functional relation on G by looking at how to build Dyck paths out of concatenations of smaller Dyck paths. Consider Figure 1.10.

Since each Dyck path of length $t \geq 2$ must contain at least one 'up' step and at least one 'down' step, we may decompose it as in Figure 1.10. Summing over all possible such decompositions, we obtain, for $t \geq 2$,

$$c_t = c_0 c_{t-2} + c_2 c_{t-4} + ... + c_{t-2} c_0. \tag{1.44}$$

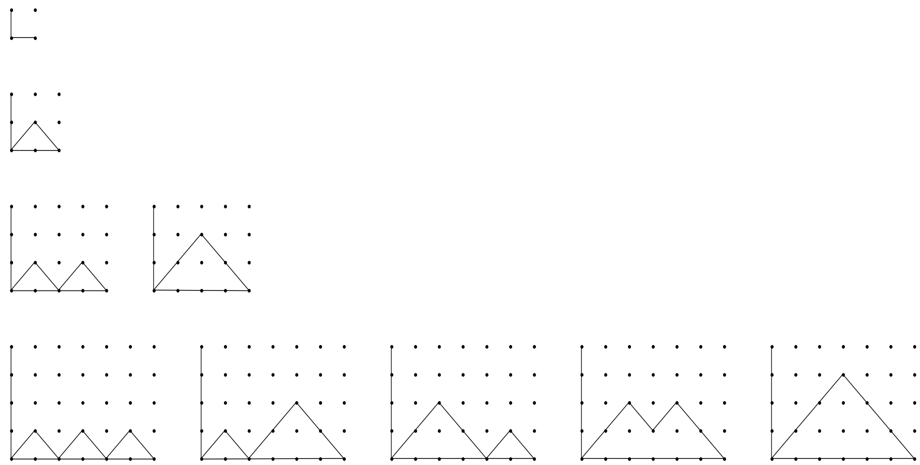


Figure 1.9: The first few Dyck paths.

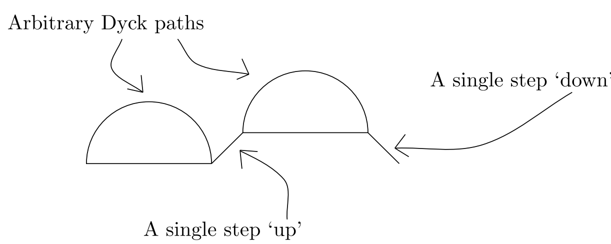


Figure 1.10: Any Dyck path of length $n \geq 2$ must take the form illustrated, where each semicircle represents an arbitrary Dyck path in its own right, which may be the zero path.

We want to construct a function which has coefficients of this form. Since

$$G(x) = c_0 + c_2x^2 + c_4x^4 + ..., \tag{1.45}$$

squaring G gives

$$G(x)^2 = c_0c_0 + (c_0c_2 + c_2c_0)x^2 + (c_0c_4 + c_2c_2 + c_4c_0)x^4 + ... \tag{1.46}$$

Now we just need to get the lengths right. Multiply by x^2 to take into account the two extra edges in each path. Then add 1 to put the zero path back in at the start. We have

$$1+x^2G(x)^2 = 1+c_0c_0x^2+(c_0c_2+c_2c_0)x^4+(c_0c_4+c_2c_2+c_4c_0)x^6+... \tag{1.47}$$

In light of relation (1.44), this is precisely $G(x)$. We have shown the first half of famous result:

Lemma 9. *Let $G(x) = \sum_{t\geq 0} c_t x^t$ be the ordinary generating function for the counting sequence for Dyck paths. Then $G(x)$ satisfies the functional relation*

$$1 + x^2G(x)^2 = G(x) \tag{1.48}$$

Furthermore,

$$G(x) = \frac{1-\sqrt{1-4x^2}}{2x^2}. \tag{1.49}$$

Proof. We need only show that Equation (1.49) holds. As Equation (1.48) is quadratic in G there are two possible solutions. Expanding the first few terms of each as follows:

$$\frac{1-\sqrt{1-4x^2}}{2x^2} = 1 + x^2 + 2x^4 + 5x^6 + ... \tag{1.50}$$

$$\frac{1+\sqrt{1-4x^2}}{2x^2} = \frac{1}{x^2} - 1 - x^2 - 2x^4 - 5x^6 + ... \tag{1.51}$$

shows that the minus sign is the correct choice. □

Extracting the coefficients

Given a generating function, $G(x)$, there are five traditional ways to extract the coefficients.

1. 'Guess': expand (by computer or by hand) the first few terms and then

(a) look up a table of sequences such as [96], or

(b) look for patterns directly: see [87] and references therein, then

'Check': use induction to check the answer.

(For an engaging general discussion of Ansatzen see [116], [117].)

2. Lagrange Inversion – only applicable when the Generating Function may be expressed in a form suitable for using the well-known Lagrange Inversion Theorem:

Theorem 10 (Lagrange Inversion)**.** *Let $f(u)$ and $\phi(u)$ be formal power series in u, with $\phi(0) = 1$. Then there is a unique formal power series $u = u(t)$ that satisfies*

$$u = t\phi(u). \tag{1.52}$$

Furthermore,

$$[t^n]\{(f(u(t)))\} = \frac{1}{n}[u^{n-1}]\{f'(u)\phi(u)^n\}. \tag{1.53}$$

3. 'Geometric Series' expansion - only works if $G(x)$ happens to be a rational function.

4. 'Taylor Series' expansion - write $G(x) = c_0 + c_1x + c_2x^2 + ...$ as:

$$G(x) = G(a) + \frac{G'(a)}{1!}(x-a) + \frac{G''(a)}{2!}(x-a)^2 + ... \tag{1.54}$$

When expansion about $a = 0$ is allowable the Taylor expansion method succeeds if a general expression can be found for the t^{th} derivative. For expansions about $a \neq 0$ there is an additional sum to be performed to collect coefficients of like powers of x.

5. 'Cauchy integral': Observe that

$$c_t = \text{the } -1^{\text{th}} \text{ coefficient of } \left[\frac{c_0 + c_1x + c_2x^2 + ... + c_tx^t + ...}{x^{t+1}}\right] \tag{1.55}$$

$$= \text{Res}_{x=0}\left[\frac{G(x)}{x^{t+1}}\right] \tag{1.56}$$

Provided we can find a small closed contour γ about the origin which excludes other poles of $G(x)/x^{t+1}$, Equation (1.56) becomes

$$c_t = \frac{1}{2\pi i}\oint_\gamma \frac{G(x)}{x^{t+1}}dx. \tag{1.57}$$

For the Dyck path generating function given by Lemma 9, only the first, second and fifth methods look tractable. It is rare that $G(x)$ turns out to be an explicitly rational generating function, and Taylor Series expansions are usually prohibitive.

The fourth 'Cauchy' method has been successfully used on the Dyck path generating function and other lattice path generating functions, with some effort and ingenuity employed in evaluating the integrals. It turns out that evaluating the necessary integrals is equivalent to carrying out the CT method developed in Chapter 10. The CT method, which had its genesis in the Cauchy integral approach, has been systematized for Ballot-like and Motzkin-like lattice paths. For this case of unweighted Dyck paths, it is by far easiest to use either the first or second method.

We close this section with an application of the second method, as reviewed in [12], to this famous sequence. Re-expressing the generating function (1.45) so that the coefficient of t^n is the number of paths of length $2n$, we have modified functional equation

$$H(t) = 1 + tH^2. \tag{1.58}$$

Then a further transformation

$$H = u + 1 \tag{1.59}$$

gives new functional equation

$$u = t(1+u)^2, \tag{1.60}$$

which we see is in the correct form for the application of the Lagrange Inversion Theorem, with

$$\phi(u) = (1+u)^2. \tag{1.61}$$

We choose f to be the identity function:

$$f(u) = u \text{ and } f'(u) = 1; \tag{1.62}$$

so that

$$\begin{aligned}
[t^n]u &= \frac{1}{n}[u^{n-1}](1+u)^{2n} && (1.63)\\
&= \frac{1}{n}[u^{n-1}]\sum_{i=0}^{2n}\binom{2n}{i}u^i && (1.64)\\
&= \frac{1}{n}\binom{2n}{n-1} && (1.65)\\
&= \frac{1}{n+1}\binom{2n}{n}. && (1.66)
\end{aligned}$$

Thus we have classic result

Lemma 11. *The counting sequence* $\{c_{2n}\}_{n\geq 0}$ *for Dyck paths of length* $2n$ *is given by*

$$c_{2n} = \frac{1}{n+1}\binom{2n}{n}. \tag{1.67}$$

The sequence $\{\frac{1}{n+1}\binom{2n}{n}\}_{n\geq 0} = 1, 1, 2, 5, 14, 42, 132, 429, ...$ is the renowned **Catalan numbers**.

1.4 Many-weighted Paths: a combinatorial answer

The more comprehensive theorem may be easier to prove, the more general problem may be easier to solve.

George Polya, 'How to Solve It' [89]

This section is a rediscovery of a method and result of Flajolet [44]. The main result may come as a surprise, in the light of the rest of the thesis, much of which is devoted to finding novel means to calculate weight polynomials for Dyck paths subject to weightings involving a small number of weights. Here we explicitly write down the weight polynomial for Dyck paths subject to a maximal number of independent weights. The method is purely combinatorial, and applies equally well whether the paths are confined to a strip or not.

We first describe a general weighting, then give the enumeration. The form of the theorem makes it clear why this general answer does not make subsequent investigations into specific weightings redundant. We follow the proof of the theorem with a brief discussion of the relationship between the form of this general answer and the forms of answers to be expected to questions pertaining to more specific weightings.

Definition 5. *A* **general downstep weighting** *for Dyck paths is a weighting for which all 'down' edges have a distinct weight assigned, and all 'up' edges are weighted '1'.*

Definition 6. *A* **general weighting** *for Dyck paths is a weighting for which all edges, both 'down' and 'up', have distinct weights assigned.*

For the purposes of the enumeration problem on Dyck paths, it is sufficient to consider a general downstep weighting. Since for Dyck paths what goes up must come down, so that 'up' and 'down' steps are paired off with each other, the result for the general weighting immediately follows.

The following theorem is a result of Flajolet [44] which he relates to expansions of continued fractions.

Theorem 12. *Let $\mathcal{W}_{2n}$ be the set of Dyck paths of length $2n$ with general downstep weighting:*

$$\begin{aligned} w(\text{edge from height `}y\text{' to height `}y+1\text{'}) &= 1, & (1.68)\\ w(\text{edge from height `}y\text{' to height `}y-1\text{'}) &= \kappa_y; & (1.69)\end{aligned}$$

in either a strip of height L or in the half plane. Then the weight polynomial for $\mathcal{W}_{2n}$ is given by

$$W_{2n}(\kappa_1,\kappa_2,...;L) = \sum_{l=0}^{\min\{n-1,L-1\}} s_l, \tag{1.70}$$

where s_l is the weight polynomial for that subset of paths in $\mathcal{W}_{2n}$ which reach but do not exceed height $l+1$. The half plane result is obtained by setting $L=\infty$. The s_l's are given by

$$s_0 = \kappa_1^n, \tag{1.71}$$

and

$$s_l = \sum_{j_1=l}^{j_0-1}\sum_{j_2=l-1}^{j_1-1}\cdots\sum_{j_l=1}^{j_{l-1}-1}\prod_{k=0}^{l-1}\binom{j_k-j_{k+2}-1}{j_k-j_{k+1}}\kappa_1^{j_0-j_1}\kappa_2^{j_1-j_2}\dots\kappa_{l+1}^{j_l-j_{l+1}}, \tag{1.72}$$

for $l \geq 1$; with

$$\begin{aligned} j_0 &:= n, & (1.73)\\ j_{l+1} &:= 0. & (1.74)\end{aligned}$$

Before proving the theorem we give an example showing the construction behind the proof.

Example 1. *We systematically list all general downstep weighted Dyck paths of length $2n = 8$.*

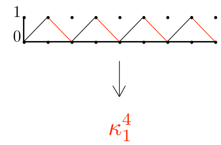


Figure 1.11: There is one path of height 1.

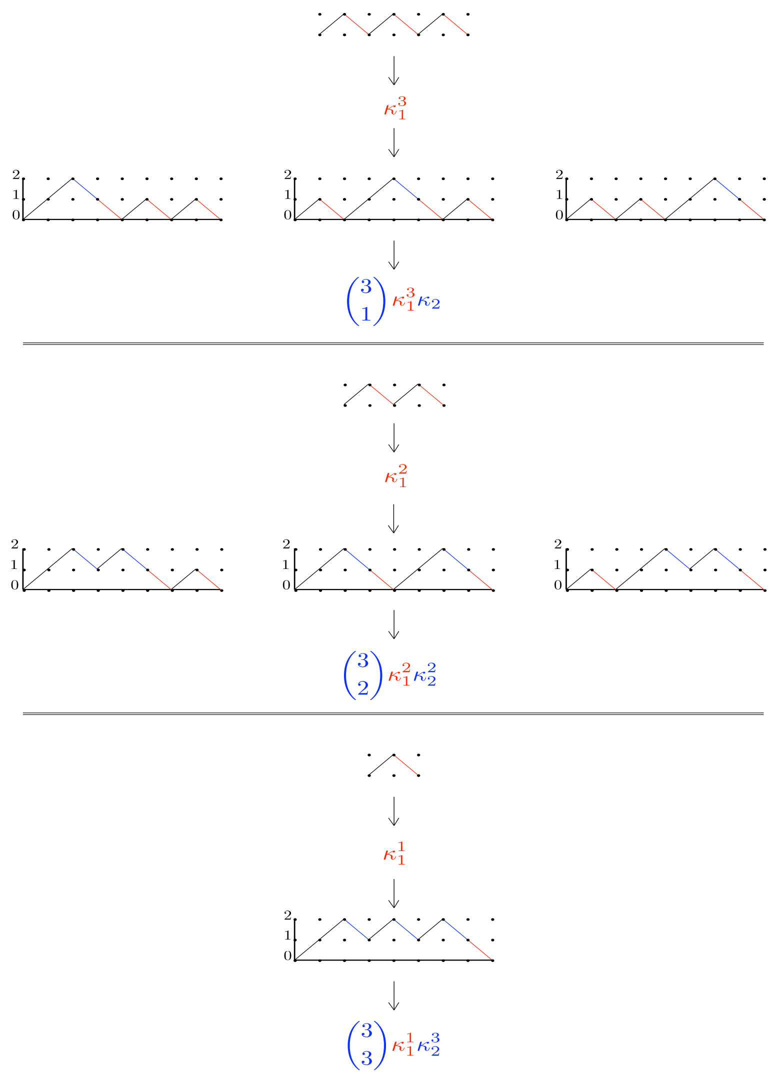


Figure 1.12: There are seven paths of height 2.

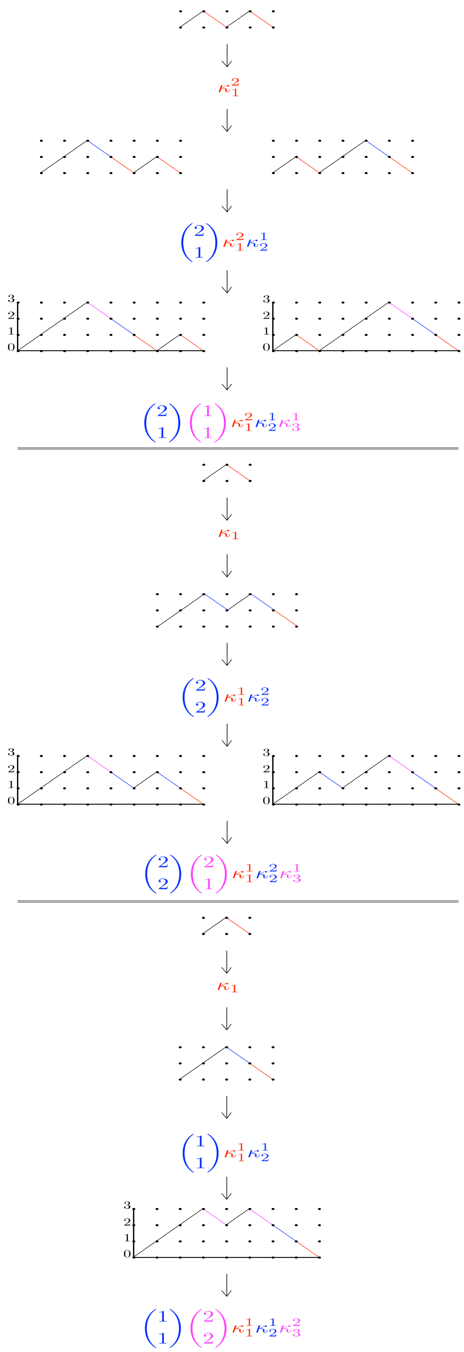
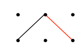


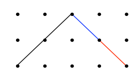
Figure 1.13: There are five paths of height 3.



$$\downarrow$$

$$\kappa_1$$

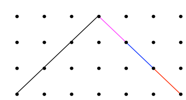
$$\downarrow$$



$$\downarrow$$

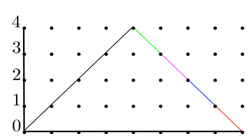
$$\binom{1}{1}\kappa_1^1\kappa_2^1$$

$$\downarrow$$



$$\downarrow$$

$$\binom{1}{1}\binom{1}{1}\kappa_1^1\kappa_2^1\kappa_3^1$$



$$\downarrow$$

$$\binom{1}{1}\binom{1}{1}\binom{1}{1}\kappa_1^1\kappa_2^1\kappa_3^1\kappa_4^1$$

Figure 1.14: There is one path of height 4

We will utilize the following classical lemma in the proof of Theorem 12.

Lemma 13. *There are exactly*

$$\binom{n+i-1}{i-1} \tag{1.75}$$

ways to place n indistinguishable objects in i distinct sets.

Proof of Theorem 12. We explicitly illustrate the first few inductive steps of this proof, as the notation is more transparent that way. First observe that

$$s_0(n-j_1) = \kappa_1^{n-j_1}, \tag{1.76}$$

for

$$0 \leq j_1 \leq n-1 \tag{1.77}$$

since there is only one path in $\mathcal{W}_{2(n-j_1)}$ which reaches but does not exceed height 1: a zig-zag which returns to the surface $n-j_1$ times.

Paths which reach but do not exceed height 2 may be constructed from paths which reach but do not exceed height 1. By Lemma 13, there are

$$\binom{n-j_2-1}{n-j_1-1} \tag{1.78}$$

ways to insert $j_1 - j_2$ peaks, shared amongst $n - j_1$ spaces. Noting that we require $j_1 \geq 1$ for a positive number of peaks, we have

$$s_1(n-j_2) = \sum_{j_1=1}^{n-1} \binom{n-j_2-1}{n-j_1-1} \kappa_1^{n-j_1} \kappa_2^{j_1-j_2}, \tag{1.79}$$

for

$$0 \leq j_2 \leq j_1 - 1. \tag{1.80}$$

Paths which reach but do not exceed height 3 may be constructed from paths which reach but do not exceed height 2. By Lemma 13, there are

$$\binom{j_1-j_3-1}{j_1-j_2-1} \tag{1.81}$$

ways to insert $j_2 - j_3$ peaks, shared amongst $j_1 - j_2$ spaces. Noting that we require $j_2 \geq 1$ hence $j_3 \geq 2$ for a positive number of peaks, we have

$$s_2(n-j_3) = \sum_{j_1=2}^{n-1} \sum_{j_2=1}^{j_1-1} \binom{n-j_2-1}{n-j_1-1} \binom{j_1-j_3-1}{j_1-j_2-1} \kappa_1^{n-j_1} \kappa_2^{j_1-j_2} \kappa_3^{j_2-j_3}, \tag{1.82}$$

for

$$0 \leq j_3 \leq j_2 - 1. \tag{1.83}$$

Induction on the height completes the proof. □

Comments on Theorem 12

1. The expression for s_l contains no minus signs. We have an explicit solution for the weight polynomial which does not require any cancellation.

2. The solution in this form is bad for performing asymptotic expansions. A generating function approach would be more useful for this purpose. Generating functions for these paths have been previously independently constructed using the 'layer at a time' idea in [27] .

3. The quote from Polya at the beginning of this chapter applies, but it does not buy us as much as it usually would. This is because the number of sums in the solution depends on L, even when we specialize to a simpler weighting by setting $\kappa_i = 1$ for some i. Thus, Theorem 12 is computationally efficient when the number of weights is close to L, and (extremely!) inefficient for sparsely weighted or unweighted paths in a wide strip. By contrast, the Method of Images is (very!) computationally efficient in the half plane when the number of weights is zero.

   Thus, Theorem 12 provides us with an upper bound on the computational complexity of the enumeration problem for Dyck paths subject to any weighting. We can do 'no weights' efficiently and we can do 'many weights' efficiently. The sequel is concerned with tackling weighted cases that lie inbetween.

1.5 Why Pavings?

The two weighting extremes for decorated paths: unweighted and 'many-weighted', as considered in the previous two sections, are the easier part of the spectrum. The more difficult problems lie in the gap inbetween: the case of a finite number of decorations.

It is firstly towards bridging this gap that the auxiliary theme of Pavings is introduced in Part II. These are 'Pavings' in the sense defined by Viennot [104], [103], and they play a critical role in both the derivation of and the practical application of the 'Constant Term' method of Chapter 10, which we use to generalize and solve a long-standing open decorated Ballot-like path enumeration problem.

Viennot's original pavings provide a way of combinatorially manipulating three term *non constant coefficient* recurrences, the solutions to which are classic and more general classes of orthogonal polynomials. Thus we are able to obtain closed-form expressions which were previously inaccessible for many orthogonal polynomials.

The other area of focus of this thesis, in additional to the enumeration of weighted Ballot and Motzkin like paths, is the enumeration of weighted paths with extended step sets, such as d-up and Jump Step paths. Investigating Viennot's pavings, we find a natural extension of the definition which relates pavings to 2-up, 3-up and general d-up step sets for lattice paths. We obtain pavings which satisfy higher order non constant coefficient recurrences which we are then able to manipulate combinatorially.

The same natural extension does *not* apply to Jump Step paths. In the latter portion of Chapters 5 and 6 we find a new way to generalize so that we may define pavings which *are* useful for Jump Step paths as well.

In Chapter 9 we see how generating functions for paths are given in terms of pavings, across all the classes of pavings we have defined.

In understanding the observed efficacy of applying pavings to paths, we note that

1. Paths satisfy partial difference equations, whereas
2. Pavings satisfy ordinary difference equations.

Since partial difference equations with boundary conditions are much harder to solve than ordinary difference equations, the ability to express a path problem in terms of pavings is an advantage.

1.6 Overview of Thesis

The thesis is divided into three parts:

1. Paths – Introduces path enumeration problems and gives some pure combinatorial results;
2. Pavings – Develops the theory of pavings;
3. More Paths – Gives path results dependent on pavings.

In more detail:

- Part I
  - Chapter 1, the introduction, sets the scene by reviewing
    * Classic path problems

    * Generic approaches to path enumeration with solutions to classic problems
    * What's not known, and where difficulties lie

  - Chapter 2 deals with unweighted paths in the plane, in the half plane, in a strip and on a cylinder. The purpose of the chapter is to show the connections between these boundary conditions and the extensions to the basic Method of Images that apply in the strip and cylinder cases. Some cylinder results therein may not have been explicitly stated before, though they are probably implicit in the existing literature.
  - Chapter 3 solves a half plane path enumeration problem in which there are infinitely many weights taking two possible values in alternation. The solution is by means of a bijection between the bibanded problem and a new combinatorial encoding of another problem, 'corner counting', the answer to which is already known in the literature. The bibanded result proves a new combinatorial interpretation of the Narayana numbers.

    We also find the generating function for banded paths in the half plane with any finite number of repeating bands, thus proving that it is always quadratic.

- Part II
  - Chapters 4–6 relate pavings in the sense of Viennot [104], [103] to cycles on various digraph. We
    * Review the known Motzkin digraph – monomer/dimer paving relationship
    * Show that the above relationship extends in a natural way to digraphs supporting d-up and Lukasiewicz paths, but not to other paths such as Jump-step.
    * Derive a different kind of extension of the path digraph to paving relationship, which is applicable to Jump Step paths.
    * Calculate many paving polynomials
  - Chapter 7 lays foundations for further work extending the two dimensional paving results to higher dimensions, as well as giving an indication of the difficulties thereof. We also give a concrete example of a phenomena which may be useful in any number of dimensions, whereby two distinct families of pavings with distinct recurrences generate the same data. By this we mean that both

sets generate the same desired set of data, as well as some extra information on which they disagree. We show a combinatorial relationship between the two sets of pavings.

- Chapter 8 develops another new kind of paving, a 'Laurent paving' which is related to the original Viennot pavings by a combinatorial change of variables, and an involution. The corresponding 'Laurent paving polynomials' are essential in Part III.

- Part III

  - Chapter 9 describes two ways in which path length generating functions may be obtained combinatorially in terms of pavings. They are

    * Using a combinatorial interpretation of determinants as sums over cycles – this section is review
    * Using a single variable constant coefficient recurrence on the path weight polynomials to derive the generating function. It is well known that the generating function may be found from the recurrence algebraically, and vice versa. Here we need the recurrence first, which we find by expressing the recurrence coefficients in terms of weighted binomial paths. We then use an original bijection between binomial paths and pavings to obtain the coefficients combinatorially. Previous combinatorial derivations have gone via the intermediate step of combinatorial objects known as 'Heaps'.

  - Chapter 10 derives a new form of 'Constant Term' method for solving lattice path enumeration problems. We use the method to solve an open problem from the 1970's, and generalizations thereof.

Chapter 2

Images for unweighted Strip paths

Paths confined to a strip in the plane may be counted by a modified version of the Method of Images which we introduced in Section 1.3.1. We deal with general Ballot-like paths in a strip, with arbitrary starting and ending heights. We begin with the classic result:

Theorem 14. *The number of Ballot-like paths of length t in a strip of height L, where the paths begin at height y' and end at height y, for $0 \leq y', y \leq L$ is*

$$W_t = \sum_j \left(W_t^+(j) - W_t^-(j)\right) \tag{2.1}$$

where $\frac{-t-y-y'-2}{2L+4} \leq j \leq \frac{t-y+y'}{2L+4}$ *and*

$$W_t^+(j) = \binom{t}{(t+y-y')/2+(L+2)j}, \tag{2.2}$$

$$W_t^-(j) = \binom{t}{(t+y+y'+2)/2+(L+2)j}. \tag{2.3}$$

We employ the convention $\binom{a}{b} := 0$ *whenever the condition* $0 \leq b \leq a$ *fails.*

Proof. The proof is by Inclusion/Exclusion, and generalizes the Involution presented in Section 1.3.1. The argument is illustrated in Figures 2.1–2.5.

Begin, as usual for Inclusion/Exclusion, by over-counting. Ballot-like paths which start at height y', end at height y and have length t, when not confined by an upper or lower wall, occupy a binomial box of size $(t + y -$

$y')/2 \times (t - y + y')/2$. We call this box the **basic binomial box**. Paths within this box are termed **basic paths**. There are

$$\# \text{ of basic paths} = \binom{t}{(t + y - y')/2} \tag{2.4}$$

paths in the basic binomial box, by Lemma 1.

As in the ordinary method of images, the basic binomial box includes (potentially) paths which fall below the line $y = 0$. Such paths need to be removed from the count. To do so, notice that they are in bijection with paths occupying a binomial box of size $(t+y+y'+2)/2\times(t-y-y'-2)/2$ (see Figure 2.2). The bijection is the same mapping indicated in Section 1.3.1, where the latter portion of any path hitting the line $y = -1$ is reflected in that line (see Figure 2.1). The number of paths (again using Lemma 1) in this new image box is

$$\begin{array}{l}\# \text{ of paths in first lower image box} = \\ \text{(from reflection in } y = -1)\end{array} \binom{t}{(t + y + y' + 2)/2}. \tag{2.5}$$

A similar argument produces another image box above the basic binomial box. These paths come from reflection in the line $y = L + 1$. They occupy of box of size $(t + y + y' + 2 - (L + 2))/2 \times (t - y - y' - 2 + (L + 2))/2$ thus they number

$$\begin{array}{l}\# \text{ of paths in first upper image box} = \\ \text{(from reflection in } y = L + 1)\end{array} \binom{t}{(t + y + y' + 2 - (L + 2))/2}. \tag{2.6}$$

But now, there may be some paths in the basic box which have been subtracted twice, since they hit both upper and lower walls, so have images in both the upper and lower image binomial boxes. Call such a path **doubly bad**. (See Figure 2.3.) We need to compensate for the extra copy of the doubly bad path. It needs to be determined which copy to cancel out.

Note, a doubly bad path must be incident both with the line $y = -1$ and $y = -1 + (L + 2)$. Given this, there are three possibilities for its images.

1. The upper image of the doubly bad path hits the line $y = -1+2(L+2)$ but the lower image does not hit the line $y = -1 - (L + 2)$.

2. The upper image of the doubly bad path does not hit the line $y = -1 + 2(L + 2)$ but the lower image does hit the line $y = -1 - (L + 2)$.

3. The upper image hits $y = -1 + 2(L+2)$ and the lower image hits $y = -1 - (L+2)$.

It is not possible for a doubly bad path that neither the upper image hits $y = -1 + 2(L+2)$ nor the lower image hits $y = -1 - (L+2)$.

- In the first case, the upper image is reflected again (see Figure 2.4) and cancelled out.
- In the second case, the lower image is reflected again and cancelled out.
- In the third case, the procedure is iterated. Both images are reflected again, and we consider whether these images of images hit lines $y = -1 + 3(L+2)$ and $-1 - 2(L+2)$. Repeating this process, it must eventually terminate with either an upper image reflected but not a lower, or vice versa. Thus a correct count is maintained.

The iterated reflections produce a sequence of 'positive' and 'negative' binomial boxes. A box is termed **positive** if it is either the basic box or if it comes from an even number of iterations of the reflection procedure. A box is termed **negative** if it comes from an odd number of iterations of the reflection procedure. Each positive box contains

$$W_t^+(j) = \binom{t}{(t+y-y')/2 + (L+2)j} \tag{2.7}$$

paths and each negative box contains

$$W_t^-(j) = \binom{t}{(t+y+y'+2)/2 + (L+2)j} \tag{2.8}$$

paths, where the positive and negative boxes are each indexed by j, with j taking integer values in the range

$$\frac{-t-y-y'-2}{2L+4} \leq j \leq \frac{t-y+y'}{2L+4} \tag{2.9}$$

The images are bounded within the triangle defined by lines $y = t$, $y = -t$ and $t =$ 'path length'. (See Figure 2.5.) Summing over positive and negative boxes gives the theorem. □

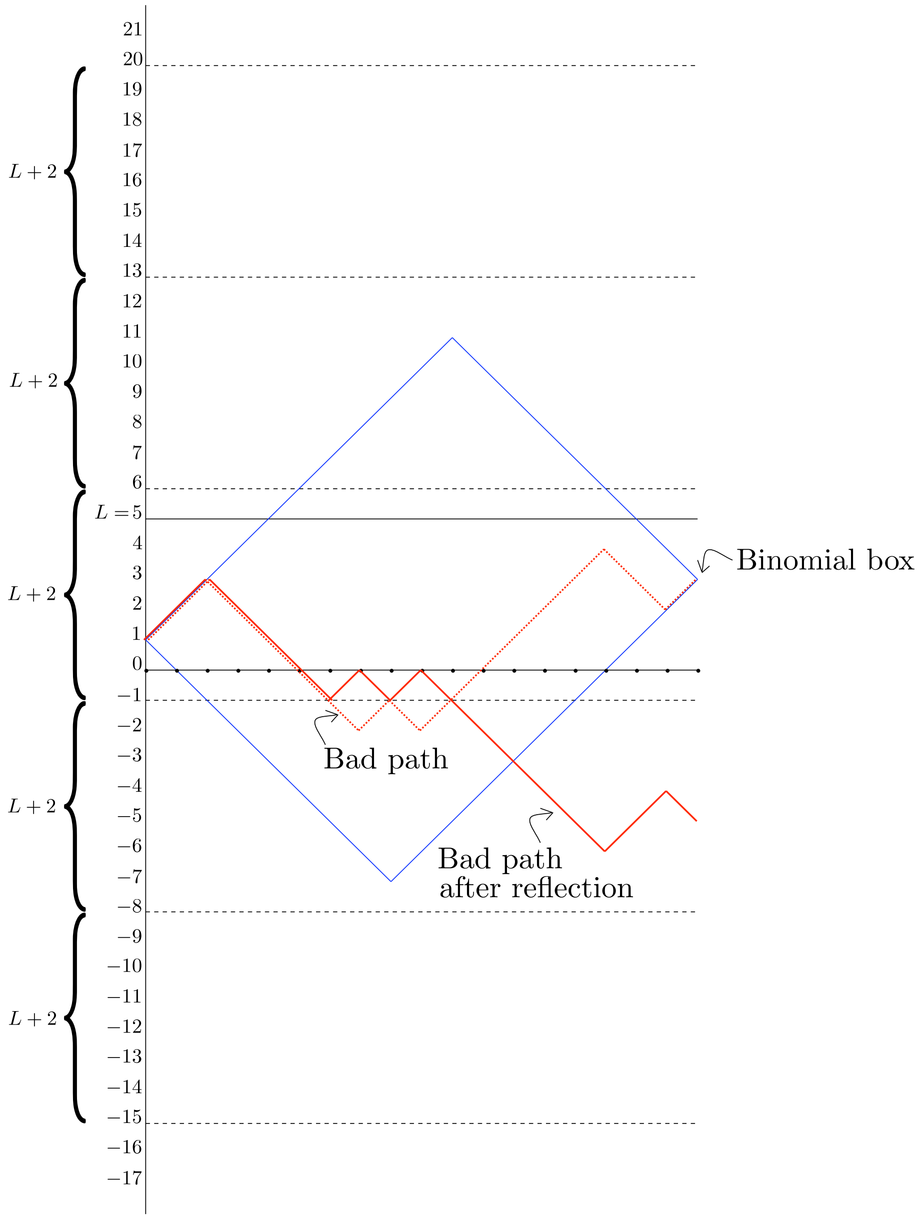


Figure 2.1: The binomial box for paths of length $t = 18$ starting at height $y' = 1$ and ending at height $y = 3$ in a strip of height $L = 5$, showing a 'bad path' which is incident upon the line $y = -1$, as well as its image.

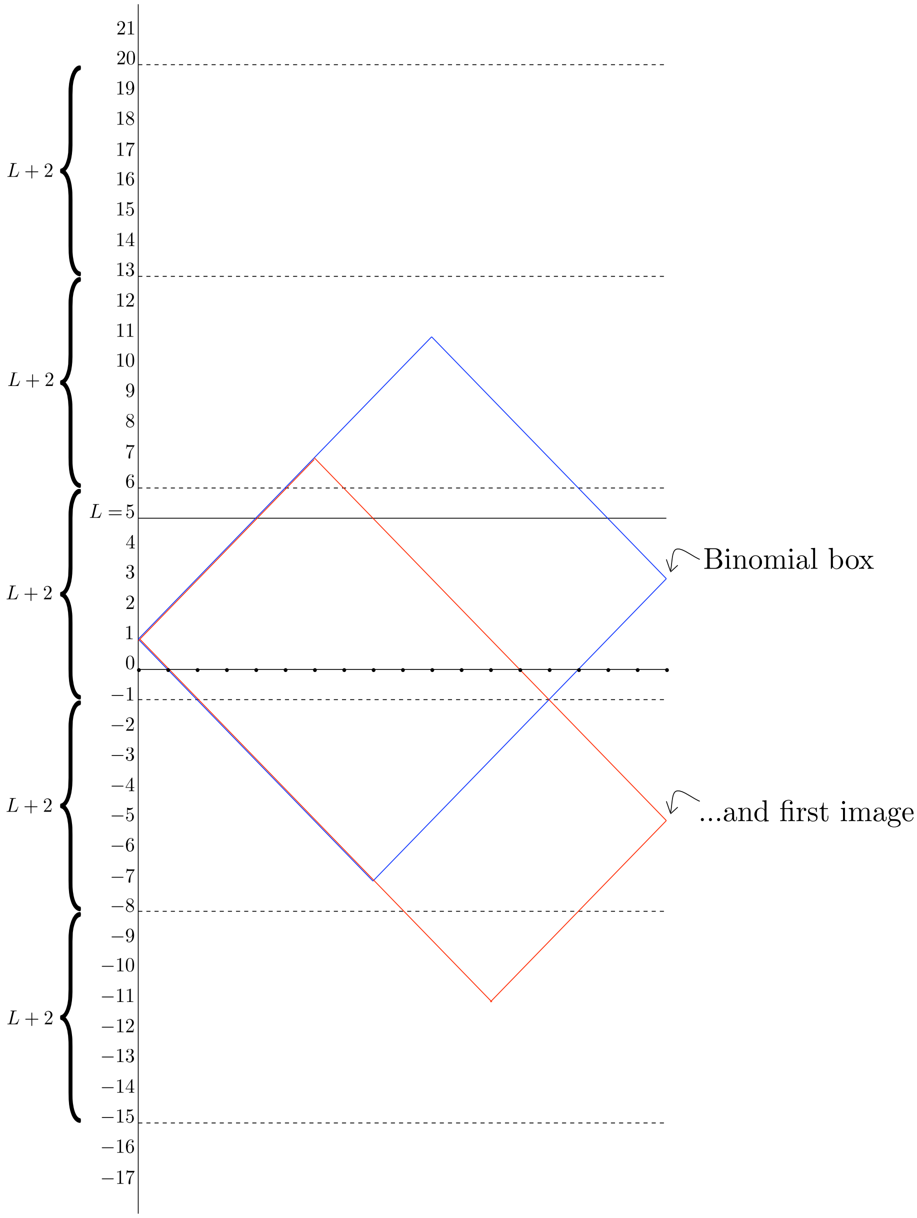


Figure 2.2: The binomial box and its first image for paths of length $t = 18$ starting at height $y' = 1$ and ending at height $y = 3$ in a strip of height $L = 5$.

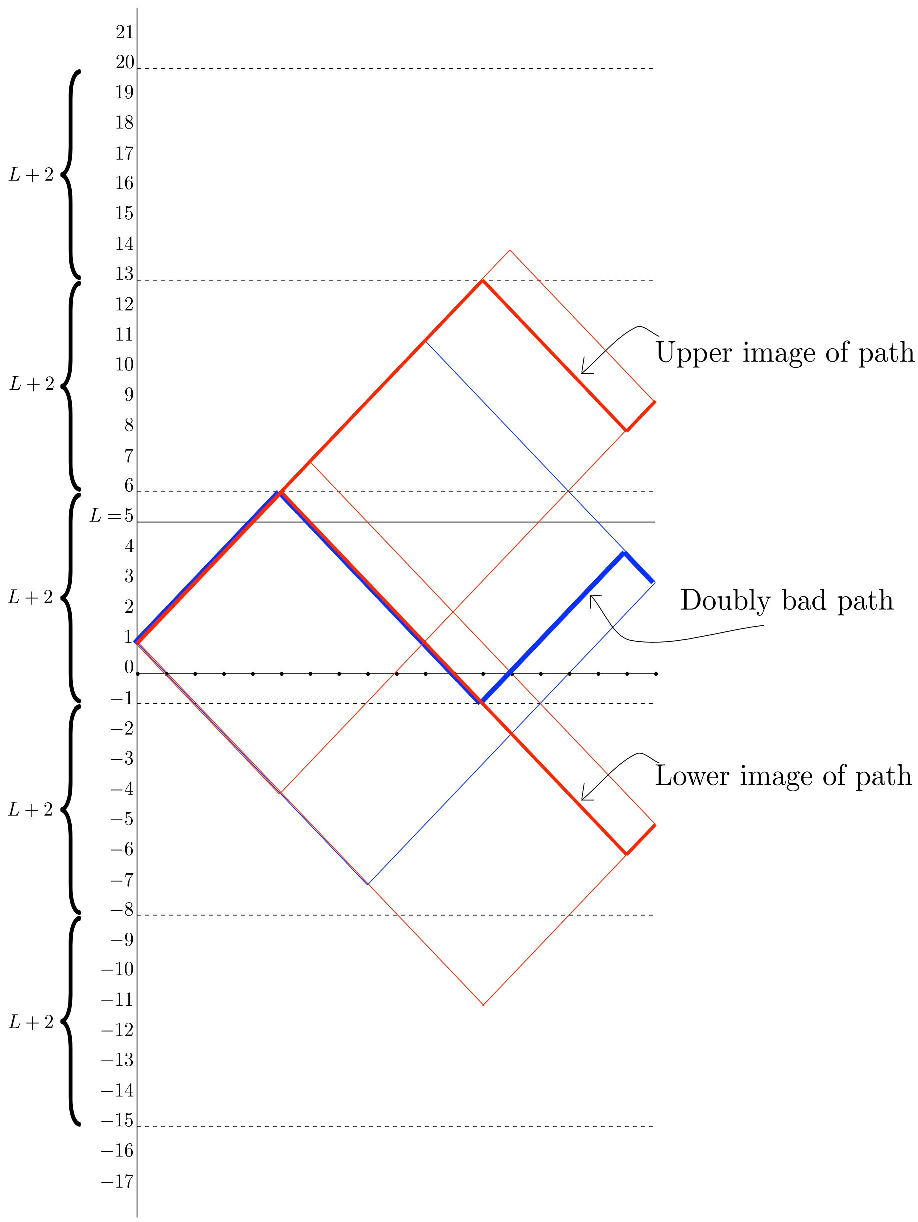


Figure 2.3: A 'doubly bad' path which exceeds its boundaries in both directions, thus producing a pair of images.

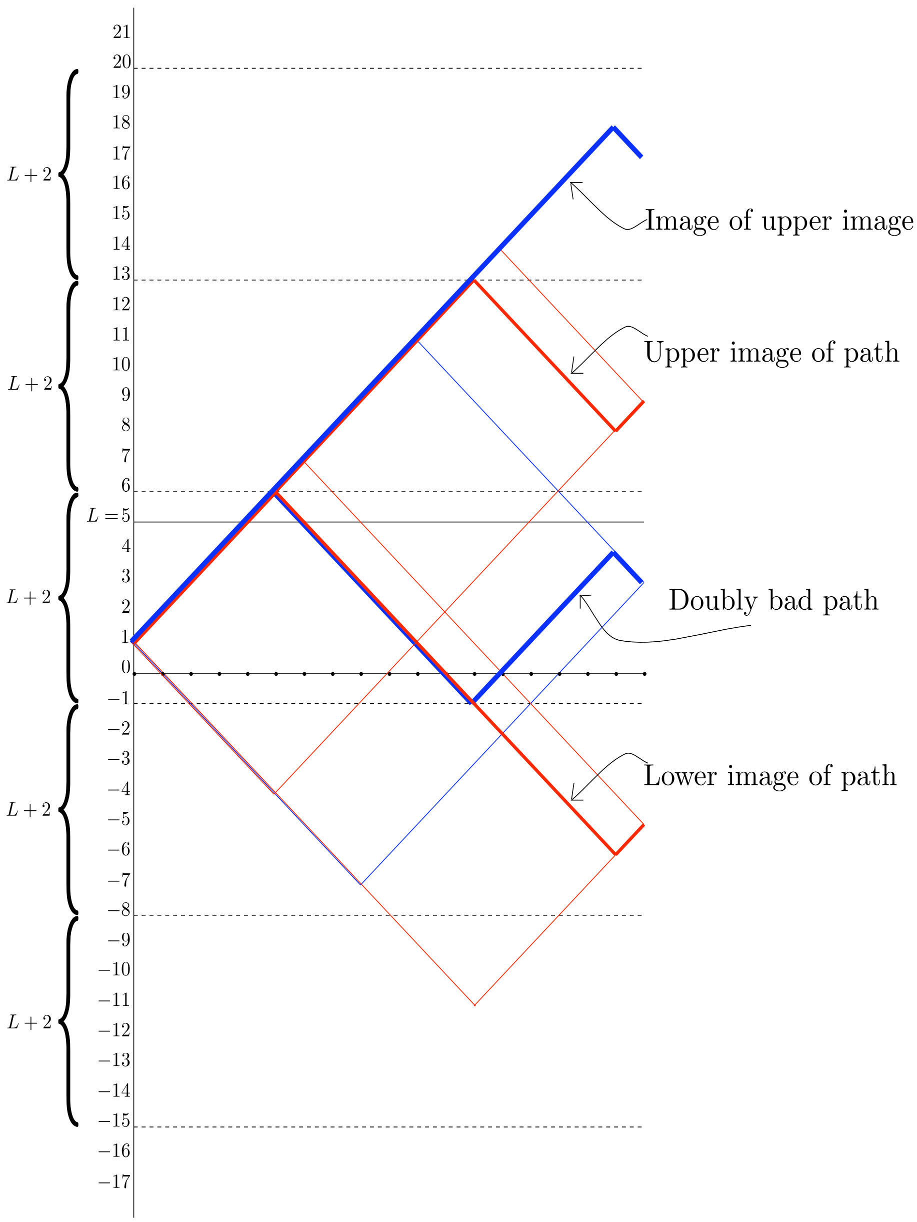


Figure 2.4: The 'doubly bad' path produces two images, but the upper image is reflected again, cancelling it out.

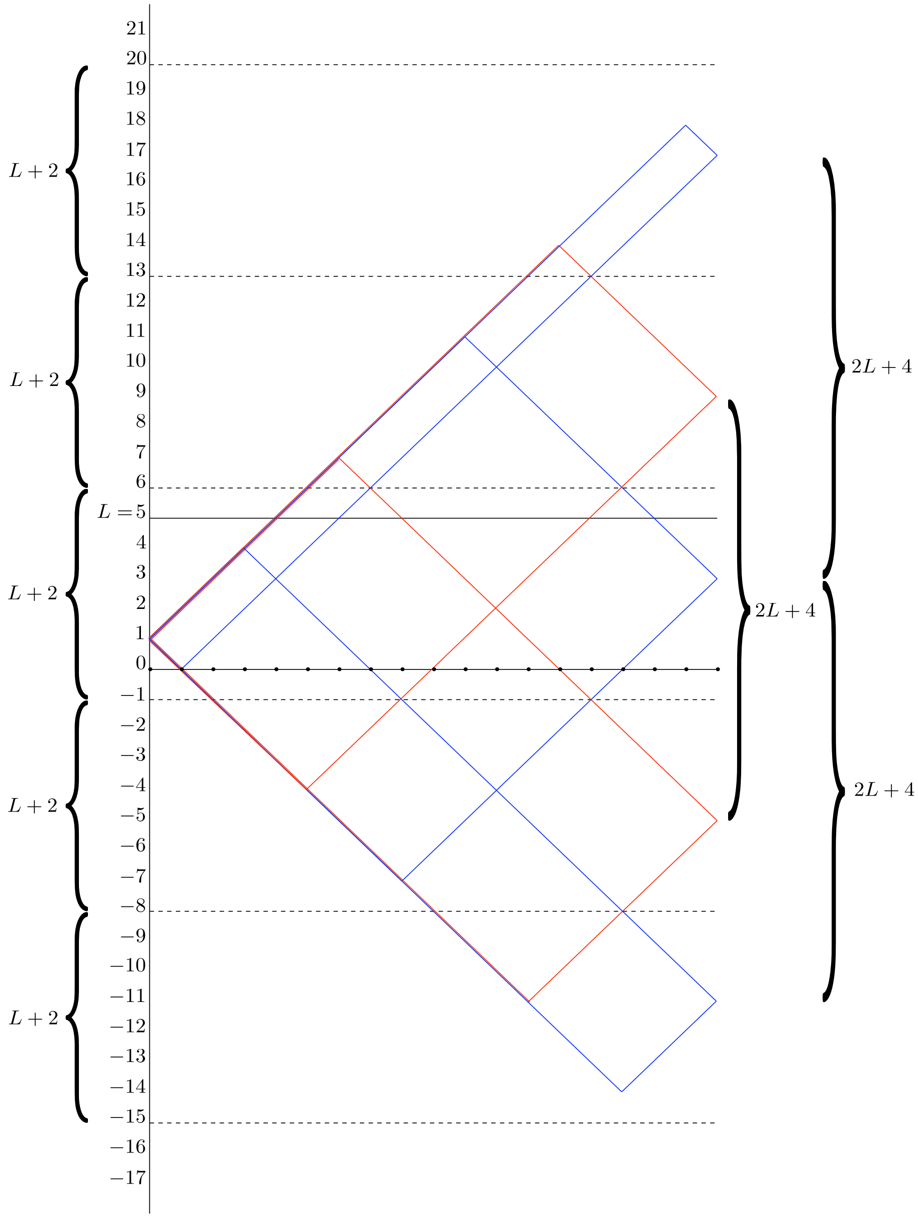


Figure 2.5: The binomial box and all its images for paths of length $t = 18$ starting at height $y' = 1$ and ending at height $y = 3$ in a strip of height $L = 5$.

2.1 Unweighted strip paths: another view

Equation 2.1 may be seen from two perspectives, corresponding to the left hand side and right hand side of the following equality

$$\sum_j W_t^+(j) - \sum_j W_t^-(j) = \sum_j \left(W_t^+(j) - W_t^-(j)\right). \tag{2.10}$$

From the left hand point of view,

$$W_t^+ := \sum_j W_t^+(j) \text{ and} \tag{2.11}$$

$$W_t^- := \sum_j W_t^-(j), \tag{2.12}$$

the sums of the positive and negative images respectively, are considered as objects in their own right. From the right hand point of view, the difference

$$W_t^+(j) - W_t^-(j), \tag{2.13}$$

for each j, is the core object.

In this section we investigate the left hand point of view, i.e. give a combinatorial meaning to each of the sums (2.11) and (2.12). [1]

We begin by looking at some specific cases.

- Dyck paths of length $2r$ in a strip of height L

L	$\left(W_{2r}^+\right)_{r\geq 0}$	Form?	$\left(W_{2r}^-\right)_{r\geq 0}$	Form?
2	$1, 2, 6, 20,$ $72, 272, ...$	$2^{r-1} + 4^{r-1}$	$0, 1, 4, 16,$ $64, 256, ...$	4^{r-1}
3	$1, 2, 6, 20,$ $70, 254, 948, ...$	A095929	$0, 1, 4, 15,$ $57, 220, 859, ...$	A095930
4	$1, 2, 6, 20, 70,$ $252, 926, 3460, ...$	A087433	$0, 1, 4, 15, 56,$ $211, 804, 3095, ...$	–

[1] It would also be an interesting piece of further work to look at the right hand point of view and give a combinatorial meaning to the difference (2.13).

The codes A095929, A095930, A087433 are sequence numbers in the Online Encyclopedia of Integer Sequences [96].

- Ballot-like paths of length $2r$, with starting and ending heights respectively $y' = 0$, $y = 2$, in a strip of height L

L	$\left(W^{+}_{2r}\right)_{r\geq 0}$	Form?	$\left(W^{-}_{2r}\right)_{r\geq 0}$	Form?
2	$0, 1, 4, 16,$ $64, 256, 1024, ...$	4^{r-1}	$0, 0, 2, 12,$ $56, 240, 992, ...$	$4^{r-1} - 2^{r-1}$
3	$0, 1, 4, 15, 57,$ $220, 859, 3381, ...$	A095930	$0, 0, 1, 7, 36,$ $165, 715, 3004, ...$	A095931

The codes A095930, A095931 are sequence numbers in [96].

2.1.1 Cylinder paths and cycle walks

We first comment that walks on a cycle graph biject to directed paths on cylinders, as illustrated in Figure 2.6. Thus we refer to 'cycle walks' and 'cylinder paths' interchangeably.

Definition 7. *Let $\mathcal{Z}_t(k,m)$ refer interchangeably to*

- *the set of walks of length t on the cycle graph C_m, where the walks start at vertex 0 and end at vertex k; and*
- *the set of paths of length t on the cylinder graph $\mathbb{Z} \times C_m$, where the paths start at vertex $(0,0)$ and end at vertex (t,k).*

Let

$$Z_t(k,m) := |\mathcal{Z}_t(k,m)| \tag{2.14}$$

be the number of such walks.

We next observe that we may obtain a formula for the number of cycle walks on C_{2L+4} (equivalently cylinder paths on $\mathbb{Z}\times C_{2L+4}$) by summing over an appropriate set of paths in the half plane.

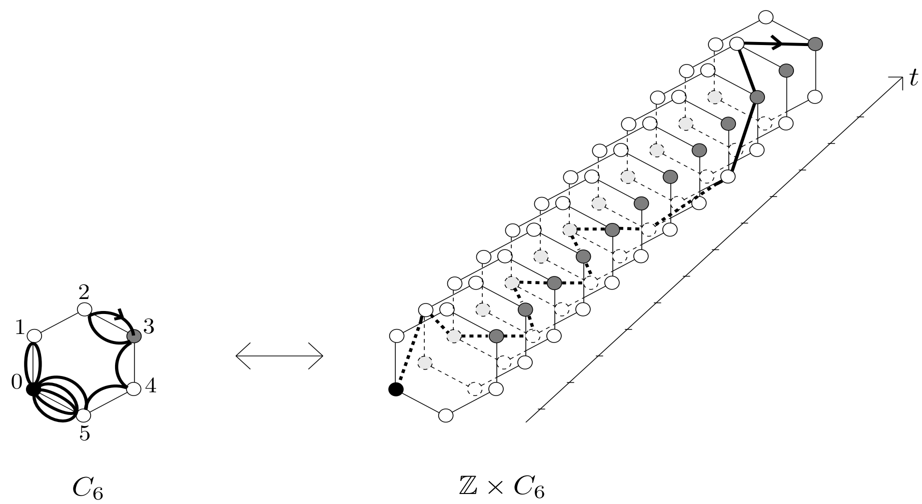


Figure 2.6: Walks on a cycle graph biject to directed paths on a cylinder.

Lemma 15. *Let $Z_t(k, 2L+4)$ be the number of cycle/cylinder walks, as defined in Definition 7, of length t on a cycle/cylinder of circumference $2L+4$. Then*

$$Z_t(k, 2L+4) = \sum_j \binom{t}{(t+k)/2 + (L+2)j}, \tag{2.15}$$

where $\frac{-t-k}{2L+4} \leq j \leq \frac{t-k}{2L+4}$.

Proof. Let C_{2L+4} be a cycle graph. Label the vertices $0, 1, 2, ..., 2L+3$ consecutively. Without loss of generality we may assume that the walk starts at vertex 0 and ends at vertex k. We may 'unwrap' the graph C_{2L+4} and map it to the number line, where the i^{th} vertex of C_{2L+4} is identified with all of the vertices congruent to i modulo $2L+4$. (See Figure 2.7.)

Walks which start at 0 and end at k on the cycle graph correspond to walks which start at 0 and end at any of the heights congruent to k modulo $2L+4$, on the number line. Hence to count walks of length t ending at k on the cycle graph, sum over walks ending at lengths congruent to k mod $2L+4$ on the number line. Stretching these later walks out in time, they become directed walks in the half plane (see Fig 2.8), and application of Lemma 1 gives the result. □

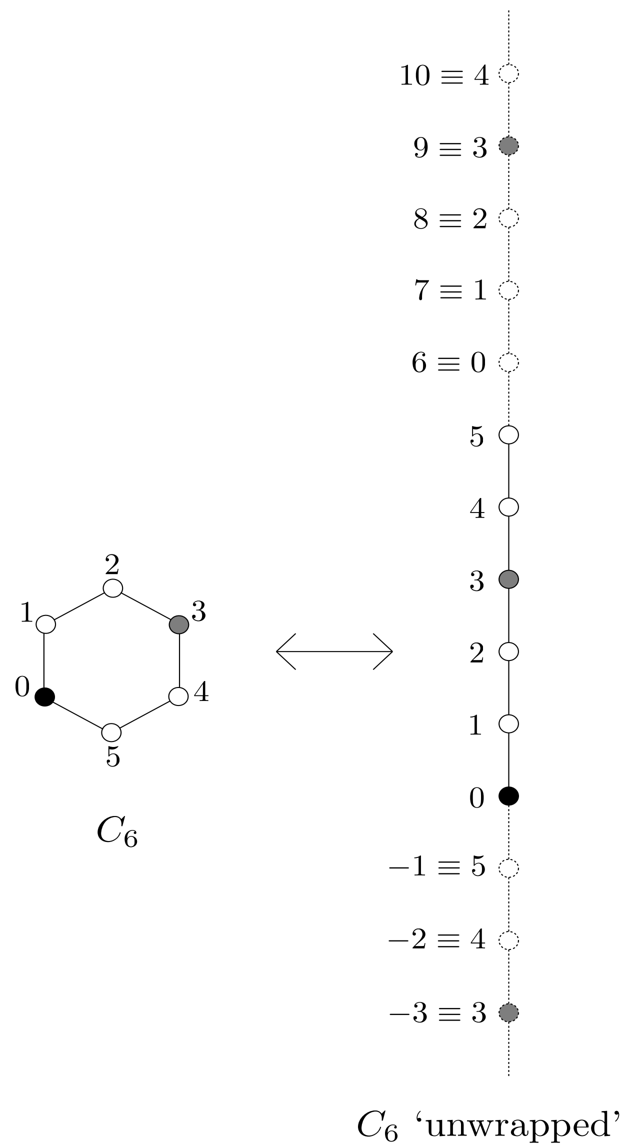


Figure 2.7: Illustrated is the cycle graph C_{2L+4}, for $L = 1$. The cycle graph 'unwraps' to a number line, where heights are identified modulo $2L + 4$. Walks which start at 0 and end at 3 on the cycle graph correspond to walks which start at 0 and end at any of the heights congruent to 3 modulo $2L + 4$ on the number line.

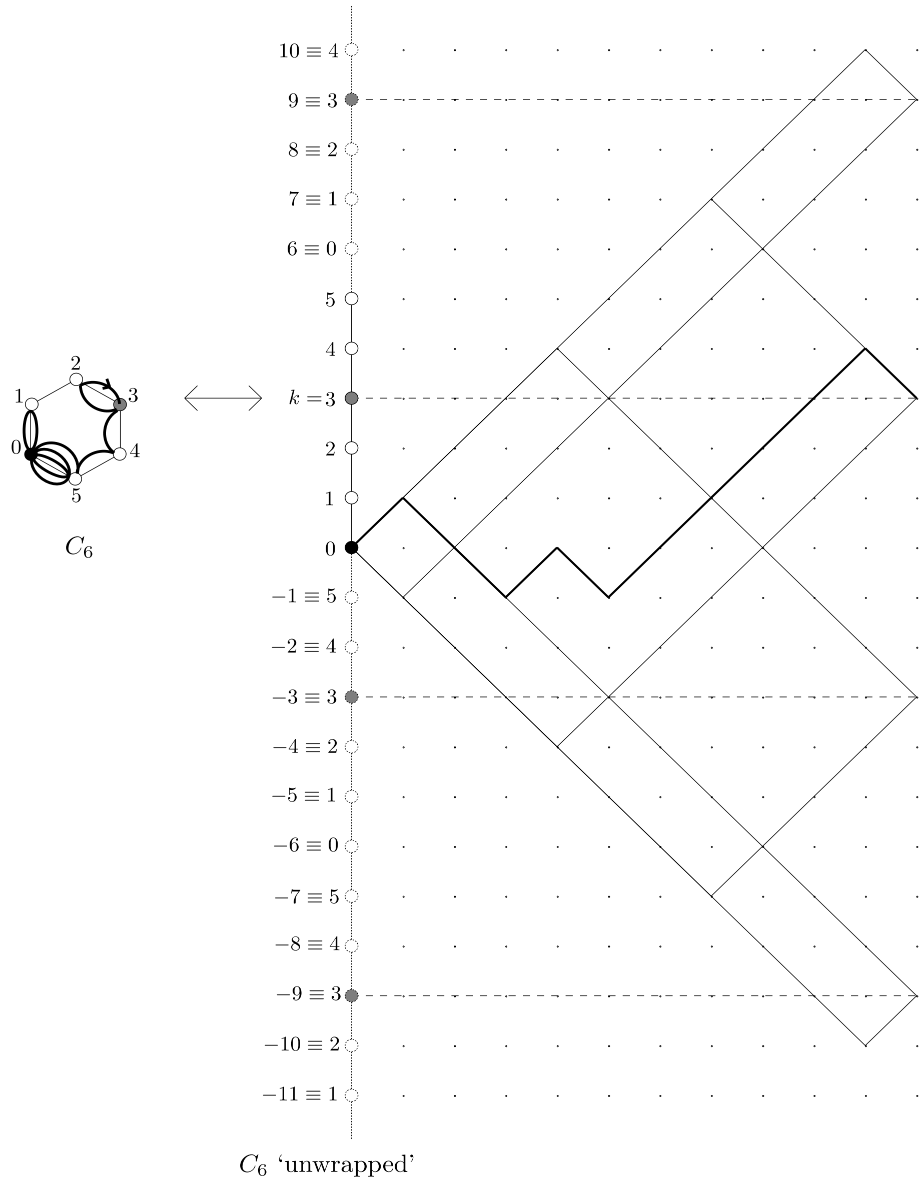


Figure 2.8: Walks of length $t = 11$ on the cycle graph C_6, starting at vertex 0 and ending at vertex $k = 3$, correspond to walks in the half plane that fall within any of the four binomial boxes shown. (Each box has rightmost corner on the vertical line $t = 11$.) A particular walk is illustrated.

2.1.2 Counting strip paths with cylinders

We have a reinterpretation of Theorem 14.

Theorem 16. *The number of Ballot-like paths of length t in a strip of height L, where the paths begin at height y' and end at height y, for $0 \leq y', y \leq L$ is*

$$W_t = W_t^+ - W_t^- \tag{2.16}$$

where

$$W_t^+ = Z_t(|y - y'|, 2L + 4) \tag{2.17}$$

is the number of walks of length t on the cycle graph C_{2L+4}, with starting and ending positions separated by $|y - y'|$; and

$$W_t^- = Z_t(|y + y' + 2|, 2L + 4) \tag{2.18}$$

is the number of walks of length t on the cycle graph C_{2L+4}, with starting and ending positions separated by $|y + y' + 2|$.

Explicitly,

$$\begin{aligned} W_t^+ &= \sum_j \binom{t}{(t + y - y')/2 + (L + 2)j}, \\ W_t^- &= \sum_j \binom{t}{(t + y + y' + 2)/2 + (L + 2)j}, \end{aligned}$$

for $\frac{-t-y+y'}{2L+4} \leq j \leq \frac{t-y+y'}{2L+4}$, and $\frac{-t-y-y'-2}{2L+4} \leq j \leq \frac{t-y-y'-2}{2L+4}$ respectively.

Alternatively, both sums may be taken over $-\infty \leq j \leq \infty$ given the convention that $\binom{a}{b} := 0$ whenever the condition $0 \leq b \leq a$ fails.

Proof. Using Theorem 14 write

$$\begin{aligned} W_t &= \sum_j \left(W_t^+(j) - W_t^-(j)\right) &(2.19)\\ &= \sum_j W_t^+(j) - \sum_j W_t^-(j), &(2.20) \end{aligned}$$

and apply Lemma 15. □

An alternative formula to Equation (2.15) is known in the literature [96], sequence number A095931, to be

Lemma 17. *The number of walks of length t on the cycle graph C_m, where the walks start at vertex 0 and end at vertex k, is given by the formula*

$$Z_t(k,m) = (2^t/m)\sum_{j=0}^{m-1}\cos(2\pi kj/m)\cos(2\pi j/m)^t. \tag{2.21}$$

Lemma 17, together with Theorem 16, immediately gives us an alternative expression for the number of strip paths.

Theorem 18. *The number of Ballot-like paths of length t in a strip of height L, where the paths begin at height y' and end at height y, for $0 \leq y', y \leq L$ is*

$$W_t = \frac{2^{t-1}}{L+2}\sum_{j=0}^{2L+3}\Big(\cos(\pi d_1 j/(L+2)) - \cos(\pi d_2 j/(L+2))\Big)\cos(\pi j/(L+2))^t \tag{2.22}$$

where $d_1 = |y-y'|$ and $d_2 = y+y'+2$.

Comment 1. *We now have several answers to the enumeration problem for Ballot-like paths of length t in a strip of height L. Which is best to use depends upon whether we have a relatively narrow strip and a long path, or a relatively wide strip and short path. A natural consideration to bear in mind is*

- *if $t >> L$, Theorem 18 may be better, since the sum therein is of order L, whereas*
- *if $L >> t$, Theorem 16 may be better, since the sum therein is of order t.*

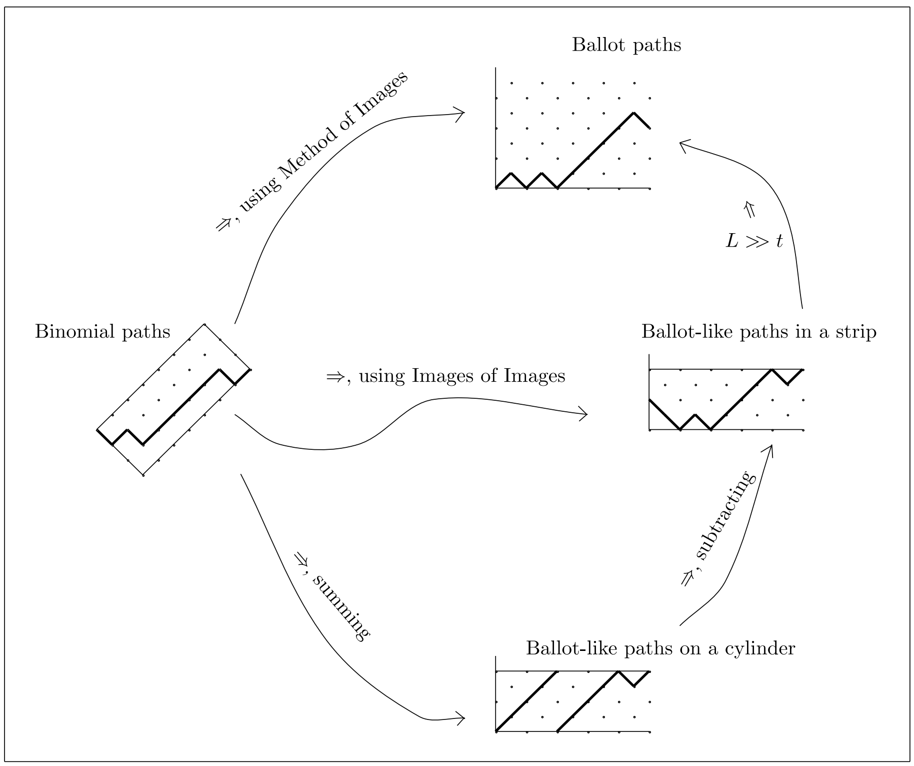


Figure 2.9: Indicated are interdependencies of methods to calculate unweighted Ballot-like paths with various boundary conditions.

Chapter 3

Counting Corners and Banded Paths

There are two kinds of weighted path enumeration problems considered in this chapter, 'corner counting' and 'banded'. These two kinds of weightings differ from each other conceptually in two aspects. The first difference refers to whether weights are assigned to vertices in the path or to edges in the path. 'Corner counting' paths have weights assigned to their vertices, whereas 'banded' paths have weights associated with their edges. The second difference refers to another aspect of how weights are designated. For banded paths, as with most other paths considered in this thesis, weights are associated with locations in the underlying lattice; so that banded paths pick up their weights as they pass through periodically weighted bands. For a 'corner counting' weighted path, the weight is independent of the underlying lattice, instead depending entirely upon the intrinsic shape of the path.

Both corner counting and banded problems are of use to physicists as well as being of inherent mathematical interest. Banded paths provide some of the simplest examples of paths for which steps at every height have non-unit weights assigned to them. They have interpretations in compact and directed damp percolation [14]. They also provide a combinatorial interpretation of one of the representations of the parallel update ASEP algebra [41], [18]. Banded paths also relate to a weighted version of the cylinder paths introduced in Subsection 2.1.1 of Chapter 2, since a weighted path on a cylinder of circumference c 'unwraps' to a banded binomial path with periodicity c. The 'corner counting' problem that we present is the most elementary in its class, and has a direct interpretation in terms of polymer physics. Paths with few corners correspond to fairly stretched-out polymers, and paths with many corners model polymers that are more compressed.

The two problems: corner counting and banded, seem at first to be unrelated. However we construct a bijection between the corner counting problem and a 'bi-banded' problem for which there are just two kinds of bands alternating odd/even with the height. This bijection provides proof of a foundational result for bi-banded paths, out of which all of the subsequent bi-banded results are constructed.

3.1 Counting Corners

We begin by counting binomial paths by corners. Recall that binomial paths have step set $\{N, E\}$, where N steps have the form $(0, 1)$ and E steps have the form $(1, 0)$.

Definition 8. *Let $p = v_0e_1v_1...e_{n+m}v_{n+m}$ be a path on a lattice such that each arc e_i is one of exactly two possible types; i.e. the allowed step set is $\{N, E\}$ for some N and E. Let $e_{i-1} = v_{i-1}v_i$ and $e_i = v_iv_{i+1}$ be a pair of consecutive arcs. Then the intermediate vertex v_i is defined to be a* **corner** *provided that (e_{i-1}, e_i) is either a (N, E) pair or an (E, N) pair. We fix the convention that the first vertex v_0 is a corner only if e_1 is an E step; and that the last vertex v_{n+m} is a corner only if e_{n+m} is a N step. (See Figure 3.1.)*

Definition 9. *Let $\mathcal{C}_{m,n}$ be the set of binomial paths on the square lattice which begin at vertex $(0, 0)$ and end at vertex (m, n). A path $p \in C_{m,n}$ has corner weight*

$$w(p) = c^{|corners|}, \tag{3.1}$$

where $|corners|$ is the number of corners in the path $p = v_0e_1v_1...e_{n+m}v_{n+m}$. The **corner weight polynomial** *for binomial paths is defined to be*

$$C_{m,n}(c) = \sum_{p \in \mathcal{C}_{m,n}} w(p). \tag{3.2}$$

The following Theorem is a rediscovery of a special case of a Theorem of Krattenthaler [70].

Theorem 19. *The corner weight polynomial defined in Equation (3.2) is*

$$C_{m,n}(c) = \sum_{i \geq 0} \binom{n}{i} \binom{m}{i} c^{2i+1}. \tag{3.3}$$

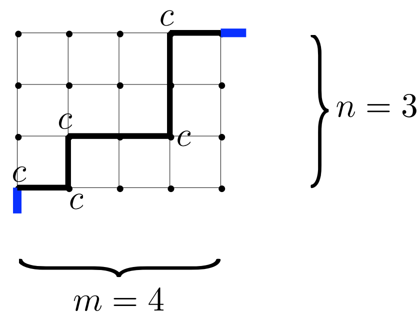


Figure 3.1: A 4×3 binomial box is shown, with an example path of weight c^5. We think of all paths as entering from the South and exiting to the East, so that of the five corners possessed by this path, the first one occurs at the origin.

Proof. Our proof depends upon a particular construction for the binomial paths. Take an $m \times n$ portion of square lattice. Such a box consists of $n+1$ horizontal lines and $m+1$ vertical lines. Label the first n horizontal lines $1, 2, ..., n$, starting at the bottom-most line and working upwards. For convenience of reference, call these the **black labels**. Label the middle $m-1$ vertical lines $1, 2, ..., m-1$ from left to right, skipping the first and last line. Call these the **red labels**. See Figure 3.2 for an example. There are $\binom{n}{i}$ possible ways to choose i of the n black labels. There are $\binom{m-1}{i-1} + \binom{m-1}{i}$ ways to choose either $i-1$ or i of the red labels.

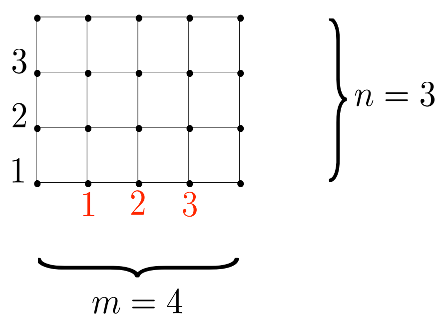


Figure 3.2: The first n of the horizontal lines are labeled $1, 2, ..., n$ in black, starting at the bottom and leaving out the topmost boundary line. The middle $m-1$ of the vertical lines are labeled from left to right $1, 2, ..., m-1$ in red, skipping both boundary lines.

For each i such that $0 \leq i \leq \min\{n, m-1\}$, independently choose i black labels and either i or $i-1$ red labels. Mark each label chosen - call these marks **check marks**. We claim that each such choice of check marks generates a binomial path with $2i+1$ corners. Conversely, each path from $(0,0)$ to (m,n) with $2i+1$ corners corresponds to exactly one choice of check marks.

To see geometrically how the check marks give a recipe defining a path, start with the assumption that the path enters the bottom left corner of the grid 'traveling' North. Repeat the following two steps until reaching the top right corner.

1. Continue in the N direction until reaching either the uppermost boundary or another horizontal line bearing a check mark. Turn East.

2. Continue in the E direction until reaching either the rightmost vertical boundary, or another vertical line whose label bears a check mark. Turn North.

See Figure 3.3 for an example. This procedure will always terminate

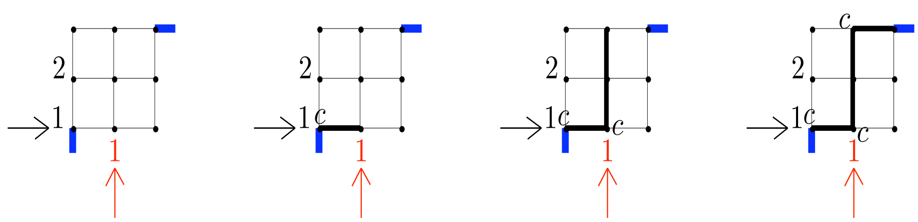


Figure 3.3: Successively interpreting check marks as corners uniquely determines a binomial path.

with the path reaching the top right corner. There are two cases.

- Where we began with i black check marks and i red check marks, each check mark generates a corner. There is one final corner generated when the path hits the upper boundary. Thus there are $i+i+1$ corners created in total.

- Where we began with i black check marks and $i-1$ red check marks, each check mark generates a corner. The path hits the right most boundary, which generates another corner, and then the topmost boundary at the final vertex, which generates the last corner. Hence there are $i+(i-1)+1+1$ corners generated.

Thus in both cases we generate $2i+1$ corners. Conversely we can take an arbitrary path with $2i+1$ corners and generate a valid set of check marks by extending each straight section of the path back until it hits either the left most or lower boundary, and placing a black or red check mark there accordingly.

Thus to count paths by corners we need only count sets of check marks, appropriately grouped. We obtain

$$\begin{aligned} C_{m,n}(c) &= \binom{n}{0}\binom{m-1}{0}c + \left(\binom{n}{1}\binom{m-1}{0} + \binom{n}{1}\binom{m-1}{1}\right)c^3 \\ &\quad + \left(\binom{n}{2}\binom{m-1}{1} + \binom{n}{2}\binom{m-1}{2}\right)c^5 + \cdots \end{aligned} \tag{3.4}$$

Collecting terms as bracketed gives the theorem. □

Example 2. *We follow the method of the proof of Theorem 19 in Figure 3.4 to construct all 10 binomial paths in the binomial box of size* 2×3.

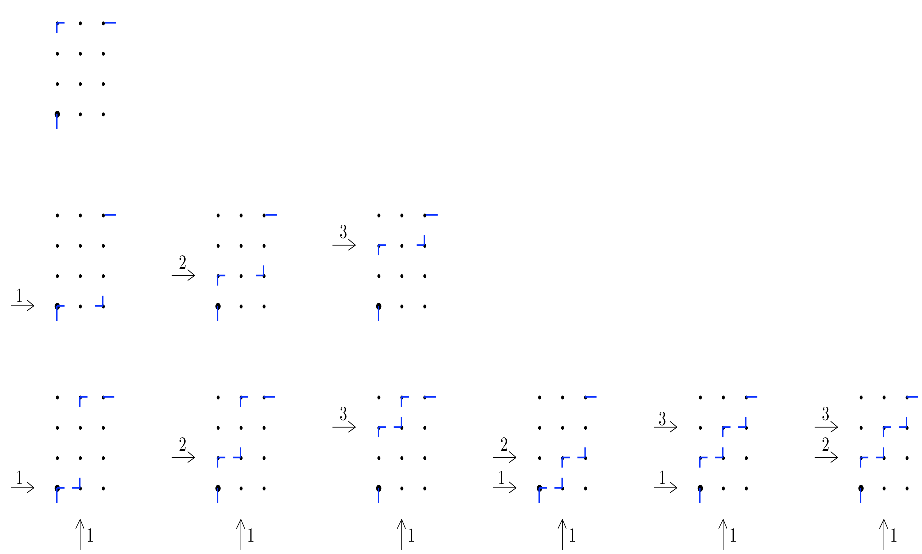


Figure 3.4: Check marks specify the corners, which determine the paths.

3.2 Bi-banded Paths

Definition 10. *A **bi-banded weighting** on a Ballot-like path is given by weights on upsteps and downsteps between odd and even heights as follows.*

$$w(\textit{upstep: even to odd}) = w(\textit{downstep: odd to even}) = a, \quad (3.5)$$
$$w(\textit{upstep: odd to even}) = w(\textit{downstep: even to odd}) = b. \quad (3.6)$$

We begin with bi-banded binomial paths. One such is illustrated in Figure 3.5.

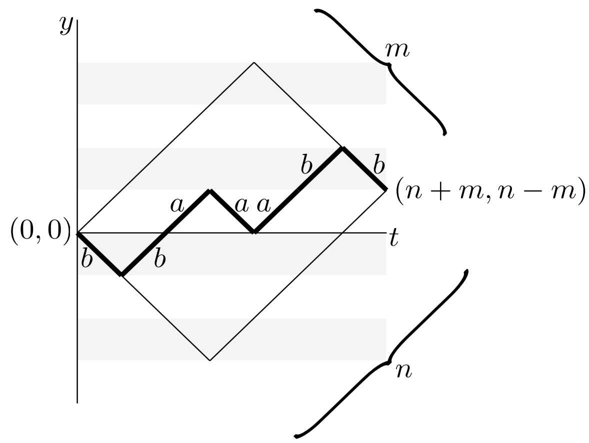


Figure 3.5: This bi-banded binomial path has weight a^3b^4.

Theorem 20. *The bi-banded weight polynomial, $B_{m,n}$, for binomial paths starting at vertex $(0,0)$ and ending at vertex $(n+m, n-m)$, in the rotated square lattice is*

$$B_{m,n} = \sum_i \binom{\lceil \frac{m+n}{2} \rceil}{n-i} \binom{\lfloor \frac{m+n}{2} \rfloor}{i} a^{\lceil \frac{m+n}{2} \rceil + n - 2i} b^{\lfloor \frac{m+n}{2} \rfloor - n + 2i}. \quad (3.7)$$

Proof. The proof depends on a construction for binomial paths in which we specify where to place up and down steps. Consider all paths from $(0,0)$ to $(n+m, n-m)$ in the rotated square lattice, where $m \le n$. Such paths contain n up steps, m down steps and have length $n+m$.

The bi-banded weighting defines horizontal bands containing, alternately, all weights a and all weights b, as in Figure 3.5. In addition to these bands, define a set of alternating vertical bands. Call the vertical bands **dark** and **light** accordingly as

$$\text{dark} \iff 2j \leq t \leq 2j+1, \text{ for } j \in \{0,1,...,\lceil \tfrac{m+n}{2} \rceil\}$$

$$\text{light} \iff 2j+1 \leq t \leq 2j+2, \text{ for } j \in \{0,1,...,\lfloor \tfrac{m+n}{2} \rfloor\}.$$

See Figure 3.6.

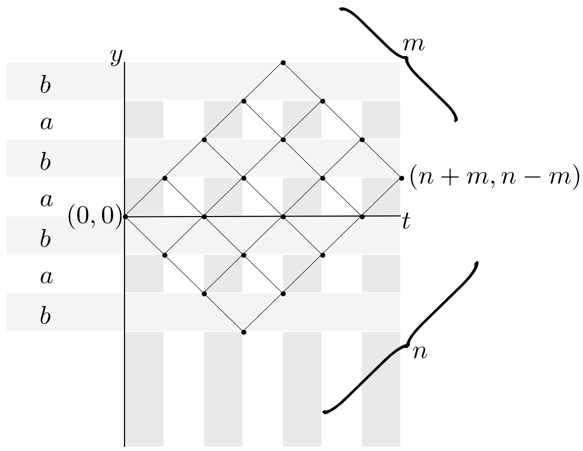


Figure 3.6: In additional to the horizontal bands which give the edge weighting for bi-banded paths, we define alternating dark and light vertical bands.

Fix i. Independently choose $n-i$ dark vertical bands and i light vertical bands, marking the chosen bands, so that a total of n vertical bands are chosen. There are

$$\binom{\lfloor \frac{m+n}{2} \rfloor}{n-i}\binom{\lceil \frac{m+n}{2} \rceil}{i} \tag{3.8}$$

ways to achieve this. We show that each of them corresponds to a distinct binomial path with weight

$$a^{\lceil \frac{m+n}{2} \rceil + n - 2i} b^{\lfloor \frac{m+n}{2} \rfloor - n + 2i}. \tag{3.9}$$

To construct the path, start at the vertex $(0,0)$, and move to the right step by step according to the rule

- Make a U step (i.e. step diagonally 'Up') if the vertical band is marked;

- Make a D step (i.e. step diagonally 'Down') if the vertical band is unmarked.

An example is illustrated in Figure 3.7. We observe that up steps weighted a always occur above arrows in the dark bands. Similarly, up steps weighted b always occur above arrows in the light bands. Furthermore, down steps weighted a occur in light bands and downsteps weighted b occur in dark bands. This is necessarily so, since same-weighted steps are separated by an even difference in their heights, hence must also be separated by an even number of time steps; which preserves the parity of the vertical bands.

Therefore the weight of the path is

$$a^{|\text{dark Up}|+|\text{light Down}|}b^{|\text{light Up}|+|\text{dark Down}|}, \tag{3.10}$$

where |dark Up| is the number of up steps in a dark vertical band, |light Up| is the number of up steps in a light vertical band, |dark Down| is the number of down steps in a dark vertical band and |light Down| is the number of down steps in a light vertical band. We have

$$|\text{dark Up}| = n - i \tag{3.11}$$

$$|\text{dark Down}| = \left\lfloor \frac{m+n}{2} \right\rfloor - (n-i) \tag{3.12}$$

$$|\text{light Up}| = i \tag{3.13}$$

$$|\text{light Down}| = \left\lceil \frac{m+n}{2} \right\rceil - i, \tag{3.14}$$

which gives expression (3.9). The theorem follows from summing on i over the product of the coefficients (3.8) and the weights (3.9). □

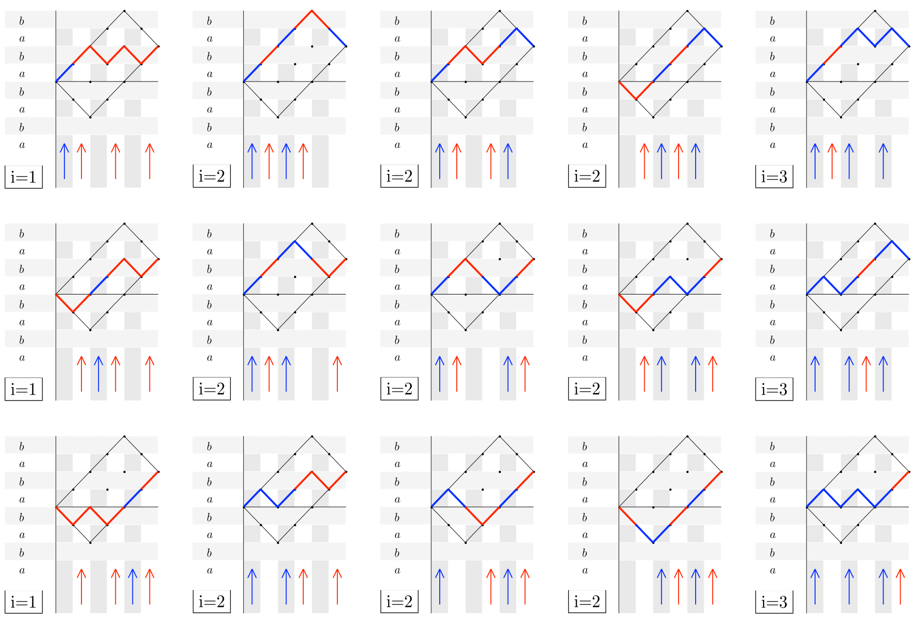


Figure 3.7: Banded paths in an $m \times n$ binomial box are constructed by specifying the location of Up steps by arrows, and Down steps by blanks. In this example, $m = 2$, $n = 4$ and the weight polynomial is $P_{2,4} = 3ab^5 + 9a^3b^3 + 3a^5b$.

3.3 Corner-counting $\leftrightarrow$ Bi-banded; in a square

The coefficients of the weight polynomials for both corner counting paths in an $m \times n$ binomial box and bi-banded paths in an $m \times n$ binomial box each determine a partition of $\binom{n+m}{m}$. Those partitions are the same when $n = m$, i.e. for paths in a square binomial box. We will show this identity bijectively. First we need to define some reversible operations on sets, as follows.

Definition 11. *Let* $\mathcal{S}_{n,i} := \{S \subseteq [n]$ *such that* $|S| = i\}$.

- **Rule 1**

$$\begin{array}{rccc} \text{Rule 1:} & \mathcal{S}_{n,i} & \longmapsto & \mathcal{S}_{n,n-i} \\ \text{by} & S & \longrightarrow & [n]\backslash S \end{array}$$

Note that Rule 1 is its own inverse. Also note that Rule 1 implies the binomial identity $\binom{n}{n-i} = \binom{n}{i}$.

- **Rule 2**
  - **Splitting**

$$\begin{array}{rccc} \text{Rule 2 (split):} & \mathcal{S}_{n,i} & \longmapsto & \mathcal{S}_{n-1,i} \cup \mathcal{S}_{n-1,i-1} \\ \text{by} & S & \longrightarrow & S\backslash\{n\} \end{array}$$

  - **Combining**

$$\begin{array}{rccc} \text{Rule 2 (combine):} & \mathcal{S}_{n-1,i} \cup \mathcal{S}_{n-1,i-1} & \longmapsto & \mathcal{S}_{n,i} \\ \text{by} & S & \longrightarrow & S \cup \{n\} \end{array}$$

Note that Rule 2 implies the binomial identity $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$.

Rules 1 and 2 give geometric interpretations of algebraic relations between binomials, and are illustrated in Figures 3.8 and Figure 3.9.

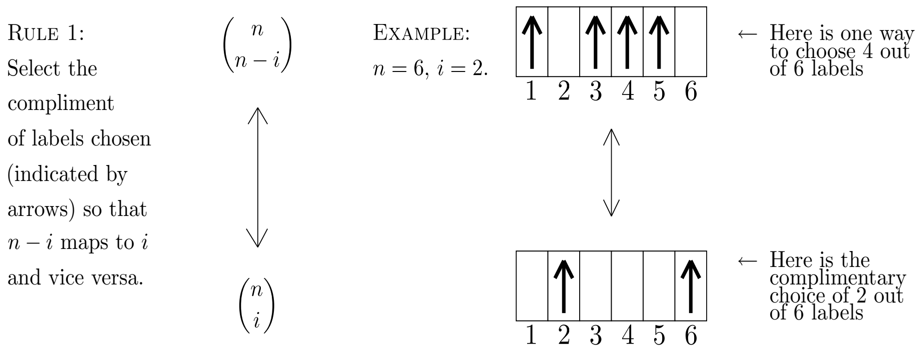


Figure 3.8: Rule 1.

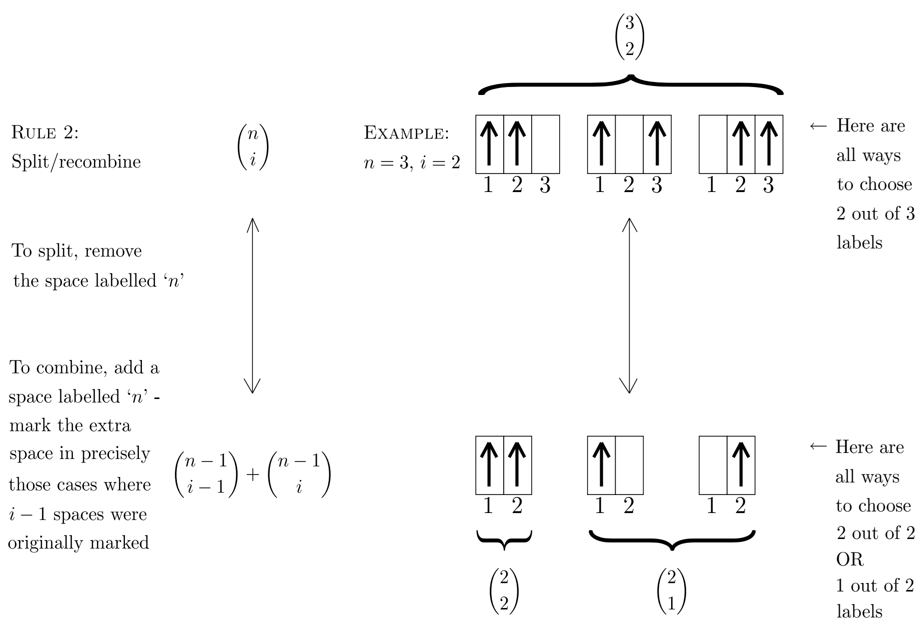


Figure 3.9: Rule 2.

We now state the theorem.

Theorem 21. *Let $C_{n,n}$ be the corner weight polynomial for binomial paths of length $2n$ in a square binomial box, and let $B_{n,n}$ be the bi-banded weight polynomial for binomial paths of length $2n$ in a square binomial box. Then*

$$C_{n,n} = \sum_i \binom{n}{i}\binom{n}{i} c^{2i+1} \tag{3.15}$$

$$B_{n,n} = \sum_i \binom{n}{n-i}\binom{n}{i} a^{2n-2i} b^{2i}. \tag{3.16}$$

Furthermore, let $\mathcal{C}_n^{(i)}$ be the set of binomial paths of length $2n$ in a square binomial box containing exactly $2i+1$ corners; and let $\mathcal{B}_n^{(i)}$ be the set of binomial paths of length $2n$ in a square binomial box whose bi-banded weighting in a and b contains exactly $n-i$ powers of a. Then there exists a bijection

$$\mathcal{C}_n^{(i)} \longleftrightarrow \mathcal{B}_n^{(i)}. \tag{3.17}$$

Proof. Equations (3.15) and (3.16) are special cases of Equations (3.3) and (3.7) respectively. The bijection is specified as follows:

Identify each path in $\mathcal{C}_n^{(i)}$ with its labelling in terms of black and red check marks, as described in the proof of Theorem 19. Identify each path in $\mathcal{B}_n^{(i)}$ with its labelling in terms of marked dark and light vertical bands, as described in the proof of Theorem 20.

Begin with a path in $\mathcal{B}_n^{(i)}$.

1. Apply Rule 1 to dark banded marks. These become the 'black labels' for corner counting paths.

2. Apply Rule 2 (splitting) to light banded marks to get the 'red labels' for corner counting paths.

To perform the inverse operation, start with a path in $\mathcal{C}_n^{(i)}$.

1. Apply Rule 1 to the 'black labels'. These become the dark banded marks for the bi-banded path.

2. Apply Rule 2 (combining) to the 'red labels'. The result is the dark banded marks for the bi-banded path.

The procedure specified is a bijection since Rules 1 and 2 of which it is composed each specify bijections on labels. (A schema of the proof of Theorem 21 is given in Figure 3.10.) □

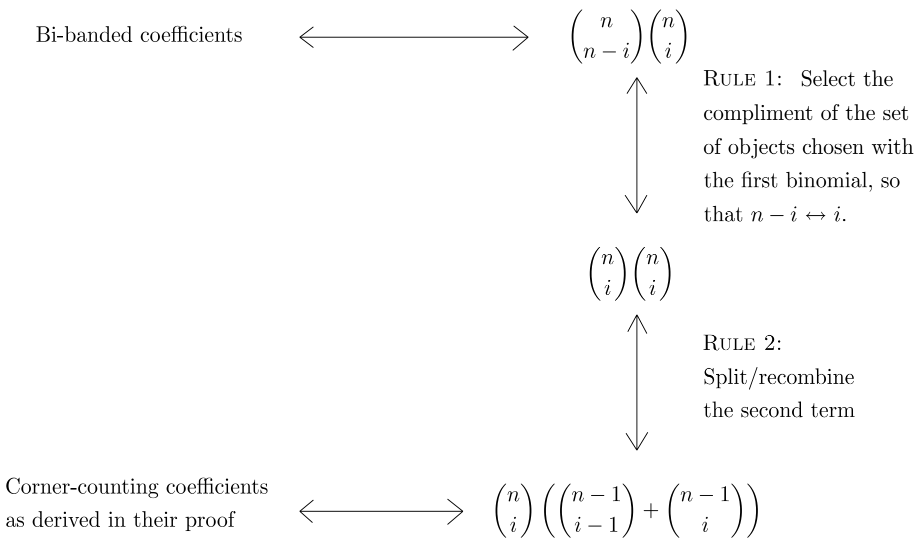


Figure 3.10: Bi-banded coefficients biject with corner-counting coefficients.

An example of the bijection between corner-counting paths in a square box and bi-banded paths in a square box is illustrated in Figures 3.11–3.12.

3.4 The bijection restricts to Dyck paths

The bijection of the last section sends Dyck paths to Dyck paths, so we have a bonus result.

Definition 12. *Let $p = v_0e_1v_1 \ldots e_tv_t$ be a Ballot-like path. Then arcs e_i are drawn from the step set $\{U, D\}$. Let $e_{i-1} = v_{i-1}v_i$ and $e_i = v_iv_{i+1}$ be a pair of consecutive arcs. Then the intermediate vertex v_i is defined to be a* **peak** *(respectively* **valley***) provided that (e_{i-1}, e_i) is an (U, D) (respectively (D, U)) pair.*

Definition 13. *Let $\mathcal{P}_{t;y',y}$ be the set of Ballot-like paths on the square lattice which begin at vertex $(0, y')$ and end at vertex (t, y). A path $\omega \in \mathcal{P}_{t;y',y}$ has peak weight*

$$w(\omega) = p^{|peaks|}, \tag{3.18}$$

where $|peaks|$ is the number of peaks in the path $p = v_0e_1v_1...e_tv_t$. The **peak**

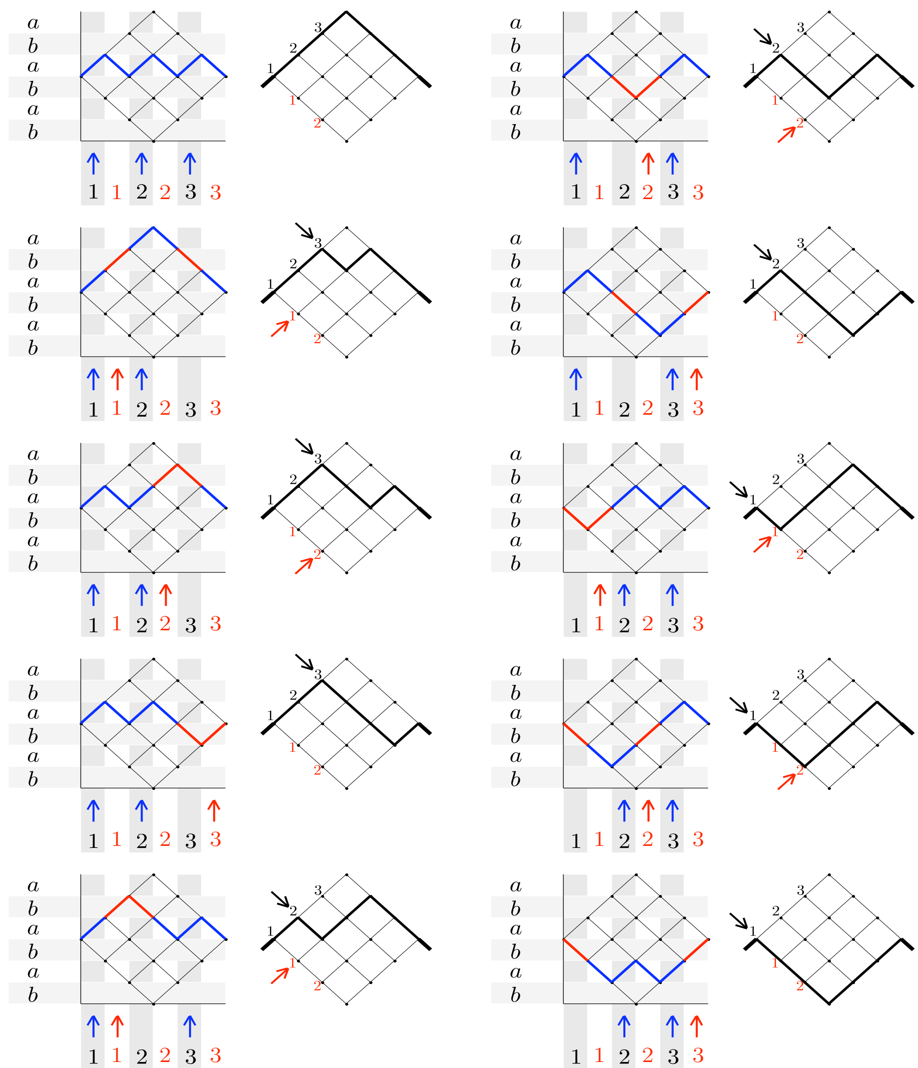


Figure 3.11: PART 'A' OF EXAMPLE: For a 3×3 binomial box, the first ten of $\binom{6}{3}$ pairs of bijecting bi-banded and corner-counting paths are shown.

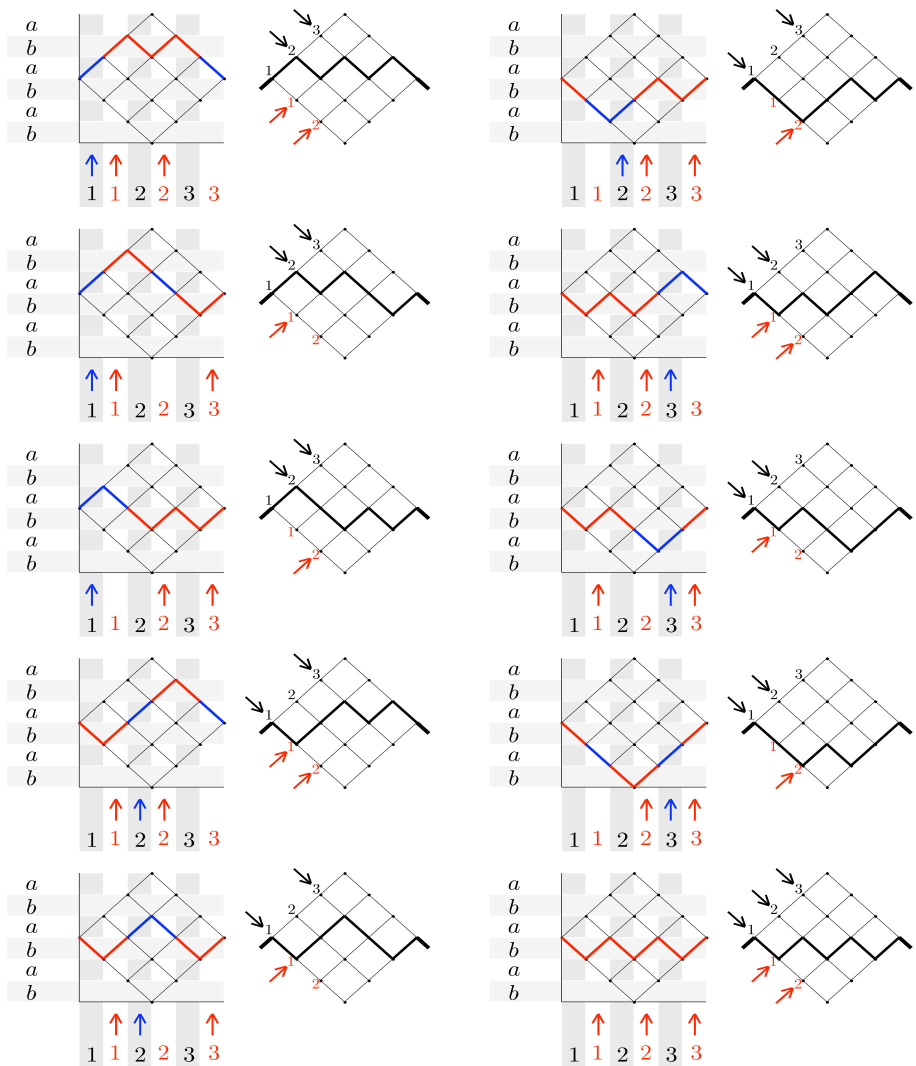


Figure 3.12: PART 'B' OF EXAMPLE: For a 3×3 binomial box, the last ten of $\binom{6}{3}$ pairs of bijecting bi-banded and corner-counting paths are shown.

weight polynomial *for Ballot-like paths is defined to be*

$$P_{t;y',y}(p) = \sum_{\omega \in \mathcal{P}_{t;y',y}} w(\omega). \tag{3.19}$$

Theorem 22. *Let $\overline{\overline{D}}_{2n}(a,b)$ be the bi-banded weight polynomial for Dyck paths of length $2n$. Let $\hat{D}_{2n}(p)$ be the peak weight polynomial for Dyck paths of length $2n$. Then*

$$\overline{\overline{D}}_{2n}(a,b) = \sum_{i=1}^{n} N_{n,i} a^{2(n+1-i)} b^{2(i-1)}, \tag{3.20}$$

$$\hat{D}_{2n}(p) = \sum_{i=1}^{n} N_{n,i} p^i, \tag{3.21}$$

where the $\{N_{n,i}\}_{n\geq 1, i\geq 1}$ are defined by

$$N_{n,i} := \frac{1}{i}\binom{n}{i-1}\binom{n-1}{i-1} \tag{3.22}$$

and are the **Narayana numbers***.*

Proof. The structure of the proof is as follows. We know from Theorem 21 that there exists a bijective mapping, call it 'ϕ', from corner-counting binomial paths in a square box to bi-banded binomial paths in a square box.

A subset of the corner-counting binomial paths are corner-counting Dyck paths. We show that under the bijection 'ϕ' Dyck paths map to Dyck paths. In other words, restricting the domain of ϕ to Dyck paths yields an induced mapping whose range consists entirely of Dyck paths, too. Conversely, restricting the domain of the inverse function ϕ^{-1} to Dyck paths yields an induced function whose range consists entirely of Dyck paths, as well.

Now corner-counting Dyck paths are identifiable with peak-counting Dyck paths. Specifically, since Dyck paths always have exactly one more peak than valley, the problem of counting corners is equivalent to that of counting peaks, up to a change of exponent in the generating variable. Paths with i peaks have $2i-1$ corners, for $1 \leq i \leq n$.

Thus the induced bijection between bi-banded Dyck paths and corner-counting Dyck paths extends to peak-counting Dyck paths; and the latter are well known to generate Narayana numbers [96], sequence number A001263.

It remains to show why the bijection of Theorem 21 between bi-banded and corner-counting binomial paths restricts to a bijection between Dyck paths. We start with bi-banded Dyck paths.

bi-banded $\Rightarrow$ corner-counting

Begin with a bi-banded Dyck path in a square box, and identify it with its labeling, as in Section 3.2. There is a marked dark vertical band for every up-step labeled 'a', a marked light vertical band for every up-step labeled 'b', an unmarked dark vertical band for every down-step labeled 'b', and an unmarked light vertical band for every down-step labeled 'a'. (See the example in Figures 3.11 and 3.12.)

Dyck paths must stay on or above the t-axis. Thus, in every horizontal band, for every down-step there is a mapping to a uniquely identifiable up-step preceding it. Hence, for every unmarked dark vertical band, there is a uniquely identifiable marked light vertical band preceding it.

We need to show that after bijecting from bi-banded to corner-counting paths, the property of staying above the t-axis is preserved. Carrying out the bijection, unmarked dark vertical bands become marked black labels and marked light vertical bands become marked red labels, except for possibly the last one – markings in the rightmost light-banded position being truncated. Thus for every marked black label at position k, there is a uniquely identifiable marked red label at some position in $\{1, 2, ..., k-1\}$.

But this is precisely what we need to ensure that the corner-counting path corresponding to the black and red labeled markings is a Dyck path. Each black check mark at position k sends the path heading downward. To stop it from descending below the t-axis, we require a corresponding red check mark at one of the positions $\{1, 2, ..., k-1\}$, which is what we have shown to be present.

corner-counting $\Rightarrow$ bi-banded

Begin with a corner-counting Dyck path, and identify it with its labeling, as in Section 3.1. To every black check mark there corresponds a unique red check mark with smaller label which prevents the path from dropping below the t-axis.

Carrying out the bijection to bi-banded paths, black check marks map to unmarked dark vertical bands and red check marks map to marked light vertical bands. Hence every down-step weighted 'b' in the bi-banded path is preceded by a uniquely identifiable up-step weighted 'b'.

Hence none of these b-weighted steps can be below the t-axis, because if they were there would have to be a down-step weighted b between heights 0 and 1 with no preceding like-weighted up-step, a contradiction.

But the position of the b-weighted steps determine the entire path, given that it must be connected. Since the lowest height that a b-weighted step can drop to and still be above the t-axis is height 1, it follows that the lowest height that a connecting a-weighted step may drop to is height 0; i.e. the path must be a Dyck path. □

Example 3. *Figure 3.13 illustrates the bijection of Figures 3.11 and 3.12 restricted to Dyck paths.*

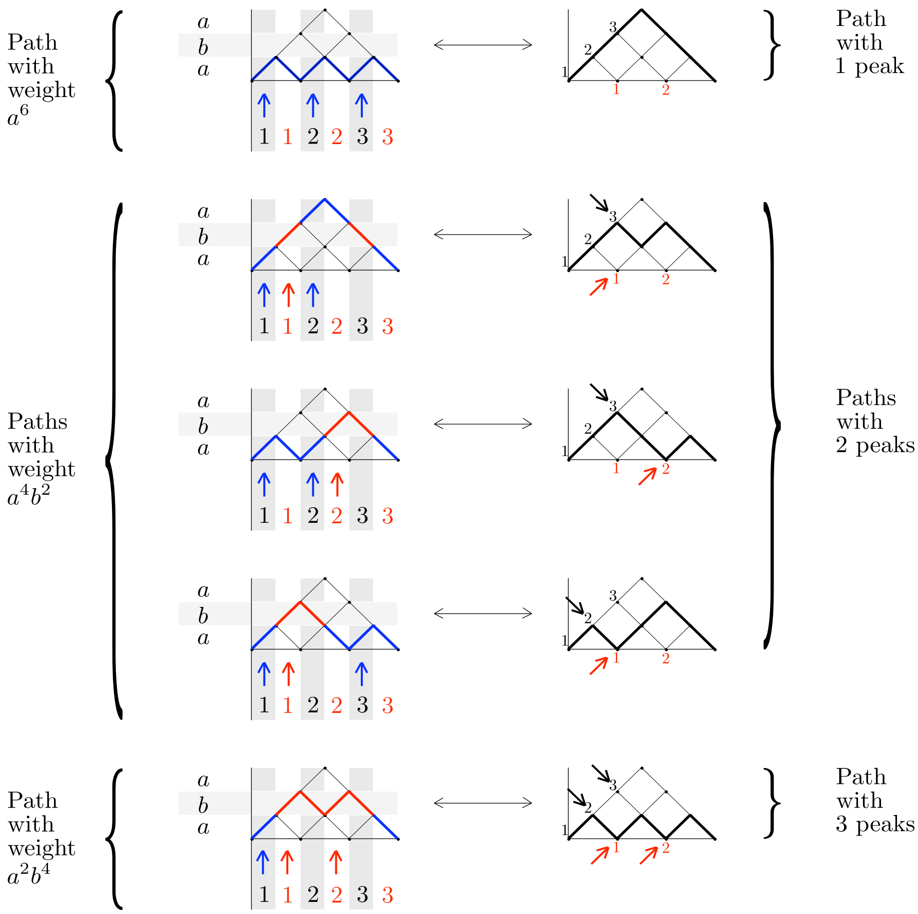


Figure 3.13: The five Dyck paths of length $2n = 6$, illustrating the bijection between bi-banded paths and Dyck paths counted by peaks.

3.5 A Bi-banded Dyck Path Generating Function

We backtrack a little here, and start by re-deriving the bi-banded weight polynomial for Dyck paths using generating functions. We define the shorthand

$$D_{2n}(a,b) := \overline{\overline{\overline{D}}}_{2n}(a,b) \tag{3.23}$$

for the bi-banded weight polynomial for Dyck paths of length $2n$. We have

Lemma 23. *Let $D_{2n}(a,b)$ be the weight polynomial for bi-banded Dyck paths of length $2n$. Then the weight polynomials satisfy recurrence*

$$D_{2n}(a,b) = a^2 \sum_{i=0}^{(n-1)/2} D_{2i}(a,b)D_{2(n-1-i)}(b,a) \tag{3.24}$$

with initial condition $D_0(a,b) = 1$. Let the generating function be

$$f(a,b;x) = \sum_{n\geq 0} D_{2n}(a,b)x^{2n}. \tag{3.25}$$

Then

$$f(a,b;x) = 1 + a^2x^2 f(a,b;x)f(b,a;x). \tag{3.26}$$

Proof. A bi-banded Dyck path of length $2n \geq 2$ may be factored into two smaller banded Dyck paths, as in Figure 3.14. Summing over all possible sizes for the subpaths gives recurrence (3.24).

Evaluating the generating function according to the banded weighting $a, b, a, ...$ and its compliment $b, a, b, ...$ gives

$$1+a^2x^2 f(a,b;x)f(b,a;x) =$$
$$1 + a^2\left[D_0(a,b)D_0(b,a)\right]x^2 + a^2\left[D_0(a,b)D_2(b,a) + D_2(a,b)D_0(b,a)\right]x^4 + ... \tag{3.27}$$

Comparing Equation (3.27) with the recurrence relation (3.24) gives the relationship (3.26). □

Theorem 24. *The generating function for bi-banded Dyck paths of length $2n$, (3.25), is*

$$f(a,b;x) = \frac{1 + x^2(b^2-a^2) - \sqrt{(a^2-b^2)^2x^4 - 2(a^2+b^2)x^2 + 1}}{2x^2b^2}. \tag{3.28}$$

Proof. Define abbreviations $A = f(a,b;x)$, $B = f(b,a;x)$, $S_1 = a^2x^2$, $S_2 = b^2x^2$. Then, by Lemma 23,

$$A = 1 + S_1AB, \tag{3.29}$$
$$B = 1 + S_2BA. \tag{3.30}$$

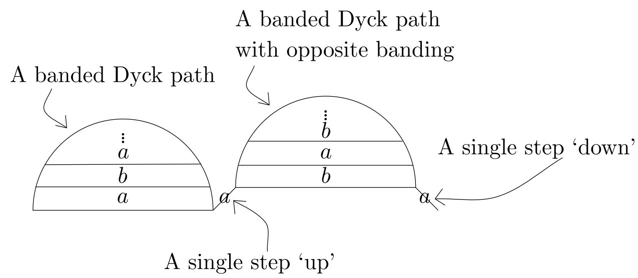


Figure 3.14: A bi-banded Dyck path of length ≥ 2 may be factored into two smaller banded Dyck paths (either of which may be the zero path), as well as an up and a down step. The rightmost subpath has opposite banding to the overall path.

Solving each equation for AB and equating gives

$$\frac{A-1}{S_1} = \frac{B-1}{S_2} \tag{3.31}$$

so that

$$B = (A-1)\frac{S_2}{S_1} + 1. \tag{3.32}$$

Substituting Equation (3.32) into Equation (3.29) yields a quadratic for A:

$$A^2(S_2) + A(S_1 - S_2 - 1) + 1 = 0. \tag{3.33}$$

Solving for A gives

$$\frac{1 + x^2(b^2 - a^2) \pm \sqrt{(a^2 - b^2)^2 x^4 - 2(a^2 + b^2)x^2 + 1}}{2x^2 b^2}. \tag{3.34}$$

We choose the minus sign so that the specialization $f(1,1;x)$ gives the correct generating function for unweighted Dyck paths. $\square$

3.5.1 Bi-banded Dyck path weight polynomials, again

To (re)discover the weight polynomials, expand Equation (3.28) as a series in x. We obtain

$$\begin{aligned} f(a,b;x) &= 1 \\ &\quad +a^2x^2 \\ &\quad +a^2(a^2+b^2)x^4 \\ &\quad +a^2(a^4+3a^2b^2+b^4)x^6 \\ &\quad +a^2(a^6+6a^4b^2+6a^2b^4+b^6)x^8 \\ &\quad +a^2(a^8+10a^6b^2+20a^4b^4+10a^2b^6+b^8)x^{10} \\ &\quad \vdots \end{aligned} \tag{3.35}$$

The triangle of coefficients is

$$\begin{array}{ccccccccccccccc} &&&&&&&1&&&&&&& \\ &&&&&&1&&1&&&&&& \\ &&&&&1&&3&&1&&&&& \\ &&&&1&&6&&6&&1&&&& \\ &&&1&&10&&20&&10&&1&&& \\ &&1&&15&&50&&50&&15&&1&& \\ &1&&21&&105&&175&&105&&21&&1& \\ 1&&28&&196&&490&&490&&196&&28&&1 \\ &&&&&&&\vdots&&&&&&& \end{array} \tag{3.36}$$

Triangle (3.36) is recognizable as the **Narayana triangle** (also known as the **Catalan triangle**) – see Sequence number A001263 of [96] – of the 'Narayana' numbers that were introduced in Theorem 22. Thus we have found an alternative route to, *but not an alternative proof of* Equation (3.20) of Theorem 22.

If we wanted an alternative proof, given the generating function way of finding the solution, the natural thing to try would be to use induction on the recurrence relation (3.24). This is not straightforward. It boils down to proving an identity which is a double sum over some binomials, where the inductive variable sits both inside the binomials and in the summation limits.

$$\sum_{i=0}^{(n-1)/2}\sum_{j=1}^{i}\sum_{k=1}^{n-1-i} N_{i,j}N_{n-1-i,k}a^{2n+4-2j-2k}b^{2j+2k-4} = \sum_{j=1}^{n} N_{n,j}a^{2(n+1-j)}b^{2(j-1)}, \tag{3.37}$$

We turn the problem around, and instead of using an identity to create an alternative proof, we use the fact that we have already proved bi-banded weight polynomials to have Narayanan coefficients combinatorially, to conclude that the identity must hold. Thus we have

Corollary 25. *The identity*

$$\sum_{i=1}^{n-2}\sum_{k=0}^{j-1}\frac{\binom{n-1-i}{k}\binom{n-1-i}{k-1}\binom{i}{j-1-i}\binom{i}{j-k}}{(n-1-i)i}=\frac{2j\binom{n-1}{j}\binom{n-2}{j}}{(n-j)(j+1)} \tag{3.38}$$

holds for that range of variables for which both sides are defined.

The bi-banded weight polynomial for Dyck paths turns out to be one of the relatively rare instances where a combinatorial proof is easier than other kinds of proofs, and precedes them.

3.6 Bi-banded Paths by generating functions

We begin with generating functions.

Theorem 26. *Let*

$$f(a,b;x)=\sum_{n\geq 0} D_{2n}(a,b)x^{2n} \tag{3.39}$$

be the generating function for Bi-banded Dyck paths which was given explicitly in Theorem 24, and let

$$g(a,b;h,x)=\sum_{t\geq 0} B_t(a,b;h)x \tag{3.40}$$

be the generating function for Bi-banded Ballot paths ending at height h. Then

$$g(a,b;h,x)=a^{\lceil h/2\rceil}b^{\lfloor h/2\rfloor}x^h f(a,b;x)^{\lceil (h+1)/2\rceil} f(b,a;x)^{\lfloor (h+1)/2\rfloor}. \tag{3.41}$$

Proof. The proof follows from a decomposition of Ballot paths into Dyck paths connected by up steps. The '$a^{\lceil h/2\rceil}b^{\lfloor h/2\rfloor}x^h$' term counts the connecting up steps, and the powers of f count the Dyck path components. See Figure 3.15. □

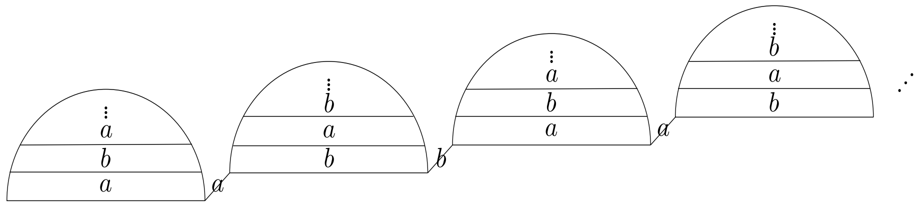


Figure 3.15: A bi-banded Ballot path may be factored into a sequence of bi-banded Dyck paths with alternating bandings, connected by a single step up between each (possibly empty) Dyck path.

Expanding the generating functions given by Theorem 26 for values of $h = 0, 1, 2, 3, ...$ gives triangles of coefficients of the same general structure as Triangle (3.36). Fitting polynomials along diagonals we conjecture forms for the coefficients.

h	Guess i^{th} coefficient, for $0 \le i \le n-1$	$\times$ weight	Path length
0	$\binom{n+1}{i}\binom{n}{i}\frac{1}{i+1}$	$a^{2(n-i)}b^{2i}$	$2n$
1	$\binom{n+1}{i}\binom{n}{i}\frac{1}{i+1}$	$a^{2(n-i)+1}b^{2i}$	$2n+1$
2	$\binom{n+1}{i}\binom{n}{i}\frac{2n-i+2}{(i+1)(i+2)}$	$a^{2(n-i)+1}b^{2i+1}$	$2n+2$
3	$\binom{n+2}{i}\binom{n+1}{i+1}\frac{2}{i+2}$	$a^{2(n-i)+2}b^{2i+1}$	$2n+3$
4	$\binom{n+2}{i}\binom{n+1}{i+1}\frac{3n-i+6}{(i+2)(i+3)}$	$a^{2(n-i)+2}b^{2i+2}$	$2n+4$
5	$\binom{n+3}{i}\binom{n+2}{i+2}\frac{3}{i+3}$	$a^{2(n-i)+3}b^{2i+2}$	$2n+5$
6	$\binom{n+3}{i}\binom{n+2}{i+2}\frac{4n-i+12}{(i+3)(i+4)}$	$a^{2(n-i)+3}b^{2i+3}$	$2n+6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The $h = 0$ coefficient is the already proven result for Dyck paths, with indices shifted so that i starts at zero instead of one. The rest follow an odd/even pattern, which, given that we know the Dyck path result as a base case, we are able to prove by induction, using the standard Ballot recurrence

$$B_t(a, b; h) = B_{t-1}(a, b; h-1) + B_{t-1}(a, b; h+1), \tag{3.42}$$

for $t, h \geq 1$; with

$$B_t(a, b; 0) = B_{t-1}(a, b; 1). \tag{3.43}$$

We have

Theorem 27. *Let $B_t(a, b; h)$ be the weight polynomial for bi-banded Ballot paths of length t, ending at height h. Then*

$$B_t(a, b; h) = \sum_{i=0}^{\lceil t/2 \rceil} \binom{\lceil t/2 \rceil}{i} \binom{\lceil t/2 \rceil - 1}{i + \lceil h/2 \rceil - 1} \frac{\lceil h/2 \rceil}{i + \lceil h/2 \rceil} a^{t - \lfloor h/2 \rfloor - 2i} b^{\lfloor h/2 \rfloor + 2i} \tag{3.44}$$

for odd t and odd height, h. Also,

$$B_t(a, b; h) = \sum_{i=0}^{t/2} \binom{t/2}{i} \binom{t/2 - 1}{i + h/2 - 1} \frac{th/4 + t/2 - i}{(i + h/2)(i + h/2 + 1)} a^{t - h/2 - 2i} b^{h/2 + 2i} \tag{3.45}$$

for even t and even height, $h \geq 2$.

Proof. The proof is a tedious but straightforward induction. □

3.7 Tri-banded generating functions

The tri-banded Dyck path generating function may be found by the same method as for bi-banded.

Definition 14. *A* **tri-banded weighting** *on a Ballot-like path is given by weights on upsteps and downsteps between adjacent heights defined as follows.*

$$w(\textit{upstep: } i \textit{ to } i+1) = w(\textit{downstep: } i+1 \textit{ to } i) = \begin{cases} a & i \equiv 0 \pmod 3 \\ b & i \equiv 1 \pmod 3 \\ c & i \equiv 2 \pmod 3. \end{cases} \tag{3.46}$$

Theorem 28. *The generating function for Tri-banded Dyck paths,*

$$f(a, b, c; x) = \sum_{n \geq 0} D_{2n}(a, b, c) x^{2n} \tag{3.47}$$

is given in closed form by $f(a,b,c;x) =$

$$\frac{(1-a^2x^2-b^2x^2+c^2x^2)-\sqrt{(a^2x^2+b^2x^2-c^2x^2-1)^2-4c^2x^2(1-a^2x^2)(1-b^2x^2)}}{2c^2x^2(1-a^2x^2)} \tag{3.48}$$

Proof. Define shorthand notation

$$A := f(a,b,c;x) \tag{3.49}$$
$$B := f(b,c,a;x) \tag{3.50}$$
$$C := f(c,a,b;x) \tag{3.51}$$

$$S_1 := a^2x^2 \tag{3.52}$$
$$S_2 := b^2x^2 \tag{3.53}$$
$$S_3 := c^2x^2 \tag{3.54}$$

Using analogous reasoning to that employed to give Equation (3.26) from Figure 3.14 in Section 3.5, write f in terms of itself with permuted arguments. We have functional relations

$$A = 1 + S_1AB \tag{3.55}$$
$$B = 1 + S_2BC \tag{3.56}$$
$$C = 1 + S_3CA \tag{3.57}$$

Substituting (3.57) into (3.56) into (3.55) and collecting terms gives

$$(S_3 - S_1S_3)A^2 + (S_1 + S_2 - S_3 - 1)A + (1 - S_2) = 0 \tag{3.58}$$

$$A = \frac{(1-S_1-S_2+S_3) \pm \sqrt{(S_1+S_2-S_3-1)^2 - 4S_3(1-S_1)(1-S_2)}}{2S_3(1-S_1)} \tag{3.59}$$

We have

$$A|_{S_1=S_2=S_3=1} = \frac{1 \pm \sqrt{1-4x^2}}{2x^2}. \tag{3.60}$$

Hence we choose the minus sign, since this gives the generating function for the Catalan numbers in the particular case $S_1 = S_2 = S_3 = 1$, as it should. □

Tri-banded Ballot path generating functions may be built out of tri-banded Dyck path generating functions analogously to the construction of (3.41) from Figure 3.15. We obtain

Theorem 29. *Let the generating function for Tri-banded Dyck paths which was given in closed form in Theorem 28 be*

$$f(a,b,c;x) = \sum_{n\geq 0} D_{2n}(a,b,c)x^{2n} \tag{3.61}$$

and let

$$g(a,b,c;h,x) = \sum_{t\geq 0} B_t(a,b,c;h)x^t \tag{3.62}$$

be the generating function for Tri-banded Ballot paths ending at height h. Then g is given in terms of powers of f as

$$g(a,b,c;h,x) = \\ a^{\lceil h/3\rceil} b^{\lceil (h-1)/3\rceil} c^{\lceil (h-2)/3\rceil} x^h f(a,b,c;x)^{\lceil (h+1)/3\rceil} f(b,c,a;x)^{\lceil h/3\rceil} f(c,a,b;x)^{\lceil (h-1)/3\rceil}. \tag{3.63}$$

In the next section, similar arguments to those used in the bi and tri-banded cases may be utilized to obtain n-banded Dyck and Ballot path generating functions. Successive n-banded Dyck path generating functions each satisfy a quadratic functional equation, as we will see.

3.8 n-banded generating functions

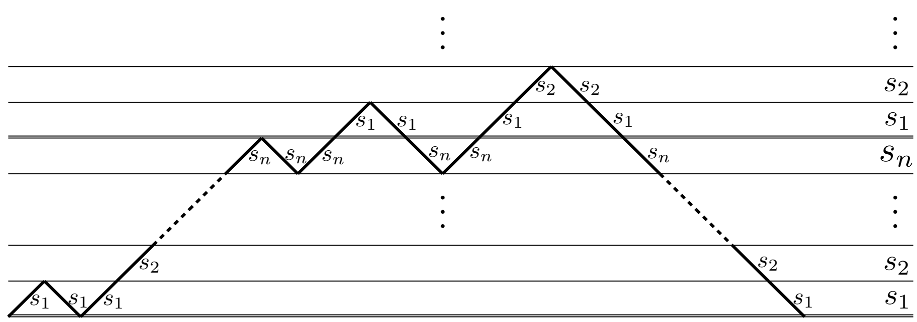
Banded paths are a special case for which we can deal with an unbounded number of decorations. We extend the work of the previous section to the case of arbitrary n-bandings, instead of just bi-banded and tri-banded paths. To give the n-banded generating function explicitly, we utilize orthogonal polynomials, which we note are the same family that shall make another appearance in Part II as Ballot paving polynomials.

Definition 15. *An n-**banded weighting** on a Ballot-like path is given by weights on upsteps and downsteps between adjacent heights as follows.*

$$w(\textit{upstep: } [i-1] \textit{ to } [i]) = w(\textit{downstep: } [i] \textit{ to } [i-1]) = s_i, \tag{3.64}$$

for $i \geq 1$, where

$$[i] \equiv i \pmod n. \tag{3.65}$$



Theorem 30. *For each $n \in \mathbb{N}$, the n-banded Dyck path generating function, $A(x)$, satisfies quadratic functional equation:*

$$S_nP_{n-1}(1)A^2+\left[\left(\sum_{i=0}^{n-2} S_{i+1}P_i(1)\right) - S_nP_{n-2}^{(1)} - 1\right]A-\left[\left(\sum_{i=0}^{n-3} S_{i+2}P_i^{(1)}(1)\right) - 1\right]=0, \tag{3.66}$$

where

$$S_{[i]} := s_{[i]}^2x^2, \tag{3.67}$$

shifted orthogonal polynomials $P_k^{(j)}(\mu)$ are defined by three term recurrence

$$P_k^{(j)}(\mu) = \mu P_{k-1}^{(j)}(\mu) - S_{k+j}P_{k-2}^{(j)}(\mu), \tag{3.68}$$

together with initial conditions

$$P_0^{(j)}(\mu) = 1 \tag{3.69}$$
$$P_1^{(j)}(\mu) = \mu; \tag{3.70}$$

and the orthogonal polynomials $P_k(\mu)$ are equated with the zero-shifted ones:

$$P_k(\mu) := P_k^{(0)}(\mu). \tag{3.71}$$

Proof. Define shorthand notation

$$A_{[i]} := f(s_{[i]}, ..., s_{[i+n-1]}; x), \tag{3.72}$$

Using analogous reasoning to that employed to give Equation (3.26) from Figure 3.14 in Section 3.5, write f in terms of itself with permuted arguments. We have functional relations

$$\begin{aligned} A_1 &= 1 + S_1 A_1 A_2 && (3.73)\\ A_2 &= 1 + S_2 A_2 A_3 && (3.74)\\ \vdots\ &\ \vdots\ \ \vdots \\ A_n &= 1 + S_n A_n A_1 && (3.75) \end{aligned}$$

We first prove that for $2 \leq k < n$,

$$\prod_{i=1}^{k} A_i = \frac{P_{k-1}(1)A_1 - P_{k-2}^{(1)}(1)}{\prod_{i=1}^{k-1} S_i}. \tag{3.76}$$

The proof is by induction. For the base case, compare (3.73) with (3.76), evaluated at $k = 2$. Now make inductive assumption (3.76), up to some $k < n$. Then

$$\begin{aligned} \prod_{i=1}^{k+1} A_i &= \left(\prod_{i=1}^{k-1} A_i\right)(A_k A_{k+1}) && (3.77)\\ &= \left(\prod_{i=1}^{k-1} A_i\right)\left(\frac{A_k - 1}{S_k}\right) && (3.78)\\ &= \frac{\left(P_{k-1}(1)A_1 - P_{k-2}^{(1)}(1)\right) - S_{k-1}\left(P_{k-2}(1)A_1 - P_{k-3}^{(1)}(1)\right)}{\prod_{i=1}^{k} S_i} && (3.79)\\ &= \frac{P_k(1)A_1 - P_{k-1}^{(1)}(1)}{\prod_{i=1}^{k} S_i}, && (3.80) \end{aligned}$$

where in line (3.78) we used functional relations (3.73)–(3.75), in line (3.79) we used the inductive assumption, and in line (3.80) we used the orthogonal polynomial recurrence relation (3.68). Hence Equation (3.76) is shown.

Notice that repeatedly substituting functional relations (3.73)–(3.75), as follows:

$$\begin{aligned} A_1 &= 1 + S_1 A_1 A_2 && (3.81)\\ &= 1 + S_1 A_1 (1 + S_2 A_2 A_3) && (3.82)\\ &= 1 + S_1 A_1 (1 + S_2 A_2 (1 + S_3 A_3 A_4)) && (3.83)\\ &\vdots \quad \vdots \\ &= 1 + S_1 A_1 (1 + S_2 A_2 (1 + \ldots (1 + S_n A_n A_1))) && (3.84) \end{aligned}$$

finally wraps to give

$$A_1 = 1 + \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i} S_j A_j \right) + \left(\prod_{j=1}^{n} S_j A_j \right) A_1. \tag{3.85}$$

Abbreviating

$$A := A_1, \tag{3.86}$$

and collecting terms in (3.85) gives the theorem. □

It is interesting to note that banded Dyck path generating functions are quadratic, whereas those in a strip (as we will see in Chapter 9) have rational generating functions. We might have expected banded paths to behave like strip paths, since a banding is just a repetition of the finite weights in a strip repeated over and over. The quadratic generating function is telling us that despite the similarity between strip paths and banded paths, the latter have more in common with un-weighted paths in the half plane.

As in the previous section, Ballot path generating functions may be built out of Dyck path generating functions. This holds for any n-banding in an analogous way to that for which the bi-banded and tri-banded Ballot path generating functions were constructed in Theorems 26 and 29. The result for general n is of just the same form as (3.63), with n copies of f corresponding to n cyclings of its arguments.

Theorem 31. *Let the generating function for n-banded Dyck paths which was given in closed form in Theorem 30 be*

$$f(s_1, \ldots, s_n; x) = \sum_{n \geq 0} D_{2n}(s_1, \ldots, s_n) x^{2n} \tag{3.87}$$

and let

$$g(s_1, \ldots, s_n; h, x) = \sum_{t \geq 0} B_t(s_1, \ldots, s_n; h) x^t \tag{3.88}$$

be the generating function for n-banded Ballot paths ending at height h. Then g is given in terms of powers of f as

$$g(s_1, \ldots, s_n; h, x) = x^h \left(\prod_{i=0}^{n-1} s_i^{\lceil (h-i)/n \rceil} f\left(s_{[i]}, ..., s_{[i+n-1]}; x\right)^{\lceil (h+1-i)/n \rceil} \right). \tag{3.89}$$

Part II

Pavings

Chapter 4

Introduction to Pavings

Directed lattice paths turn out to be intimately connected to underlying pavings, where looking at pavings instead of paths can, by several distinct methods, strip away at least one dimension from the problem. In particular, two dimensional directed lattice path enumeration problems can be solved utilizing pavings on a one dimensional path graph. We begin with the necessary definitions.

4.1 Definitions

A **monomer** is a single distinguished vertex in a graph. A **dimer** is a distinguished pair of adjacent vertices. A **trimer** is a distinguished triple of vertices $v_1v_2v_3$ such that v_1 is adjacent with v_2 and v_2 is adjacent with v_3. An **n-mer** is a distinguished set of n distinct vertices $v_1v_2...v_n$ such that v_i is adjacent with v_{i+1} for $1 \leq i < n$. A **non-covered** vertex is a vertex which is not part of any n-mer for $n \geq 1$. A non-covered vertex is also called a **0-mer**.

A **paver** is an n-mer, for any $n \in \mathbb{N}_{\geq 0}$. A **paving**, (G, p), is a graph, G, together with a specified collection of pavers, p, such that no two distinct pavers share a vertex. We abbreviate $p \equiv (G, p)$ when it is clear from the context which is the underlying graph. Visually, we represent a paving on a graph by colouring the pavers a different colour from the rest of the graph, or else by thickening the corresponding edges/vertices.

This use of the terms 'paving' and 'paver' is derived from the more standard description of a paving as a tiling by the bijection indicated in Figure 4.1. 0 The special case of a paving which utilizes only dimers and non-covered vertices is commonly called a **matching**.

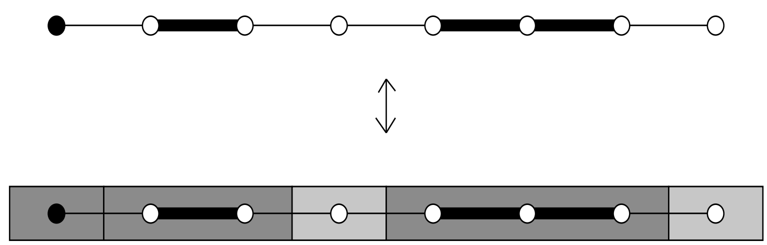


Figure 4.1: A paving with a monomer, a dimer, a trimer and two non-covered vertices. The upper picture is the representation we use. The lower picture is a more standard representation as a tiling.

The pavings required for this thesis all occur on a 'path graph'. A **path graph** is a graph whose vertices may be listed in order such that each vertex is adjacent to the vertex immediately preceding it in the list (if there is a preceeding vertex), the vertex immediately following it in the list (if there is a following vertex), and to no other vertices.

Similarly, a **path digraph** is a digraph whose vertices may be listed in order such that each vertex is adjacent to the vertex immediately preceding it in the list (if there is a preceeding vertex), the vertex immediately following it in the list (if there is a following vertex), and to no other vertices. The **path digraph** (V, A) **associated with the path graph** (V, E) is that digraph with vertex set V and arc set A where $A = \cup_{uv \in E}\{uv, vu\}$.

We commonly use the same name for a path (di)graph as for its underlying vertex set. Examples of underlying vertex sets for path graphs which we often use are the set $\mathbb{Z}$, which we term the **(number) line**, the set $\mathbb{Z}_{\geq 0}$, which we term the **half (number) line** and the set $\{0, 1, 2, ..., n-1\}$, which we term a **(number) line segment**. We denote this last mentioned finite path graph $\mathcal{P}_n$. In each of the three named cases the edges are pairs of adjacent integers.

Recall that an arc uv is said to be 'of the form' of the vector w provided that $v - u = w$. We refer to arcs of the form $(k, k \pm i)$ as being **(based at) height** k. Similarly, we refer to vertices labelled k as being **at height** k.

Recall that a 'weight function' is a function which maps a set of combinatorial objects into a field; the field usually being $\mathbb{C}$, $\mathbb{R}$ or extensions thereof. Let $\alpha \in p$ be a paver. Then $w(\alpha)$ is the **weight of the paver**. Unless otherwise specified, the weight of a non-covered vertex is always μ;

i.e.

$$w(\alpha_0) = \mu \tag{4.1}$$

for α_0 a 0-mer.

The **weight of a paving**, p, on a graph is the product of the weights of the pavers, α, in the paving; i.e.

$$w(p) = \prod_{\alpha \in p} w(\alpha). \tag{4.2}$$

A **uniform weighting on a paving** is a weighting such that all pavers of the same length have the same weight, independently of location, i.e. for fixed n,

$$w(n\text{-mer}) = \text{constant}. \tag{4.3}$$

A **decorated paving** is a weighted paving whose weighting is not uniform.

A **paving (weight) polynomial** is a polynomial in μ which is the sum of the weights of all of the pavings on a graph. In particular, the paving weight polynomial on the path graph $\mathcal{P}_k$ is indexed by the order of the path graph, so that

$$P_k(\mu) = \sum_{p \text{ a paving on } \mathcal{P}_k} w(p). \tag{4.4}$$

Pavings are closely connected to cycles. Recall that a 'cycle' on a (di)graph G is a walk $v_0 v_1 \ldots v_{n-1} v_n$ on G such that $v_0 v_1 \ldots v_{n-1}$ is a path, where we also recall that a path is a walk with no repeated vertices. A **cycle configuration** on a (di)graph G is any set of cycles on G. In particular, a **basic configuration (of cycles)** on a (di)graph, G, is a set of cycles on G such that no two cycles in the set share a vertex. An **uncovered vertex** in a basic configuration is a vertex not contained in any cycle of the configuration. A **covering configuration (of cycles)** on a (di)graph, G, is a basic configuration such that there are no uncovered vertices in G.

The **weight of an n-cycle**, for positive n, on (di)graph G, is the product of the weights of the vertices and edges/arcs in the cycle, multiplied by -1; i.e.

$$w(c) = -\prod_{v,e \in c} w(v)w(e) \tag{4.5}$$

for c the cycle on G, each v a vertex and each e an edge/arc. When vertices are unweighted, i.e. implicitly taken to have weight 1, Equation (4.5) simplifies to

$$w(c) = -\prod_{e \in c} w(e). \tag{4.6}$$

A 0-**cycle** is a cycle consisting of just one vertex, i.e. an uncovered vertex. We define the **weight of a** 0-**cycle** separately, to be

$$w(c_0) = \mu, \tag{4.7}$$

where c_0 is a 0-cycle / uncovered vertex. Note that the definition of the weight of an n-cycle differs from the definition of the weight of a generic walk by the factor of -1. The **weight of a cycle configuration** on a (di)graph G is the product of the weights of the cycles in the configuration

$$w(C) = \prod_{c \in C} w(c), \tag{4.8}$$

where c's are cycles in the cycle configuration C and we recall that $w(c_0) = \mu$ for c_0 a 0-cycle.

4.2 Motivation

4.2.1 Paths biject to walks

Start with an example. Consider the walk on the half line that is specified by vertex list: 0, 1, 2, 3, 2, 3, 4, 5, 4, 5, 4, 3, 2, 1, 0. This walk has length 14, starts at 0 and finishes at 0. The walk is difficult to illustrate on the half line (see Fig 4.2, left), because it repeatedly visits the same vertices. However creating an axis for time, and stretching the walk out in time, means that each vertex has a distinct time coordinate. The new vertex list is (0,0), (1,1), (2,2), (3,3), (4,2), (5,3), (6,4), (7,5), (8,4), (9,5), (10,4), (11,3), (12,2), (13,1), (14,0), and now we have a path in the half plane, which is easy to illustrate (see Fig 4.2, right).

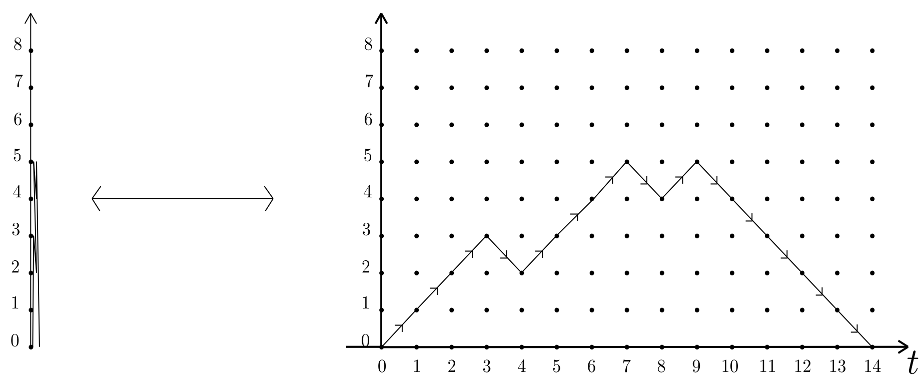


Figure 4.2: A walk on the half line becomes a path in the half plane.

The principle is general: walks on a graph with vertex set V biject to paths on a directed graph with vertex set $V \times \mathbb{Z}_{\geq 0}$. Walks become paths because the extra coordinate distinguishes between vertices, and the new graph is a digraph whether or not the old one was, because paths on this new graph must always advance in time.

Definition 16. *Let $G = (V, E)$ be a graph. For each edge $e = \{u, v\} \in E$, create a pair of arcs*

$$a = ((u,t),(v,t+1)), \tag{4.9}$$
$$b = ((v,t),(u,t+1)). \tag{4.10}$$

Let

$$A = \cup_{e \in E}\{a, b\} \tag{4.11}$$

be the union of the arcs thus created. Then the digraph $D = (V \times \mathbb{Z}_{\geq 0}, A)$ is called the the **(time) extended digraph** *of G. Let $H = (V, E')$ be the associated digraph of G. Then we call H the* **(time) compressed digraph** *of D.*

The following well-known observations follows from Definition 16.

Lemma 32. *Any set of walks $\mathcal{W}$ on a graph G is in bijection with a set of directed paths $\mathcal{P}$ on the time extended digraph of G. The bijection $\phi : \mathcal{W} \mapsto \mathcal{P}$ is defined by*

$$\phi(v_0 e_1 v_1 \ldots e_m v_m) = \phi(v_0)\phi(e_1)\phi(v_1) \ldots \phi(e_m)\phi(v_m) \tag{4.12}$$

where

$$\phi(v_t) = (v_t, t) \tag{4.13}$$

and

$$\phi(e_t) := \phi(v_{t-1}v_t) = ((v_{t-1}, t-1), (v_t, t)). \tag{4.14}$$

Thus the enumeration problem of counting walks on a graph is the same as the enumeration problem of counting paths on a new, directed graph. In particular, counting walks on the half line is the same as counting directed paths in the half plane.

One more piece of terminology will be useful in the sequel.

Definition 17. *Let $\mathcal{W}$ be a set of walks (either directed or undirected) on a (di)graph G. Let $\mathcal{P}$ be the isomorphic set of directed paths (defined by Lemma 32) on the extended digraph, D, of G. Let H be the associated digraph of G. Then we say that*

1. *G is the* **graph underlying** *$\mathcal{W}$.*

2. *H is the* **digraph underlying** *$\mathcal{W}$.*

3. *D is the* **digraph underlying** *$\mathcal{P}$.*

4. *H is the* **compressed digraph underlying** *$\mathcal{P}$.*

So now we have paths as walks. The next thing is to relate walks to cycles.

4.2.2 Walking or cycling?

The walk $0, 1, 2, 3, 2, 3, 4, 5, 4, 5, 4, 3, 2, 1, 0$ of Figure 4.2 is made up of cycles. To see this explicitly, successively remove cycles as follows.
(i) Parse the vertex list from left to right.
(ii) When a vertex, v, occurs which has been visited before, extract the portion of the walk between the two copies of v, up to and including the second copy, and add it to the list of cycles.
(iii) Repeat from step (i) until there are no more duplicated vertices in the walk.

$$\begin{array}{c}
0, 1, \underline{2, 3, 2}, 3, 4, 5, 4, 5, 4, 3, 2, 1, 0 \\
\downarrow \\
0, 1, \mathbf{2}, 3, \underline{4, 5, 4}, 5, 4, 3, 2, 1, 0 \\
\downarrow \\
0, 1, 2, 3, \underline{\mathbf{4}, 5, 4}, 3, 2, 1, 0 \\
\downarrow \\
0, 1, 2, \underline{3, \mathbf{4}, 3}, 2, 1, 0 \\
\downarrow \\
0, 1, \underline{2, \mathbf{3}, 2}, 1, 0 \\
\downarrow \\
0, \underline{1, \mathbf{2}, 1}, 0 \\
\downarrow \\
\underline{0, \mathbf{1}, 0} \\
\downarrow \\
\mathbf{0}
\end{array} \tag{4.15}$$

Because this walk ended at the same height it started at, the procedure terminates with just that initial vertex,

$$\{0\}, \tag{4.16}$$

together with the list of cycles

$$(2, 3), (4, 5), (4, 5), (3, 4), (2, 3), (1, 2), (0, 1). \tag{4.17}$$

Had the walk ended up somewhere else from where it started, we would have obtained a (shortest) path between beginning and ending points; as well as a list of cycles extracted.

4.2.3 'Cycle paths paved'

The representation of a walk as a shortest path together with a sequence of cycles has been used by Viennot [105] and subsequently others to biject directed lattice paths to combinatorial objects called ‘heaps’. ‘Heaps’ are then enumerated in terms of ‘trivial heaps’ which in turn are in bijection with pavings. In Part III of this thesis we introduce a number of new methods to utilize pavings in the enumeration of paths.

In all methods where pavings are used to enumerate directed lattice paths, the structure of the paving reflects the structure of cycles permissible in the underlying compressed digraph. We wish to be able to enumerate many different kinds of directed paths, specified with many different allowed step sets. These correspond to walks on a large collection of different digraphs. In the next chapter we define suitable classes of pavings to capture the essential cycle information for almost all digraphs of interest.

This work builds upon the framework laid down by Viennot, [104], [103], in mapping cycles on Ballot digraphs to dimers, an example of which is illustrated in Figure 4.3.

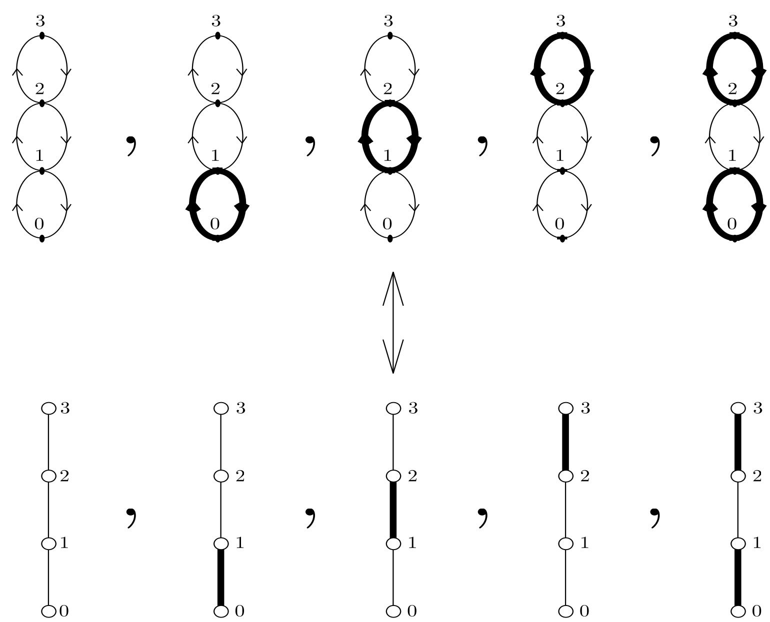


Figure 4.3: All basic configurations of cycles on the path digraph with four vertices are illustrated, along with a bijection to pavings.

Chapter 5

Pavings for uniformly weighted digraphs

In this chapter we introduce a large class of uniformly weighted digraphs. Each section introduces a new variety of digraph, which is the compressed underlying digraph for a directed lattice path enumeration problem of interest.

Within each section the following agenda is pursued.

1. Investigate the cycle structure suppported by the variety of digraph, finding the nature of the individual cycles as well as determining all possible basic configurations of cycles on such digraphs of arbitrary finite order.
2. Define suitably (uniformly) weighted pavers, hence a family of paving polynomials indexed by the order of the digraph, and a mapping between cycles and pavers such that the paving polynomials encode the information we will need in the sequel about the basic cycle structures of the given variety of digraph.
3. Explicitly list the first few paving polynomials, for small order digraphs.
4. Use combinatorics of pavings to find a recurrence on the paving polynomials.
5. Find the roots of the characteristic polynomial of the recurrence.
6. Write an explicit closed form expression for the family of paving polynomials.

With each new variety of digraph introduced, some new combinatorial or algebraic feature arises. Depending on the structure of the digraph, the difficulty of steps 1–6 may individually lie anywhere in the range trivial to formidable. The paving polynomials arising from uniformly weighted Ballot-like and Motzkin-like path enumeration problems are the classic Fibonacci and Motzkin orthogonal polynomials, as discovered and explicitly solved by Viennot [104], [103]. In contrast Jump-step and Lukasiewicz paths give rise to paving polynomials which are no longer orthogonal, for which step 5 in particular may be intractable.

We begin with the compressed underlying digraph for Ballot paths.

5.1 Ballot digraphs

This section on Ballot digraphs and associated pavings, as well as the next section, on Motzkin digraphs and pavings, is a review of the fundamental idea of Viennot, upon which we will build the rest of our paving theory.

Definition 18. *Let* $D_k = (V, A)$ *be a digraph of order* k *with vertex set* $V = \{0, 1, \ldots k-1\}$, *and arc set* A *composed of*

- **up arcs** *of the form* $(i, i+1)$, *and*
- **down arcs** *of the form* $(i, i-1)$.

Then any digraph isomorphic to D_k *is a* **Ballot digraph**.

We note that Ballot digraphs are path digraphs, as was indicated in Figure 4.3. In Figure 5.1 we show a generic Ballot digraph drawn horizontally, illustrating some typical cycles. Of the several possible different uniform weightings on Ballot-like paths, we choose one consistent with the path enumeration problems considered in Part I and define pavings according to that choice.

Definition 19. **Standard uniform Ballot arc weights** *are*

$$w(\textit{up arc}) = 1 \quad (5.1)$$
$$w(\textit{down arc}) = \lambda \quad (5.2)$$

Note that when $\lambda = 1$ *this is called the* **trivial (uniform) Ballot arc weighting**.

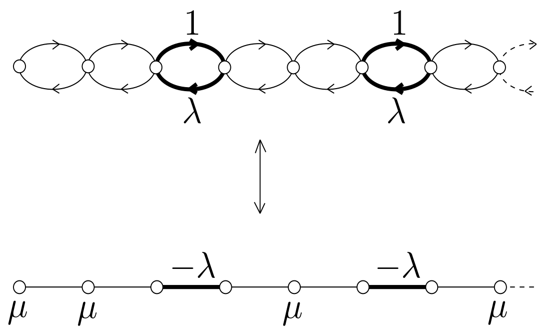


Figure 5.1: A basic cycle configuration on a Ballot digraph bijects to a paving with dimers on a path graph.

Recall that a paving on a graph is a set of pavers such that no two pavers share a vertex. We see in Figure 5.1 that only 2-cycles are possible on a Ballot digraph, so all basic cycle configurations project to dimers. Thus the pavers for Ballot pavings are 0-mers (i.e. non-covered vertices) and dimers.

Definition 20. Standard uniform Ballot paver weights *are*

$$w(\textit{dimer}) = -\lambda \tag{5.3}$$

$$w(0\textit{-mer}) = \mu. \tag{5.4}$$

These weights, in conjunction with the definition of a paving polynomial, give **standard uniform Ballot paving polynomials***. Note that for $\lambda = 1$ the weights are termed* **trivial (uniform) paver weights** *and the polynomials* **trivial (uniform) Ballot paving polynomials***.*

The first few standard uniform Ballot paving polynomials are as shown in Figure 5.2. There is a linear recurrence relation on these paving polynomials, which Figure 5.3 allows us to see. Having found the three term constant coefficient recurrence, we solve it by the standard ansatz. Let

$$P_k = z^k. \tag{5.5}$$

Substitute (5.5) into recurrence

$$P_{k+1} = \mu P_k - \lambda P_{k-1} \tag{5.6}$$

to get characteristic equation

$$z^2 - \mu z + \lambda = 0. \tag{5.7}$$

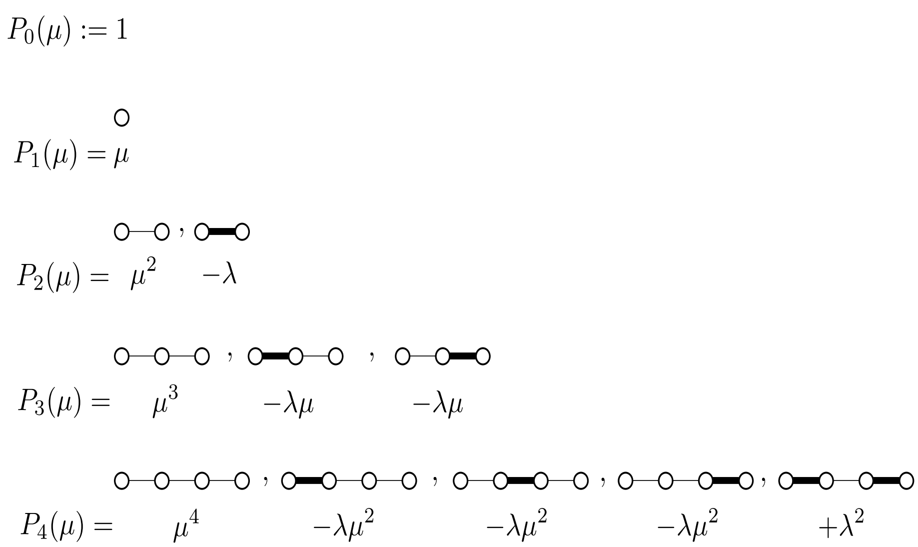


Figure 5.2: The first few Ballot polynomials.

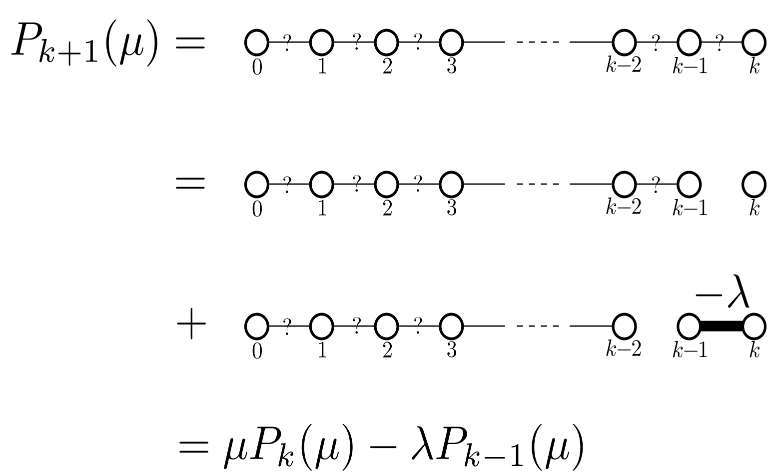


Figure 5.3: A three term recurrence for Ballot paving polynomials may be derived using the pavings. The question marks indicate that the corresponding edge may or may not be paved, thus the first line of the figure indicates that $P_{k+1}(\mu)$ is a sum over all possible pavings on the line segment.

Solve for z to get

$$z_{\pm} = \frac{\mu \pm \sqrt{\mu^2 - 4\lambda}}{2}. \tag{5.8}$$

Thus, solving $P_k = Az_+^k + Bz_-^k$ for constants A and B determined by the initial conditions $P_0(\mu) = 1$ and $P_1(\mu) = \mu$, we obtain

Lemma 33. *The family of standard uniform Ballot paving polynomials,* $\{P_k(\mu)\}_{k\geq 0}$*, satisfies 3-term recurrence*

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda P_{k-1}(\mu) \tag{5.9}$$

with characteristic equation

$$z^2 - \mu z + \lambda = 0. \tag{5.10}$$

The polynomials are given in closed form by

$$P_k(\mu) = \frac{(\mu + \sqrt{\mu^2 - 4\lambda})^{k+1} - (\mu - \sqrt{\mu^2 - 4\lambda})^{k+1}}{2^{k+1}\sqrt{\mu^2 - 4\lambda}}. \tag{5.11}$$

Note that subject to trivial uniform weighting $\lambda = 1$, this family of orthogonal polynomials are those termed **Fibonacci polynomials***.*

5.2 Motzkin digraphs

This section on Motzkin digraphs and associated pavings is, as in the Ballot case, a review of Viennot's basic insight. The modification required in the shift from Ballot to Motzkin pavings allows loops on the digraph, and is the same modification that will be required in all of our later generalizations in order to allow loops in the digraphs (or, equivalently, horizontal steps in the paths).

Definition 21. *Let $D_k = (V, A)$ be a digraph of order k with vertex set $V = \{0, 1, \ldots k-1\}$, and arc set A composed of*

- **up arcs** *of the form $(i, i+1)$,*
- **down arcs** *of the form $(i, i-1)$, and*
- **loops** *of the form (i, i).*

Then any digraph isomorphic to D_k is a **Motzkin digraph***.*

Note that a Motzkin digraph is the same as a Ballot digraph except for the addition of loops on the vertices. These loops mean that directed walks on the time extended digraph may take horizontal steps in addition to up and down steps.

We choose to develop pavings for the following uniform weighting.

Definition 22. Standard uniform Motzkin arc weights *are*

$$w(\textit{up arc}) = 1 \tag{5.12}$$
$$w(\textit{down arc}) = \lambda \tag{5.13}$$
$$w(\textit{loop}) = b \tag{5.14}$$

The Motzkin digraph supports 2-cycles and 1-cycles, as illustrated in Figure 5.4. We define weightings on Motzkin-pavers as follows.

Definition 23. Standard uniform Motzkin paver weights *are*

$$w(\textit{dimer}) = -\lambda \tag{5.15}$$
$$w(\textit{monomer}) = -b \tag{5.16}$$
$$w(\textit{0-mer}) = \mu. \tag{5.17}$$

These weights, in conjunction with the definition of a paving polynomial, give **standard uniform Motzkin paving polynomials**.

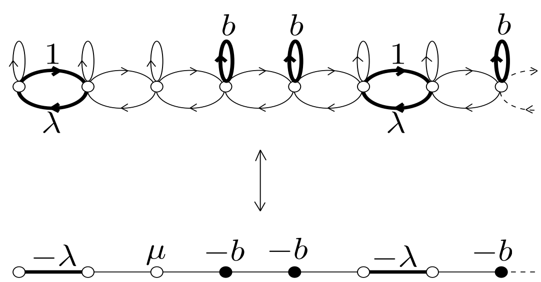


Figure 5.4: On a Motzkin digraph, cycles biject to monomer-dimer pavings.

The first few standard uniform Motzkin paving polynomials are given in Figure 5.5.

Motzkin polynomials satisfy a three term recurrence as illustrated in Figure 5.6. We have Lemma:

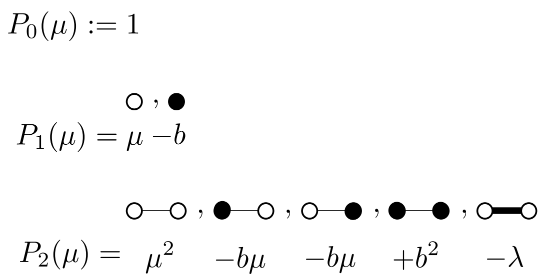


Figure 5.5: The first few standard uniform Motzkin polynomials.

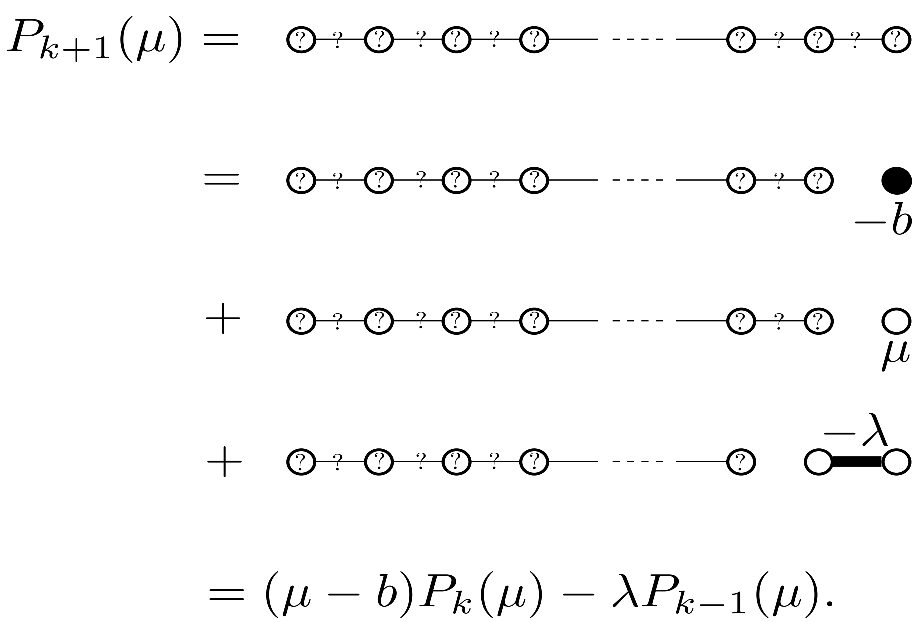


Figure 5.6: A three term recurrence for Motzkin paving polynomials derived using pavings. The question marks indicate that the corresponding edge or vertex may or may not be paved, and we are summing over all such possibilities. With regard to the last vertex, there are three ways in which this can happen. Either the last vertex is a monomer, the last vertex is uncovered, or the last vertex is part of a dimer.

Lemma 34. *The family of standard uniform Motzkin paving polynomials* $\{P_k(\mu)\}_{k\geq 0}$ *satisfies 3-term recurrence*

$$P_{k+1}(\mu) = (\mu - b)P_k(\mu) - \lambda P_{k-1} \tag{5.18}$$

with characteristic equation

$$z^2 - (\mu - b)z + \lambda = 0 \tag{5.19}$$

The polynomials are given in closed form by

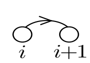
$$P_k(\mu) = \frac{(\mu - b + \sqrt{(\mu-b)^2 - 4\lambda})^{k+1} - (\mu - b - \sqrt{(\mu-b)^2 - 4\lambda})^{k+1}}{2^{k+1}\sqrt{(\mu-b)^2 - 4\lambda}}. \tag{5.20}$$

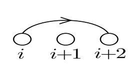
Note that for $\lambda = 1$*, these are the family of orthogonal polynomials called* **Motzkin polynomials**.

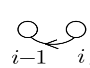
Comment 2. *We see that the form of the solution for the Motzkin paving polynomial is very similar that for the Ballot paving polynomial, with just a shift in the argument from* μ *to* $\mu - b$*. This observation is general: adding loops (all with the same weight) to the vertices of a digraph, or, equivalently, allowing (identically weighted) monomers in a paving, changes the paving polynomial only by a shift in argument; since loops must always map to either monomers or uncovered vertices, independently of the rest of the paving.*

5.3 '2-up' digraphs

Definition 24. *Let* $D_k = (V, A)$ *be a digraph of order* k *with vertex set* $V = \{0, 1, \ldots k-1\}$*, and arc set* A *composed of*

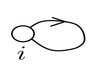
- **short up arcs** *of the form* $(i, i+1)$*, i.e.* 

,
- **long up arcs** *of the form* $(i, i+2)$*, i.e.* 

and
- **down arcs** *of the form* $(i, i-1)$*, i.e.* 

.

Then any digraph isomorphic to D_k *is a* **2-up digraph**. *When* A *also includes an extra set of arcs:*

- **loops** *of the form* (i, i)*, i.e.* 

then digraphs isomorphic to D_k are called **Lukasiewicz 2-up digraphs**.

By Comment 2 of the previous section, it is sufficient to find 2-up paving polynomials, since Lukasiewicz 2-up paving polynomials are obtainable from the former by a shift in argument.

In Figure 5.7 we illustrate an unweighted 2-up digraph, with a pair of generic cycles shown. There are two kinds: those of the form (short up step, down step) and those of the form (long up step, down step, down step); which map to dimers and trimers respectively.

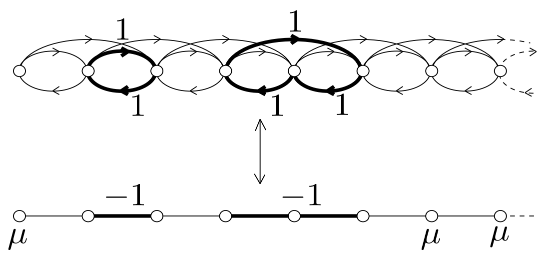


Figure 5.7: An unweighted 2-up digraph with a cycle of both possible types shown, with bijection to dimers and trimers indicated.

We define three different uniform arc weightings on 2-up digraphs, from which follow three uniform paver weightings on the path graph, as illustrated in Figures 5.8, 5.9 and 5.10.

Definition 25. Uniform 2-up standard arc weighting '1' *is*

$$w(\textit{short up arc}) = 1 \quad (5.21)$$
$$w(\textit{long up arc}) = 1 \quad (5.22)$$
$$w(\textit{down arc}) = \lambda \quad (5.23)$$

Definition 26. Uniform 2-up standard paver weighting '1' *is*

$$w(\textit{trimer}) = -\lambda^2 \quad (5.24)$$
$$w(\textit{dimer}) = -\lambda \quad (5.25)$$
$$w(\textit{0-mer}) = \mu. \quad (5.26)$$

These weights, in conjunction with the definition of a paving polynomial, give **Uniform 2-up standard weighting '1' paving polynomials**.

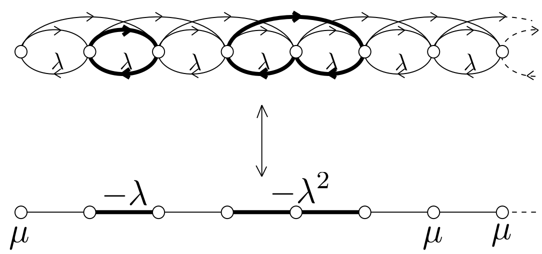


Figure 5.8: Uniform weighting '1' on a 2-up digraph with cycles bijecting to dimers and trimers.

Definition 27. Uniform 2-up standard arc weighting '2' *is*

$$w(\textit{short up arc}) = \eta \tag{5.27}$$
$$w(\textit{long up arc}) = 1 \tag{5.28}$$
$$w(\textit{down arc}) = 1 \tag{5.29}$$

Definition 28. Uniform 2-up standard paver weighting '2' *is*

$$w(\textit{trimer}) = -1 \tag{5.30}$$
$$w(\textit{dimer}) = -\eta \tag{5.31}$$
$$w(0\textit{-mer}) = \mu. \tag{5.32}$$

These weights, together with the definition of a paving polynomial, give **Uniform 2-up standard weighting '2' paving polynomials**.

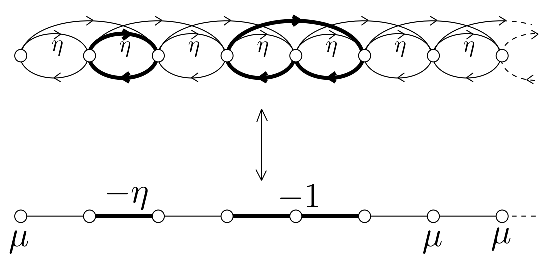


Figure 5.9: Uniform weighting '2' on a 2-up digraph with cycles bijecting to dimers and trimers.

Definition 29. Uniform 2-up standard arc weighting '3' *is*

$$w(\textit{short up arc}) = 1 \tag{5.33}$$
$$w(\textit{long up arc}) = \gamma \tag{5.34}$$
$$w(\textit{down arc}) = 1 \tag{5.35}$$

Definition 30. Uniform 2-up standard paver weighting '3' *is*

$$w(\textit{trimer}) = -\gamma \tag{5.36}$$
$$w(\textit{dimer}) = -1 \tag{5.37}$$
$$w(0\textit{-mer}) = \mu. \tag{5.38}$$

These weights, together with the definition of a paving polynomial, give **Uniform 2-up standard weighting '3' paving polynomials**.

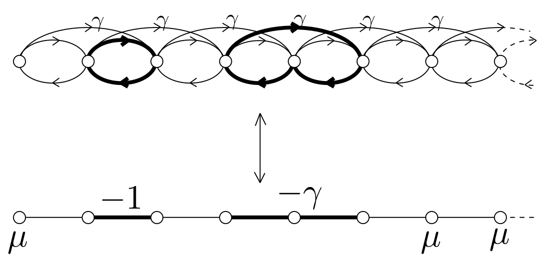


Figure 5.10: Uniform weighting '3' on a 2-up digraph with cycles bijecting to dimers and trimers.

We develop the paving recurrence for Uniform weighting '1'. The first few such 2-up paving polynomials are illustrated in Figure 5.11. Figure 5.12 illustrates the derivation of the four-term recurrence in the following Lemma.

Lemma 35. *The family, $\{P_k(\mu)\}_{k\geq 0}$, of uniform 2-up paving polynomials subject to standard weighting '1', satisfies four-term recurrence*

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda P_{k-1}(\mu) - \lambda^2 P_{k-2}(\mu) \tag{5.39}$$

with characteristic equation

$$z^3 - \mu z^2 + \lambda z + \lambda^2 = 0. \tag{5.40}$$

Thus

$$P_k = A z_1^k + B z_2^k + C z_3^k \tag{5.41}$$

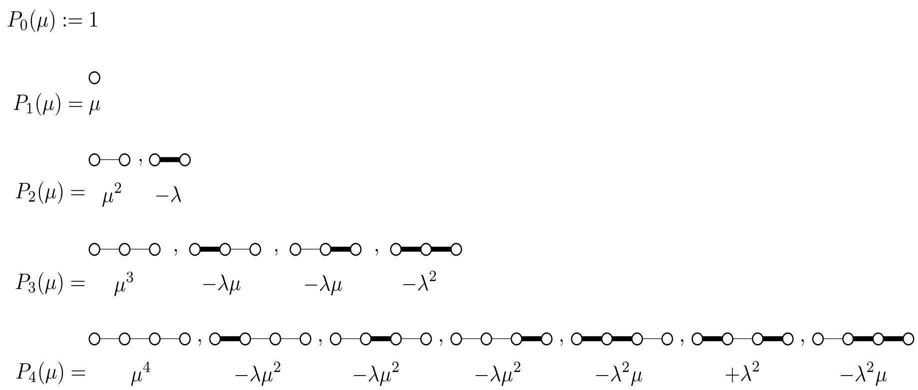
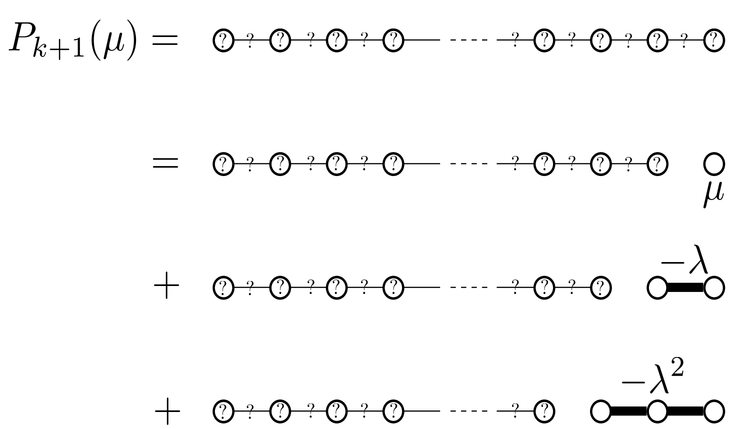


Figure 5.11: The first few 2-up polynomials under Uniform weighting '1'.



$$= \mu P_k(\mu) - \lambda P_{k-1}(\mu) - \lambda^2 P_{k-2}(\mu)$$

Figure 5.12: A four term recurrence for '2-up' paving polynomials derived using pavings subject standard weighting '1'. The question marks indicate that we are summing over all possibilities such that edges so marked are either paved or not paved.

where A, B and C are constants obtainable using the the initial conditions provided by Figure 5.11, and z_1, z_2, z_3 *are the roots of characteristic equation (5.41). Further, sending*

$$P_k(\mu) \mapsto P_k(\mu - b) \tag{5.42}$$

yields the family of uniform Lukasiewicz 2*-up paving polynomials subject to* 2*-up standard weighting '1', with the additional arc weighting on the digraph of*

$$w(\textit{loop arc}) = b; \tag{5.43}$$

and the corresponding additional paver weighting

$$w(\textit{monomer}) = -b. \tag{5.44}$$

We note that equation (5.41) may trivially be found explicitly using a computer algebra system, in terms of powers of sums of fractions of sums of cube roots of sums of square roots. But we can do better than write down the resulting cumbersome expression. Looking ahead to Chapter 8, we will find that μ is not the most convenient variable, and that 2-up paving polynomials may be written in other more 'natural' variables which yield more concise expressions.

5.4 '3-up' digraphs

Definition 31. *Let* $D_k = (V, A)$ *be a digraph of order* k *with vertex set* $V = \{0, 1, \ldots k-1\}$*, and arc set* A *composed of*

- **short up arcs** *of the form* $(i, i+1)$*,*
- **medium up arcs** *of the form* $(i, i+2)$*,*
- **long up arcs** *of the form* $(i, i+3)$ *and*
- **down arcs** *of the form* $(i, i-1)$*.*

Then any digraph isomorphic to D_k *is a* **3-up digraph**. *When* A *also includes an extra set of arcs:*

- **loops** *of the form* (i, i)*,*

then digraphs isomorphic to D_k *are called* **Lukasiewicz 3-up digraphs**.

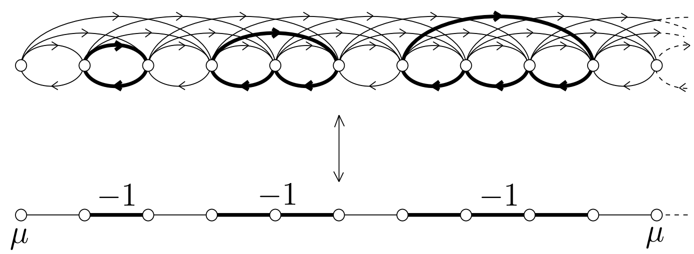


Figure 5.13: An unweighted 3-up digraph with the three generic types of cycle shown, bijecting to dimers, trimers and 4-mers.

Figure 5.13 indicates the three kinds of cycle possible on a 3-up digraph. These biject to dimers, trimers and 4-mers. Four more choices of uniform weightings are defined below and illustrated in figures 5.14 – 5.17, with more obtainable by combining those possibilities.

Definition 32. Uniform 3**-up standard arc weighting '1'** *is*

$$w(\textit{short up arc}) = 1 \quad (5.45)$$
$$w(\textit{medium up arc}) = 1 \quad (5.46)$$
$$w(\textit{long up arc}) = 1 \quad (5.47)$$
$$w(\textit{down arc}) = \lambda \quad (5.48)$$

Definition 33. Uniform 3**-up standard paver weighting '1'** *is*

$$w(\textit{4-mer}) = -\lambda^3 \quad (5.49)$$
$$w(\textit{trimer}) = -\lambda^2 \quad (5.50)$$
$$w(\textit{dimer}) = -\lambda \quad (5.51)$$
$$w(\textit{0-mer}) = \mu. \quad (5.52)$$

These weights, together with the definition of a paving polynomial, give **Uniform** 3**-up standard weighting '1' paving polynomials***.*

Definition 34. Uniform 3**-up standard arc weighting '2'** *is*

$$w(\textit{short up arc}) = \eta \quad (5.53)$$
$$w(\textit{medium up arc}) = 1 \quad (5.54)$$
$$w(\textit{long up arc}) = 1 \quad (5.55)$$
$$w(\textit{down arc}) = 1 \quad (5.56)$$

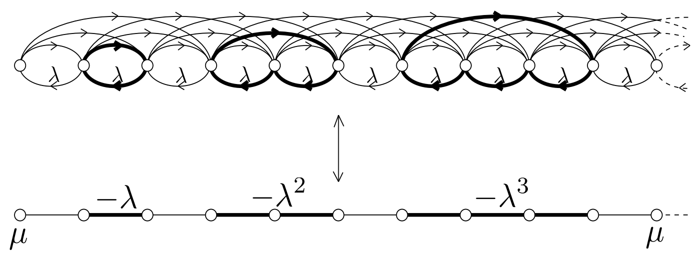


Figure 5.14: Uniform weighting '1' on a 3-up digraph with cycles bijecting to pavers.

Definition 35. Uniform 3-up standard paver weighting '2' *is*

$$w(\textit{4-mer}) = -1 \quad (5.57)$$

$$w(\textit{trimer}) = -1 \quad (5.58)$$

$$w(\textit{dimer}) = -\eta \quad (5.59)$$

$$w(\textit{0-mer}) = \mu. \quad (5.60)$$

These weights, in conjunction with the definition of a paving polynomial, give **Uniform 3-up standard weighting '2' paving polynomials**.

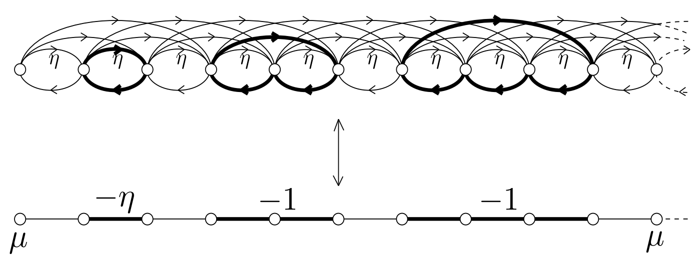


Figure 5.15: Uniform weighting '2' on a 3-up digraph with cycles bijecting to pavers.

Definition 36. Uniform 3-up standard arc weighting '3' *is*

$$w(\textit{short up arc}) = 1 \quad (5.61)$$

$$w(\textit{medium up arc}) = \gamma \quad (5.62)$$

$$w(\textit{long up arc}) = 1 \quad (5.63)$$

$$w(\textit{down arc}) = 1 \quad (5.64)$$

Definition 37. Uniform 3-up standard paver weighting '3' *is*

$$\begin{aligned} w(\textit{4-mer}) &= -1 && (5.65)\\ w(\textit{trimer}) &= -\gamma && (5.66)\\ w(\textit{dimer}) &= -1 && (5.67)\\ w(\textit{0-mer}) &= \mu. && (5.68) \end{aligned}$$

These weights, in conjunction with the definition of a paving polynomial, give **Uniform 3-up standard weighting '3' paving polynomials**.

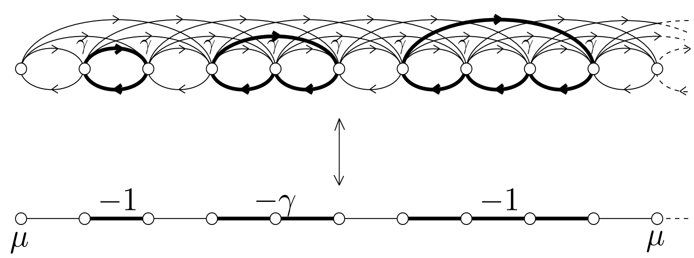


Figure 5.16: Uniform weighting '3' on a 3-up digraph with cycles bijecting to pavers.

Definition 38. Uniform 3-up standard arc weighting '4' *is*

$$\begin{aligned} w(\textit{short up arc}) &= 1 && (5.69)\\ w(\textit{medium up arc}) &= 1 && (5.70)\\ w(\textit{long up arc}) &= \delta && (5.71)\\ w(\textit{down arc}) &= 1 && (5.72) \end{aligned}$$

Definition 39. Uniform 3-up standard paver weighting '4' *is*

$$\begin{aligned} w(\textit{4-mer}) &= -\delta && (5.73)\\ w(\textit{trimer}) &= -1 && (5.74)\\ w(\textit{dimer}) &= -1 && (5.75)\\ w(\textit{0-mer}) &= \mu. && (5.76) \end{aligned}$$

These weights, in conjunction with the definition of a paving polynomial, give **Uniform 3-up standard weighting '4' paving polynomials**.

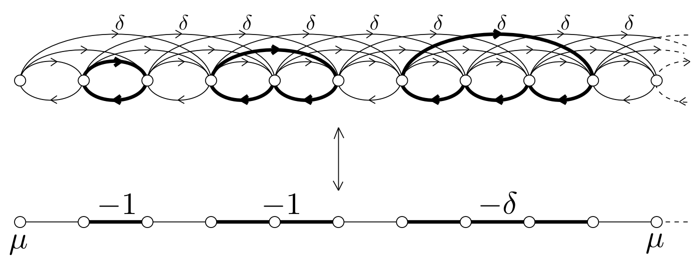


Figure 5.17: Uniform weighting '4' on a 3-up digraph with cycles bijecting to pavers.

We develop the paving recurrence for Uniform weighting '1'. Proceeding by analogy with Figure 5.12, we derive

Lemma 36. *The family,* $\{P_k(\mu)\}_{k\geq 0}$*, of uniform* 3*-up paving polynomials subject to standard weighting '1', satisfies five-term recurrence*

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda P_{k-1}(\mu) - \lambda^2 P_{k-2}(\mu) - \lambda^3 P_{k-3}(\mu) \tag{5.77}$$

with characteristic equation

$$z^4 - \mu z^3 + \lambda z^2 + \lambda^2 z + \lambda^3 = 0 \tag{5.78}$$

Thus

$$P_k = Az_1^k + Bz_2^k + Cz_3^k + Dz_4^k \tag{5.79}$$

where A*,* B*,* C *and* D *are constants and* z_1, z_2, z_3, z_4 *are the roots of characteristic equation (5.78). Further, sending*

$$P_k(\mu) \mapsto P_k(\mu - b) \tag{5.80}$$

yields the family of uniform Lukasiewicz 3*-up paving polynomials subject to* 3*-up standard weighting '1', with the additional arc weighting on the digraph of*

$$w(\textit{loop arc}) = b; \tag{5.81}$$

and the corresponding additional paver weighting

$$w(\textit{monomer}) = -b. \tag{5.82}$$

Note that solving characteristic equation (5.78) for z gives four roots, each of which is a long expression in μ (and λ) involving sums, fractions, square and cubic roots. The roots z_1, z_2, z_3 and z_4 may be simplified by the methods of Chapter 8.

5.5 'd-up' digraphs

Definition 40. *Let $D_k = (V, A)$ be a digraph with vertex set $V = \{0, 1, \dots k-1\}$, and arc set A composed of*

- ***u-up arcs** of the form $(i, i+u)$, for values of u in $\{1, \dots d\}$; and*
- ***down arcs** of the form $(i, i-1)$.*

*Then any digraph isomorphic to D_k is a **d-up digraph**. When A also includes an extra set of arcs:*

- ***loops** of the form (i, i),*

*then digraphs isomorphic to D_k are called **Lukasiewicz d-up digraphs**.*

We define one of many possible uniform weightings.

Definition 41. Uniform d-up standard arc weighting '1' *is*

$$w(u\text{-}up\ arc) = 1 \tag{5.83}$$
$$w(down\ arc) = \lambda \tag{5.84}$$

Definition 42. Uniform d-up standard paver weighting '1' *is*

$$w(u\text{-}mer) = -\lambda^{u-1} \text{ for } u \geq 2 \tag{5.85}$$
$$w(0\text{-}mer) = \mu. \tag{5.86}$$

*These weights, in conjunction with the definition of a paving polynomial, give **Uniform d-up standard weighting '1' paving polynomials**.*

Then

Lemma 37. *The family, $\{P_k(\mu)\}_{k\geq 0}$, of uniform d-up paving polynomials subject to standard weighting '1', satisfies $(d+2)$-term recurrence*

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda P_{k-1}(\mu) - \lambda^2 P_{k-2}(\mu) - \dots - \lambda^d P_{k-d}(\mu) \tag{5.87}$$

with characteristic equation

$$z^{d+1} - \mu z^d + \lambda z^{d-1} + \lambda^2 z^{d-2} + \dots + \lambda^d = 0. \tag{5.88}$$

Thus

$$P_k = \sum_{i=1}^{d+1} A_i z_i^k \tag{5.89}$$

where the A_i are constants and the z_i are the roots of the characteristic equation (5.88). Further, sending

$$P_k(\mu) \mapsto P_k(\mu - b) \tag{5.90}$$

yields the family of uniform Lukasiewicz d-up paving polynomials subject to d-up standard weighting '1', with the additional arc weighting on the digraph of

$$w(\textit{loop arc}) = b; \tag{5.91}$$

and the corresponding additional paver weighting

$$w(\textit{monomer}) = -b. \tag{5.92}$$

Comment 3. *For $d \geq 4$ the characteristic polynomial of Equation (5.88) is a quintic or higher degree polynomial. By Abel's Impossibility Theorem, there is no finite algorithm using just additions, subtractions, multiplications, divisions, and root extractions by which to find all roots of such polynomials, [106]. Thus, although we may be lucky in certain special cases, in general the best we can expect is an approximate family of uniform d-up paving polynomials $\{P_k(\mu)\}_{k\geq 0}$, built out of numerical estimates for $A_i(\mu)$ and $z_i(\mu)$.*

5.6 'Mixed-up' digraphs

Definition 43. *Let $\{h_1, \ldots, h_l\}$ be any set of positive integers ordered such that $h_1 < \ldots < h_l$. Let $D_k = (V, A)$ be a digraph with vertex set $V = \{0, 1, \ldots k-1\}$, and arc set A composed of*

- ***u*-up arcs** *of the form $(i, i+u)$, for values of u in $\{h_1, h_2, \ldots h_l\}$; and*
- **down arcs** *of the form $(i, i-1)$.*

Then any digraph isomorphic to D_k is a **mixed-up digraph**. *When A also includes an extra set of arcs:*

- **loops** *of the form (i, i),*

digraphs isomorphic to D_k are called **Lukasiewicz mixed-up digraphs**.

The significance of mixed-up digraphs and their cousins Lukasiewicz mixed-up digraphs is that this is the most general class that still bijects to pavings - see the example in Figure 5.18.

These classes of digraphs generalize those in Sections 5.1 – 5.5, and are the last of the digraphs in this chapter with only one kind of down arc. The preceding special cases comprise most of those instances of mixed-up digraphs for which calculating a closed form for the paving polynomial is tractable. A more generic example is illustrated in Figure 5.18. In this

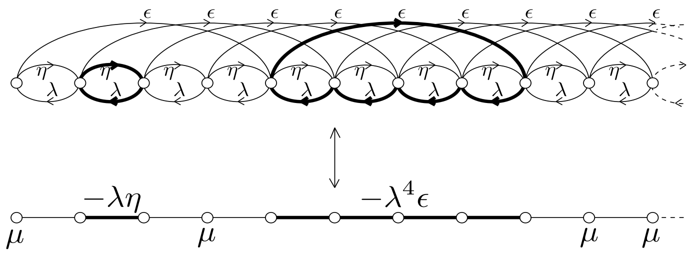


Figure 5.18: A mixed-up digraph with up edges of the form $(k, k+1)$ and $(k, k+4)$. The two kinds of cycles that may exist on this digraph map to dimers and 5-mers. The most general background weighting on the edges of the digraph is shown, from which dimers and 5-mers inherit independent weights $-\alpha := -\lambda\eta$ and $-\beta := -\lambda^4\epsilon$ respectively.

example, arcs are of the forms $(k, k+1)$, $(k, k+4)$ and $(k, k-1)$. These result in two kinds of cycles which biject to dimers with weight $-\alpha$ and 5-mers with weight $-\beta$. Reasoning by analogy with Figure 5.12 we derive paving polynoimal recurrence:

$$P_{k+1}(\mu) = \mu P_k(\mu) - \alpha P_{k-1}(\mu) - \beta P_{k-4}(\mu) \tag{5.93}$$

and characteristic equation

$$z^5 - \mu z^4 + \alpha z^3 + \beta = 0 \tag{5.94}$$

for this mixed-up digraph. This quintic characteristic equation is amongst the class which cannot be solved by a finite number of additions, subtractions, multiplications, divisions, and root extractions. In general,

Lemma 38. *Any family, $\{P_k(\mu)\}_{k\geq 0}$, of mixed-up paving polynomials subject to any uniform arc weighting satisfies some (h_l+2)-term recurrence, where h_l is the maximum value of u such that non-zero weighted up arcs of the form $(i, i+u)$ exist in the mixed up digraph. The recurrence has structure*

$$P_{k+1}(\mu) = \mu P_k(\mu) - c_{k-1} P_{k-1}(\mu) - \ldots - c_{k-h_l} P_{k-h_l}(\mu) \tag{5.95}$$

for constants $c_0, \dots c_{k-h_l}$*, and characteristic equation*

$$z^{h_l+2} - \mu z^{h_l+1} + c_{h_l} z^{h_l} \dots + c_0 = 0. \tag{5.96}$$

Notice that the characteristic polynomials of Equation (5.96) always contain a term of degree one less than their order. This feature tends to make finding exact roots difficult, for higher order polynomials.

5.7 Jump 2-step digraphs

With 'jump step' digraphs come a proliferation of cycles. This is where the mapping to pavings becomes something better than just a conveniently flattened notation for cycles, as we can now combine several cycles into a single weighted paver, which cuts the complexity of combinatorial calculations.

Definition 44. *Let* $D_k = (V, A)$ *be a digraph of order* k *with vertex set* $V = \{0, 1, \dots k-1\}$*, and arc set* A *composed of*

- **short up arcs** *of the form* $(i, i+1)$*,*
- **long up arcs** *of the form* $(i, i+2)$*,*
- **short down arcs** *of the form* $(i, i-1)$ *and*
- **long down arcs** *of the form* $(i, i-2)$*,*

Then any digraph isomorphic to D_k *is a* **Jump 2-step digraph***. When* A *also includes an extra set of arcs:*

- **loops** *of the form* (i, i)*,*

then digraphs isomorphic to D_k *are called* **L-Jump 2-step digraphs***.*

The Jump 2 digraph can support an infinity of distinct kinds of cycles, of increasing length, the first few of which are illustrated in Figure 5.19. The figure shows unweighted arcs on the digraph, which we call the 'trivial weighting'.

Definition 45. The trivial uniform Jump-2 arc weighting*, also called* **standard weighting '0'***, is*

$$w(\text{short up arc}) = 1 \tag{5.97}$$
$$w(\text{long up arc}) = 1 \tag{5.98}$$
$$w(\text{short down arc}) = 1 \tag{5.99}$$
$$w(\text{long down arc}) = 1 \tag{5.100}$$

The trivial weighting on arcs leads to a weighting with a more interesting structure on pavers. We have

Definition 46. Standard uniform Jump-2 paver weighting '0' *is*

$$w(0\text{-}mer) = \mu \tag{5.101}$$
$$w(dimer) = -1 \tag{5.102}$$
$$w(trimer) = -2-\mu \tag{5.103}$$
$$w(4\text{-}mer) = -1 \tag{5.104}$$
$$w(n\text{-}mer) = -2 \text{ for } n \geq 5. \tag{5.105}$$

These paver weights, together with the definition of a paving polynomial, give **uniform standard weighting '0' Jump-2 paving polynomials**.

The derivation of this weightings on pavers comes from the projection of cycles illustrated in Figure 5.19 onto a path graph.

- There is just one cycle structure, namely (short up, short down), which projects to a dimer. The cycle has weight $(-1)(1)(1)$, thus dimers are assigned weight -1.
- There are three cycle structures which project to a trimer. These have weights $(-1)(1)(1)(1)$, $(-1)(1)(1)(1)$ and $(-1)(1)(1)(\mu)$ respectively; where the last of the trio collects weight μ from the uncovered vertex in the middle. Adding together the three possibilities, trimers are assigned weight $-2-\mu$, so that the weights of all three cycle structures are counted at once.
- There are three cycle configurations which project to a 4-mer. The first two consist of a single cycle each, and both carry weight -1. The third consists of a pair of cycles both of the form (long up, long down) which overlap as illustrated, but are a pair of distinct cycles since they share no vertices. Each member of the pair has weight -1 so combined the pair has weight $(-1)^2 = +1$. Summing together the weights of the three configurations, 4-mers are assigned weight $-1-1+1=-1$.
- There are precisely two cycle structures which project to an n-mer, for $n \geq 5$. Both consist of a single cycle with weight -1. Adding these together gives weight $-1-1=-2$ for any n-mer such that $n \geq 5$.

The first few Jump-2 paving weight polynomials (subject to standard uniform weighting '0') are calculated combinatorially in Figure 5.20. Collecting

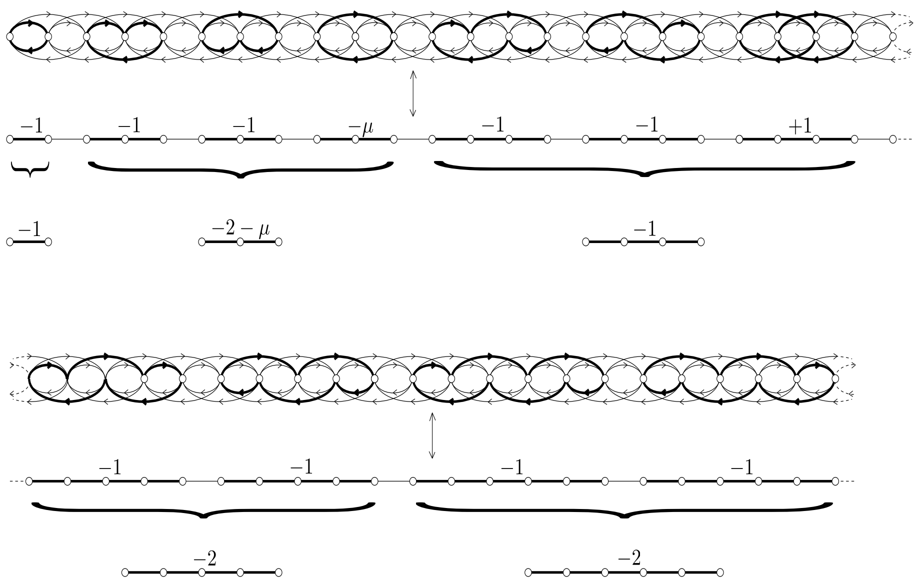


Figure 5.19: A Jump 2 step digraph. Cycles of arbitrary length are supported by this digraph.

terms from Figure 5.20, and extending the list a little, gives

$$P_0(\mu) := 1 \tag{5.106}$$
$$P_1(\mu) = \mu \tag{5.107}$$
$$P_2(\mu) = \mu^2 - 1 \tag{5.108}$$
$$P_3(\mu) = \mu^3 - 3\mu - 2 \tag{5.109}$$
$$P_4(\mu) = \mu^4 - 5\mu^2 - 4\mu \tag{5.110}$$
$$P_5(\mu) = \mu^5 - 7\mu^3 - 6\mu^2 + 3\mu + 2 \tag{5.111}$$
$$P_6(\mu) = \mu^6 - 9\mu^4 - 8\mu^3 + 10\mu^2 + 12\mu + 3 \tag{5.112}$$

The presence of pavers of arbitrary length would seem to bode ill for a fixed order recurrence relation for Jump 2 paving polynomials, but there is a pleasant surprise in store. Start by getting a $k+1$ term recurrence using Figure 5.21. Write this result down twice, with staggered coefficients:

$$P_{k+1} = \mu P_k - P_{k-1} - (2+\mu)P_{k-2} - P_{k-3} - 2P_{k-4} - 2P_{k-5} - \ldots - 2 \tag{5.113}$$
$$P_k = \mu P_{k-1} - P_{k-2} - (2+\mu)P_{k-3} - P_{k-4} - 2P_{k-5} - 2P_{k-6} - \ldots - 2 \tag{5.114}$$

Take the difference between Equations (5.113) and (5.114). We obtain a short recurrence for the P_k's in Lemma 39.

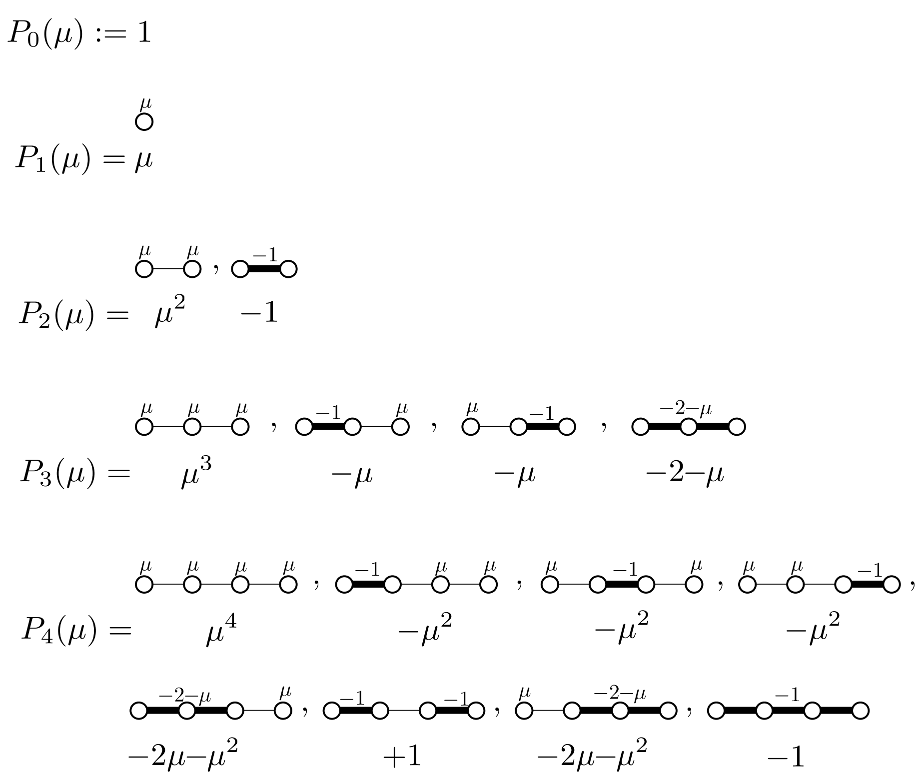
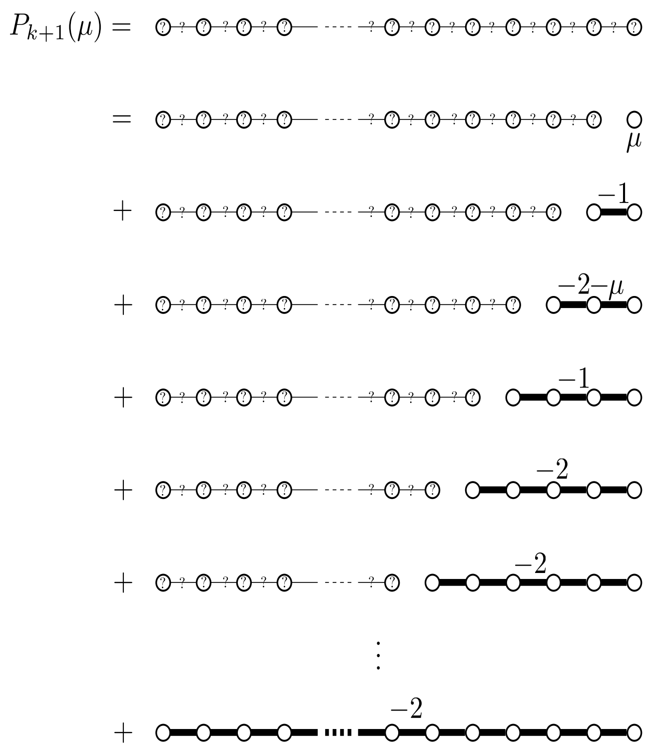


Figure 5.20: The first few jump 2 polynomials.



$$= \mu P_k(\mu) - P_{k-1}(\mu) - (2+\mu)P_{k-2}(\mu) - P_{k-3}(\mu)$$
$$-2P_{k-4}(\mu) - 2P_{k-5}(\mu) - ... - 2$$

Figure 5.21: Breaking up the paving according to the possibilities for the last vertex gives a $(k+2)$ - term linear recurrence for Jump 2 step paving polynomials.

Lemma 39. *The family, $\{P_k(\mu)\}_{k\geq 0}$, of uniform jump-2 paving polynomials subject to standard weighting '0', satisfies a 6-term recurrence:*

$$P_{k+1} = (\mu+1)P_k - (\mu+1)P_{k-1} - (\mu+1)P_{k-2} + (\mu+1)P_{k-3} - P_{k-4} \quad (5.115)$$

with characteristic equation

$$z^5 - (\mu+1)z^4 + (\mu+1)z^3 + (\mu+1)z^2 - (\mu+1)z + 1 = 0, \quad (5.116)$$

which has five solutions

$$z = -1, z = \frac{1}{4}\left(2 \pm \sqrt{\mu^2 - 4\mu} \pm \sqrt{2}\sqrt{-10 - 4\mu - \sqrt{\mu^2 - 4\mu}(2+\mu) + (2+\mu)^2}\right). \quad (5.117)$$

Thus

$$P_k = \sum_{i=1}^{5} A_i z_i^k \quad (5.118)$$

where the roots z_i are the members of Equation (5.117), and the A_i are constants obtainable from the initial conditions given by Equations (5.106) – (5.112). Further, sending

$$P_k(\mu) \mapsto P_k(\mu - b) \quad (5.119)$$

yields the family of uniform Lukasiewicz 2*-up paving polynomials subject to* 2*-up standard weighting '1', with the additional arc weighting on the digraph of*

$$w(loop\ arc) = b; \quad (5.120)$$

and the corresponding additional paver weighting

$$w(monomer) = -b. \quad (5.121)$$

Comment 4. *The characteristic polynomial of Equation (5.116) is noteworthy in two ways.*

1. *The polynomial is a quintic for which we may nonetheless find roots exactly; and*

2. *The polynomial is linear in μ. We will use this fact in Chapter 8 to find a new variable in which the above five roots have terser incarnation.*

5.8 Jump 3-step digraphs

Definition 47. *Let* $D_k = (V, A)$ *be a digraph of order* k *with vertex set* $V = \{0, 1, \ldots k-1\}$*, and arc set* A *composed of*

- **short up arcs** *of the form* $(i, i+1)$*,*
- **medium up arcs** *of the form* $(i, i+2)$*,*
- **long up arcs** *of the form* $(i, i+3)$*,*
- **short down arcs** *of the form* $(i, i-1)$*,*
- **medium down arcs** *of the form* $(i, i-2)$ *and*
- **long down arcs** *of the form* $(i, i-3)$*,*

Then any digraph isomorphic to D_k *is a* **Jump 3-step digraph***. When* A *also includes an extra set of arcs:*

- **loops** *of the form* (i, i)*,*

then digraphs isomorphic to D_k *are called* **L-Jump 3-step digraphs***.*

As in the Jump-2 case, we consider here only the trivial arc weighting.

Definition 48. The trivial uniform Jump-3 arc weighting*, or* **standard weighting '0'***, is*

$$w(\textit{short up arc}) = 1 \quad (5.122)$$
$$w(\textit{medium up arc}) = 1 \quad (5.123)$$
$$w(\textit{long up arc}) = 1 \quad (5.124)$$
$$w(\textit{short down arc}) = 1 \quad (5.125)$$
$$w(\textit{medium down arc}) = 1 \quad (5.126)$$
$$w(\textit{long down arc}) = 1 \quad (5.127)$$

Jump-3 step digraphs support a multitude of kinds and configurations of cycles. The process of grouping together and adding up the weights of all the cycles which project onto a specific n-mer, becomes a combinatorial problem in its own right, which we have not solved for general n. We merely calculate the weights that the n-mers inherit from the cycles for $n = 1, 2, ..., 7$.

Definition 49. *The first few paver weights for* **Standard uniform Jump-3 paver weighting '0'** *are*

$$\begin{aligned}
w(0\text{-}mer) &= \mu & (5.128)\\
w(dimer) &= -1 & (5.129)\\
w(trimer) &= -2-\mu & (5.130)\\
w(4\text{-}mer) &= -4-4\mu-\mu^2 & (5.131)\\
w(5\text{-}mer) &= -4-3\mu & (5.132)\\
w(6\text{-}mer) &= -8-8\mu-\mu^2 & (5.133)\\
w(7\text{-}mer) &= -14-20\mu-6\mu^2 & (5.134)
\end{aligned}$$

We define higher order n-mers consistently with Equations (5.128)–(5.134) procedurally, by

$$w(n\text{-}mer) = \sum_{C\in R_n} w(C), \tag{5.135}$$

where R_n is the set of configurations of cycles, C, which project to an n-mer. The weight of a configuration, $w(C)$, is the product of weights of cycles in the configuration, i.e.

$$W(C) = \prod_{c\in C} w(c), \tag{5.136}$$

where we recall that the weight of a cycle $w(c)$, is given by the product of the weights of the arcs in the cycle (including uncovered vertices which are 0-arcs) multiplied by -1. These paver weights, together with the definition of a paving polynomial, specify the **uniform standard weighting '0' Jump-3 paving polynomials**.

Comment 5. *Paver weights for standard uniform jump-3 paver weighting '0' are polynomials in μ which are at most quadratic, since there can be no cycle configuration in the jump-3 digraph which both projects to an n-mer and contains 3 or more uncovered vertices.*

Figures 5.22, 5.23 and 5.24 illustrate the derivation of the weights for dimers, trimers, 4-mers and 5-mers. Comparison between them shows the rapid growth, with n, in numbers of cycle configurations that map to a given n-mer. This growth makes the problem of assigning weights to general n-mers more difficult, but would also mean that if a general expression were found then the combinatorial complexity saved by counting pavings instead of cycles would be large.

In Figure 5.25 we generate the first few paving polynomials.

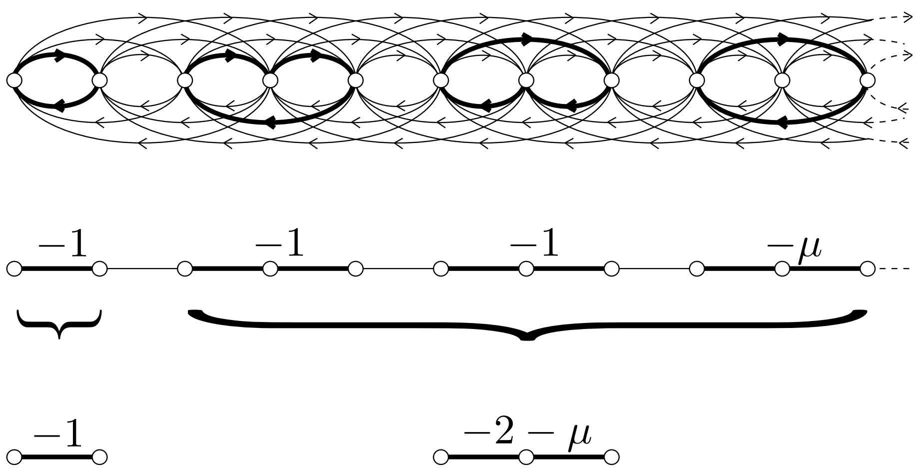


Figure 5.22: Cycles that map to dimers and trimers are the same as for Jump 2 step digraphs.

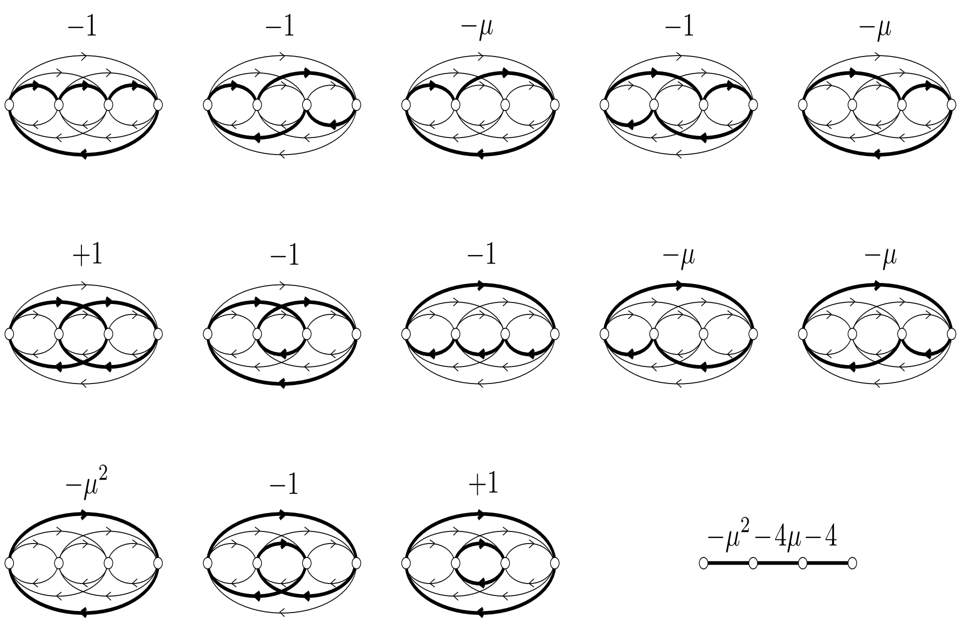


Figure 5.23: These thirteen cycle configurations on the Jump 3 step digraph coalesce into a single 4-mer with weight $-\mu^2 - 4\mu - 4$.

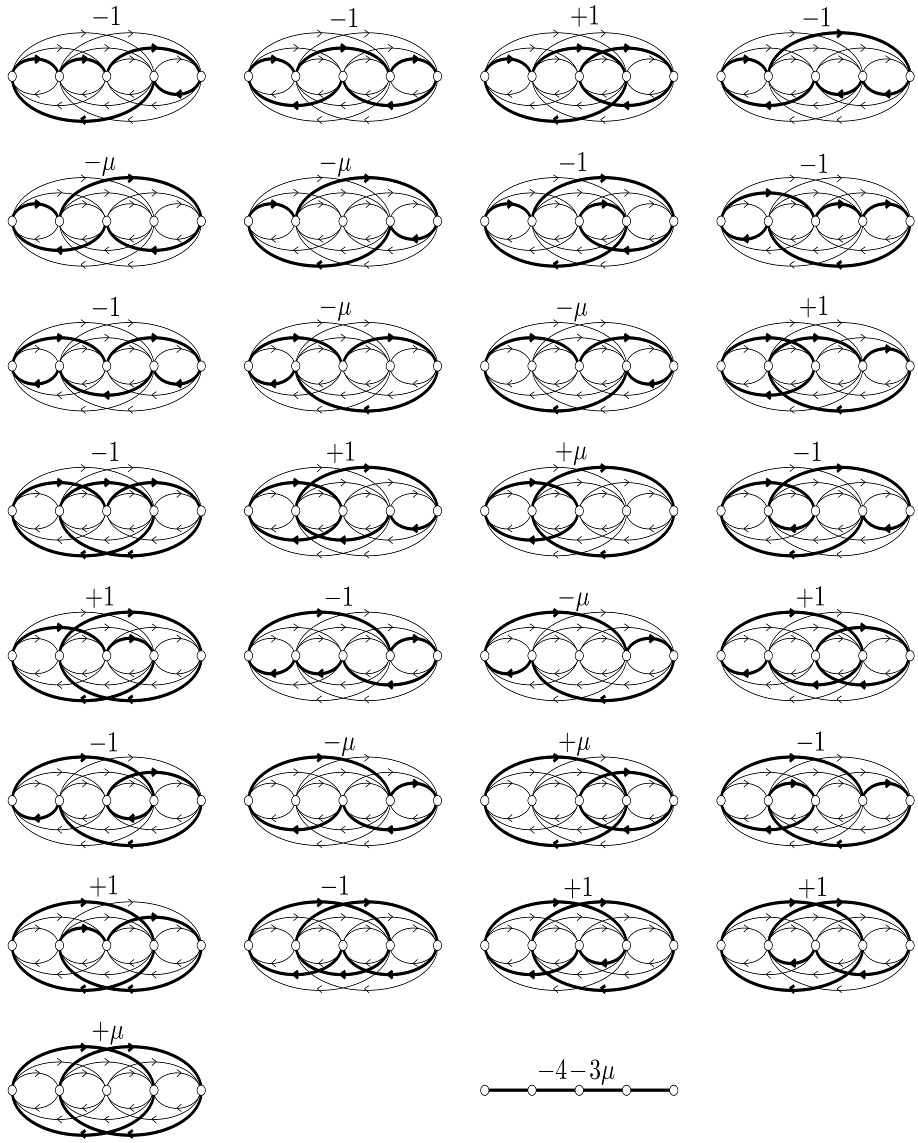


Figure 5.24: To make sure of capturing all twenty-nine of these cycle configurations Mathematica code was written which generated all permutations of $\{1, 2, 3, 4, 5\}$ (since each such permutation labels a configuration of cycles on the Jump 3 step digraph) and then pared away those which did not project to a 5-mer.

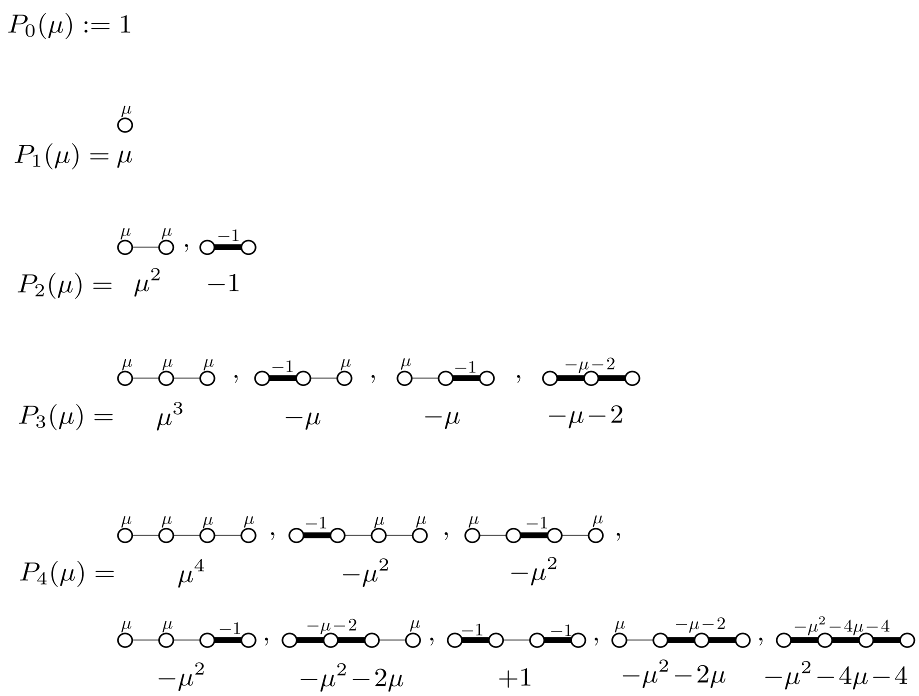


Figure 5.25: The first few Jump 3 step paving polynomials.

Lemma 40. *Let* $\{P_k(\mu)\}_{k\geq 0}$ *be the family of uniform jump-3 paving polynomials subject to standard uniform weighting '0'. Then the first few such polynomials are*

$$\begin{aligned}
P_0(\mu) &= 1 && (5.137)\\
P_1(\mu) &= \mu && (5.138)\\
P_2(\mu) &= \mu^2 - 1 && (5.139)\\
P_3(\mu) &= \mu^3 - 3\mu - 2 && (5.140)\\
P_4(\mu) &= \mu^4 - 6\mu^2 - 8\mu - 3 && (5.141)\\
P_5(\mu) &= \mu^5 - 9\mu^3 - 14\mu^2 - 6\mu && (5.142)\\
P_6(\mu) &= \mu^6 - 12\mu^4 - 20\mu^3 - 4\mu^2 + 8\mu + 3 && (5.143)\\
P_7(\mu) &= \mu^7 - 15\mu^5 - 26\mu^4 + 6\mu^3 + 38\mu^2 + 23\mu + 4. && (5.144)
\end{aligned}$$

Note that these polynomials factor as follows.

$$\begin{aligned}
P_2(\mu) &= (\mu - 1)(\mu + 1) && (5.145)\\
P_3(\mu) &= (\mu - 2)(\mu + 1)^2 && (5.146)\\
P_4(\mu) &= (\mu - 3)(\mu + 1)^3 && (5.147)\\
P_5(\mu) &= (\mu + 1)^2\mu(\mu^2 - 2\mu - 6) && (5.148)\\
P_6(\mu) &= (\mu + 1)(\mu^2 + \mu - 1)(\mu^3 - 2\mu^2 - 8\mu - 3) && (5.149)\\
P_7(\mu) &= (\mu^3 + \mu^2 - 2\mu - 1)(\mu^4 - \mu^3 - 12\mu^2 - 15\mu - 4). && (5.150)
\end{aligned}$$

The pattern which emerges and then ceases between orders 2 and 4 occurs in a more sustained way in the section on 'Jump-any digraphs', where its origins are explicated.

5.9 Jump Any-step digraphs and an Involution

Definition 50. *Let* $D_k = (V, A)$ *be a digraph of order* k *with vertex set* $V = \{0, 1, \ldots k-1\}$, *and arc set* A *composed of*

- u**-up arcs** *of the form* $(i, i+u)$ *and*
- d**-down arcs** *of the form* $(i, i-d)$,

for $u, d \in \mathbb{N}$. *Then any digraph isomorphic to* D_k *is a* **Jump Any-step digraph**. *When* A *also includes an extra set of arcs:*

- **loops** *of the form* (i, i),

then digraphs isomorphic to D_k are called **L-Jump Any-step digraphs***.*

Here we consider only the trivial arc weighting.

Definition 51. *The* **trivial uniform Jump-Any arc weighting***, or* **standard weighting '0'** *is*

$$w(u\text{-}up\ arc) = 1 \quad (5.151)$$
$$w(d\text{-}down\ arc) = 1. \quad (5.152)$$

The trivial uniform Jump-Any arc weighting extends to the **trivial uniform L-Jump-Any arc weighting***, or* **standard weighting '0' for L-Jump-Any arcs***, by the addition of the uniform weight on loops*

$$w(arc\ of\ the\ form\ (i,i)) = b. \quad (5.153)$$

One might expect Jump Any-step digraphs to pose arduous difficulties, given that it took only three kinds of up and down edges in the last section to make cycles proliferate and counting them prohibitive. It turns out that adding more complexity lets us cancel a lot of it out, so that Jump Any-step digraphs are easier than all the rest. We derive paver-weighting consistent with the trivial arc weighting in Subsection 5.9.1. We obtain

Definition 52. Standard uniform Jump-Any paver weighting '0' *is*

$$w(0\text{-}mer) = \mu \quad (5.154)$$
$$w(n\text{-}mer) = -(\mu+2)^{n-2}\ for\ n \geq 2. \quad (5.155)$$

These paver weights, together with the definition of a paving polynomial, give **uniform standard weighting '0' Jump-Any paving polynomials***.*

5.9.1 An iterated set of involutions to find paving weights

We begin with the 2-cycle on a pair of adjacent vertices. This cycle projects to a dimer. We can extend it to project to a trimer in three different ways. These are illustrated in Figure 5.26.

The same trick applied again in Figure 5.27 gives most of the cycle configurations the next size up, but not all. The configurations we didn't get are interesting in two ways. The first is that their weights cancel in pairs. The second is that they are more complicated-looking than those which were generated by the three extension operations.

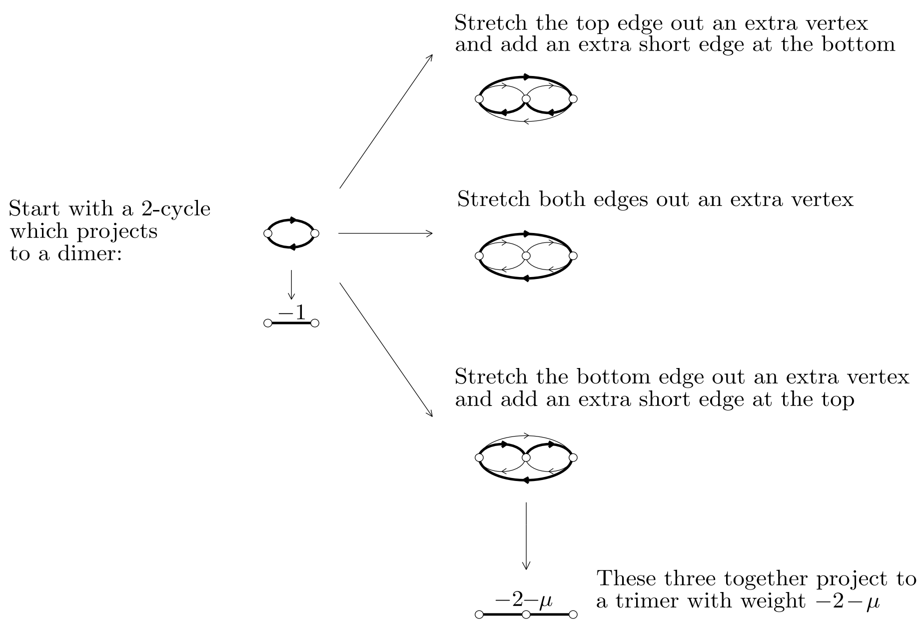


Figure 5.26: Starting with a 2-cycle on adjacent vertices, we create all those cycles which project to a trimer by operating in three possible ways on our initial cycle.

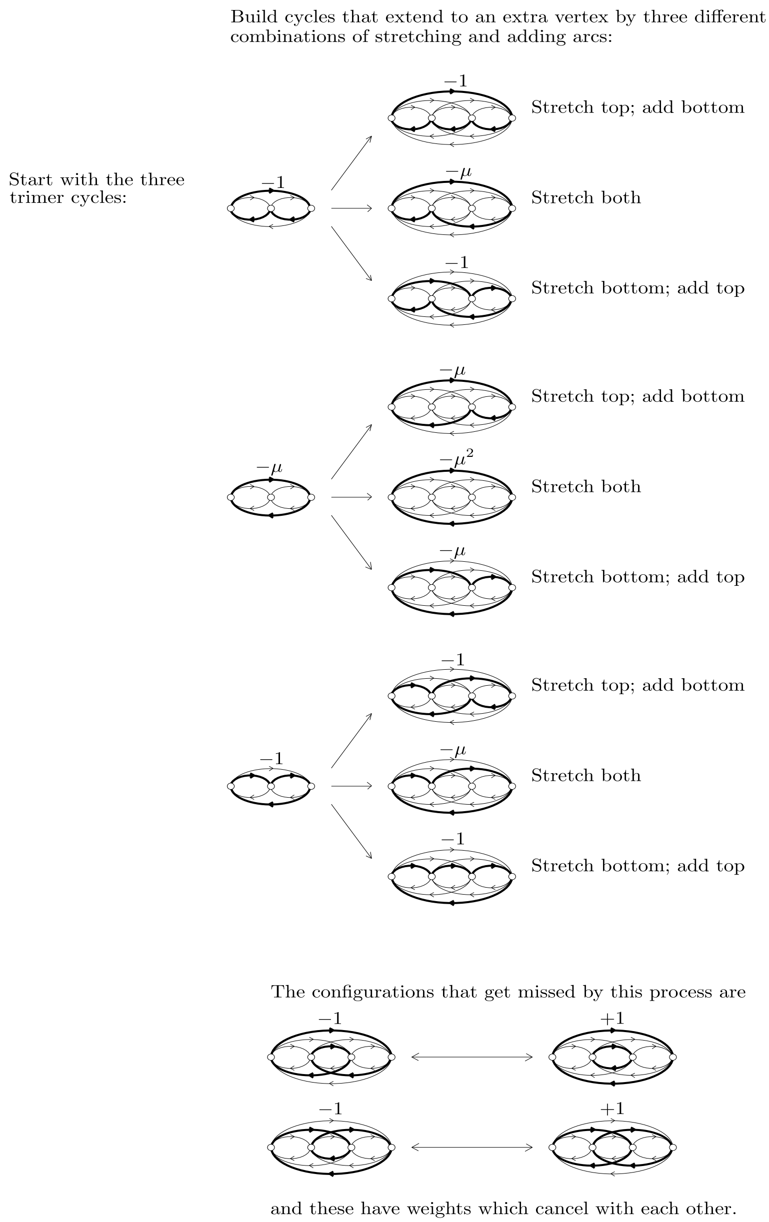


Figure 5.27: The cycles that project to 4-mers come in two groups: those which may be generated by extending spanning cycle on one fewer vertices, and the exceptional remainder. This remainder group cancel with each other.

It turns out that the features present in Figure 5.27 hold across cycle configurations projecting to n-mers for all n. The simple-looking configurations, which will always be single cycles of the form (up, ..., up, down, ..., down), will always come from stretching simple configurations of the same form on one less vertex. The more complicated configurations will always cancel in pairs. Thus three stretching operations ensure that the number of cycles contributing to the weight of an n-mer, after the oppositely weighted ones have been cancelled out, will be 3^{n-2} for all $n \geq 2$.

The next few pages are devoted to checking these claims and consequently showing that the weight of an n-mer is $-(\mu + 2)^{n-2}$ for all $n \geq 2$.

We define a **spanning (cycle) configuration on n vertices** to be a cycle configuration on a digraph whose vertices can be labeled $1, 2, ..., n$, such that the cycle configuration projects to an n-mer, i.e. if there are multiple cycles in the configuration then they overlap so that there are no gaps between them. We define a **set of contributing configurations to an n-mer**, C_n, to be a subset of the set of spanning configurations on n vertices, S_n, such that the sum of the weights of the elements of C_n is equal to the sum of the weights of the elements of S_n but no element of C_n has opposite weight to any other element of C_n. In analogy with the concept of a basis, a set of contributing configurations is not necessarily unique.

We formally define the stretch operations as follows. These operations apply to spanning configurations on n vertices; and produce spanning configurations on $n+1$ vertices. The **Stretch top; add bottom** operation replaces the unique edge of the form (p, n) with the edge $(p, n+1)$, as well as inserting the edge $(n+1, n)$ to re-complete the cycle. Since we are on a Jump Any-step digraph the underlying edges are there to chose. The **Stretch both** operation replaces the unique edges of the form (p, n) and (n, q) respectively with $(p, n+1)$ and $(n+1, q)$. The **Stretch bottom; add top** operation replaces the unique edge of the form (n, q) with the edge $(n+1, q)$, as well as inserting the edge $(n, n+1)$ to re-complete the cycle.

We want to show that the members of the set of spanning cycle configurations on $k+1$ vertices always either come from stretching a spanning configuration on k vertices, or they cancel in pairs. To this end, consider all spanning configurations on $k+1$ vertices labeled $1, 2, ..., k+1$. Focus on the second last vertex. The cases are:

1. The vertex k is not part of a cycle in the configuration:

   Thus edges $(k, k+1)$ and $(k+1, k)$ are not included in the configuration. But since the configuration is spanning, we know that edges $(p, k+1)$ and $(k+1, q)$ must be present for some $p < k$ and some $q < k$.

Thus we may transform the configuration by 'shrinking' the last two mentioned edges. i.e. Replace marked edge $(p, k+1)$ with (p, k); and replace marked edge $(k+1, q)$ with (k, q). This new configuration is a spanning configuration of cycles on the Jump Any-step digraph with vertices labeled $1, 2, ..., k$.

Conversely, any spanning configuration of cycles on vertices $1, 2, ..., k$ may be extended to a spanning configuration on $k+1$ vertices by 'stretching' the edges; i.e. replacing k where it appears with $k+1$.

Thus configurations which fall into this case may all be generated by the 'Stretch both' operation indicated in Figure 5.27, applied to cycles which span k vertices.

2. The edge $(k, k+1)$ is part of a cycle in the configuration:

   Therefore the edge $(k+1, k)$ cannot be present unless $k = 1$, because if it was, there would be a 'gap' between vertices $k-1$ and k, and the configuration would not be a spanning configuration. Thus for $k > 1$, there must be an edge present of the form $(k+1, q)$, where $q < k$.

   Hence we may perform the inverse of the "Stretch bottom; add top" operation indicated in Figure 5.27 to obtain a cycle configuration which spans one fewer vertices. Conversely, applying the "Stretch bottom; add top" operation to any spanning configuration on k vertices gives a spanning configuration with the edge $(k, k+1)$ present in a cycle.

   Thus, for $k > 1$, configurations which fall into this case may all be generated by the "Stretch bottom; add top" operation indicated in Figure 5.27, applied to cycles which span k vertices.

3. The edge $(k+1, k)$ is part of a cycle in the configuration:

   By reasoning mutatis mutandis to the previous case, for $k > 1$, configurations which fall into this case may all be generated by the "Stretch top; add bottom" operation indicated in Figure 5.27, applied to cycles which span k vertices.

4. An edge of the form (j, k), for $j \leq k-1$, is part of the cycle configuration and the edges $(k, k+1)$ and $(k+1, k)$ are not:

   For this case to occur we must have $k+1 \geq 4$.

   Then an edge of the form (k, i) must be present for $i \leq k-1$. Reason: the edge exiting k has got to go backwards because it can't go forwards.

Also, edges of the form $(p, k+1)$ and $(k+1, q)$ must be present for $p \leq k-1$ and $q \leq k-1$. Reason: edges entering and exiting '$k+1$' have got to come from and go to somewhere.

These four cases comprise all of the possibilities. Cases 1–3 were all shown to be derived by stretching operations. We need to show that all members of Case 4 cancel in pairs.

To this end, define an operation **crossuncross** on cycle configurations belonging to Case 4. Such configurations always contain an edge of the form (k, i), for $i \leq k-1$; and one of the form $(k+1, q)$, for $q \leq k-1$. 'Crossuncross' replaces this pair of edges in the cycle configuration by a new pair

$$(k, i), (k+1, q) \mapsto (k+1, i), (k, q). \tag{5.156}$$

Clearly,

$$\text{crossuncross}^2 = \text{Identity} \tag{5.157}$$

We argue that 'crossuncross' always increases or decreases the number of cycles by one as follows. Suppose that the edges (k, i) and $(k+1, q)$ are each part of disjoint cycles. Then there exists a path $p_{i,k}$ from i to k as well as a path $p_{q,k+1}$ from q to $k+1$. After applying 'crossuncross', $p_{i,k}(k, q)p_{q,k+1}(k+1, i)$ constitutes a single cycle. Hence the number of cycles has been decreased by one. A similar argument shows that if edges (k, i) and $(k+1, q)$ start out part of the same cycle, 'crossuncross' splits them into exactly two distinct cycles.

Since 'crossuncross' changes only the parity of the number of cycles and does not change the number of uncovered vertices, it multiplies the weight of a configuration by -1. Since each configuration maps to one whose weight is the negative of its own, the involution 'crossuncross' cancels the weights of all configurations in Case 4.

Having shown that all spanning configurations either came from stretching operations or else cancel with each other, we have shown that iteratively applying the three stretching operations, starting with $n = 2$ and the 2-cycle will for each iteration generate a set of contributing configurations for any n-mer.

To check the weights, note that Figures 5.26 and 5.27 show that for $n = 2, 3, 4$ the contributing configurations have total weights respectively -1, $-(\mu+2)$ and $-(\mu+2)^2$. All that remains is to check that the application of the three stretching operations increases the weight of the set by a factor of $\mu + 2$ for each iteration. This follows directly from the definition of the stretch operations. The first and last of these three operations don't change the number of uncovered vertices, and applied to a single cycle they generate a single cycle. Hence these two leave the weight invariant. The middle operation introduces an extra uncovered vertex, but does not change the number of cycles either. Therefore it produces a cycle with weight μ times the weight of the originating cycle. Hence together the three operations generate three new cycles of combined weight $\mu + 2$ times the weight of the cycle they extended.

5.9.2 Jump-Any Paving polynomials

The first few standard uniform weight '0' Jump-Any paving polynomials factorize nicely:

$$
\begin{aligned}
P_1(\mu) &= \mu && (5.158)\\
P_2(\mu) &= (\mu-1)(\mu+1) && (5.159)\\
P_3(\mu) &= (\mu-2)(\mu+1)^2 && (5.160)\\
P_4(\mu) &= (\mu-3)(\mu+1)^3 && (5.161)\\
\vdots \quad &\ \vdots \quad \ \vdots
\end{aligned}
$$

We need a recurrence relation to prove the pattern. We obtain a long recurrence relation in Figure 5.28. A trick inspired by the one that worked in the Jump 2 step case works in the Jump Any-step case. Write down the long recurrence relation twice, with coefficients staggered by one. Multiply the second equation by $(\mu + 2)$:

$$P_{k+1} = \mu P_k - P_{k-1} - (\mu+2)P_{k-2} - \dots - (\mu+2)^{k-1} \quad (5.162)$$

$$(\mu+2)P_k = \mu(\mu+2)P_{k-1} - (\mu+2)P_{k-2} - (\mu+2)^3 P_{k-4} - \dots - (\mu+2)^{k-1} \quad (5.163)$$

Take the difference between the two and simplify to get

Lemma 41. *The family, $\{P_k(\mu)\}_{k\geq 0}$, of uniform jump-any paving polynomials subject to standard weighting '0', satisfies*

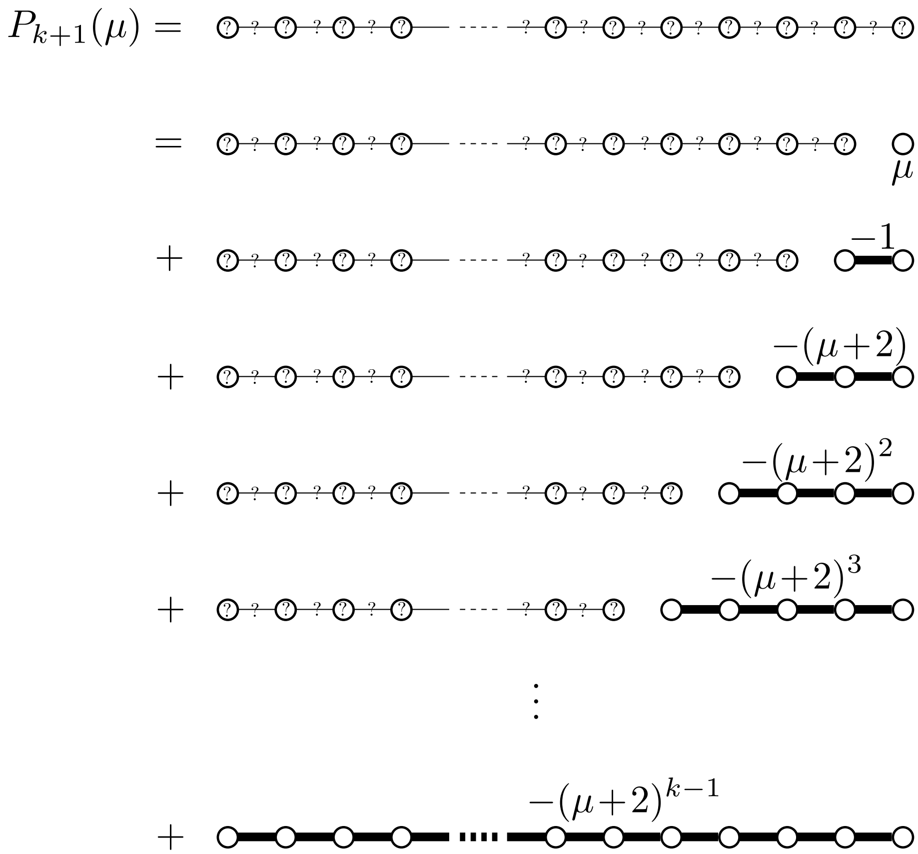
$$P_{k+1} = 2(\mu+1)P_k - (\mu+1)^2 P_{k-1}. \quad (5.164)$$

This 3-term recurrence relation has characteristic equation which factors as

$$(z - (\mu+1))^2 = 0 \quad (5.165)$$

so that

$$P_k(\mu) = (\mu - (k-1))\,(\mu+1)^{k-1}. \quad (5.166)$$



$$= \mu P_k - P_{k-1} - (\mu+2)P_{k-2} - (\mu+2)^2 P_{k-3}$$
$$-(\mu+2)^3 P_{k-4} - ... - (\mu+2)^{k-1}$$

Figure 5.28: Breaking up the paving according to the possibilities for the last vertex gives a $(k+2)$ - term linear recurrence for Jump Any-step paving polynomials.

Comment 6. *Since equation (5.164) is a 3-term recurrence, the family of* $P_k(\mu)$*'s are orthogonal polynomials. Furthermore, the closed form solution (5.166) is explicitly rational. This is a more tractable form even than that obtained in Section 5.1 for Ballot paving polynomimals!*

Comment 7. *Note that this result has connections to the theory of Toeplitz matrices, since the path enumeration problem associated with Jump-Any paving polynomials has a transfer matrix of Toeplitz form.*

Another unheralded nicety appears in the extension to L-Jump Any-step digraphs. As was explained in Comment 2, adding loops whose edge carries uniform weight b shifts the weight of pavers and paving polynomials via the substitution $\mu \mapsto \mu - b$. Here, choosing $b = 2$ transforms Equation (5.155) into a set of monomial weights for pavers. Alternatively, choosing $b = 1$ transforms the paving polynomials (5.166) into the especially simple form $P_k(\mu - 1) = (\mu - k)\mu^{k-1}$.

5.10 Summary of Recurrences

For comparison, we list a paving polynomial recurrence for some choice of standard uniform weighting on various classes of digraph considered in this chapter.

Digraph	Paving Polynomial Recurrence
Ballot	$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda P_{k-1}(\mu)$
Motzkin	$P_{k+1}(\mu) = (\mu - b)P_k(\mu) - \lambda P_{k-1}(\mu)$
2-up	$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda P_{k-1}(\mu) - \lambda^2 P_{k-2}(\mu)$
3-up	$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda P_{k-1}(\mu) - \lambda^2 P_{k-2}(\mu) - \lambda^3 P_{k-3}(\mu)$
d-up	$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda P_{k-1}(\mu) - \lambda^2 P_{k-2}(\mu) - \ldots - \lambda^d P_{k-d}(\mu)$
Jump 2	$P_{k+1} = (\mu+1)P_k - (\mu+1)P_{k-1} - (\mu+1)P_{k-2} + (\mu+1)P_{k-3} - P_{k-4}$
Jump 3	unsolved
Jump Any	$P_{k+1}(\mu) = 2(\mu+1)P_k(\mu) - (\mu+1)^2 P_{k-1}(\mu)$.

Chapter 6

Decorated pavings

Humpty Dumpty sat on a wall
Humpty Dumpty had a great fall
All the king's horses and all the king's men
Couldn't put Humpty together again.

'What makes solving non constant coefficient linear recurrence relations hard?', is a question to motivate this chapter. The situation stands in contrast to the constant coefficient case, for which there is a known algorithm that will in principal always give a solution. Even then, difficulties may arise in the step which requires finding the roots of the characteristic polynomial, since as we know from the work of Gauss and Abel on the quintic [107], there is no general algorithm for finding said roots. However, modulo that difficulty, a general method exists for solving a constant coefficient linear recurrence.

When even a finite set of coefficients are permitted to stray from constancy, however, the usual method for solving a constant coefficient linear recurrence breaks down. The problem is that a recurrence relation tells you what is happening to the nth term in relation to the previous few, whereas the deviations from constancy are hidden somewhere early in the sequence. We solve this problem by seeing non-constant coefficient recurrences as 'decorated pavings', which we can break apart at an arbitrary point, and then, unlike Humpty Dumpty, put the pieces back together again.

The decorated pavings we're most interested in are those which arise from decorated digraphs, since it is these which relate to the lattice paths we wish to enumerate. We begin by recalling the essential definitions for digraphs from ‘Path Definitions’, page xxv, and for pavings from Section 4.1, page 89:

1. A ‘uniformly weighted digraph’ is a weighted digraph for which arcs of the same form all carry the same weighting.

2. A ‘decorated digraph’ is a weighted digraph whose weighting is not uniform.

3. A ‘uniformly weighted paving’ is a weighted paving for which pavers of the same size all carry the same weighting.

4. A ‘decorated paving’ is a weighted paving whose weighting is not uniform.

A decorated weighting on a digraph induces a decorated weighting on a paving, in just the same way as a uniform weighting on a digraph induced a uniform weighting on a paving in the previous chapter. In each of the following sections we introduce decorations on digraphs which correspond to walk problems of interest, and use them to define suitable decorated pavings. Thence we work directly with the decorated pavings to calculate decorated paving polynomials.

Within this chapter, **decorated paving polynomials** shall be denoted $P_k(\mu)$, whilst the **undecorated paving polynomials** that we found in the last chapter shall be denoted $M_k(\mu)$.

6.1 Decorated Ballot pavings

The definition of decorated Ballot pavings which follows, together with the bijection between weighted pavings and weighted cycles, is review of the key idea of Viennot. We then illustrate the principal that paving polynomials may be explicitly calculated by breaking and recombining pavings – an idea due to Richard Brak – to the closed-form calculation of a class of decorated paving polynomials.

Definition 53. General downstep decorated Ballot arc weights *on up arcs and down arcs are*

$$w(\textit{up arc}) = 1 \quad (6.1)$$

$$w((k, k-1)) = \lambda_k \quad (6.2)$$

Figure 6.1 indicates the weighting induced by this arc-weighting upon pavers in the path graph.

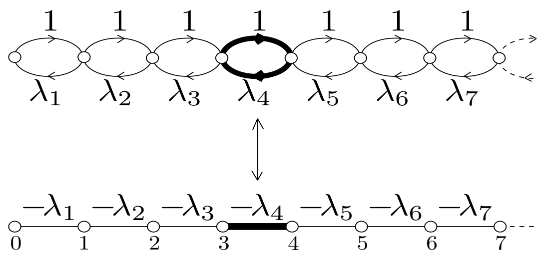


Figure 6.1: A decorated Ballot digraph leads to a decorated path graph. A 2-cycle on the digraph is shown, projecting to a dimer with decorated weight $-\lambda_4$.

Definition 54. General decorated Ballot paver weights *for uncovered vertices and dimers are*

$$w(0\text{-}mer) = \mu \tag{6.3}$$
$$w(\{k-1,k\}) = -\lambda_k. \tag{6.4}$$

These paver weights, together with the definition of a paving polynomial, give **general decorated Ballot paving polynomials**

We recall the result of Figure 5.3, and modify the labeling slightly to get

Lemma 42. *Given a set of non-constant weights, $\{\lambda_k\}$, (i.e. it is not the case that $\lambda_k = \lambda$ for some λ and all k) the family, $\{P_k(\mu)\}_{k\geq 0}$, of general decorated Ballot paving polynomials satisfies non-constant coefficient 3-term recurrence*

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda_k P_{k-1}(\mu) \tag{6.5}$$

as compared with the family, $\{M_k(\mu)\}_{k\geq 0}$, of standard uniform Ballot paving polynomials, which satisfies constant coefficient 3-term recurrence

$$M_{k+1}(\mu) = \mu M_k(\mu) - \lambda M_{k-1}(\mu). \tag{6.6}$$

Comment 8. *Since all Ballot paving polynomials, decorated or undecorated, satisfy a 3-term recurrence as shown in Lemma 42, they are all orthogonal polynomials [104], [103].*

Specific decorated paving polynomials are expressed in terms of the M_k's by breaking apart the path graph representing the set of decorated pavings. In particular, Figure 6.2 illustrates how to find the decorated paving polynomial for a paving with decorations near each end of the path graph.

This class of pavings is of particular interest because it includes pavings relevant to some of the simplest 'difficult' path problems, and also because these enumerations are important in a number of applications [27], [19], [15].

Theorem 43. *Let the family of* l**-wall weights decorated Ballot paving polynomials**, $\{P_k(\mu)\}_{k\geq 0}$, *be defined by weights on monomers and dimers on a path graph as follows.*

$$w(0\text{-}mer) = \mu \tag{6.7}$$

$$w(\{i-1,i\}) = \begin{cases} \kappa_i & 1 \leq i \leq l \\ \lambda & l+1 \leq i \leq L-l \\ \omega_{L-i+1} & L-l+1 \leq i \leq L \end{cases}. \tag{6.8}$$

Then

$$\begin{aligned} P_{L+1}(\mu) = {} & D^{\kappa}_{l+1}(\mu)D^{\omega}_{l+1}(\mu)M_{L-2l-1}(\mu) \\ & -\lambda\left(D^{\kappa}_{l}(\mu)D^{\omega}_{l+1}(\mu) + D^{\kappa}_{l+1}(\mu)D^{\omega}_{l}(\mu)\right)M_{L-2l-2}(\mu) \\ & +\lambda^2 D^{\kappa}_{l}(\mu)D^{\omega}_{l}(\mu)M_{L-2l-3}(\mu), \end{aligned} \tag{6.9}$$

where the D's are general decorated Ballot polynomials in μ *defined by initial conditions*

$$D^{\kappa}_0(\mu) = 1 \tag{6.10}$$

$$D^{\kappa}_1(\mu) = \mu \tag{6.11}$$

with recurrence relation

$$D^{\kappa}_{i+1}(\mu) = \mu D^{\kappa}_i(\mu) - \kappa_i D^{\kappa}_{i-1}(\mu), \tag{6.12}$$

similarly for $D^{\omega}_i(\mu)$*; and* $M_i(\mu)$*'s are the standard uniform Ballot polynomials.*

Comment 9. *Many of the weighted path enumeration problems we are most interested in are examples of an* l*-wall weighting, for small values of* l*. For instance, the DiMazio/Rubin problem posed in 1971 [37] and solved in 2006 [15], [25] is such a wall weighting for* $l = 1$*. For fixed small* l*, the D's of Theorem 43 are short polynomials easily calculated using their defining recurrences.*

Theorem 43 is a special case of the following more general observation.

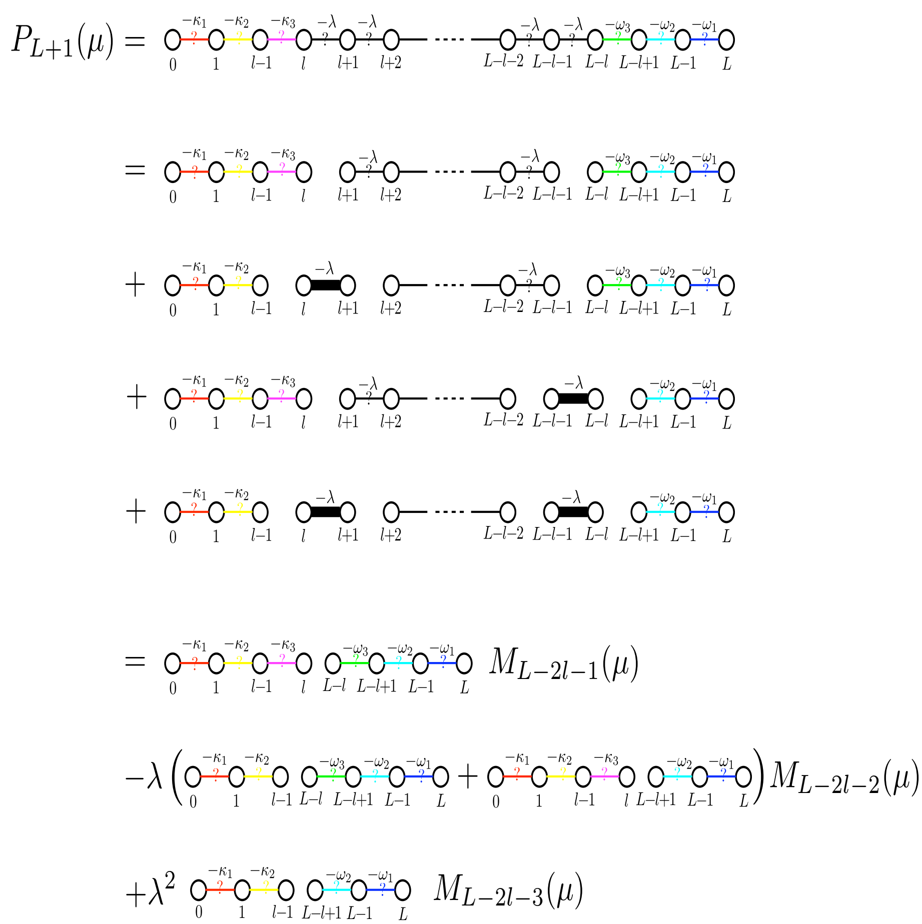


Figure 6.2: A wall-weights decorated Ballot paving polynomial is shown for $l = 3$. We break it in two places, in all 2^2 possible ways. In the last three lines we collect terms.

Lemma 44. *Let $\{P_k(\mu)\}_{k\geq 0}$ be a family of decorated Ballot paving polynomials with a finite number of decorations, i.e. the sequence of weights $\{\lambda_k\}_{k\geq 0}$ in recurrence (6.5) has the property that*

$$\lambda_k \neq \lambda \tag{6.13}$$

for only a finite number of values of k. Let the number of such decorations be d. Then the P_k's may be expressed as a finite sum of products of uniform Ballot paving polynomials. That is,

$$P_k(\mu) = \sum_{i=1}^{m} c_i \prod_{j=1}^{n} M_j(\mu) \tag{6.14}$$

where

$$1 \leq m \leq 2^d, \tag{6.15}$$

$$1 \leq n \leq d+1 \tag{6.16}$$

and c_i's are polynomials in μ (with coefficient λ_k's) of degree at most $2d$.

We defer the proof of Lemma 44 until stating a similar result for Motzkin pavings in the next section. Note that the bound on the number of sums, 2^d, is only tight when the decorations are sufficiently isolated from each other. To illustrate the potential difference between the bound and the reality, in Theorem 43, $d = 2l$ may be arbitrarily large but there are still only 3 terms in the sum (right hand side of Equation (6.9)). Also the bound of the degree of the c_i's is only tight when the decorations come in a single contiguous block. Consideration of the possible block structures of decorations would lead to more detailed theorems with tighter bounds. In practice, complexity considerations should be carried out of a case-by-case basis depending on the 'block structure' of the weights – the more bunched together the weights are, the better for lower complexity.

6.2 Decorated Motzkin pavings

The definition of decorated Motzkin pavings which follows, together with the bijection between weighted pavings and weighted cycles, is review of Viennot's idea, as it was for the Ballot case. Once again, the observation due to Brak that paving polynomials may be explicitly calculated by breaking and recombining pavings is what gives this definition it's power in the sense which we go on to use it in Chapter 10.

Definition 55. General downstep decorated Motzkin arc weights *on up arcs, loops and down arcs are*

$$w(\textit{up arc}) \;=\; 1 \tag{6.17}$$
$$w((k,k)) \;=\; b_k \tag{6.18}$$
$$w((k,k-1)) \;=\; \lambda_k \tag{6.19}$$

Figure 6.3 indicates the weighting induced by this arc-weighting upon pavers in the path graph.

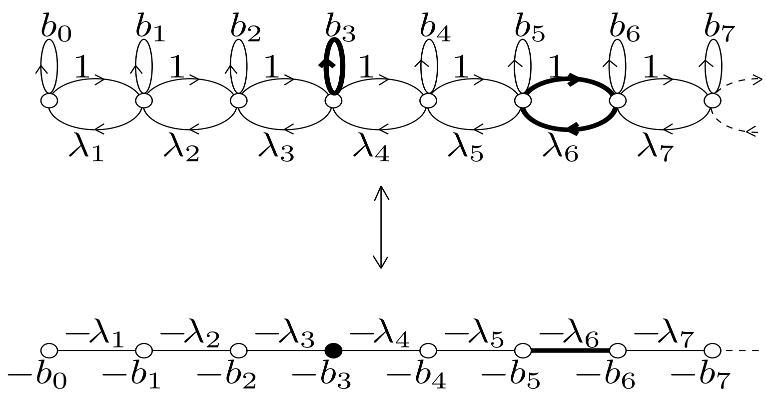


Figure 6.3: A decorated Motzkin digraph leads to a decorated path graph. A 1-cycle and a 2-cycle on the digraph are shown, projecting to a monomer and a dimer respectively with decorated weights $-\,b_3$ and $-\,\lambda_6$.

Definition 56. General decorated Motzkin paver weights *for uncovered vertices, monomers and dimers are*

$$w(0\text{-}mer) \;=\; \mu \tag{6.20}$$
$$w(\{k\}) \;=\; -b_k \tag{6.21}$$
$$w(\{k-1,k\}) \;=\; -\lambda_k. \tag{6.22}$$

These paver weights, together with the definition of a paving polynomial, give **general decorated Motzkin paving polynomials**

We recall the result of Figure 5.6, and modify the labeling slightly to get

Lemma 45. *The family, $\{P_k(\mu)\}_{k\geq 0}$, of general decorated Motzkin paving polynomials satisfies non-constant coefficient 3-term recurrence*

$$P_{k+1}(\mu) = (\mu - b_k)P_k(\mu) - \lambda_k P_{k-1}(\mu) \tag{6.23}$$

as compared with the family, $\{M_k(\mu)\}_{k\geq 0}$, of standard uniform Motzkin paving polynomials, which satisfies constant coefficient 3-term recurrence

$$M_{k+1}(\mu) = (\mu - b)M_k(\mu) - \lambda M_{k-1}(\mu). \tag{6.24}$$

Comment 10. *Since all Motzkin paving polynomials, whether decorated or undecorated, satisfy a 3-term recurrence as shown in Lemma 45, they are all orthogonal polynomials [104], [103].*

Specific decorated paving polynomials are expressed in terms of the M_k's by breaking apart the path graph representing the set of decorated pavings in the same generic fashion as was indicated for Ballot pavings in the previous section. A similar result to Theorem 43 is obtainable by similar methods. We also have general result

Lemma 46. *Let $\{P_k(\mu)\}_{k\geq 0}$ be a family of decorated Motzkin paving polynomials with a finite number of decorations, i.e. the sequences of weights $\{\lambda_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ in recurrence (6.23) have the property that*

$$\lambda_k \neq \lambda \tag{6.25}$$

for only a finite number, d_λ, of values of k;

$$b_k \neq b \tag{6.26}$$

for only a finite number, d_b, of values of k. Then the P_k's may be expressed as a finite sum of products of uniform Motzkin paving polynomials. That is,

$$P_k(\mu) = \sum_{i=1}^{m} c_i \prod_{j=1}^{n} M_j(\mu) \tag{6.27}$$

where

$$1 \leq m \leq 2^{d_\lambda} 3^{d_b}, \tag{6.28}$$

$$1 \leq n \leq d_\lambda + d_b + 1 \tag{6.29}$$

and c_i's are polynomials in μ (with coefficient λ_k's and b_k's) of degree at most $2d_\lambda + d_b$.

Note, analogously with Lemma 44, this bound is far from tight when blocks of weights are bunched together.

Proof. Fix $n \in \mathbb{N}$. Let $P_n(\mu)$ be a paving polynomial associated with a paving on a path graph of order n. Let d_b be the number of decorations on monomers and d_λ be the number of decorations on dimers.

Suppose there is a decorated monomer at height k. Then the path graph splits into at most two smaller path graphs in one of at most four different ways. (If the monomer happens to be located at an extreme position on each end of the path graph, then we obtain only one smaller path graph in each of three ways.) Of these ways, two group together, as shown in Figure 6.4. Thus there are at most three terms created by the extraction of

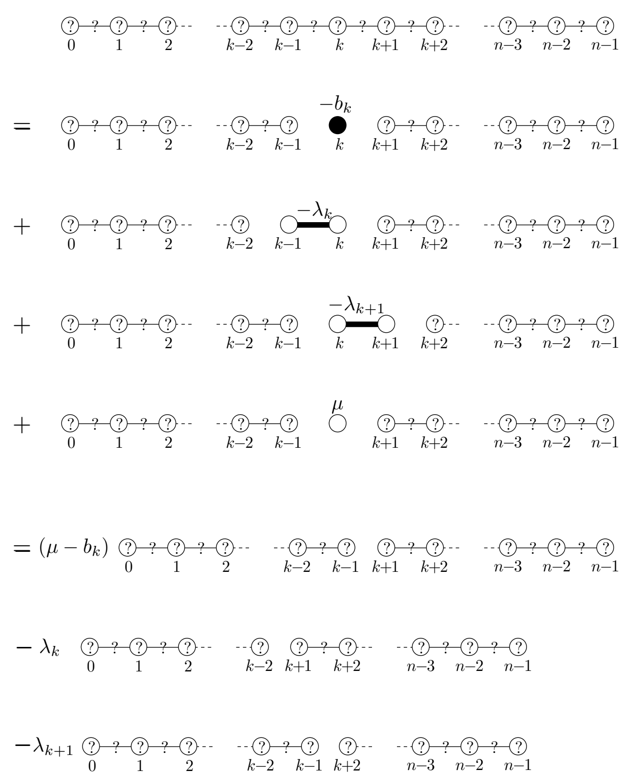


Figure 6.4: Extracting a decoration on a monomer by breaking apart the path graph gives a sum of at most three distinct terms, each of which is made up of a product of shorter paving polynomials.

one decorated monomer. After repeating the procedure d_b times, a sum of at most 3^{d_b} terms has been created.

Now suppose the dimer $\{j-1, j\}$ carries decorated weight $-\lambda_j$. Then a path graph of order m splits into pieces in at most two ways, as indicated in Figure 6.5. Repeating this procedure at most d_λ times multiplies the

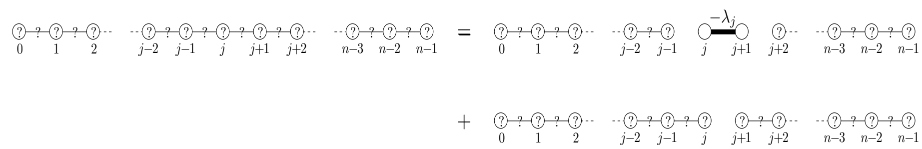


Figure 6.5: Extracting a decoration on a dimer by breaking apart the path graph gives a sum of at most two distinct terms, each of which is made up of a product of shorter paving polynomials.

number of terms by at most 2^{d_λ}. Thus bound (6.28) has been shown.

Notice also that the first time that the path graph is broken, at most two separate pieces are produced. With each subsequent break, at most one extra piece is created. When all breaks at decorations have been made, all the pieces inbetween are uniformly weighted. Thus there are are most $d_\lambda + d_b + 1$ terms in the product of Motzkin paving polynomials; i.e. bound (6.29) is shown.

Finally observe that in breaking the path where there are decorations, uncovered vertices are created only in the positions of the decorations. Each decorated dimer involved two vertices and decorated vertices are singlets, so that the degrees of the c_i's are at most equal to $2d_\lambda + d_b$.

Thus the lemma has been shown. Note that this argument has also proven Lemma 44. □

6.3 Decorated d-up pavings

We develop some decorated d-up paving results explicitly for the case $d = 2$.

Definition 57. General decorated 2-up standard arc weighting '1' *for short up arcs, long up arcs and down arcs is*

$$w(\textit{short up arc}) = 1 \tag{6.30}$$
$$w(\textit{long up arc}) = 1 \tag{6.31}$$
$$w(\{k, k-1\}) = \lambda_k \tag{6.32}$$

Figure 6.6 indicates the weighting upon pavers in the path graph induced by this arc weighting.

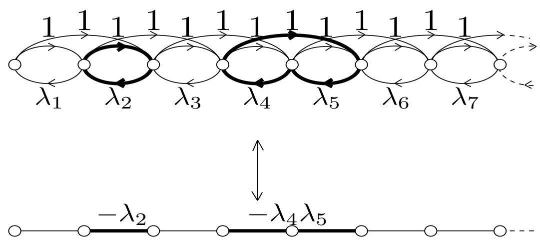


Figure 6.6:

Definition 58. General decorated 2-up standard paver weighting '1' *for 0-mers, dimers and trimers is*

$$w(0\text{-}mer) = \mu \quad (6.33)$$

$$w(\{k-1,k\}) = -\lambda_k \quad (6.34)$$

$$w(\{k-1,k,k+1\}) = -\lambda_k\lambda_{k+1} \quad (6.35)$$

These weights, in conjunction with the definition of a paving polynomial, give **General decorated 2-up standard weighting '1' paving polynomials**.

We recall the result of Figure 5.12, and make minor modifications to obtain

Lemma 47. *The family, $\{P_k(\mu)\}_{k\geq 0}$, of 2-up paving polynomials subject to general decorated weighting '1' satisfies non-constant coefficient 4-term recurrence*

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda_k P_{k-1}(\mu) - \lambda_{k-1}\lambda_k P_{k-2}(\mu) \quad (6.36)$$

as compared with the family, $\{M_k(\mu)\}_{k\geq 0}$, of standard uniform '1' 2-up paving polynomials, which satisfies constant coefficient 4-term recurrence

$$M_{k+1}(\mu) = \mu M_k(\mu) - \lambda M_{k-1}(\mu) - \lambda^2 M_{k-2}(\mu). \quad (6.37)$$

Specific decorated paving polynomials are expressed in terms of the M_k's by breaking apart the path graph representing the set of decorated pavings in the same generic fashion as was indicated for Ballot and Motzkin pavings in the previous sections. We consider an example for which weights are clustered near the lower end of the path graph in figure 6.7 to derive the following theorem.

Theorem 48. *Let the family of* r**-return weights decorated** 2**-up paving polynomials**,
$\{P_k(\mu)\}_{k\geq 0}$, *be defined by weights on* 0*-mers, dimers and trimers on a path graph as follows.*

$$w(0\text{-}mer) \;=\; \mu \tag{6.38}$$

$$w(\{i-1,i\}) \;=\; \begin{cases} \kappa_i & 1 \leq i \leq r \\ \lambda & r+1 \leq i \leq L \end{cases} \tag{6.39}$$

$$w(\{i-1,i,i+1\}) \;=\; \begin{cases} \kappa_{i-1}\kappa_i & 1 \leq i \leq r-1 \\ \lambda & r+1 \leq i \leq L-1 \end{cases} \tag{6.40}$$

Then

$$P_{L+1}(\mu) = D_{r+1}(\mu)M_{L-r}(\mu) - \big(\lambda D_r(\mu) + \lambda\kappa_r D_{r-1}\big)M_{L-r-1}(\mu) - \lambda^2 D_r(\mu)M_{L-r-2}(\mu) \tag{6.41}$$

where the D's are general decorated Ballot polynomials in μ *defined by initial conditions*

$$D_0(\mu) \;=\; 1 \tag{6.42}$$

$$D_1(\mu) \;=\; \mu \tag{6.43}$$

$$D_2(\mu) \;=\; \mu^2 - \kappa_1 \tag{6.44}$$

with recurrence relation

$$D_{i+1}(\mu) = \mu D_i(\mu) - \kappa_i D_{i-1}(\mu) - \kappa_i\kappa_{i-1}D_{i-2}(\mu) \tag{6.45}$$

and $M_i(\mu)$*'s are 2-up paving polynomials subject to standard uniform weight '1'.*

Decorated paving polynomials for d-up pavings may also be expressed as a sum over uniform d-up polynomials, though with an increasing penalty in number of summands as d grows.

6.4 Decorated Mixed-up pavings

Decorations on mixed-up pavings may be defined after the same fashion as those on d-up pavings. The same principles also apply to breaking up the pavings and expressing the decorated polynomials in terms of uniform ones. Bounds on numbers of sums may be found by considering the maximal number of times the the paving needs to be 'broken' to isolate the decorations from the rest of the paving.

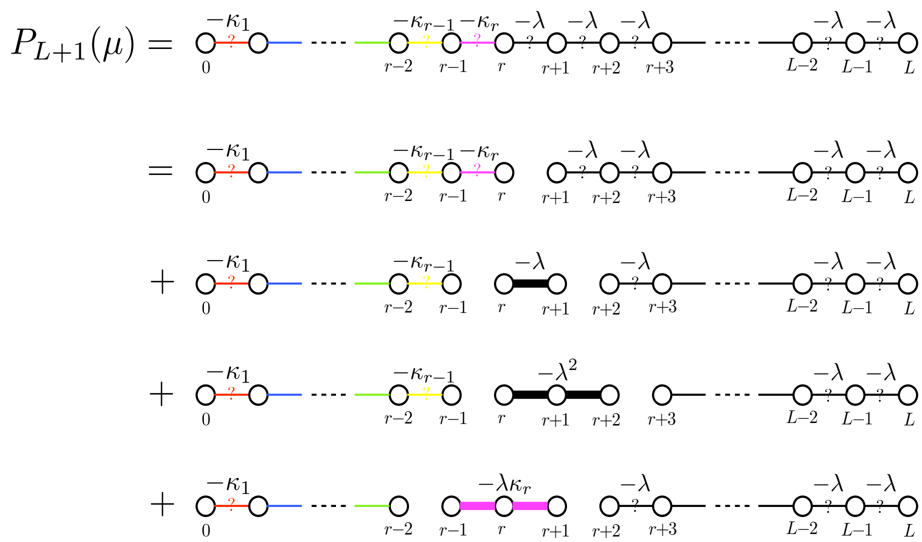


Figure 6.7: 2-up paving polynomials with r-return weight decorations can be written in terms of undecorated 2-up paving polynomials. As usual, question marks indicate that the given edge may or may not be paved, and that we are taking the sum over all such pavings.

6.5 Decorated Jump 2-step pavings

Jump step pavings are special since they contain arbitrarily long pavers. Nonetheless we may sometimes find expressions for the decorated paving polynomials in terms of short sums of uniformly weighted paving polynomials. As an example, we develop the paving polynomial for a 'short return' weighting.

Definition 59. *The* **short return-weight decorated Jump-2 arc weighting** *is*

$$\begin{aligned} w(\textit{long up arc}) &= 1 && (6.46)\\ w(\textit{short up arc}) &= 1 && (6.47)\\ w(\textit{long down arc}) &= 1 && (6.48)\\ w(\textit{short down arc}) &= \begin{cases} \kappa & \textit{for arc } (1,0) \\ 1 & \textit{otherwise} \end{cases} && (6.49) \end{aligned}$$

As indicated by Figures 6.8–6.9, this induces a decorated weighting on pavers as follows.

Definition 60. *The* **short return weight decorated Jump-2 paving weighting** *is*

$$w(0\text{-}mer) = \mu, \qquad (6.50)$$

$$\begin{aligned} w(\{0,1\}) &= -\kappa && (6.51)\\ w(\{0,1,2\}) &= -(\kappa+1)-\mu && (6.52)\\ w(\{0,1,2,3\}) &= -\kappa && (6.53)\\ w(\{0,1,\ldots,n\}) &= -(\kappa+1) \textit{ for } n \geq 4 && (6.54) \end{aligned}$$

and, for $i \geq 1$*,*

$$\begin{aligned} w(\{i,i+1\}) &= -1 && (6.55)\\ w(\{i,i+1,i+2\}) &= -2-\mu && (6.56)\\ w(\{i,i+1,i+2,i+3\}) &= -1 && (6.57)\\ w(\{i,i+1,\ldots,i+n\}) &= -2 \textit{ for } n \geq 4. && (6.58) \end{aligned}$$

These paver weights, together with the definition of a paving polynomial, give **short return weight decorated Jump-2 paving polynomials**

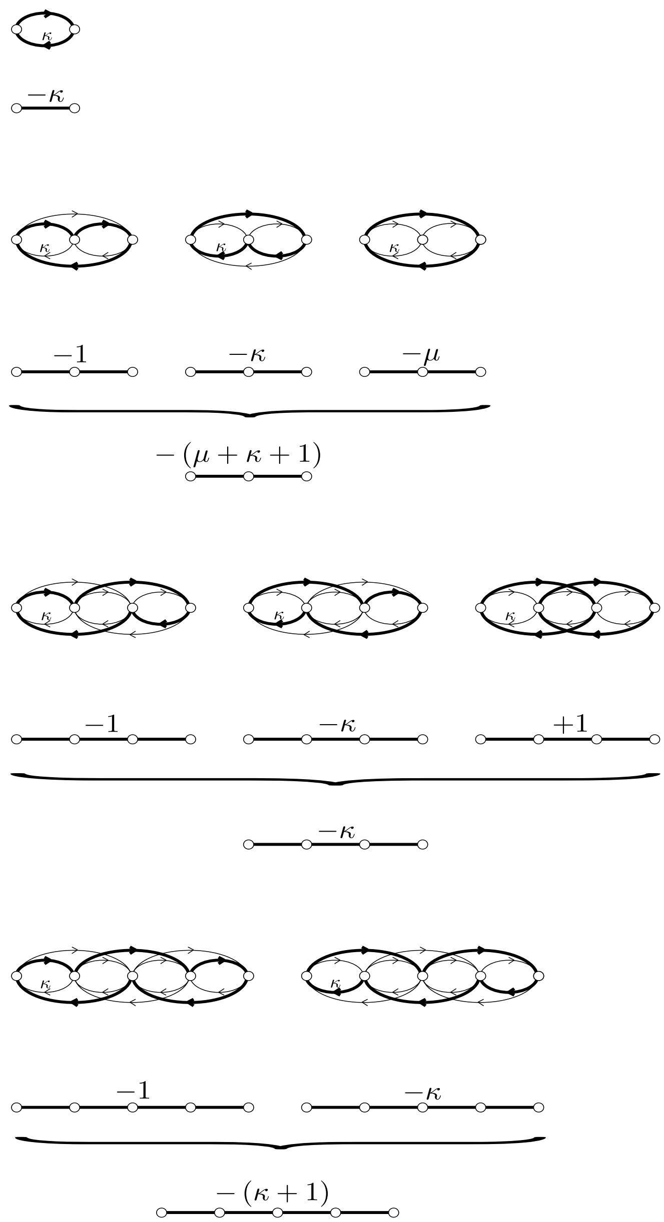


Figure 6.8: A return weight decoration on a Jump-2 digraph induces decorated pavers of increasing length. The first few are shown above. The sequence is continued in Figure 6.9.

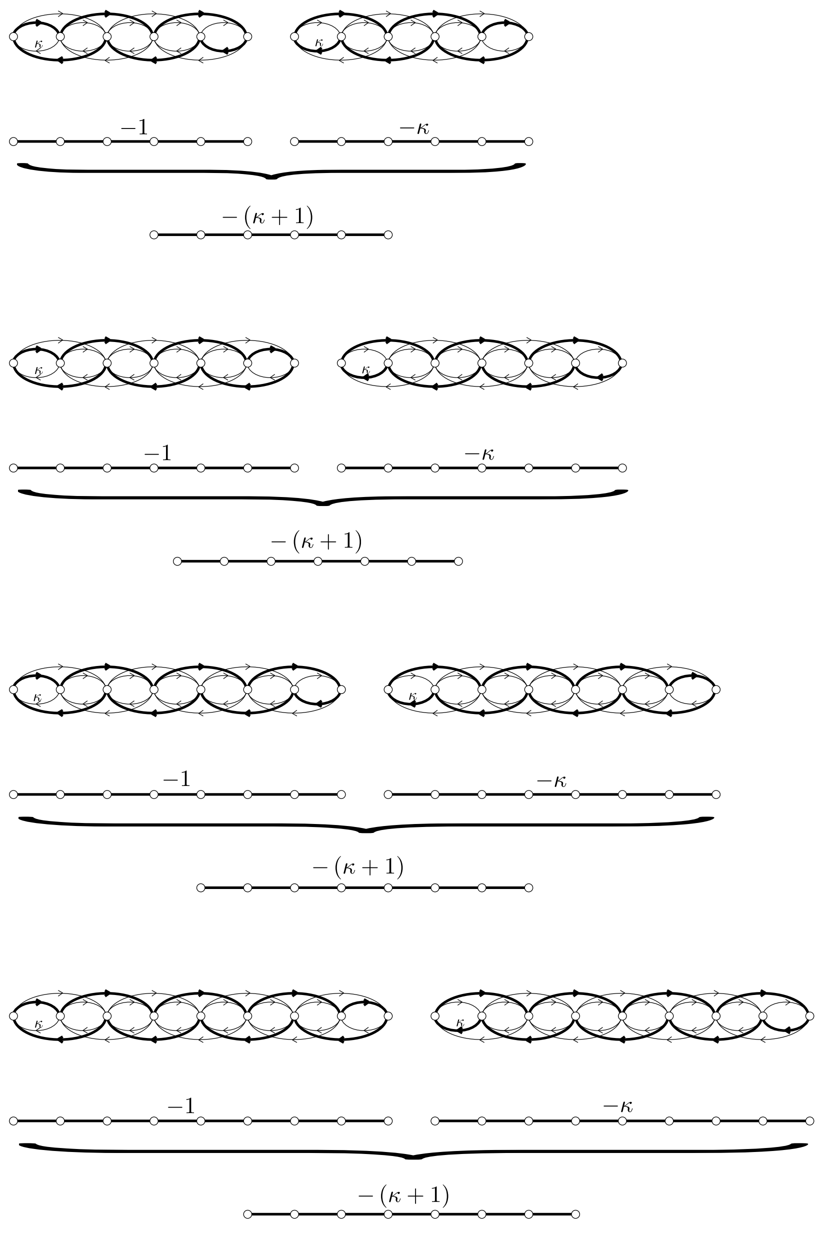


Figure 6.9: The sequence of decorated pavers induces by a return weight decoration on a Jump-2 digraph is continued from Figure 6.8.

Lemma 49. *Let* $\{P_k(\mu)\}_{k\geq 0}$ *be the family of Jump 2-step short return weight decorated paving polynomials of Definition 60. Let* $\{M_k(\mu)\}_{k\geq 0}$ *be the family of uniform Jump-2 paving polynomials of Definition 46. Then*

$$2P_{k+1} = (\kappa+1)M_{k+1} - \mu(\kappa-1)M_k - (\kappa-1)M_{k-1} + (\mu\kappa+\kappa+1)M_{k-2} - (\kappa-1)M_{k-3}. \tag{6.59}$$

Proof. Breaking apart the path graph in the usual way in Figure 6.10 gives a long expression for $P_{k+1}(\mu)$ in terms of shorter uniform graphs. Comparing this expression, Equation (6.60), with the long recurrence for unweighted pavings, Equation (6.61) that we had previously calculated in Section 5.7:

$$P_{k+1} = \mu M_k - \kappa M_{k-1} - (\mu+\kappa+1)M_{k-2} - \kappa M_{k-3} - (\kappa+1)M_{k-4} - (\kappa+1)M_{k-5} - \ldots \tag{6.60}$$

$$M_{k+1} = \mu M_k - M_{k-1} - (\mu+2)M_{k-2} - M_{k-3} - 2M_{k-4} - 2M_{k-5} - \ldots \tag{6.61}$$

we see that the difference

$$2P_{k+1} - (\kappa+1)M_{k+1} \tag{6.62}$$

gives the theorem. □

6.6 Decorated Jump Any-step pavings

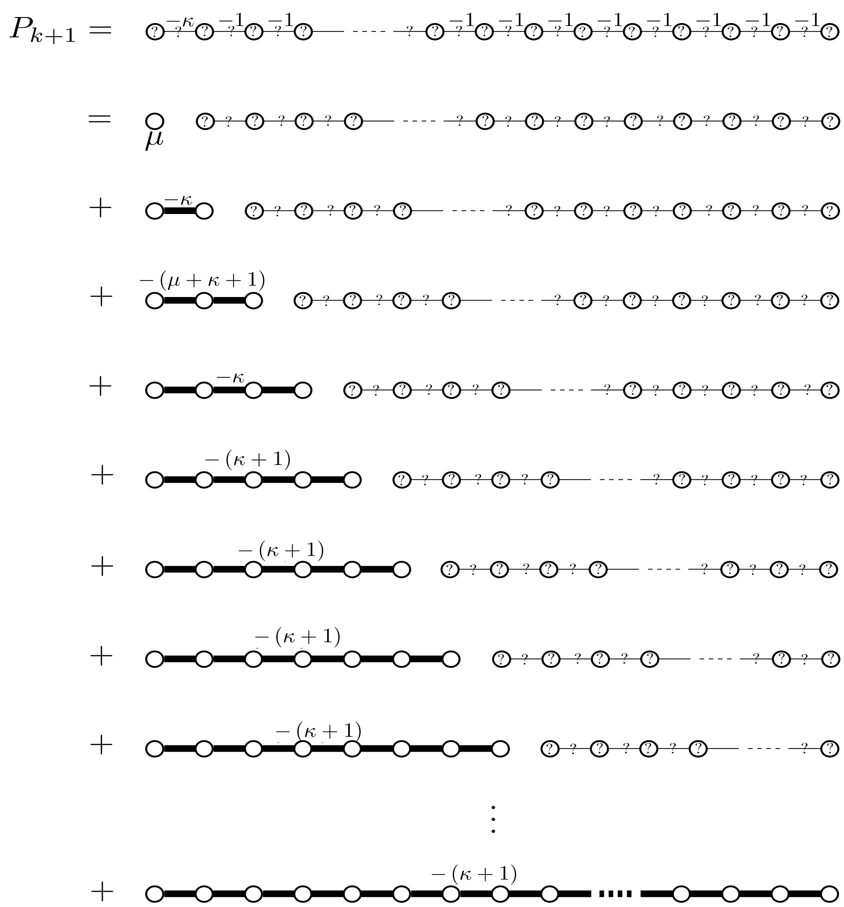
Jump Any-step pavings are special just as Jump 2-step pavings are, because they contain arbitrarily long pavers, but may still sometimes have concise paving polynomials. We develop the paving polynomial for a Jump Any-step 'short return' weighting.

Definition 61. *The* **short return weight decorated Jump-Any arc weighting** *is*

$$w(\textit{u-up arc}) = 1 \tag{6.63}$$

$$w(\textit{d-down arc}) = \begin{cases} \kappa & \textit{for arc } (1,0) \\ 1 & \textit{otherwise} \end{cases} \tag{6.64}$$

An argument precisely analogous to that carried out in Subsection 5.9.1 for Jump Any-step paths under the trivial uniform weighting gives decorated paver weights as in Definition 62. A sketch of this argument is summarized in Figure 6.11. The only modification to the reasoning required is to check that pairs of cycle weights that cancelled in the uniform case still cancel in the decorated case. This holds because the 'cross-uncross' involution only effects arcs to the right of arc $(1,0)$ which holds the κ weight. Thus we have



$$= \mu M_k - \kappa M_{k-1} - (\mu+\kappa+1)M_{k-2} - \kappa M_{k-3}$$

$$-(\kappa+1)M_{k-4} - (\kappa+1)M_{k-5} - (\kappa+1)M_{k-6}\dots$$

Figure 6.10: The Jump-2 return weight decorated paving polynomial may be expressed in terms of a long sum of shorter uniform Jump-2 paving polynomials.

Definition 62. *The* **short return weight decorated Jump-Any paving weighting** *is*

$$\begin{aligned}
w(0\text{-}mer) &= \mu & (6.65)\\
w(\{0,1\}) &= -\kappa & (6.66)\\
w(\{i,i+1\}) &= -1 \text{ for } i \geq 1 & (6.67)\\
w(\{0,1,\ldots,n\}) &= -(\mu+(\kappa+1))(\mu+2)^{n-2} \text{ for } n \geq 2 & (6.68)\\
w(\{i,i+1,\ldots i+n\}) &= -(\mu+2)^{n-1} \text{ for } i \geq 1,\ n \geq 2 & (6.69)
\end{aligned}$$

and $i \geq 1$. These paver weights, together with the definition of a paving polynomial, give **short return weight decorated Jump-Any paving polynomials***.*

A suitable modification of the labeling of Figure 6.10 gives long expression for $P_{k+1}(\mu)$ in terms of shorter M's:

$$P_{k+1} = \mu M_k - \kappa M_{k-1} - \sum_{i=2}^{k}(\mu+\kappa+1)(\mu+2)^{i-2}M_{k-i} \tag{6.70}$$

Comparing this with the long recurrence for $M_{k+1}(\mu)$ in terms of shorter M's that was found in Section 5.9:

$$M_{k+1} = \mu M_k - M_{k-1} - \sum_{i=2}^{k}(\mu+2)^{i-1}M_{k-i} \tag{6.71}$$

we derive

Lemma 50. *Let $\{P_k(\mu)\}_{k\geq 0}$ be the family of Jump Any-step short return weight decorated paving polynomials of Definition 62. Let $\{M_k(\mu)\}_{k\geq 0}$ be the family of uniform Jump Any-step paving polynomials of Definition 52, with formula given by Equation (5.166). Then*

$$P_{k+1}(\mu) = \left(\frac{\mu+\kappa+1}{\mu+2}\right)M_{k+1}(\mu) - \left(\frac{\mu(\kappa-1)}{\mu+2}\right)M_k(\mu) - \left(\frac{(\mu+1)(\kappa-1)}{\mu+2}\right)M_{k-1}(\mu). \tag{6.72}$$

Corollary 51. *Let $\{P_k(\mu)\}_{k\geq 0}$ be as in Lemma 50. Then*

$$P_k(\mu) = (\mu+1)^{k-2}(\mu^2-(k-2)(\mu+1)-\kappa). \tag{6.73}$$

Comment 11. *Corollary 51 gives a neat closed form* explicitly rational *paving polynomial on short-return weight decorations for Jump-Any pavings. Setting $\kappa = 1$ gives back the similarly neat result of Lemma 41 which comes from the uniform digraph weighting. These results reflect the special properties of the determinants of: Toeplitz matrices and small perturbations of Toeplitz matrices; with which weighted Jump-Any pavings are associated.*

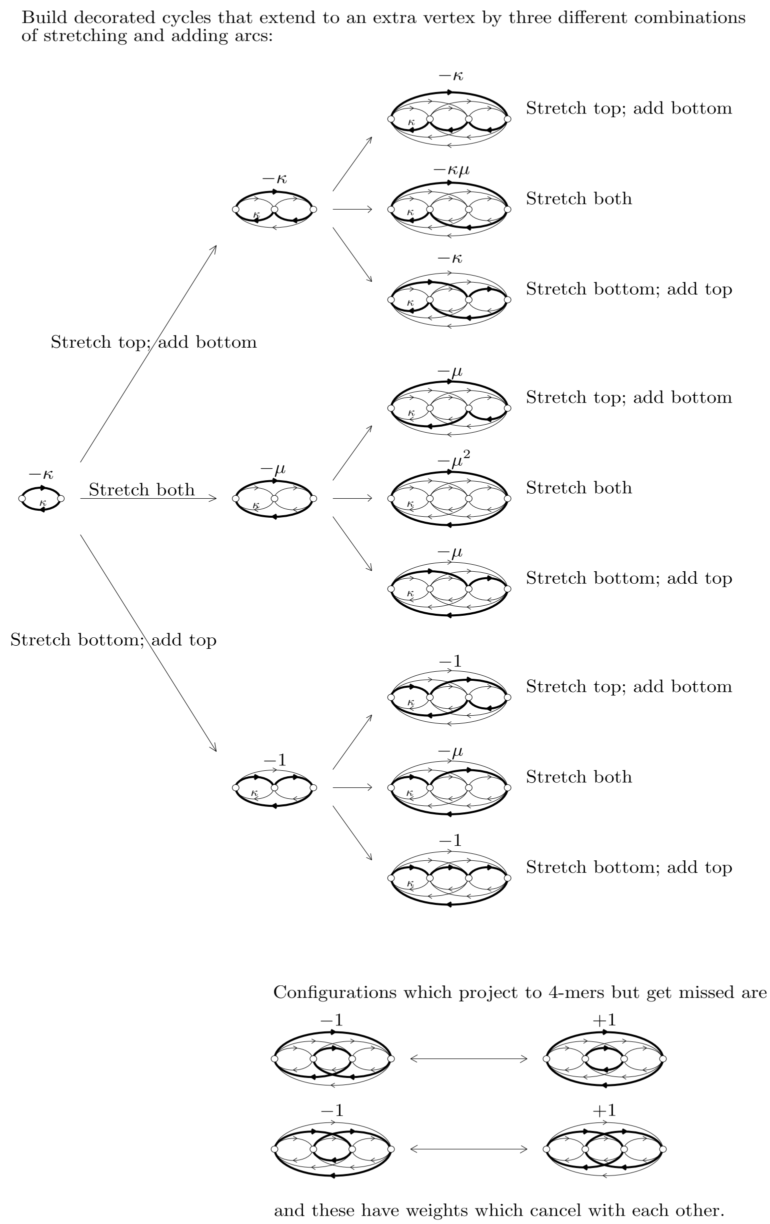


Figure 6.11: Decorated cycles that project to $(n+1)$-mers may be built from decorated cycles that project to n-mers in just the same way as we did in the uniform case. Fortunately, those configurations which cancelled in the uniform case still cancel here.

6.7 Summary of Recurrences

For comparison, we list a paving polynomial recurrence for some choice of decorated weightings on various classes of digraph considered in this chapter.

Digraph	Paving Polynomial Recurrence
Ballot	$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda_k P_{k-1}(\mu)$
Motzkin	$P_{k+1}(\mu) = (\mu - b_k) P_k(\mu) - \lambda_k P_{k-1}(\mu)$
2-up	$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda_k P_{k-1}(\mu) - \lambda_{k-1}\lambda_k P_{k-2}(\mu)$
Jump 2	$2P_{k+1} = (\kappa+1)M_{k+1} - \mu(\kappa-1)M_k - (\kappa-1)M_{k-1} + (\mu\kappa+\kappa+1)M_{k-2} - (\kappa-1)M_{k-3}$, where M_k's are uniformly weighted Jump 2 polynomials.
Jump Any	$P_{k+1}(\mu) = \left(\frac{\mu+\kappa+1}{\mu+2}\right) M_{k+1}(\mu) - \left(\frac{\mu(\kappa-1)}{\mu+2}\right) M_k(\mu) - \left(\frac{(\mu+1)(\kappa-1)}{\mu+2}\right) M_{k-1}(\mu)$, where M_k's are uniformly weighted Jump Any polynomials.

Chapter 7

Higher dimensional musings

7.1 Higher dimensional walks as motivation

We would like to count paths in three or higher dimensions in slabs of cubic or hyper-cubic lattice, as a natural extension to the work counting paths in a strip on the two-dimensional square lattice.

So far, in Chapters 5 and 6, we've developed one-dimensional pavings which we will use to count two dimensional directed paths. The most straightforward generalization in higher dimensions would be to develop n-dimensional pavings which we could use to count $(n+1)$-dimensional directed paths.

This is not the approach we take. Instead we explore the feasibility of developing 1-dimensional pavings (as before) which encode information about directed paths in dimensions higher than 2.

The reason for attempting this route requires some explanation. We expect it to be difficult, since there is a lot of 'room to move' in a high number of dimensions which will need to be compressed into a single dimension. The reason is that one-dimensional pavings are special, in a way that, for instance, tilings of the plane are not. As illustrated throughout Chapter 6, one-dimensional pavings give us a powerful means to isolate decorated parts of a paving from uniform parts, so that we may obtain closed-form expressions for pavings with finitely many decorations – see for instance Figures 6.2, 6.4, 6.7, 6.10.

The crucial difference between a one-dimensional paving and a higher dimensional paving in this context is in its connectivity properties. To isolate one part of a path graph from another we may split the graph into two pieces by removing a single dimer, whereas removing a single tile from a tiling of a higher dimensional space will typically leave it connected – for definiteness think about removing one triangle from a regular tiling of the plane with equilateral triangles.

Summarizing, the a priori justification for this chapter is to try to find

- one-dimensional pavings which encode information about paths in higher dimensions

As it turns out, we also find an a posteriori justification, discovering

- 'equivalent pavings'. That is, we find that distinct paving problems with distinct paving polynomials and recurrences may arise from the same path problem and encode the same set of desired information (as well as possibly some extra irrelevant information on which they disagree).

The notion of 'equivalent pavings' is of interest both in the high and low dimensions. It may be important because we have found examples where some of the paving polynomial recurrences (and corresponding characteristic equations for the uniform cases) are very much simpler than others, for the same path problem.

The best we could hope for is the good situation in which, given a difficult recurrence, one is able to find a paving representation for it, and then an equivalent paving representation encoding the same information (as well as possibly some extra, irrelevant information) but with a simpler recurrence.

All of this work is very preliminary. What follows is a presentation of some basic ideas, with room for further explorations.

Figure 7.1 shows a cube. The arrows indicate that the cube may be considered to be a directed graph, with vertices at the corners and arcs connecting them. One may trace a path from the near vertex (indicated) to the far vertex (also indicated). There are six such walks, corresponding to three choices of which arc to leave by, times two choices at the next vertex, and then only one choice available at the following vertex. Each of the six paths, one of which is shown, have length three.

The same figure is drawn, slightly rotated so that in this two dimensional projection the vertices at which the path begins and ends are superimposed, in Figure 7.2. Thus the path on the three dimensional cube may be seen as a walk that closes upon itself, on a two dimensional directed triangular lattice.

By stacking cubes together, or else partitioning the cube we started with into smaller cubes, and then in either case projecting as in Figure 7.2, we

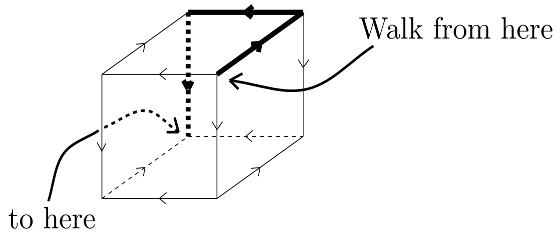


Figure 7.1: A cube, with a path from the near to the far vertex shown.

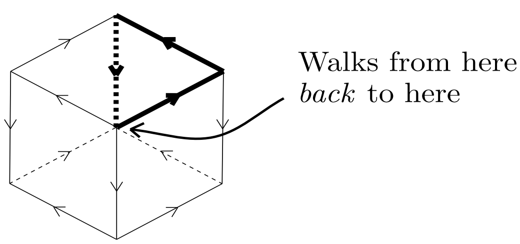


Figure 7.2: The same path and cube that were shown in Figure 7.1, slightly rotated so that the projection is a closed walk on a piece of directed triangular lattice.

obtain a larger piece of this directed triangular lattice, which is illustrated in Figure 7.3. Formally we define

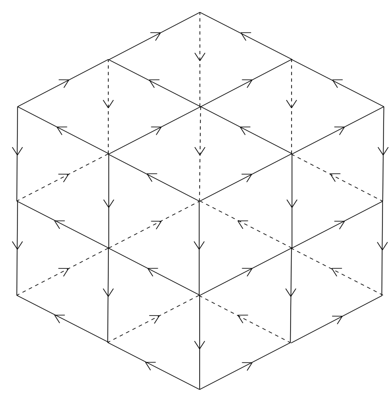


Figure 7.3: A larger piece of the 'flat cube' lattice.

Definition 63. *Let $\mathcal{L}$ be a lattice with vertex set*

$$V = \{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3 | \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}\} \tag{7.1}$$

defined as the span over $\mathbb{Z}$ of allowed step set

$$\{s_1, s_2, s_3\} \tag{7.2}$$

where

$$s_1 + s_2 + s_3 = (0, 0) \tag{7.3}$$

and the pair

$$\{s_i, s_j\} \tag{7.4}$$

are linearly independent over $\mathcal{Z}$ for any $i \neq j \in \{1, 2, 3\}$. Then $\mathcal{L}$ is a **flat cube lattice**. *If, in addition, $\{s_1, s_2, s_3\}$ are vectors in $\mathbb{R}^2$ at angles of 120 degrees to each other, $\mathcal{L}$ is termed an* **equilateral flat cube lattice**. *A particular choice of allowed step set which gives the equilateral flat cube lattice of Figure 7.3 is*

$$\begin{aligned} s_1 &= (\sqrt{3}/2, 1/2), & (7.5) \\ s_2 &= (-\sqrt{3}/2, 1/2), & (7.6) \\ s_3 &= (0, -1). & (7.7) \end{aligned}$$

7.2 Cycles on a planar digraph

As usual, we are interested in the cycles on the time compressed (recall Definition 16) digraph shown in Figure 7.3. Several cycles are illustrated in Figure 7.4. It is clear that in this representation, cycles must all be depicted

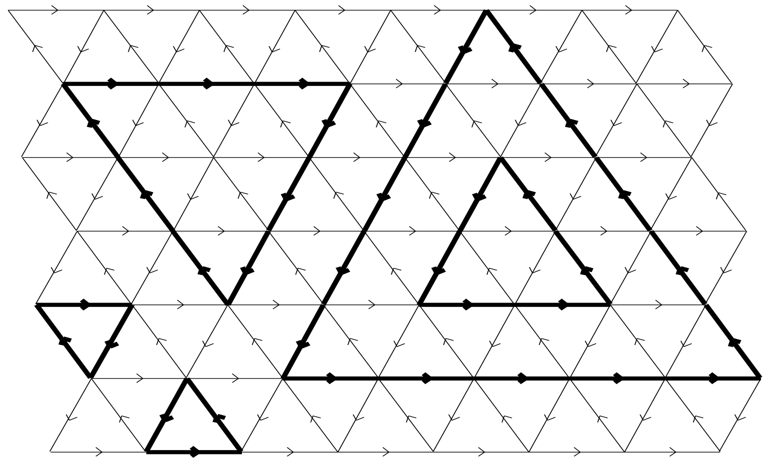


Figure 7.4: A configuration of cycles on the 'flat cube' lattice.

as triangles of varying sizes. No more elaborate shapes are possible.

We want to count signed cycles. As noted in Section 7.1, one approach is to map the cycle configurations to tilings of the plane. The possibility we explore instead is to map the cycle problem to a one-dimensional tiling of a path graph, as we have done in earlier chapters.

The basic approach and some expected difficulties are explored via a simple example in the next section. This example also leads us to define the concept of 'equivalent pavings', which may turn out to be independently useful.

7.3 Isomorphic digraphs

We begin with a standard definition.

Definition 64. *A pair of (di)graphs* $G_1 = (V_1, A_1)$*,* $G_2 = (V_2, A_2)$ *are called* **isomorphic** *provided that there exists a bijective mapping* $\phi : V_1 \mapsto V_2$ *such that for* uv *an arc/edge in* A_1

$$\phi(uv) := \phi(u)\phi(v) \tag{7.8}$$

is an arc/edge in A_2. The mapping ϕ is called an **isomorphism**.

Next we describe/define a way of drawing digraphs which is suited to our purposes.

Definition 65. *Let $\mathcal{D}$ be a digraph of order n. Then we say that a* **linear representative of** *$\mathcal{D}$ is a digraph $\mathcal{R}$ which is an isomorphic copy of $\mathcal{D}$ drawn in a certain way, as follows.*

- *Vertices of $\mathcal{R}$ are labeled $0, 1, \ldots n-1$.*
- *Vertices of $\mathcal{R}$ are indicated visually by a line of dots drawn in a line from left to right such that the dot representing vertex $i+1$ is to the right of the dot representing vertex i for all i.*
- *Arcs are drawn as curved lines on either side of the row of vertices (with arrows indicating direction).*
- *Arcs from i to j with $i \leq j$ are drawn above the row of vertices, and arcs from j to i with $j > i$ are drawn below the row of vertices.*

7.4 An example

We consider a single rowed piece of the flat cube lattice, as illustrated in Figure 7.5. There are many ways in which its vertices may be labeled,

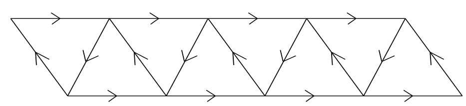


Figure 7.5: A narrow slice of 'flat cube' lattice.

each generating an isomorphic copy of the digraph. We choose three such labelings, and illustrate the linear representative for each. For the first two we develop families of paving polynomials and compare them.

7.4.1 Labeling A

Labeling 'A', of the single-rowed flat cube digraph, leads to a linear representative as shown in Figure 7.6, supporting cycles all of the same appearance independently of location. All cycles project to trimers; and all basic configurations of cycles project to pavings with trimers.

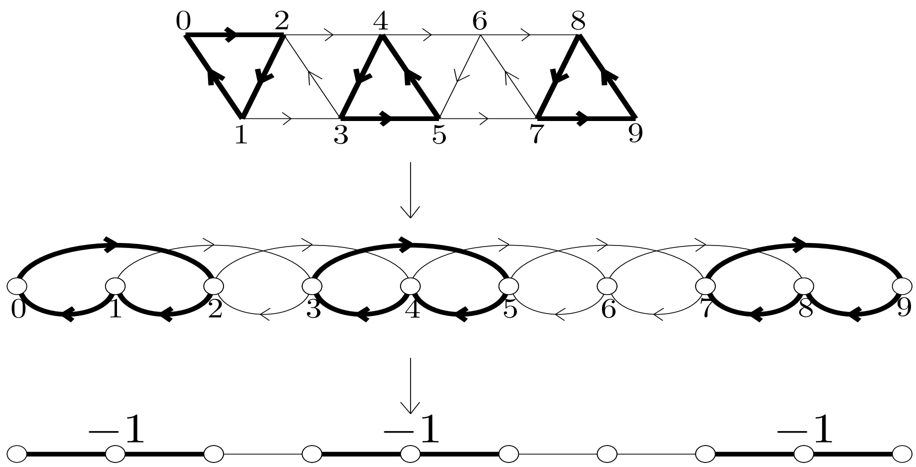


Figure 7.6: Labeling A. The digraph has linear representative such that each cycle projects to a trimer. Under the trivial weighting on the digraph, trimers inherit weight -1.

Under the trivial weighting on the digraph, all arcs have weight 1, thus pavers inherit weight $(-1)(1)(1)(1)$. Labeling 'A' naturally extends to single-rowed sections of the flat cube digraph of arbitrary order. The corresponding path graph is also extended to arbitrary order, supporting pavings with trimers along its length. In the usual way, such pavings define an infinite family of paving polynomials $\{P_k(\mu)\}_{k\geq 0}$.

The first few such paving polynomials are derived in Figure 7.7. We find a recurrence relation on these paving polynomials in Figure 7.8. Thus we have

Lemma 52. *Let $\{P_k(\mu)\}_{k\geq 0}$ be the family of trivially uniformly weighted paving polynomials generated by labeling 'A' on a single rowed piece of 'flat cube' lattice, as in Figure 7.6. Then the paving polynomials satisfy third order recurrence*

$$P_k(\mu) = \mu P_{k-1} - P_{k-3}, \tag{7.9}$$

with characteristic equation

$$z^3 - \mu z^2 + 1 = 0 \tag{7.10}$$

and subject to initial conditions $P_0(\mu) := 1$, $P_1(\mu) = \mu$ and $P_2(\mu) = \mu^2$.

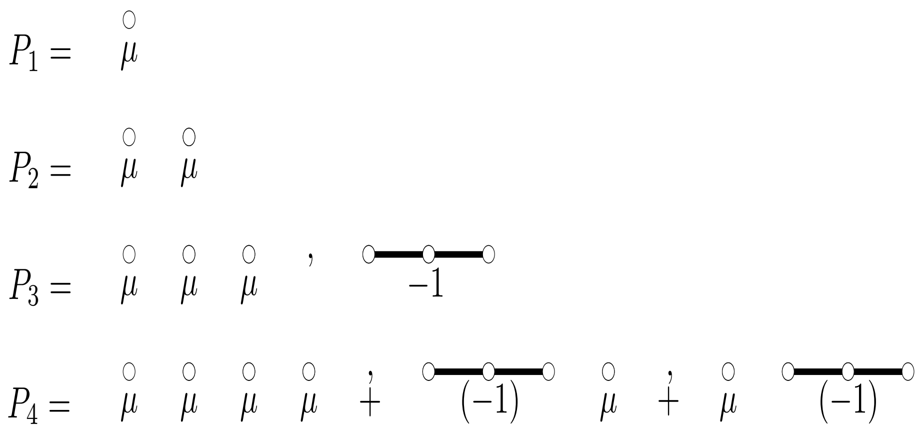
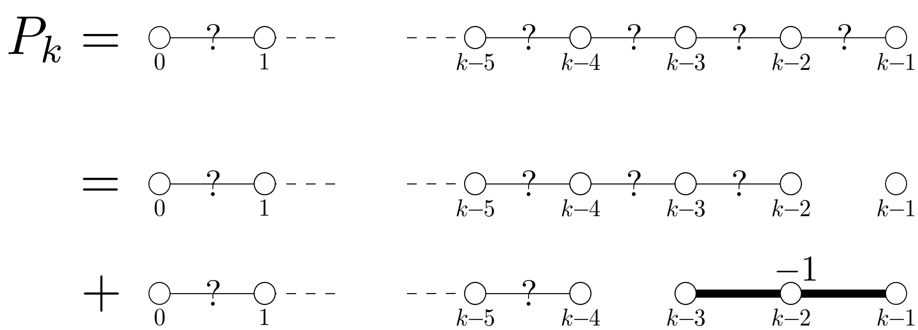


Figure 7.7: The first few trivially uniformly weighted paving polynomials generated by labeling 'A' on a single rowed piece of 'flat cube' lattice.



$$= \mu P_{k-1} - P_{k-3}$$

Figure 7.8: Trivially uniformly weighted paving polynomials generated by labeling 'A' on a single rowed piece of 'flat cube' lattice satisfy a third order recurrence relation.

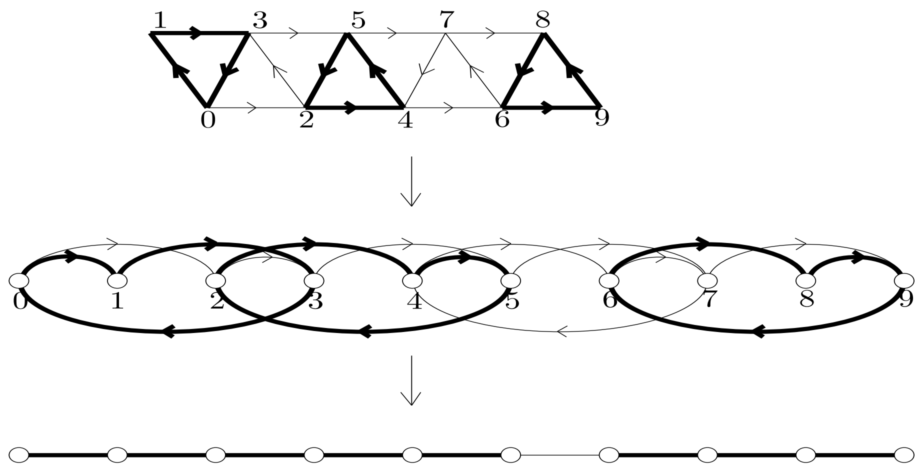


Figure 7.9: Labeling 'B'. The digraph has linear representative such that single cycles project to 4-mers and pairs of overlapping cycles project to 6-mers.

7.4.2 Labeling B

Labeling 'B' of the single-rowed flat cube digraph gives a linear representative as shown in Figure 7.9. This representative supports cycles with one of two possible appearances, as well as the possibility that cycles overlap. Both shapes of cycles project to 4-mers, as shown in Figure 7.10. Thus the trivial arc weighting on the digraph leads to a weighting of $-1-1=-2$ applied to 4-mers on the path graph.

It is also possible for (at most) a pair of such cycles to overlap, so that their combined projection is a 6-mer. Thus the trivial arc weighting on the digraph leads to a weighting of $(-1)(-1) = +1$ applied to 6-mers on the path graph.

All basic configurations of cycles on the digraph project to pavings on the path graph with 4-mers and 6-mers. However not all pavings with 6-mers and 4-mers come from a basic cycle configuration. Looking carefully we see that only every second vertex in the path graph may be the least (leftmost) vertex in a paver.

Labeling 'B' may be naturally extended to an arbitrarily long row of the flat cube lattice, with corresponding linear representation and path graph. The path graph supports pavers along its length, subject to the constraint that the leftmost vertex in a paver must always be of even parity.

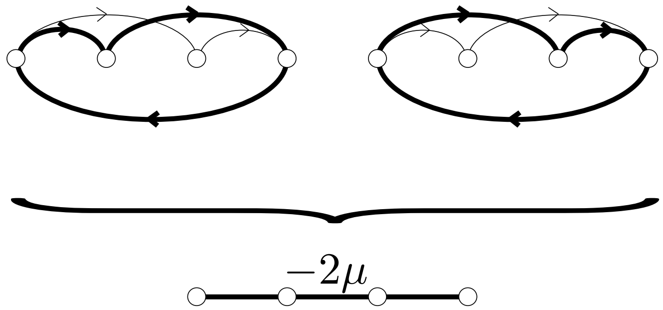


Figure 7.10: Labeling 'B'. Two kinds of cycle which both project to 4-mers means that the paver inherits the weight -2μ under the trivial arc weighting.

The first few paving polynomials are given in Figure 7.11. A recurrence relation is derived in Figures 7.12 and 7.13.

We have

Lemma 53. *Let* $\{P_k(\mu)\}_{k\geq 0}$ *be the family of trivially uniformly weighted paving polynomials generated by labeling 'B' on a single rowed piece of 'flat cube' lattice, as in Figure 7.9. Then the paving polynomials with even indices satisfy recurrence*

$$P_{2n}(\mu) = \mu^2 P_{2n-2}(\mu) - 2\mu P_{2n-4}(\mu) + P_{2n-6}(\mu). \tag{7.11}$$

Paving polynomials with odd indices are obtainable from those with even indices as follows.

$$P_{2n+1}(\mu) = \mu P_{2n}(\mu) \tag{7.12}$$

The even index recurrence Equation (7.11) has characteristic equation

$$z^6 - \mu^2 z^4 + 2\mu z^2 - 1 = 0. \tag{7.13}$$

The first few paving polynomials are

$$P_0(\mu) := 1 \tag{7.14}$$
$$P_1(\mu) = \mu \tag{7.15}$$
$$P_2(\mu) = \mu^2 \tag{7.16}$$
$$P_3(\mu) = \mu^3 \tag{7.17}$$
$$P_4(\mu) = \mu^4 - 2\mu \tag{7.18}$$
$$P_5(\mu) = \mu^5 - 2\mu^2 \tag{7.19}$$
$$P_6(\mu) = \mu^6 - 2\mu^3 + 1. \tag{7.20}$$

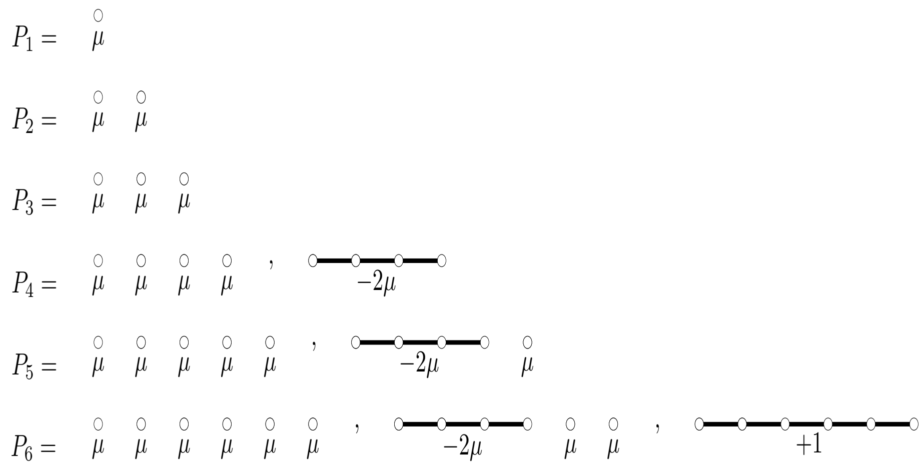
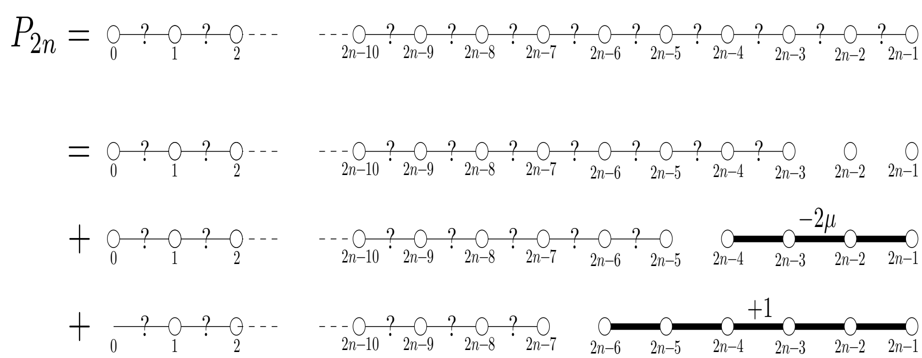


Figure 7.11: The first few trivially uniformly weighted paving polynomials generated by labeling 'A' on a single rowed piece of 'flat cube' lattice.



$$= \mu^2 P_{2n-2} - 2\mu P_{2n-4} + P_{2n-6}$$

Figure 7.12: The recurrence relation for trivially uniformly weighted paving polynomials with even indices, generated by labeling 'B' on a single rowed piece of 'flat cube' lattice, is derived.

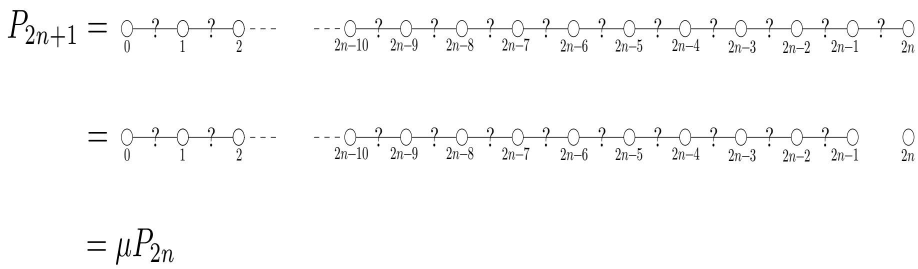


Figure 7.13: Trivially uniformly weighted paving polynomials who indices are odd numbers, as defined by labeling 'B' on a single rowed piece of 'flat cube' lattice, may be obtained from those with even indices.

Notice that the *even* index paving polynomials subject to labeling 'B' (as in Lemma 53) are identical with the *even* index paving polynomials subject to labeling 'A' (as in Lemma 52). As expected odd index paving polynomials differ in the two labelings since they correspond to different boundaries on the original slice of flat cube lattice.

7.4.3 Labeling C

Cycles on the digraph drawn linearly with the labeling of Figure 7.14 present with one of two possible appearances. Both of these shapes project to 6-mers. A pair of overlapping cycles may project to an 8-mer, 9-mer or 10-mer depending on where they overlap. Triples may also overlap and project to a 10-mer.

One natural extension of the labeling of Figure 7.14 to an arbitrarily long row from a flat cube lattice is illustrated in Figure 7.15. This labeling makes determining induced pavings more difficult than with labelings 'A' and 'B'. The problem is that vertex labels '$n-1$' and 'n' are close together as integers but far apart as vertices on the digraph. (Distance between a pair of vertices on a graph or digraph is the length of the shortest path between the vertices, where the length of a path is the number of edges in the path.)

Labeling 'C' is indicative of the difficulties to expect in attempting to find pavings of path graphs which characterize cycles on portions of the flat cube lattice which consist of more than one row of small triangles. An intuition is that labelings 'A' and 'B' take advantage of the fact that a single-rowed slice is almost a one-dimensional problem; whereas as we take thicker strips the two-dimensionality of the entire flat cube lattice comes more into play.

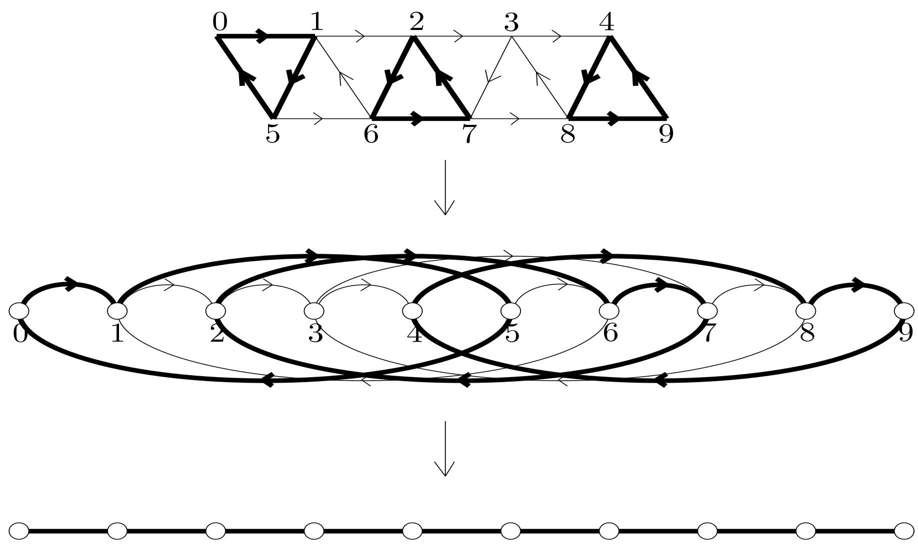


Figure 7.14: Labeling 'C'. The digraph has linear representative such that single cycles project to 6-mers. Pairs and triples of overlapping cycles project to higher order pavers.

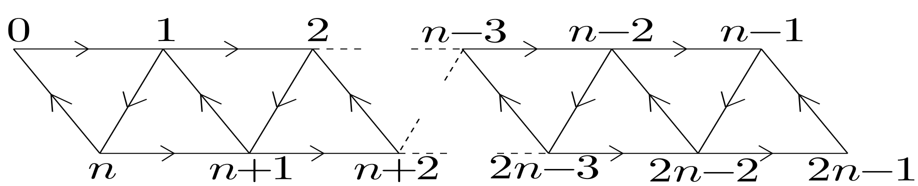


Figure 7.15: Labeling 'C'. An extension of the labeling indicated in Figure 7.14 to higher order digraphs.

7.5 Equivalent pavings

To define 'equivalant pavings' we need to formalize notions of 'overlapping cycles' and 'induced pavings' which we have used implicitly up until here.

Definition 66. *Let $\mathcal{D}$ be a digraph and $\mathcal{R}$ be a linear representative (see Definition 65) of $\mathcal{D}$. Then a pair of cycles $\{c_1, c_2\}$ is said to* **overlap in** $\mathcal{R}$ *provided that the following condition holds.*

- *There exists a triple of vertices u, v, w in $\mathcal{R}$ such that $u < v < w$ and either*
  - *$u, w \in c_1$ and $v \in c_2$; or*
  - *$u, w \in c_2$ and $v \in c_1$.*

A set of cycles $S = \{c_1, \ldots c_k\}$ is said to **overlap in** $\mathcal{R}$ *provided that for every cycle $c_i \in S$, there exists a distinct $c_j \in S$ (i.e. $c_i \neq c_j$) such that the pair $\{c_i, c_j\}$ overlap in $\mathcal{R}$. A set of cycles is called* **non-overlapping in** $\mathcal{R}$ *provided that they do not overlap in $\mathcal{R}$.*

Definition 67. *Let $\mathcal{D}$ be a digraph of order n. Let $\mathcal{R}$ be a linear representative of $\mathcal{D}$. Let $\mathcal{P}$ be a path graph of order n with vertex labeling $0, 1, \ldots n-1$. Let S be a basic configuration of cycles on $\mathcal{R}$. Then S projects to an 'induced paving' on $\mathcal{P}$ by the following steps.*

1. *Partition the set S into disjoint subsets*

$$S = \cup_{1 \leq i \leq k} S_i, \tag{7.21}$$
$$\emptyset = \cap_{1 \leq i \leq k} S_i; \tag{7.22}$$

   such that

   (a) *each S_i is a set of overlapping cycles, and*

   (b) *$S_i \cup S_j$ is non-overlapping for all $i \neq j$.*

   Note that this partitioning exists and is unique, as follows from the definition of an overlapping set of cycles.

2. *For each set S_i (which may contain one or more cycles) create a single paver on the path graph $\mathcal{P}$, as follows. Let p be the least vertex in S_i, and q be the greatest vertex, where least and greatest are defined according to the ordinary ordering of the integers. Then the* **paver induced on** $\mathcal{P}$ **by** $(S_i, \mathcal{R})$*, is the set $\{p, p+1, \ldots q\}$. The collection of pavers on $\mathcal{P}$ is called the* **paving induced on** $\mathcal{P}$ **by** $(S, \mathcal{R})$.

Further, the set of pavings *induced on $\mathcal{P}$ by $(\mathcal{S}, R)$, where*

$$\mathcal{S} = \{S | S \textit{ is a basic configuration of cycles}\} \tag{7.23}$$

is called a **complete set of induced pavings**.

We may now define 'equivalent pavings'.

Definition 68. *Let $\mathcal{D}$ be a digraph of fixed order n, and let S be a basic configuration of cycles on $\mathcal{D}$. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two linear representatives of $\mathcal{D}$, and let $\mathcal{P}$ be a path graph with vertices $0, 1, \ldots, n-1$. Let*

- *p_1 be the paving induced on $\mathcal{P}$ by $(S, \mathcal{R}_1)$; and*
- *p_2 be the paving induced on $\mathcal{P}$ by $(S, \mathcal{R}_2)$.*

Then p_1 and p_2 are termed **equivalent pavings**. *A set of equivalent pavings induced by $\mathcal{S}$ defined as in Equation (7.23), is called a* **complete set of equivalent pavings**.

For example, the pavings induced by Labelings 'A' and 'B' of the previous section are equivalent for any fixed even order of the inducing digraph. It follows from the definitions that

Lemma 54. *Let $\mathcal{P}$ be a path graph; with pavings thereon induced by some digraph $\mathcal{D}$. Then*

1. *A pair of equivalent pavings on $\mathcal{P}$ have the same paving monomial.*
2. *A pair of complete sets of equivalent pavings on $\mathcal{P}$ have the same paving polynomial.*

The implication of Lemma 54 is that in tackling a paving or cycle counting problem, we may potentially switch back and forth between different paving representations as convenient.

Further, if a given paving problem has a difficult recurrence, we may be able to replace it (perhaps by interpolating other polynomials, as arose in the example considered in Section 7.4) by an equivalent problem with a simpler recurrence. For instance, the recurrence

$$P_{2n}(\mu) = \mu^2 P_{2n-2}(\mu) - 2\mu P_{2n-4}(\mu) + P_{2n-6}(\mu) \tag{7.24}$$

of labeling 'B' from Section 7.4 is more complicated than the recurrence

$$P_k(\mu) = \mu P_{k-1} - P_{k-3} \tag{7.25}$$

of labeling 'A' from Section 7.4. Working with Equation (7.25) instead of Equation (7.24) loses no information, but it means the advantage of replacing characteristic polynomial

$$z^6 - \mu^2 z^4 + 2\mu z^2 - 1 \tag{7.26}$$

with

$$z^3 - \mu z^2 + 1. \tag{7.27}$$

The latter polynomial is more directly amenable to the root-simplifying methods of Chapter 8.

Chapter 8

Laurent Pavings

If a problem is symmetric in some ways we may derive some profit from noticing its interchangeable parts and it often pays to treat those parts which play the same role in the same fashion. Try to treat symmetrically what is symmetrical, and do not destroy wantonly any natural symmetry.

George Polya, 'How to Solve It' [89]

In this chapter we discover that recasting Ballot and Motzkin paving polynomials in terms of a well known change of variables, when interpreted combinatorially, leads to a new kind of paving which is unexpectedly simpler than the original. The relevant variable change has the form

$$\mu \mapsto a\rho + b + c\rho^{-1}, \tag{8.1}$$

with a, b and c constants and $b = 0$ for Ballot polynomials. We term the new kind of paving a 'Laurent paving' because the negative powers of ρ in Equation (8.1) lead to representation by a Laurent polynomial. When we need to distinguish between Laurent pavings (respectively Laurent paving polynomials) and our original pavings (respectively paving polynomials), we refer to the originals as **Viennot pavings** (respectively **Viennot paving polynomials**.)

The simplicity of Laurent pavings (which we will formally define below) was unexpected because the change of variables replaces one object with either two or three, so that one would expect an exponential proliferation of configurations. It turns out that the anticipated abundance is compensated for by a cascade of cancellation.

An advance clue to the simplicity of Laurent pavings could have been found in the simplicity of those Laurent polynomials which are obtained by making a substitution of the form (8.1) into Viennot Ballot and Motzkin paving polynomials; and which we later term ‘Laurent paving polynomials’ – see Theorems 56 and 63 below. These explicitly rational expressions are essential for the new variant of ‘Constant Term’ method which is introduced in Chapter 10.

In light of the new efficacies we have found for classic variable change (8.1), we look a little more deeply at the substitution, both from a combinatorial and an algebraic point of view, to see why this choice works well. In thinking about why, one perspective is that using *positive*, *zero* and *negative* powers of ρ reflects the geometry of walks on the underlying digraph, which allows a ‘disentangling’, both combinatorial and algebraic, of types of steps: *up*, *across* and *down*.

Next we apply, a priori, to other step sets the same considerations from which we derived, a posteriori, the Ballot/Motzkin substitution (8.1). By this approach we achieve partial success, in that we obtain a useful simplification but not complete rationalization of 2-up and Jump 2-step paving polynomials.

8.1 Disentangling Ballot pavings

We begin this section by defining Ballot Laurent pavings.

8.1.1 Laurent pavings

We first specify the pavers which shall be used in Ballot Laurent pavings, and some useful associated notation.

Definition 69. *Let $\mathcal{P}_k$ be a path graph of order k, with vertices $0, 1, \ldots, k-1$. Then a* **plus-mer** *is a distinguished vertex of $\mathcal{P}_k$ carrying weight ρ. A plus-mer is represented by the symbol*

$$\oplus \tag{8.2}$$

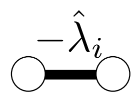
A **minus-mer** *is a distinguished vertex of $\mathcal{P}_k$ carrying weight $\lambda\rho^{-1}$. The coefficient λ is termed a* **background weight***. A minus-mer is represented by the symbol*

$$\ominus \tag{8.3}$$

The term **sign-mer** *is used to indicate either a plus-mer or a minus-mer. The symbol we use for a signmer, i.e. to represent a vertex for which it has not been determined whether it is paved as a plus-mer or minus-mer is*

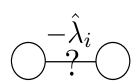
$$\oplus\!\!\!\!\!\pm \tag{8.4}$$

A **Laurent-weighted dimer** *is the dimer* $\{i-1, i\}$ *carrying weight* $-\hat{\lambda}_i$. *The symbol we use for a Laurent-weighted dimer is*



(8.5)

When $\hat{\lambda}_i = 0$, *the Laurent-weighted dimer is termed* **vanishing**, *and when* $\hat{\lambda}_i \neq 0$, *the Laurent-weighted dimer is termed* **non-vanishing**. *The symbol we use to represent an edge for which it has not been determined whether it is paved or not is*



(8.6)

Definition 70. *Let* $\{i_1 - 1, i_1\}$ *and* $\{i_2 - 1, i_2\}$ *be a pair of non-vanishing dimers such that* $i_2 - 1 > i_1$ *and such that there are no non-vanishing dimers inbetween. Then the set of vertices* $\{i_1+1, i_1+2, \ldots, i_2-2\}$ *is called the* **gap (between non-vanishing dimers)**. *A* **plus-minus run** *is a collection of plus-mers and minus-mers in a gap such that every dimer in the gap is paved with either a plus-mer or a minus-mer; and for each plus-mer in the gap, its vertex label is smaller than that for any minus-mer in the gap.*

Definition 71. *A* **Laurent Ballot paving** *of a path graph* $\mathcal{P}_k$ *is a set of*

- *non-vanishing dimers*
- *plus-mers, and*
- *minus-mers*

such that no two pavers share a vertex, and, furthermore, the gaps between non-vanishing dimers are each paved with plus-minus runs.

Definition 72. *Given a path graph* $\mathcal{P}_k$ *and set (possibly empty) of non-vanishing Laurent-weighted dimers* $\hat{\lambda}_k$, *the* **Laurent Ballot paving polynomial** *is the sum over the weights of Laurent pavings:*

$$\tilde{P}_k(\rho) := \sum_{r \in R_k} w(r), \tag{8.7}$$

where R_k is the set of Laurent Ballot pavings on $\mathcal{P}_k$. The **weight of a Laurent Ballot paving**, *$w(r)$, is the product*

$$w(r) = \prod_{m \in r} w(m), \tag{8.8}$$

with each m a paver in r and

$$w(m) = \begin{cases} \rho & \text{for } m \text{ a plus-mer} \\ \lambda\rho^{-1} & \text{for } m \text{ a minus-mer; and} \\ -\hat{\lambda}_k & \text{for } m \text{ the Laurent dimer } \{k-1, k\}. \end{cases} \tag{8.9}$$

An example of a Laurent Ballot Paving with associated Laurent Ballot Paving Polynomial is given in Figure 8.1.

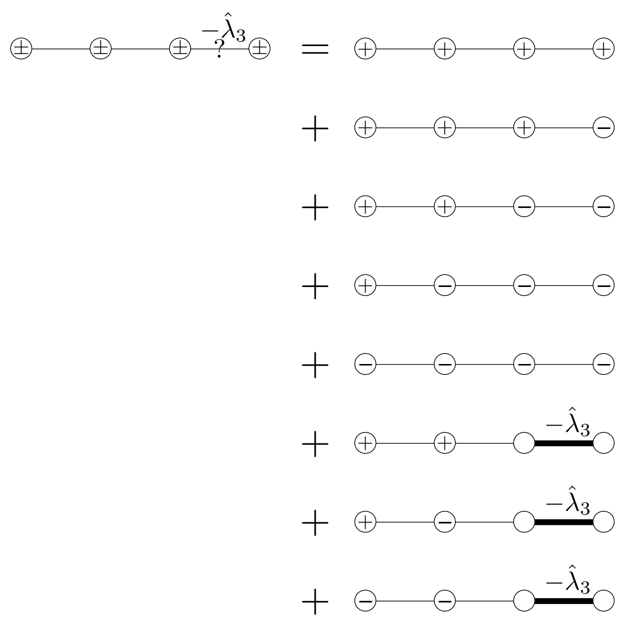


Figure 8.1: A Laurent Ballot paving with background weight λ, and single non-vanishing decoration $\hat{\lambda}_3$. The Laurent paving polynomial is $\tilde{P}_4(\rho) = \left(\rho^4 + \lambda\rho^2 + \lambda^2 + \lambda^3\rho^{-2} + \lambda^4\rho^{-4}\right) - \left(\rho^2 + \lambda + \lambda^2\rho^{-2}\right)\hat{\lambda}_3$.

We have defined Laurent Ballot paving polynomials as a weighted sum over Laurent Ballot pavings. The same algebraic objects may be obtained by making a substitution into the Viennot Ballot paving polynomials. This result is the content of the following key Theorem.

Theorem 55. *Fix λ. Let $\{P_k(\mu)\}_{k\geq 0}$ be a family of (Viennot) Ballot paving polynomials satisfying recurrence*

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda_k P_{k-1}(\mu) \tag{8.10}$$

for some sequence of λ_k's. Then under rational change of variables,

$$\mu \mapsto \rho + \lambda\rho^{-1}. \tag{8.11}$$

we have

$$P_k(\mu(\rho)) = \tilde{P}_k(\rho), \tag{8.12}$$

where $\tilde{P}_k(\rho)$'s are Laurent Ballot polynomials with Laurent weights on dimers, $\hat{\lambda}_k$, related to the coefficients in Equation (8.10) by

$$\hat{\lambda}_k := \lambda_k - \lambda. \tag{8.13}$$

Note, as we will see in the proof of Theorem 55, any choice of λ will give the same Laurent paving polynomial, but choosing λ such that the $\hat{\lambda}_k$'s equal zero for as many values of k as possible will reduce the complexity of the calculation. We first give an example of Theorem 55, and then follow with the proof.

A Laurent paving example

Fix λ. We consider the family of (Viennot) Ballot polynomials defined by the usual recurrence

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda_k P_{k-1}(\mu), \tag{8.14}$$

where in this instance the $\lambda_3 \neq \lambda$ and $\lambda_k = \lambda$ for all other values of k. Then we may calculate the corresponding family of Laurent polynomials (with respect to λ) defined by

$$\tilde{P}_k(\rho) = P_k(\rho + \lambda\rho^{-1}) \tag{8.15}$$

as a sum over Laurent pavings, as is illustrated for $\tilde{P}_4(\rho)$ in the right hand side of Figure 8.1. Alternatively one may calculate $P_4(\mu)$ using a Viennot paving as in the left hand side of the same figure, and then substitute

$$\mu = \rho + \lambda\rho^{-1}. \tag{8.16}$$

Carrying out this substitution shows that many terms cancel to give

$$
\begin{aligned}
P_4(\mu) &= \mu^4 - 2\lambda\mu^2 - \lambda_3\mu^2 + \lambda\lambda_3 && (8.17)\\
&= (\rho + \lambda\rho^{-1})^4 - 2\lambda(\rho + \lambda\rho^{-1})^2 - \lambda_3(\rho + \lambda\rho^{-1})^2 + \lambda\lambda_3 && (8.18)\\
&= \left(\rho^4 + \lambda\rho^2 + \lambda^2 + \lambda^3\rho^{-2} + \lambda^4\rho^{-4}\right) - \left(\rho^2 + \lambda + \lambda^2\rho^{-2}\right)(\lambda_3 - \lambda) && (8.19)\\
&= \tilde{P}_4(\rho). && (8.20)
\end{aligned}
$$

The left hand side of Figure 8.2 gives the Viennot paving polynomial, $P_4(\mu)$, in terms of Viennot pavings. The right hand side of Figure 8.2 indicates the calculation of the Laurent paving polynomial, $\tilde{P}_4(\rho)$, in terms of Laurent pavings.

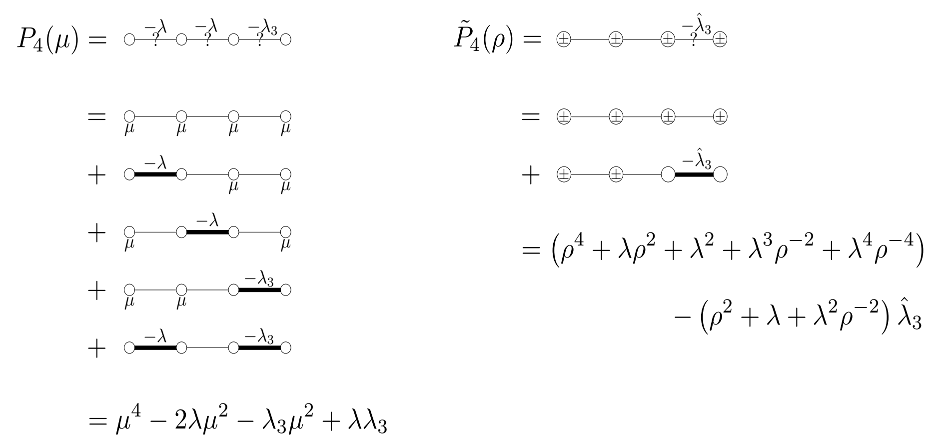
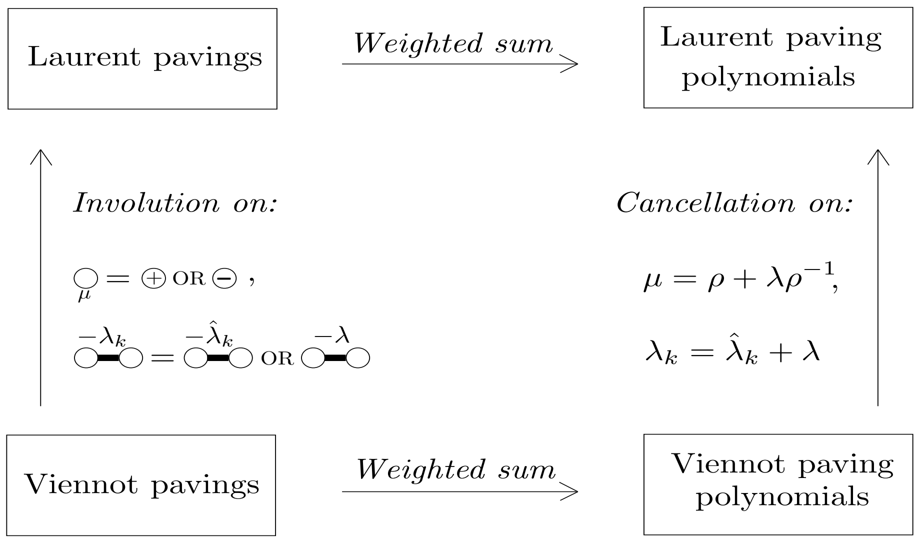


Figure 8.2: A Viennot paving and a Laurent paving give respectively the Viennot paving polynomial and the Laurent paving polynomial for edge weighting $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 \neq \lambda$; for some λ. The Laurent paving is shown in abbreviated form, with all possible plus-minus runs indicated by the $\pm$ symbols. Compare with the non-abbreviated version in Figure 8.1.

Proof of Theorem 55

Proof. Theorem 55 asserts the commutativity of the following diagram.



The proof of the theorem is by construction of the involution indicated. Let $\{P_k(\mu)\}_{k\geq 0}$ be a family of Viennot paving polynomials defined by the recurrence

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda_k P_{k-1}(\mu), \tag{8.21}$$

with

$$\{\lambda_k\}_{k\geq 0} \tag{8.22}$$

some sequence of weights. Fix λ. We will later see why it is advantageous to choose λ such that

$$\hat{\lambda}_k := \lambda_k - \lambda \tag{8.23}$$

is equal to zero for as many values of k as possible. We know by Lemma 42 that

$$P_k(\mu) = \sum_p w(p), \tag{8.24}$$

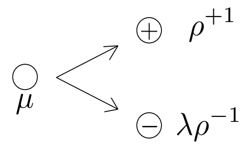
where p's are Viennot pavings with dimers weighted

$$w(\{k-1,k\}) = -\lambda_k \tag{8.25}$$

and uncovered vertices weighted

$$w(0\text{-mer}) = \mu. \tag{8.26}$$

Thus a first (inefficient) way to calculate the Laurent polynomial is to start with Viennot pavings, and create (and sum over) many new pavings by replacing instances of vertices weighted μ with plus-mers and minus-mers:



(8.27)

in all possible ways. Thus each Viennot paving containing n uncovered vertices spawns 2^n of the new pavings. We have

$$P_k(\rho + \lambda\rho^{-1}) = \sum_{p'} w(p'), \tag{8.28}$$

where p''s are pavings with plus-mers, minus-mers, and dimers (still) weighted as in Equation (8.25). At this stage, the plus-mers and minus-mers are interspersed amongst each other, and not restricted to plus-minus runs between dimers. We also have the 'wrong' kind of dimers for the theorem, and need to make another substitution to include the 'right' kind. A rearrangment of Equation (8.23) gives

$$\lambda_k = \hat{\lambda}_k + \lambda \tag{8.29}$$

which substitution can be performed combinatorially by replacing pavings containing instances of λ_k with two pavings, the first with $\hat{\lambda}_k$ in the position of the original λ_k and the second with λ in the position of the original λ_k. Now we have $P_k(\rho + \lambda\rho^{-1})$ in terms of $\hat{\lambda}_k$'s (and λ's) instead of λ_k's, via

$$P_k(\rho + \lambda\rho^{-1}) = \sum_{p''} w(p''), \tag{8.30}$$

where p'''s are pavings with plus-mers, minus-mers, and two kinds of dimers, those weighted $-\lambda$ and those weighted $-\hat{\lambda}_k$. The distribution of plus-mers and minus-mers is still general. Next we describe an involution which we show precisely cancels out by weight all pavings with plus-mers and minus-mers not in plus-minus runs, against all pavings containing dimers weighted $-\lambda$; leaving exactly the Laurent pavings described in Definition 70. We need some terminology.

- Let a **minus-plus pair** be adjacent vertices $\{i, i+1\}$ such that i is paved as a minus-mer and $i+1$ is paved as a plus-mer.
- Let a dimer, $\{i, i+1\}$, with weight $-\lambda$ be termed a **plain dimer**.
- Let a pair of adjacent vertices, $\{i, i+1\}$ which are either a minus-plus pair or a plain dimer be termed a **bad pair**.

- Let any paving containing a bad pair be called a **bad paving**.

We may now define the sets upon which the involution shall act.

- Let Ω be the set of pavings of the path graph $\mathcal{P}_k$ with plus-mers, minus-mers, plain dimers and Laurent weighted dimers (denoted p'' in Equation (8.30)).
- Let $\Omega^+ \subseteq \Omega$ be that subset of Ω consisting of pavings containing an even number of minus-plus pairs.
- Let $\Omega^- \subseteq \Omega$ be that subset of Ω consisting of pavings containing an odd number of minus-plus pairs.

It follows that

$$\Omega = \Omega^+ \cup \Omega^-, \text{ and} \tag{8.31}$$
$$\emptyset = \Omega^+ \cap \Omega^-. \tag{8.32}$$

Schematically we have Figure 8.3.

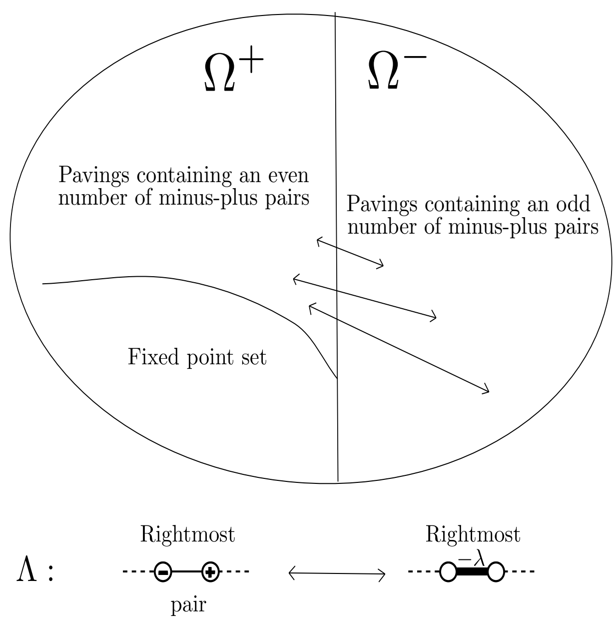


Figure 8.3: An involution exchanging minus-plus pairs for plain dimers.

It remains only to describe the involution. Let

$$\Lambda : \Omega \to \Omega \tag{8.33}$$

by the following rule.

- If $r \in \Omega$ is not a bad paving, then $\Lambda(r) = r$.
- Otherwise:
  - if the rightmost bad pair in r is a plain dimer, replace that pair with a minus-plus pair;
  - if the rightmost bad pair in r is a minus plus pair, replace that pair with a plain dimer.

We see that

$$\Lambda^2 = \textit{identity}, \tag{8.34}$$

as it must for an involution. Furthermore, since any bad paving must contain a rightmost bad pair, each bad paving in Ω^+ (respectively Ω^-) is mapped to a bad paving in Ω^- (respectively Ω^+). We note that Λ restricted to the class of bad pavings is a bijective mapping, as we see by the following considerations. (See Figure 8.4.)

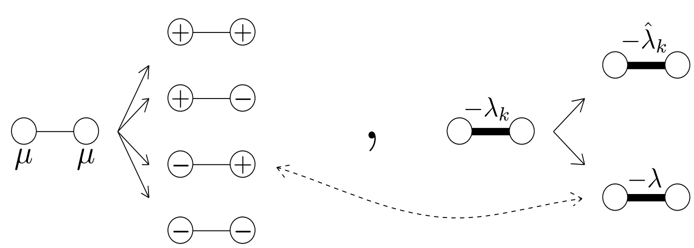


Figure 8.4: There is a bijection between minus-plus pairs and plain dimers.

- Within the context of Viennot pavings,
  - for each paving with an adjacent pair, $\{i, i+1\}$, of uncovered vertices, there exists another paving, equal in all other respects except that the pair $\{i, i+1\}$ are paved with a dimer weighted $-\lambda_i$;
  - for each paving with adjacent vertex pair $\{i, i+1\}$ paved with a dimer weighted $-\lambda_i$, there exists another paving, equal in all other respects except that $\{i, i+1\}$ are both uncovered.
- In the mapping from Viennot pavings to pavings in Ω,

– each pair of adjacent uncovered pavings generates exactly one minus-plus pair; and

– each dimer weighted $-\lambda_i$ generates exactly one plain dimer. (Note that this holds even when $-\lambda_i = -\lambda$. In this situation the Laurent dimer is vanishing, but the plain dimer is still created.)

Thus all pavings containing plain dimers are canceled out, and only those with Laurent dimers remain. Also, all pavings containing minus-plus pairs are canceled out, so that only those whose plus-mers and minus-mers form plus-minus runs, remain. Hence the fixed point set of the involution lies entirely within Ω^+ and is precisely the set of Laurent pavings.

It remains now only to check that the *weights* of bad pavings cancel. We see that they do, since

$$\begin{aligned} w(\text{minus-plus pair}) &= \rho\lambda\rho^{-1} && (8.35) \\ &= \lambda, && (8.36) \end{aligned}$$

whilst

$$w(\text{plain dimer}) = -\lambda. \tag{8.37}$$

As the mapping Λ alters only the right-most bad pair in a paving and leaves the rest intact, Equations (8.36) and (8.37) show that

$$w(\Lambda(r)) = -w(r), \tag{8.38}$$

as required. □

Comments on the involution proving Theorem 55

An example of the involution in the proof of Theorem 55 is illustrated in Figures 8.5–8.6. We now see why choosing λ such that

$$\hat{\lambda}_k := \lambda_k - \lambda \tag{8.39}$$

equals zero as often as possible is an advantage. This maximizes the number of vanishing Laurent dimers. Since vanishing dimers, by definition, have weight zero, they set to zero the weight of any paving they occur in, which means that those pavings need not be generated in the first place.

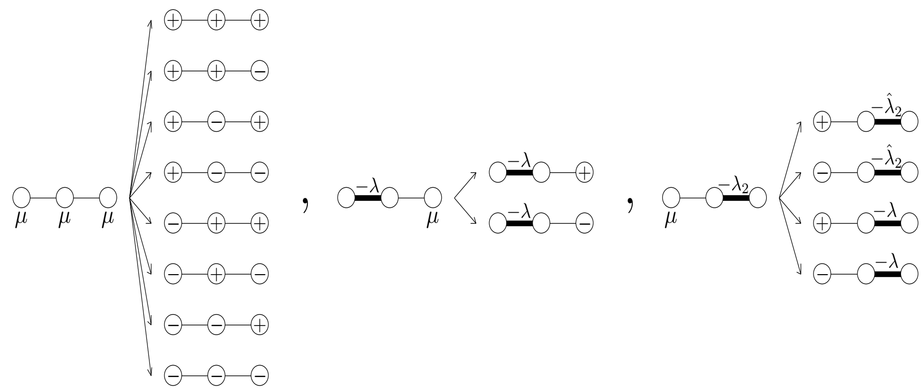


Figure 8.5: Viennot pavings of the path graph $\mathcal{P}_2$ generate pavings in Ω (as defined in the proof of Theorem 55). The choice $\lambda_1 = \lambda$, $\lambda_2 \neq \lambda$ is illustrated.

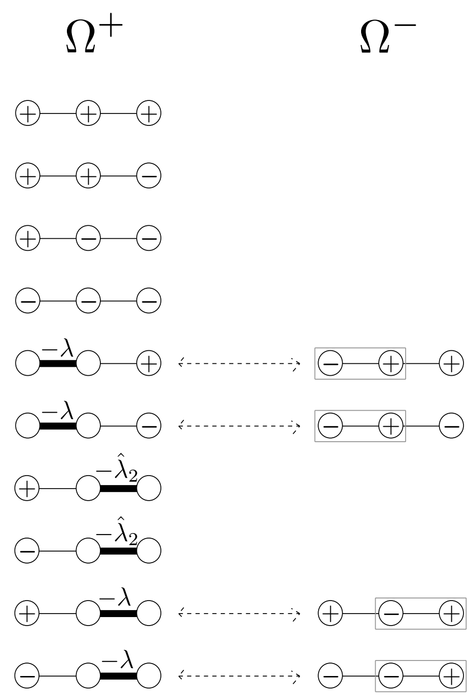


Figure 8.6: Ω-pavings of Figure 8.5 cancel under the involution of the proof of Theorem 55, leaving only Laurent pavings.

8.1.2 Laurent polynomials

The well-known part of this section is the simplifying effect of the change of variables on paving polynomials. Given the previous Theorem 55, we see that the simplified Viennot paving polynomial for uniformly weighted Ballot paths, after change of variables, is the Laurent paving polynomial for the same class of paths; and that it is explicitly rational. We use this classical change of variables, along with the combinatorics of pavings we have developed, to give Lemma 57, to also obtain explicitly rational expressions for decorated Laurent paving polynomials. It is on this lemma that the 'CT' method of Chapter 10 critically relies.

Theorem 56. *Let $\{M_k(\mu)\}_{k\geq 0}$ be the family of standard uniform Viennot Ballot paving polynomials and let $\{\tilde{M}_k(\rho)\}_{k\geq 0}$ be the associated family of standard uniform Ballot Laurent paving polynomials, under change of variables*

$$\mu \mapsto \rho + \lambda\rho^{-1}. \tag{8.40}$$

Then the closed form expression for uniform Viennot Ballot paving polynomials:

$$M_k(\mu) = \frac{(\mu + \sqrt{\mu^2 - 4\lambda})^{k+1} - (\mu - \sqrt{\mu^2 - 4\lambda})^{k+1}}{2^{k+1}\sqrt{\mu^2 - 4\lambda}} \tag{8.41}$$

becomes closed form expression for uniform Laurent Ballot paving polynomials:

$$\tilde{M}_k(\rho) = \frac{\rho^{k+1} - \lambda^{k+1}\rho^{-(k+1)}}{\rho - \lambda\rho^{-1}}. \tag{8.42}$$

Proof. By Theorem 55, the Laurent paving polynomial is obtained from the Viennot paving polynomial by direct substitution. □

We also obtain an immediate corollary to Lemma 44, giving decorated Laurent Ballot polynomials in explicitly rational closed form.

Lemma 57. *Let $\{\tilde{P}_k(\rho)\}_{k\geq 0}$ be a family of decorated Laurent Ballot paving polynomials with a finite number of non-vanishing weights, i.e.*

$$\hat{\lambda}_k \neq 0 \tag{8.43}$$

for only a finite number, d, of values of k. Then the $\tilde{P}_k$'s may be expressed as a finite sum of products of uniform Laurent Ballot paving polynomials, $\tilde{M}_j$. That is,

$$\tilde{P}_k(\rho) = \sum_{i=1}^{m} \tilde{c}_i \prod_{j=1}^{n} \tilde{M}_j(\rho), \tag{8.44}$$

where $\tilde{M}_j(\rho)$ *is given by Equation (8.42) of Theorem 56,*

$$1 \leq m \leq 2^d, \tag{8.45}$$

$$1 \leq n \leq d+1 \tag{8.46}$$

and $\tilde{c}_i$*'s are Laurent polynomials in* ρ *(with coefficient* $\hat{\lambda}_k$*'s). Further, the absolute value of the highest power of* ρ *in any* $\tilde{c}_i$ *is less than or equal to* $2d$*.*

An example we will make use of in the sequel is the Laurent polynomial for the l-wall weighting introduced in Section 6.1.

Theorem 58. *Let* $\{\tilde{P}_k(\rho)\}_{k\geq 0}$ *be the family of* ***l-wall weights decorated Ballot Laurent paving polynomials****, given by non-vanishing weights on dimers as follows.*

$$w(\{i-1,i\}) = \begin{cases} \hat{\kappa}_i & 1 \leq i \leq l \\ \hat{\omega}_{L-i+1} & L-l+1 \leq i \leq L \end{cases}. \tag{8.47}$$

Then

$$\begin{aligned} \tilde{P}_{L+1}(\rho) &= \left(\frac{\tilde{D}^{\kappa}_{l+1}(\rho) - \lambda\rho^{-1}\tilde{D}^{\kappa}_{l}(\rho)}{\rho^l}\right)\left(\frac{\tilde{D}^{\omega}_{l+1}(\rho) - \lambda\rho^{-1}\tilde{D}^{\omega}_{l}(\rho)}{\rho^l}\right)\frac{\rho^L}{\rho - \lambda\rho^{-1}} \\ &\quad - \left(\frac{\tilde{D}^{\kappa}_{l+1}(\rho) - \rho\tilde{D}^{\kappa}_{l}(\rho)}{\lambda^l\rho^{-l}}\right)\left(\frac{\tilde{D}^{\omega}_{l+1}(\rho) - \rho\tilde{D}^{\omega}_{l}(\rho)}{\lambda^l\rho^{-l}}\right)\frac{\lambda^L\rho^{-L}}{\rho - \lambda\rho^{-1}} \end{aligned} \tag{8.48}$$

where the $\tilde{D}$*'s are general decorated Laurent Ballot polynomials defined by*

$$\tilde{D}^{\kappa}_i(\rho) := D^{\kappa}_i(\mu), \tag{8.49}$$

where

$$D^{\kappa}_{i+1}(\mu) = \mu D^{\kappa}_i(\mu) - \kappa_i D^{\kappa}_{i-1}(\mu), \tag{8.50}$$

initial conditions $D^{\kappa}_0(\mu) = 1$ *and* $D^{\kappa}_1(\mu) = \mu$ *hold; and similarly for* $D^{\omega}_i(\mu)$*.*

Proof. Start with the result of Theorem 43. Then

$$\begin{aligned} \tilde{P}_{L+1}(\rho) &= \tilde{D}^{\kappa}_{l+1}(\rho)\tilde{D}^{\omega}_{l+1}(\rho)\tilde{M}_{L-2l-1}(\rho) \\ &\quad -\lambda\left(\tilde{D}^{\kappa}_{l}(\rho)\tilde{D}^{\omega}_{l+1}(\rho) + \tilde{D}^{\kappa}_{l+1}(\rho)\tilde{D}^{\omega}_{l}(\rho)\right)\tilde{M}_{L-2l-2}(\rho) \\ &\quad +\lambda^2\tilde{D}^{\kappa}_{l}(\rho)\tilde{D}^{\omega}_{l}(\rho)\tilde{M}_{L-2l-3}(\rho) \end{aligned} \tag{8.51}$$

$$\begin{aligned} &= \tilde{D}^{\kappa}_{l+1}(\rho)\tilde{D}^{\omega}_{l+1}(\rho)\left(\frac{\rho^{L-2l} - \lambda^{L-2l}\rho^{-L+2l}}{\rho - \lambda\rho^{-1}}\right) \\ &\quad -\lambda\left(\tilde{D}^{\kappa}_{l}(\rho)\tilde{D}^{\omega}_{l+1}(\rho) + \tilde{D}^{\kappa}_{l+1}(\rho)\tilde{D}^{\omega}_{l}(\rho)\right)\left(\frac{\rho^{L-2l-1} - \lambda^{L-2l-1}\rho^{-L+2l+1}}{\rho - \lambda\rho^{-1}}\right) \\ &\quad +\lambda^2\tilde{D}^{\kappa}_{l}(\rho)\tilde{D}^{\omega}_{l}(\rho)\left(\frac{\rho^{L-2l-2} - \lambda^{L-2l-2}\rho^{-L+2l+2}}{\rho - \lambda\rho^{-1}}\right) \end{aligned} \tag{8.52}$$

Collecting ρ^L and ρ^{-L} terms gives the theorem. □

This capacity to separate out the large L part of the expression from the rest of the Laurent polynomial will be a distinct advantage, come Chapter 10.

8.1.3 The 'disentangling' variable

We now go back and re-examine the classic change of variables which has worked so well in simplifying both pavings and paving polynomials. We first note the the non-isotropic version we use,

$$\mu \mapsto \rho + \lambda\rho^{-1} \tag{8.53}$$

is not the only possibility. It is perhaps more common to use the isotropic variant

$$\mu \mapsto \sqrt{\lambda}\rho + \sqrt{\lambda}\rho^{-1}. \tag{8.54}$$

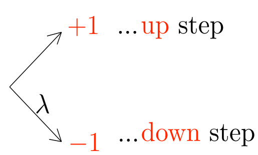
In fact any substitution of the form

$$\mu \mapsto \alpha\rho + \beta\rho^{-1} \tag{8.55}$$

where α and β are chosen so that

$$\alpha\beta = \lambda \tag{8.56}$$

will do the required job. We choose version (1.25), almost arbitrarily, as it most completely reflects the geometry of Ballot paths, as in (8.57).



$$\Longrightarrow \qquad \mu \mapsto \rho^{+1} + \lambda\rho^{-1} \tag{8.57}$$

Or, this choice could have arisen from seeking to find an explicitly rational solution to the Viennot Ballot paving recurrence (5.9). With this in mind, solve the characteristic equation (5.10)

$$z^2 - \mu z + \lambda = 0$$

for μ to obtain

$$\mu = z + \lambda z^{-1}. \tag{8.58}$$

Replacing z with ρ gives new characteristic polynomial

$$z^2 - (\rho + \lambda\rho^{-1})z + \lambda = 0. \tag{8.59}$$

This new polynomial factorizes, as

$$(z-\rho)(z-\lambda\rho^{-1}) = 0. \tag{8.60}$$

The factorization corresponds to the splitting of the second order paving polynomial recurrence for the uniform Ballot paving weighting into a pair of coupled first order recurrences, which we investigate in the next section.

8.1.4 Pairs of coupled recurrences for Laurent polynomials

The factorization of the characteristic polynomial (8.60) for uniform Viennot Ballot polynomials points to a capacity to split the second order recurrence for uniform Ballot polynomials into a pair of coupled first order recurrences. Said first order recurrences turn out to correspond to simply characterizable subsets of the Laurent pavings.

When we extend the analysis to decorated pavings, we retain the new pair of coupled recurrences but lose their first-order property. These 'split' Laurent polynomials are of independent interest in Section 8.1.5. Here we define them and investigate the coupled recurrences.

Definition 73. *Recall that*

$$\tilde{P}_k(\rho) = \sum_{r\in L_k} w(r) \tag{8.61}$$

where L_k *is the set of Laurent pavings on the path graph* $\mathcal{P}_k$ *with vertices* $0, 1, \ldots k-1$.

Given a Laurent polynomial $\tilde{P}_k(\rho)$*, the pair of* **signed Laurent polynomials** *are* $\tilde{P}_k^+(\rho)$ *and* $\tilde{P}_k^-(\rho)$ *as follows. The* **plus-suffixed Laurent paving polynomial** *is*

$$\tilde{P}_k^+(\rho) = \sum_{r\in L_k^+} w(r) \tag{8.62}$$

where $L_k^+ \subseteq L_k$ *is the set of Laurent pavings on the path graph such that either*

- *the last vertex, '*$k-1$*', is a plus-mer, or*
- *the last pair of vertices, '*$\{k-2, k-1\}$*', is a (Laurent) dimer.*

The **minus-suffixed Laurent paving polynomial** *is*

$$\tilde{P}_k^-(\rho) = \sum_{r \in L_k^-} w(r) \tag{8.63}$$

where $L_k^- \subseteq L_k$ is the set of Laurent pavings on the path graph such that

- *the last vertex, 'k − 1', is a minus-mer.*

The signed Laurent polynomials $\tilde{P}_k^+(\rho)$ and $\tilde{P}_k^-(\rho)$ are referred to as the **positive and negative parts**, *respectively, of $\tilde{P}_k(\rho)$.*

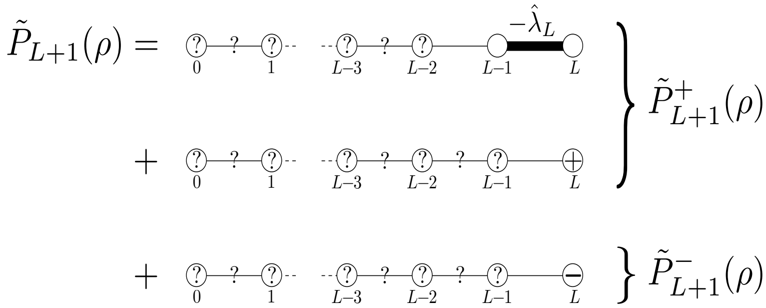


Figure 8.7: All pavings split into two groups: plus-suffixed and minus-suffixed, which give plus-suffixed and minus-suffixed paving polynomials, respectively.

Theorem 59. *Let $\{\tilde{P}_k(\rho)\}_{k \geq 0}$ be a family of decorated Laurent paving polynomials, and let $\{\tilde{P}_k^+(\rho)\}_{k \geq 0}$, $\{\tilde{P}_k^-(\rho)\}_{k \geq 0}$ be the associated pair of families of signed Laurent polynomials. Then*

$$\tilde{P}_k^+(\rho) = \rho \tilde{P}_{k-1}^+(\rho) - \hat{\lambda}_{k-1} \tilde{P}_{k-2}^+(\rho) - \hat{\lambda}_{k-1} \tilde{P}_{k-2}^-(\rho) \tag{8.64}$$

$$\tilde{P}_k^-(\rho) = \lambda \rho^{-1} \tilde{P}_{k-1}^+(\rho) + \lambda \rho^{-1} \tilde{P}_{k-1}^-(\rho) \tag{8.65}$$

subject to initial conditions

$$\tilde{P}_1^+(\rho) = \rho \tag{8.66}$$

$$\tilde{P}_0^+(\rho) = 1 \tag{8.67}$$

$$\tilde{P}_0^-(\rho) = 0. \tag{8.68}$$

Also,

$$\tilde{P}_k(\rho) = \tilde{P}_k^+(\rho) + \tilde{P}_k^-(\rho) \tag{8.69}$$

In particular, the family of uniform Laurent paving polynomials $\{\tilde{M}_k(\rho)\}_{k\geq 0}$*; has signed parts:* $\{\tilde{M}_k^+(\rho)\}_{k\geq 0}$, $\{\tilde{M}_k^-(\rho)\}_{k\geq 0}$*; for which coupled recurrences (8.64)–(8.65) become a pair of first order coupled recurrences*

$$\tilde{M}_k^+(\rho) = \rho\tilde{M}_{k-1}^+(\rho) \tag{8.70}$$
$$\tilde{M}_k^-(\rho) = \lambda\rho^{-1}\tilde{M}_{k-1}^+(\rho) + \lambda\rho^{-1}\tilde{M}_{k-1}^-(\rho), \tag{8.71}$$

subject to initial conditions

$$\tilde{M}_0^+(\rho) = 1 \tag{8.72}$$
$$\tilde{M}_0^-(\rho) = 0. \tag{8.73}$$

We have

$$\tilde{M}_k(\rho) = \tilde{M}_k^+(\rho) + \tilde{M}_k^-(\rho). \tag{8.74}$$

The occurrence of first order *coupled recurrences (8.70)–(8.71) coincides with the factorization of the characteristic polynomial for the classic uniform recurrence. That is,*

$$M_{k+1}(\mu) = \mu M_k(\mu) - \lambda M_{k-1}(\mu) \tag{8.75}$$

has characteristic polynomial

$$z^2 - \mu z + \lambda = 0 \tag{8.76}$$

which factors under variable change

$$\mu \mapsto \rho + \lambda\rho^{-1} \tag{8.77}$$

as

$$(z-\rho)(z-\lambda\rho^{-1}) = 0. \tag{8.78}$$

Proof. Equation (8.69) follows directly from the definition of signed paving polynomials, as illustrated in Figure 8.7. Coupled recurrences (8.64)–(8.65) follow from Figure 8.8, with equation (8.69) applied to the result. The initial conditions (8.66)–(8.68) are those required for consistency with classic paving initial conditions. The first order coupled recurrences (8.70)–(8.71) for the uniform case follow from (8.64)–(8.65), with all dimers vanishing. Equations (8.72)–(8.74) are also special cases of (8.67)–(8.69). Finally, (8.78) is immediate given Equations (8.75) and (8.76) from Lemma 33. □

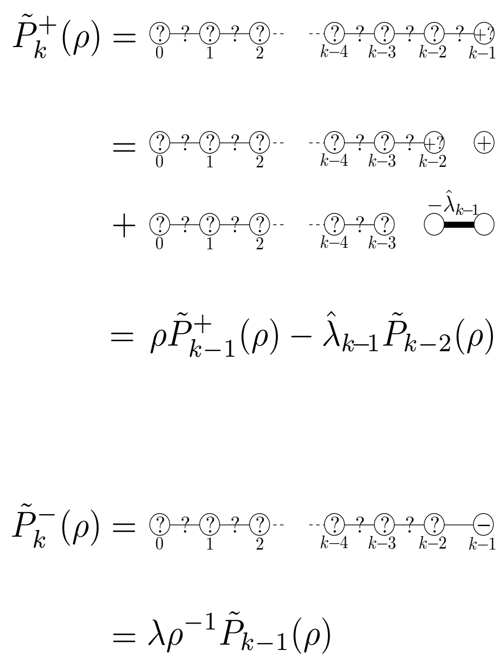


Figure 8.8: Signed Laurent polynomials are expressible in terms of shorter signed and unsigned Laurent polynomials.

8.1.5 Further combinatorics of the 'Wall Weights' example

An application of signed Laurent polynomials which will be of importance to us for calculations, in Chapter 10, is the 'wall-weighting'. The differences in Equation (8.48) of Theorem 58 have combinatorial interpretations.

Lemma 60. *Let*

$$\tilde{D}_l^+(\rho) = \sum_{p \in \mathcal{L}_l^+} w(p), \tag{8.79}$$

where $\mathcal{L}_l^+$ *is the set of Laurent pavings on the path graph of order* l *with vertices* $\{0, 1, \ldots, l-1\}$*, such that the paving ends in either a plus-mer or a dimer; and dimers* $\{i-1, i\}$ *are weighted* $-\hat{\kappa}_i$*. Let*

$$^-\tilde{D}_l(\rho) = \sum_{p \in {}^-\mathcal{L}_l} w(p), \tag{8.80}$$

where $^-\mathcal{L}_l$ *is the set of Laurent pavings on the path graph of order* l*, with vertices* $\{0, 1, \ldots, l-1\}$ *such that the paving begins in either a minus-mer or a dimer; and dimers* $\{i-1, i\}$ *are weighted* $-\hat{\kappa}_i$*. Then*

$$\tilde{D}_{l+1}(\rho) - \lambda\rho^{-1}\tilde{D}_l(\rho) = \tilde{D}_{l+1}^+(\rho) \tag{8.81}$$

and

$$\tilde{D}_{l+1}(\rho) - \rho\tilde{D}_l(\rho) = {}^-\tilde{D}_{l+1}(\rho). \tag{8.82}$$

Proof. The idea is illustrated in Figure 8.9. For equation (8.81), consider the term

$$\lambda\rho^{-1}\tilde{D}_l(\rho). \tag{8.83}$$

The factor $\tilde{D}_l(\rho)$ corresponds to the set of all Laurent pavings on l vertices. The factor $\lambda\rho^{-1}$ is the weight of a minus-mer. Appending a minus-mer to the end of any Laurent paving gives a new Laurent paving, since a minus-mer may follow either a dimer or a plus-minus run. Thus expression (8.83) corresponds to that subset of Laurent pavings on $l+1$ vertices which end with a minus-mer. Hence performing the difference $\tilde{D}_{l+1}(\rho) - \lambda\rho^{-1}\tilde{D}_l(\rho)$ combinatorially cancels off those terms and leaves precisely those Laurent pavings on $l+1$ vertices which end with either a plus-mer or a dimer; i.e. $\tilde{D}_{l+1}^+(\rho)$. Thus Equation (8.81) is shown.

The proof of Equation (8.82) is similar, except in this case we append a plus-mer to the *start* of a Laurent paving on l vertices, since a plus-mer may precede either a plus-minus run or a dimer in a Laurent paving. □

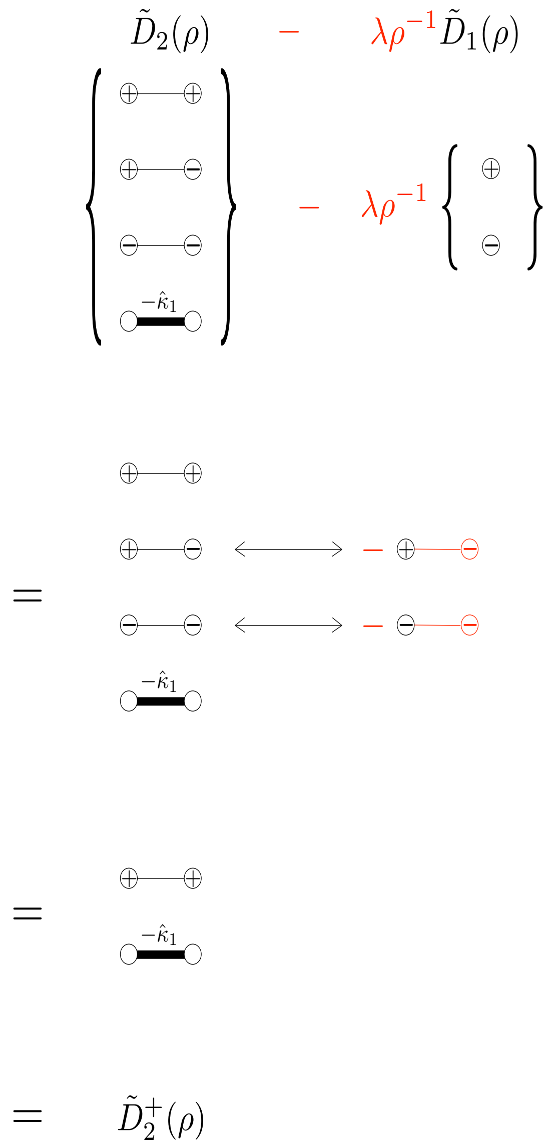


Figure 8.9: The difference $\tilde{D}_{l+1}(\rho) - \lambda\rho^{-1}\tilde{D}_l(\rho) = \tilde{D}^+_{l+1}(\rho)$; an example.

8.1.6 Viennot and Laurent pavings: complexity

Finally we compare and contrast Viennot and Laurent pavings with regard to which is the most practical to use, assuming that the algebraic object we really want is the Laurent paving polynomial.

Some relevant questions we could ask about pavings and paving polynomials, Veinnot and Laurent, are, can we find them:

1. reasonably quickly?
2. in closed form?
3. in a concise and explicitly rational closed form?

The latter two questions have been answered in the affirmative for the Ballot problem with a finite number of decorations, in Lemma 33 (uniform Viennot), Lemma 44 (decorated Viennot), Theorem 56 (uniform Laurent) and Lemma 57 (decorated Laurent). The remainder of this section is an outline of some broad considerations around the first question.

We restrict ourselves to the pavings for which we know the answer to the second and third questions, i.e. those for which the path graph may be arbitrarily long but there is a fixed finite number of decorations. We recall that the Viennot paving polynomials for such Ballot pavings satisfy three term recurrence

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda_k P_{k-1}(\mu),$$

for which only a finite number of the λ_k's differ from some background weight λ.

A baseline for complexity comparisons is given by considering how long it would take to find Viennot paving polynomials directly from the recurrence, without any of the shortcuts that we have developed. We observe that

1. Iterating the defining recurrence gives the k^{th} paving polynomial in $O(k)$ time.
2. Listing the pavings directly takes $O(g^k)$, time, where $g = (1+\sqrt{5})/2$ is the *golden ratio*.

It is an immediate consequence of the Viennot paving polynomial recurrence that Ballot paving polynomials and are counted by Fibonacci numbers, as is well-known in the literature of 'Matchings' on a path graph. (Note that a 'Matching' on a path graph is precisely a Ballot paving, i.e. a selection of dimers.)

Lemma 61. *Let $\mathcal{R}_k$ be the set of Viennot Ballot pavings on path graph $\mathcal{P}_k$. Then the number of Viennot Ballot pavings of $\mathcal{P}_k$ is*

$$|\mathcal{R}_k| = \frac{(1+\sqrt{5})^{k+1} - (1-\sqrt{5})^{k+1}}{2^{k+1}\sqrt{5}}. \tag{8.84}$$

Asymptotically,

$$|\mathcal{R}_k| \sim \left(\frac{1+\sqrt{5}}{2}\right)^k. \tag{8.85}$$

Expression (8.84) is the sequence of Fibonacci numbers.

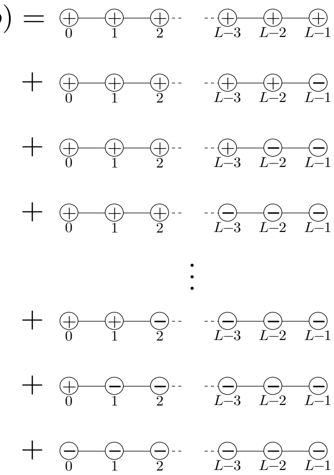
As is evident from Lemma 61, the utility of Viennot pavings as a computational tool does not derive from listing all possibilities directly, since this is an exponential-time process.

The alternative described in Chapter 6, of breaking up the Viennot pavings into decorated and undecorated stretches, submitting the latter to the standard algebra of constant coefficient recurrences, solving the former using either the recurrence or the pavings directly, and reassembling the constituent parts, is likely to be much faster, especially for pavings with long stretches of uniform weighting.

This 'break and recombine' approach yields Viennot paving polynomials directly, as sums of products of smaller uniform Viennot paving polynomials. This same approach may also yield Laurent polynomials, either by simply substituting the change of variables at the end, or by replacing the classic paving segments with Laurent paving segments in the penultimate step.

What of using Laurent pavings directly from the beginning? Firstly we note a difference from classic pavings of aesthetic if not computational importance. That is, for Laurent pavings we need not 'outsource' solution of the uniform stretches – we read their weight immediately off the combinatorics of plus-minus runs:

$$\tilde{M}_L(\rho) =$$



$$= \rho^L + \lambda\rho^{L-2} + \lambda^2\rho^{L-4} + \ \ldots + \lambda^L\rho^{-L}$$

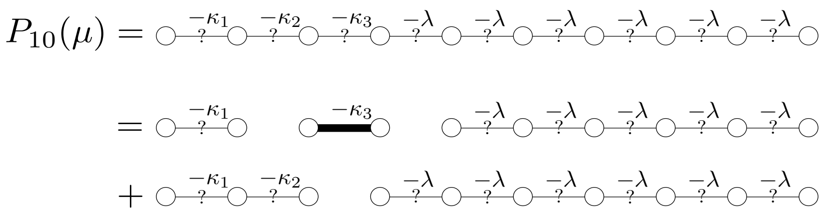
$$= \frac{\rho^{L+1} - \lambda^{L+1}\rho^{-L-1}}{\rho - \lambda\rho^{-1}} \tag{8.86}$$

Also, using Laurent pavings to find Laurent polynomials is optimally efficient in the sense that there is no cancellation of terms needed in transcribing the polynomial from the paving, whereas in getting the Laurent polynomial from the Viennot paving segments that are generated in the 'break and recombine' approach, some collecting of terms or cancellation may be required. Yet Viennot pavings are convenient in a way that Laurent pavings are not: we cannot directly and arbitrarily 'break and recombine' Laurent pavings as we can Viennot. Thus Laurent pavings are less convenient for compartmentalizing a calculation. We illustrate the trade-off with the example in Figures 8.10–8.11. The two pieces that the Viennot paving in Figure 8.10 breaks into become two parts of a Laurent polynomial under the familiar variable change $\mu \mapsto \rho + \lambda\rho^{-1}$. We have, for the weighting illustrated,

$$\begin{aligned} \tilde{P}_{10}(\rho) \quad = \quad & -(\hat{\kappa}_3 + \lambda)\left((\rho + \lambda\rho^{-1})^2 - (\hat{\kappa}_1 + \lambda)\right)\tilde{M}_6(\rho) \\ & + \left((\rho + \lambda\rho^{-1})^3 - (\hat{\kappa}_1 + \hat{\kappa}_2 + 2\lambda)(\rho + \lambda\rho^{-1})\right)\tilde{M}_7(\rho). \end{aligned} \tag{8.87}$$

Whereas the Laurent paving gives directly

$$\tilde{P}_{10}(\rho) = \tilde{M}_{10}(\rho) - \hat{\kappa}_1\tilde{M}_8(\rho) - \hat{\kappa}_2\tilde{M}_1(\rho)\tilde{M}_7(\rho) - \hat{\kappa}_3\tilde{M}_2(\rho)\tilde{M}_6(\rho) + \hat{\kappa}_1\hat{\kappa}_3\tilde{M}_6(\rho). \tag{8.88}$$



$$= -\kappa_3 \left(\mu^2 - \kappa_1\right) M_6(\mu) + \left(\mu^3 - (\kappa_1 + \kappa_2)\mu\right) M_7(\mu)$$

Figure 8.10: This Viennot paving breaks into two pieces.

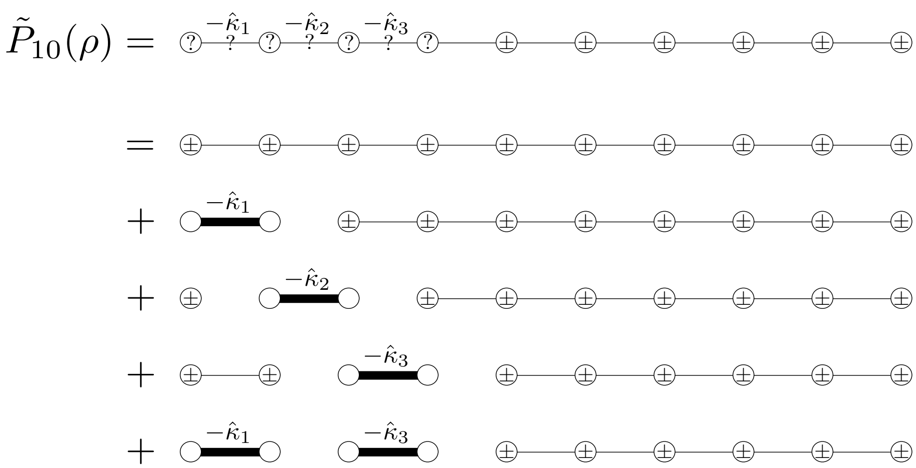


Figure 8.11: This Laurent paving problem splits into 5 terms.

We conclude that for finding Laurent polynomials, Laurent pavings are more elegant, Viennot pavings are more flexible; and the optimal choice of tool depends on the detail of the application.

8.2 Disentangling Motzkin pavings

8.2.1 Laurent pavings

'Laurent Motzkin pavings' are composed of the same objects as defined in Definition 69 for Ballot pavers, but with the following additions.

Definition 74. *Let $\mathcal{P}_k$ be a path graph of order k, with vertices $0, 1, \ldots, k-1$. Then a* **flat-mer** *is a distinguished vertex of $\mathcal{P}_k$ carrying weight b. A flat-mer is represented by the symbol*

$$\textcircled{b} \tag{8.89}$$

A **plain monomer** *is the monomer $\{i\}$ carrying weight $-b$. The symbol for a plain monomer is*

$$\overset{-b}{\bullet} \tag{8.90}$$

A **Laurent-weighted monomer** *is the monomer $\{i\}$ carrying weight $-\hat{b}_i$. The symbol for a Laurent-weighted monomer is*

$$\overset{-\hat{b}_i}{\bullet} \tag{8.91}$$

When $\hat{b}_i = 0$, the Laurent-weighted monomer is termed **vanishing***, and when $\hat{\lambda}_i \neq 0$, the Laurent-weighted monomer is termed* **non-vanishing***.*

Definition 75. *Let $\{i_1, i_2\}$ and $\{i_3, i_4\}$ be a pair of non-vanishing pavers (where i_1 may equal i_2, if the first is monomer; and i_3 may equal i_4, if the second is monomer) such that $i_3 > i_2$ and such that there are no non-vanishing pavers inbetween. Then the set of vertices $\{i_2 + 1, \ldots, i_3 - 1\}$ is called the* **gap (between non-vanishing pavers)***. A* **plus-minus run** *is a collection of plus-mers and minus-mers in a gap such that for each plus-mer in the gap, its vertex label is smaller than that for any minus-mer in the gap.*

Definition 76. *A* **Laurent Ballot paving** *of a path graph $\mathcal{P}_k$ is a set of*

- *non-vanishing dimers*
- *non-vanishing monomers*
- *plus-mers, and*
- *minus-mers*

such that no two pavers share a vertex, and, furthermore, the gaps between non-vanishing dimers are each paved with plus-minus runs.

Definition 77. *Given a path graph $\mathcal{P}_k$ and (possibly empty) sets of non-vanishing Laurent-weighted dimers $\hat{\lambda}_i$ and non-vanishing Laurent-weighted monomers $\hat{b}_j$ the*
Laurent Motzkin paving polynomial *is the sum over the weights of Laurent pavings:*

$$\tilde{P}_k(\rho) := \sum_{r \in R_k} w(r), \tag{8.92}$$

where R_k is the set of Laurent Motzkin pavings on $\mathcal{P}_k$; the **weight of a Laurent Ballot paving***, $w(r)$, is the product*

$$w(r) = \prod_{m \in r} w(m), \tag{8.93}$$

with each m a paver in r and

$$w(m) = \begin{cases} \rho & \text{for } m \text{ a plus-mer} \\ \lambda\rho^{-1} & \text{for } m \text{ a minus-mer} \\ -\hat{b}_k & \text{for } m \text{ the Laurent monomer } \{k\}\text{; and} \\ -\hat{\lambda}_k & \text{for } m \text{ the Laurent dimer } \{k-1, k\}. \end{cases} \tag{8.94}$$

Theorem 62. *Fix λ. Let $\{P_k(\mu)\}_{k \geq 0}$ be a family of (Viennot) Motzkin paving polynomials satisfying recurrence*

$$P_{k+1}(\mu) = (\mu - b_k)P_k(\mu) - \lambda_k P_{k-1}(\mu) \tag{8.95}$$

for some sequences of λ_i's and b_j's. Then under rational change of variables,

$$\mu \mapsto \rho + b + \lambda\rho^{-1}. \tag{8.96}$$

we have

$$P_k(\mu(\rho)) = \tilde{P}_k(\rho), \tag{8.97}$$

where $\tilde{P}_k(\rho)$'s are Laurent Motzkin polynomials with Laurent weights, $\hat{\lambda}_i$ and $\hat{b}_j$ related to the coefficients in Equation (8.10) by

$$\hat{\lambda}_i := \lambda_i - \lambda, \tag{8.98}$$
$$\hat{b}_j := b_j - b. \tag{8.99}$$

Proof. The proof is almost identical to that given for Ballot pavings. The necessary modification is indicated in Figure 8.12, where we see that b's generated by μ's in Equation (8.96) cancel exactly with b's generated by $\hat{b}_j$'s in Equation (8.99). □

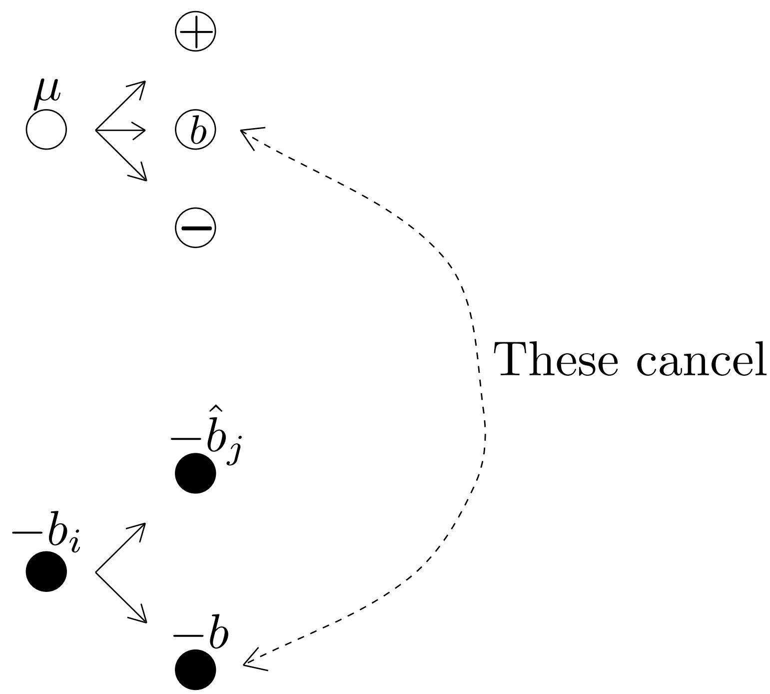


Figure 8.12: b's generated by μ's cancel exactly with $-b$'s generated by $-b_j$'s.

8.2.2 Laurent polynomials

Motzkin polynomials simplify in the same way that Ballot polynomials did, under an analogous classical change of variables.

Theorem 63. *Let $\{M_k(\mu)\}_{k\geq 0}$ be the family of standard uniform Viennot Motzkin paving polynomials and let $\{\tilde{M}_k(\rho)\}_{k\geq 0}$ be the corresponding family of standard uniform Motzkin Laurent paving polynomials, under change of variables*

$$\mu \mapsto \rho + b + \lambda\rho^{-1}. \tag{8.100}$$

Then the closed form expression for uniform Viennot Ballot paving polynomials:

$$M_k(\mu) = \frac{(\mu - b + \sqrt{(\mu-b)^2 - 4\lambda})^{k+1} - (\mu - b - \sqrt{(\mu-b)^2 - 4\lambda})^{k+1}}{2^{k+1}\sqrt{(\mu-b)^2 - 4\lambda}} \tag{8.101}$$

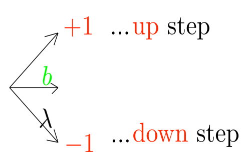
becomes closed form expression for uniform Laurent Ballot paving polynomials:

$$\tilde{M}_k(\rho) = \frac{\rho^{k+1} - \lambda^{k+1}\rho^{-(k+1)}}{\rho - \lambda\rho^{-1}}. \tag{8.102}$$

Notice that Equation (8.102) is also *identical* with Equation (8.42) for Ballot paths – the shift in variable obliterates b entirely. Also as with Ballot pavings, this Theorem may be used in conjugation with the combinatorics of pavings to obtain explicitly rational Laurent paving polynomials for decorations weightings containing any finite number of decorations.

8.2.3 The 'disentangling' variable

The disentangling variable for Motzkin pavings obeys the same considerations as did the one for Ballot pavings, and is a natural generalization of that transformation. Geometrically, we see



$$\Longrightarrow \qquad \mu \mapsto \rho^{+1} + b + \lambda\rho^{-1}$$

Algebraically, characteristic equation

$$z^2 - (\mu - b)z + \lambda = 0 \tag{8.103}$$

implies

$$\mu = z + b + \lambda z^{-1}. \tag{8.104}$$

Replacing z with ρ in Equation (8.104), and substituting back into Equation (8.103) gives

$$z^2 - (\rho + \rho^{-1})z + \lambda = 0, \tag{8.105}$$

which is identical to the corresponding equation, (8.59), for Ballot paths.

8.2.4 Viennot and Laurent pavings: complexity

Essentially the same considerations hold for ordinary and Laurent Motzkin pavings as did for Viennot and Laurent Ballot pavings. The main practical difference is that to use Viennot Motzkin pavings directly, one pays an even higher exponential price than for Viennot Ballot pavings, as shown by the following lemma, which quotes the classically fast growth of Pell numbers; and as illustrated in the subsequent example.

Lemma 64. *Let $\mathcal{R}_k$ be the set of Viennot Motzkin pavings on path graph $\mathcal{P}_k$. Then the number of Viennot Motzkin pavings of $\mathcal{P}_k$ is*

$$|\mathcal{R}_k| = \left(\frac{2+\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^k + \left(\frac{2-\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^k. \tag{8.106}$$

The first few terms in the sequence are $1, 2, 5, 12, 29, 70, \ldots$. *Asymptotically,*

$$|\mathcal{R}_k| \sim \left(1+\sqrt{2}\right)^k. \tag{8.107}$$

Expression (8.106) gives the sequence of Pell *numbers.*

Proof. The proof follows from the paving recurrence:

$$P_{k+1}(\mu) = (\mu - b_k)P_k(\mu) - \lambda_k P_{k-1}(\mu)$$

which implies that

$$|\mathcal{R}_{k+1}| = 2|\mathcal{R}_k| + |\mathcal{R}_{k-1}| \tag{8.108}$$

subject to $|\mathcal{R}_1| = 1$, $|\mathcal{R}_2| = 2$; the solution of which is the Pell numbers. □

Note that

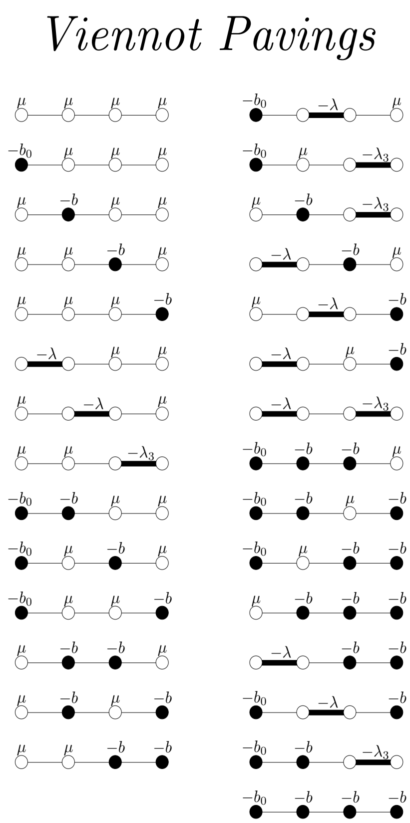
$$\begin{aligned} \text{Pell numbers} &\sim (2.41421...)^k, \\ \text{Fibonacci numbers} &\sim (1.61803...)^k; \end{aligned}$$

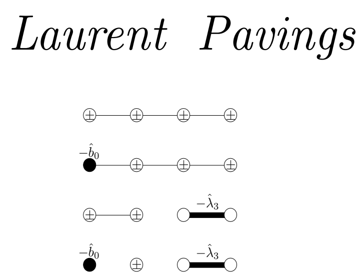
so that the penalty for going to Motzkin from Ballot paths, when directly listing all Viennot pavings, is quite severe.

A Laurent paving example

Fix λ and b. We consider the family of Viennot paving polynomials defined by the usual Viennot Motzkin recurrence

$$P_{k+1}(\mu) = (\mu - b_k)P_k(\mu) - \lambda_k P_{k-1}(\mu), \tag{8.109}$$





$$\tilde{P}_4(\rho) = \left(\rho^4 + \lambda\rho^2 + \lambda^2 + \lambda^3\rho^{-2} + \lambda^4\rho^{-4}\right)$$
$$-\hat{b}_0\left(\rho^3 + \lambda\rho + \lambda^2\rho^{-1} + \lambda^3\rho^{-3}\right)$$
$$-\hat{\lambda}_3\left(\rho^2 + \lambda + \lambda^2\rho^{-2}\right)$$
$$+\hat{b}_0\hat{\lambda}_3\left(\rho + \lambda\rho^{-1}\right)$$

$$P_4(\mu) = \mu^4 - (b_0 + 3b)\mu^3 + (3b_0 b + 3b^2 - 2\lambda - \lambda_3)\mu^2$$
$$+(b_0\lambda + b_0\lambda_3 + b\lambda_3 + 3b\lambda - 3b_0 b^2 - b^3)\mu$$
$$+\lambda\lambda_3 - b^2\lambda - b_0 b\lambda - b_0 b\lambda_3 + b_0 b^3$$

Figure 8.13: An example with two decorations, b_0 and λ_3; against background weights b on monomers and λ on dimers.

where we have decorations

$$\begin{aligned}
\lambda_3 &\neq \lambda, && (8.110)\\
\lambda_k &= \lambda \text{ otherwise}, && (8.111)\\
b_0 &\neq b, && (8.112)\\
b_k &= b \text{ otherwise}. && (8.113)
\end{aligned}$$

Then Viennot paving polynomials $\{P_k(\mu)\}_{k\geq 0}$ are given by Viennot pavings, as illustrated in the left hand side of Figure 8.13, and Laurent paving polynomials $\{\tilde{P}_k(\rho)\}_{k\geq 0}$, by the right hand side of Figure 8.13. Note that

$$\textcircled{\pm}\!\!-\!\!-\!\!-\!\!\textcircled{\pm}\cdots\quad\cdots\textcircled{\pm}\!\!-\!\!-\!\!-\!\!\textcircled{\pm} \qquad (8.114)$$

strings are shorthand for summing over all plus-minus runs on the selected vertices, *not* over arbitrary configurations of plus-mers and minus-mers. The Viennot paving polynomial we obtain is

$$\begin{aligned}
P_4(\mu) \;=\; & \mu^4 - (b_0 + 3b)\mu^3 + (3b_0 b + 3b^2 - 2\lambda - \lambda_3)\mu^2 \\
& +(b_0\lambda + b_0\lambda_3 + b\lambda_3 + 3b\lambda - 3b_0 b^2 - b^3)\mu \\
& +\lambda\lambda_3 - b^2\lambda - b_0 b\lambda - b_0 b\lambda_3 + b_0 b^3 \qquad (8.115)
\end{aligned}$$

Substituting for μ gives Laurent paving polynomial

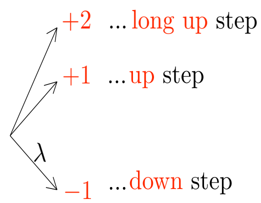
$$\begin{aligned}
P_4(\rho + b + \lambda\rho^{-1}) \;=\; & (\rho^4 + \lambda\rho^2 + \lambda^2 + \lambda^3\rho^{-2} + \lambda_4\rho^{-4}) \\
& -(b_0 - b)(\rho^3 + \lambda\rho + \lambda^2\rho^{-1} + \lambda^3\rho^{-3}) \\
& -(\lambda_3 - \lambda)(\rho^2 + \lambda + \lambda^2\rho^{-2}) \\
& +(b_0 - b)(\lambda_3 - \lambda)(\rho + \lambda\rho^{-1}) \qquad (8.116) \\
=\; & \tilde{P}_4(\rho). \qquad (8.117)
\end{aligned}$$

8.3 Partially disentangling 2-up pavings

8.3.1 Disentangling variables

The trick that works beautifully for Ballot and Motzkin pavings, of recasting the paving polynomial in another variable, works partially for other paving polynomials. Here, the geometric and algebraic ways we used previously to derive the appropriate parameter, are no longer in perfect agreement.

We consider Uniform 2-up standard paver weighting '1', as given by Definition 26. Considering the geometry of the corresponding path problem (see Definition 25), we expect parametrization



$$\Longrightarrow \qquad \mu \mapsto \rho^{+2} + \rho^{+1} + \lambda\rho^{-1} \tag{8.118}$$

On the other hand, solving characteristic equation (5.40)

$$z^3 - \mu z^2 + \lambda z + \lambda^2 = 0$$

for μ and relabeling gives alternative parametrization

$$\mu \mapsto r + \lambda r^{-1} + \lambda^2 r^{-2}. \tag{8.119}$$

Under either of the closely related parametrizations (8.118) or (8.119), the characteristic polynomial factorizes, though not completely. We have

- 'Geometric parametrization' of the characteristic equation (5.40) gives

  $$\left(z - \lambda\rho^{-1}\right)\left(z^2 - \left(\rho + \rho^2\right) z - \lambda\rho\right) = 0 \tag{8.120}$$

  with roots

  $$z = \lambda\rho^{-1}, \qquad z = \frac{\rho + \rho^2 \pm \sqrt{\rho}\sqrt{\rho^3 + 2p^2 + \rho + 4\lambda}}{2}. \tag{8.121}$$

- 'Algebraic parametrization' of the characteristic equation (5.40) gives

  $$(z - r)\left(z^2 - \left(\lambda r^{-1} + \lambda^2 r^{-2}\right) z - \lambda^2 r^{-1}\right) = 0 \tag{8.122}$$

  with roots

  $$z = r, \qquad z = \frac{r\lambda + \lambda^2 \pm \lambda\sqrt{4r^3 + r^2 + 2\lambda r + \lambda^2}}{2r^2}. \tag{8.123}$$

Both sets of roots (8.121) and (8.123) are much simpler than the roots of equation (5.40) in terms of μ, the latter expression being a cumbersome triad of sums and differences involving square and cube roots as well as imaginary numbers. However (8.121) and (8.123) still contain quadratic terms, so to find an explicitly rational paving polynomial we require the following lemma.

Lemma 65. *The following combination of powers is expressible in sum form as follows.*

$$(\alpha+\beta)\left(\gamma+\delta\right)^{k}+(\alpha-\beta)\left(\gamma-\delta\right)^{k}=2\sum_{i\geq 0}\gamma^{k-2i}\delta^{2i}\left(\alpha\binom{k}{2i}+\beta\binom{k}{2i+1}\gamma^{-1}\delta\right) \tag{8.124}$$

Proof. The proof is straightforward algebraic manipulation.

$$\begin{aligned}
& (\alpha+\beta)\left(\gamma+\delta\right)^{k}+(\alpha-\beta)\left(\gamma-\delta\right)^{k} && (8.125)\\
= \;& (\alpha+\beta)\left(\sum_{i\geq 0}\binom{k}{i}\gamma^{k-1}\delta^{i}\right)+(\alpha-\beta)\left(\sum_{i\geq 0}(-1)^{i}\binom{k}{i}\gamma^{k-1}\delta^{i}\right) && (8.126)\\
= \;& \alpha\left(\binom{k}{0}\gamma^{k}+\binom{k}{1}\gamma^{k-1}\delta+\binom{k}{2}\gamma^{k-2}\delta^{2}+\binom{k}{3}\gamma^{k-3}\delta^{3}+\dots\right)\\
& +\alpha\left(\binom{k}{0}\gamma^{k}-\binom{k}{1}\gamma^{k-1}\delta+\binom{k}{2}\gamma^{k-2}\delta^{2}-\binom{k}{3}\gamma^{k-3}\delta^{3}+\dots\right)\\
& +\beta\left(\binom{k}{0}\gamma^{k}+\binom{k}{1}\gamma^{k-1}\delta+\binom{k}{2}\gamma^{k-2}\delta^{2}+\binom{k}{3}\gamma^{k-3}\delta^{3}+\dots\right)\\
& +\beta\left(-\binom{k}{0}\gamma^{k}+\binom{k}{1}\gamma^{k-1}\delta-\binom{k}{2}\gamma^{k-2}\delta^{2}+\binom{k}{3}\gamma^{k-3}\delta^{3}-\dots\right) && (8.127)\\
= \;& 2\alpha\sum_{i\geq 0}\binom{k}{2i}\gamma^{k-2i}\delta^{2i}+2\beta\sum_{i\geq 0}\binom{k}{2i+1}\gamma^{k-2i-1}\delta^{2i+1} && (8.128)\\
= \;& 2\sum_{i\geq 0}\gamma^{k-2i}\delta^{2i}\left(\alpha\binom{k}{2i}+\beta\binom{k}{2i+1}\gamma^{-1}\delta\right). && (8.129)
\end{aligned}$$

□

Parametrizations (8.121) and (8.123) are similar. A feature of algebraic parametrization (8.123) is that there is symmetry under $z \leftrightarrow r$. In other words, solving for r in terms of z gives the same functional form as solving for z in terms of r. We choose to find paving polynomials for the algebraic parametrization.

8.3.2 Paving polynomials

Theorem 66. *Let $\{P_k(r)\}_{k\geq 0}$ be the family of 2-up uniform paving polynomials which satisfy*

$$P_k = \mu P_k - \lambda P_{k-1} - \lambda^2 P_{k-2} \tag{8.130}$$

subject to initial conditions

$$P_0 = 1 \tag{8.131}$$
$$P_1 = \mu \tag{8.132}$$
$$P_2 = \mu^2 - \lambda. \tag{8.133}$$

under parametrization

$$\mu \mapsto r + \lambda r^{-1} + \lambda^2 r^{-2}. \tag{8.134}$$

Let

$$z_1 := r \tag{8.135}$$
$$z_2 := \frac{\lambda(r+\lambda)}{2r^2} + \frac{\lambda\sqrt{4r^3 + r^2 + 2\lambda r + \lambda^2}}{2r^2} \tag{8.136}$$
$$z_3 := \frac{\lambda(r+\lambda)}{2r^2} - \frac{\lambda\sqrt{4r^3 + r^2 + 2\lambda r + \lambda^2}}{2r^2} \tag{8.137}$$

and

$$a_1 = \frac{r^3}{r^3 - \lambda r - 2\lambda^2} \tag{8.138}$$
$$a_2 = \frac{-\lambda(r+2\lambda)}{2(r^3 - \lambda r - 2\lambda^2)} + \frac{\lambda\left(2r^3 + r^2 + 3\lambda r + 2\lambda^2\right)}{2(r^3 - \lambda r - 2\lambda^2)\sqrt{4r^3 + r^2 + 2\lambda r + \lambda^2}} \tag{8.139}$$
$$a_3 = \frac{-\lambda(r+2\lambda)}{2(r^3 - \lambda r - 2\lambda^2)} - \frac{\lambda\left(2r^3 + r^2 + 3\lambda r + 2\lambda^2\right)}{2(r^3 - \lambda r - 2\lambda^2)\sqrt{4r^3 + r^2 + 2\lambda r + \lambda^2}}. \tag{8.140}$$

Then

$$P_k(r) = a_1 z_1^k + a_2 z_2^k + a_3 z_3^k. \tag{8.141}$$

Furthermore, as an explicitly rational function, we have expression

$$\begin{aligned} P_k &= \frac{r^{k+3}}{r^3 - \lambda r - 2\lambda^2} \\ &+ \frac{\lambda^{k+1}}{2^k r^{2k}(r^3 - \lambda r - 2\lambda^2))} \sum_{i \geq 0} (r+\lambda)^{k-2i} (4r^3 + r^2 + 2\lambda r + \lambda^2)^i \times \\ & \left\{ -(r+2\lambda)\binom{k}{2i} + \frac{2r^3 + r^2 + 3\lambda r + 2\lambda^2}{r+\lambda}\binom{k}{2i+1} \right\}. \end{aligned} \tag{8.142}$$

Proof. Solving characteristic equation (5.40)

$$z^3 - \mu z^2 + \lambda z + \lambda^2 = 0$$

under parametrization (8.134) gives roots (8.135)–(8.137). Application of initial conditions (8.131)–(8.133) gives coefficients (8.138)–(8.140) in the standard ansatz (8.141). It remains only to show that (8.141) may be expressed as (8.142). This requires straightforward but tedius algebra, as follows.

Abbreviate

$$\begin{aligned} A &= -(r+2\lambda) && (8.143)\\ B &= r^3-\lambda r-2\lambda^2 && (8.144)\\ C &= 2r^3+r^2+3\lambda r+2\lambda^2 && (8.145)\\ D &= 4r^3+r^2+2\lambda r+\lambda^2 && (8.146)\\ E &= r+\lambda && (8.147) \end{aligned}$$

Therefore,

$$\begin{aligned} a_2z_2^k+a_3z_3^k = {} & \left(\frac{\lambda A}{2B}+\frac{\lambda C}{2B\sqrt{D}}\right)\left(\frac{\lambda E}{2r^2}+\frac{\lambda\sqrt{D}}{2r^2}\right)^k \\ & +\left(\frac{\lambda A}{2B}-\frac{\lambda C}{2B\sqrt{D}}\right)\left(\frac{\lambda E}{2r^2}-\frac{\lambda\sqrt{D}}{2r^2}\right)^k \qquad (8.148) \end{aligned}$$

so that

$$\begin{aligned} \frac{2^{k+1}r^{2k}B\left(a_2z_2^k+a_3z_3^k\right)}{\lambda^{k+1}} = {} & \left(A+\frac{C}{\sqrt{D}}\right)\left(E+\sqrt{D}\right)^k \\ & +\left(A-\frac{C}{\sqrt{D}}\right)\left(E-\sqrt{D}\right)^k \qquad (8.149) \end{aligned}$$

Hence, by Lemma 65,

$$\begin{aligned} & \frac{2^{k+1}r^{2k}B\left(a_2z_2^k+a_3z_3^k\right)}{\lambda^{k+1}} \\ & = 2\sum_{i\geq 0}E^{k-2i}D^i\left\{A\binom{k}{2i}+\left(\frac{C\sqrt{D}}{E\sqrt{D}}\right)\binom{k}{2i+1}\right\}. \qquad (8.150) \end{aligned}$$

Thus

$$\begin{aligned} & a_2z_2^k+a_3z_3^k \\ = {} & \frac{\lambda^{k+1}}{2^kr^{2k}(r^3-\lambda r-2\lambda^2))}\sum_{i\geq 0}(r+\lambda)^{k-2i}(4r^3+r^2+2\lambda r+\lambda^2)^i\times \\ & \left\{-(r+2\lambda)\binom{k}{2i}+\frac{2r^3+r^2+3\lambda r+2\lambda^2}{r+\lambda}\binom{k}{2i+1}\right\}. \qquad (8.151) \end{aligned}$$

Adding in the term

$$a_1 z_1^k = \frac{r^{k+3}}{r^3 - \lambda r - 2\lambda^2} \tag{8.152}$$

gives Equation (8.142).

□

We comment that paving polynomial expression (8.142) is both explicitly rational and much terser than the corresponding expression would be in the original variable μ. Nonetheless, it is a much more complicated expression that the analogue for Ballot paths, which was

$$\sum_{i=0}^{k} \lambda^i \rho^{k-2i}$$

and in turn was abbreviated as ratio

$$\frac{\rho^{k+1} - \lambda^{k+1}\rho^{-k-1}}{\rho - \lambda\rho^{-1}}.$$

There is further work in determining whether expression (8.142) can be simplified more; and in particular whether it can be expressed as a sum with fewer than $O(k)$ terms.

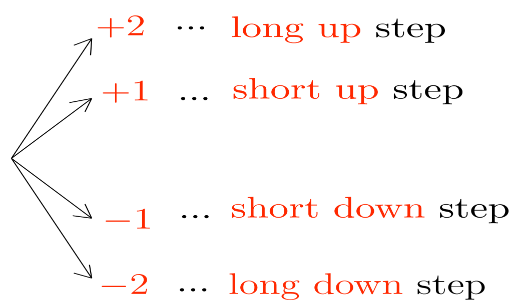
We conclude this chapter by making a rudimentary reconnaissance of Jump 2-step pavings.

8.4 Partially disentangling Jump 2-step pavings

8.4.1 Disentangling variables

Jump 2-step pavings are another natural case to try to find disentangling variables for. On the one hand, they are more symmetrical than 2-up pavings, so one might expect neater results. On the other hand, they are in a sense more entangled, if one thinks about the associated digraph from Figure 5.19 and the multiplicity of cycles thereon. It turns out that both these features are reflected in the partially disentanglements we obtain.

We consider the trivial Uniform Jump 2-step paver weighting, as given by Definition 46. Considering the geometry of the corresponding path problem (see Definition 45), we expect parametrization



(8.153)

On the other hand, solving characteristic equation (5.116)

$$z^5 - (\mu+1)z^4 + (\mu+1)z^3 + (\mu+1)z^2 - (\mu+1)z + 1 = 0$$

for μ and relabeling gives alternative parametrization

$$\mu \mapsto \frac{r^2 - 2r + 3 - 2r^{-1} + r^{-2}}{r - 2 + r^{-1}}. \tag{8.154}$$

Despite being quintic instead of cubic, the Jump-2 characteristic polynomial (5.116) compares favourably with 2-up characteristic polynomial (5.40) of the previous section. The Jump-2 characteristic polynomial (5.116) has much simpler roots, and factors into two pieces even before we change variables, as

$$(z+1)\left(z^4 - (\mu+2)z^3 + (2\mu+3)z^2 - (\mu+2)z + 1\right) = 0, \tag{8.155}$$

which we recall gives roots (5.117)

$$z = -1, z = \frac{1}{4}\left(2 \pm \sqrt{\mu^2 - 4\mu} \pm \sqrt{2}\sqrt{-10 - 4\mu - \sqrt{\mu^2 - 4\mu}(2+\mu) + (2+\mu)^2}\right).$$

Under either parametrization (8.153) or (8.154), the characteristic polynomial factorizes further, though still not completely. We have

- 'Geometric parametrization' of characteristic equation (5.116) implies

$$(z+1)\left(z^2 - \left(1 + \rho^{-1} + \rho^{-2}\right)z + \rho^{-2}\right)\left(z^2 - \left(\rho^2 + \rho + 1\right)z + \rho^2\right) = 0 \tag{8.156}$$

with roots

$$z_1 = -1 \tag{8.157}$$

$$z_{2,3} = \frac{1+\rho+\rho^2 \pm \sqrt{1+2\rho-\rho^2+2\rho^3+\rho^4}}{2} \tag{8.158}$$

$$z_{4,5} = \frac{1+\rho+\rho^2 \pm \sqrt{1+2\rho-\rho^2+2\rho^3+\rho^4}}{2\rho^2}. \tag{8.159}$$

- 'Algebraic parametrization' of characteristic equation (5.116) implies

$$(z+1)(z-r)\left(z-r^{-1}\right)\left(z^2-\left(\frac{2r-3+2r^{-1}}{r-2+r^{-1}}\right)z+1\right)=0 \tag{8.160}$$

  with roots

$$z_1 = -1 \tag{8.161}$$

$$z_{2,3} = r^{\pm 1} \tag{8.162}$$

$$z_{4,5} = \frac{2r-3+2r^{-1} \pm \sqrt{4r-7+4r^{-1}}}{2(r-2+r^{-1})}. \tag{8.163}$$

  Note that equations (8.160) and (8.163) have been written in such a fashion as to emphasize the

$$r \leftrightarrow r^{-1} \tag{8.164}$$

  symmetry. For computational purposes when utilizing a computer algebra system, it may be better to work in ratios of positive powers of r, only.

Both the 'geometric' and 'algebraic' parametrizations result in quadratic roots. Thus Lemma 65 could be used, as in the previous section, to give an explicitly rational paving polynomial. As with 2-up pavings, the answer is not nearly so aesthetically pleasing as it was for Ballot/Motzkin pavings. There is room for further work in determining whether yet greater simplification is possible. In this vein we note that roots z_4 and z_5 in Equation (8.163) are inverses of each other. Also, it would be interesting to find a geometric interpretation of the 'algebraic' parametrization.

Part III

More Paths

Chapter 9

Generating functions and Recurrences

The difficulty in lattice path enumeration problems increases rapidly with the introduction of weights which vary with height. Whilst techniques to solve the unweighted versions of various enumerations were developed in the nineteen hundreds and even the late eighteen hundreds (see Chapter 1), the corresponding weighted problems, for arbitrary weightings, have resisted until much more recently. For example, the problem of counting Ballot-like paths in a strip with a single independent weight associated with each wall was posed in 1971 [37] and only solved in 2006 [15], [25]; the solution of which we describe in Chapter 10.

The heart of the difficulty is that the recurrence relation which defines the desired weight polynomial is

1. a partial difference equation with boundary conditions,

2. with in general, non-constant coefficients.

For instance weight polynomials for Ballot paths with distinct weights for downsteps from each distinct height have recurrence relation:

$$B_t(y) = B_{t-1}(y-1) + \lambda_{y+1} B_{t-1}(y+1) \tag{9.1}$$

with boundary conditions $B_t(0) = \lambda_1 B_{t-1}(1)$, $B_t(L) = B_{t-1}(L-1)$, and initial condition $B_t(0) = \delta_{y,0}$. Three combinatorial approaches to finding generating functions for such paths are

1. Transfer matrices interpreted as adjacency matrices of digraphs give generating functions.

2. Generating functions may be derived from single variable constant coefficient recurrences, and conversely.

3. 'Heaps' give generating functions.

The transfer matrix method of Item 1 is classic, and we briefly review it. The original material of this chapter is contained in Item 2, and relates to another well-known result: the algebraic derivation of generating functions from constant coefficient recurrences or constant coefficient recurrences from generating functions. The original contribution we make is the discovery of a purely combinatorial method for finding constant coefficient recurrences directly. Finally for completeness we mention Item 3, and refer the reader to the extensive literature on the combinatorial objects known as 'Heaps' [105] for a third combinatorial derivation of generating functions.

9.1 Transfer Matrices

Let $T = (T_{i,j})_{0\leq i,j\leq L}$ be a transfer matrix, where the $(i,j)^{\text{th}}$ entry is the weight of a step from height i to height j. For example, the transfer matrix for paths in a strip of height L and satisfying recurrence relation (9.1) is

$$T_L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ \lambda_1 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & 1 & 0 & & \vdots \\ 0 & 0 & \lambda_3 & 0 & 1 & \ddots & \\ 0 & 0 & 0 & \lambda_4 & 0 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & & 0 & \lambda_L & 0 \end{pmatrix}. \tag{9.2}$$

The weight function which we ultimately want for paths of length t, starting at height y' and ending at height y, is

$$g_{y',y;L}(t) = \left(T_L^t\right)|_{y',y} \tag{9.3}$$

Then the path length generating function, by definition, is

$$G_{y',y;L}(x) = \sum_{n\geq 0} (T_L^n)\big|_{y',y} x^n. \tag{9.4}$$

For small x, this series may be written

$$G_{y',y;L}(x) = (I - xT_L)^{-1}\big|_{y',y}. \tag{9.5}$$

We note in passing that an expression of the form '$(I - xT)^{-1}$' is called a **resolvent** (of T), and that these are well-studied objects [69]. Here we need the elementary result expressing the inverse of a matrix in terms of its cofactors and determinant, to conclude another classic result:

Lemma 67. *The path length generating function, $G_{y',y;L}(x)$, for paths starting at height y', ending at height y, confined to a strip of height L and described by transfer matrix T_L as in Equation 9.2, is*

$$G_{y',y;L}(x) = \frac{\mathrm{cof}_{y,y'}(I - xT_L)}{\det(I - xT_L)}. \tag{9.6}$$

This is still of limited practical use, without a means to find cofactors and determinants as functions of T_L for general L. Here a classic combinatorial result comes to the rescue.

9.1.1 Determinants and cofactors as sums over cycles and paths in digraphs

We start by describing determinants and cofactors combinatorially for an arbitrary matrix, say 'M', and will later consider M of the form '$I - xT$'. We begin with classic theorem

Theorem 68. *Let D be a psuedo-digraph with n vertices and arc weights $M_{i,j}$ and M its weighted adjacency matrix, then*

$$\det M = \sum_{c \in \mathcal{C}} (-1)^{N_e(c)} \prod_{M_{i,j} \in A(c)} M_{i,j}, \tag{9.7}$$

where $\mathcal{C}$ is the set of all spanning sets of cycles of D, N_e is the number of even length cycles in c and $A(c)$ the arc set of $\mathcal{C}$.

The theorem states that the determinant of a matrix is a sum over weighted cycles on the digraph for which that matrix is the adjacency matrix. For a proof see for instance [12] or [7]. The basic idea of the proof is to start with the definition of a determinant as a sum over products given by permutations – it turns out that cycles on the digraph give precisely those permutations for which the summand does not vanish. The detail in the proof is in keeping careful track of the signs.

Another classic but less often stated result gives the $(i, j)^{\text{th}}$ cofactor of M as a sum over weighted cycles, together with a path from the i^{th} to the j^{th} vertex, in the digraph whose adjacency matrix is M.

Corollary 69. *Let D be a psuedo-digraph with n vertices and arc weights $M_{i,j}$ and M its weighted adjacency matrix, then*

$$\mathrm{cof}_{y,y'} M = \sum_{c \in \mathcal{C}} (-1)^{N_e(c)} \prod_{M_{i,j} \in A(c)} M_{i,j}, \tag{9.8}$$

where $\mathcal{C}$ is the set of all spanning walks, w, on D such that

- *w is a cycle; or*
- *w is a path from y to y'.*

Further, N_e is the number of even length cycles in c and $A(c)$ the arc set of $\mathcal{C}$.

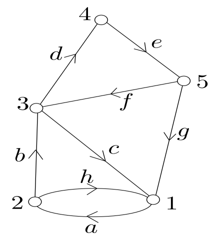
A proof of a generalization of Corollary 69 may be found in [12]. We sketch the core idea here, without carrying out the detail of keeping track of the signs.

Sketch proof. Let M be a matrix entries $M_{i,j}$ which are arc weights for some digraph D. Then the $(y, y')^{\text{th}}$ cofactor is the (signed) determinant of that matrix which is obtained by deleting the y^{th} row and the y'^{th} column from M, as in Equation 9.9.

$$\mathrm{cof}_{y,y'} M = \begin{vmatrix} M_{1,1} & \dots & M_{1,y'-1} & M_{1,y'} & M_{1,y'+1} & \dots & M_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{y-1,1} & \dots & M_{y-1,y'-1} & M_{y-1,y'} & M_{y-1,y'+1} & \dots & M_{y-1,n} \\ M_{y,1} & \dots & M_{y,y'-1} & M_{y,y'} & M_{y,y'+1} & \dots & M_{y,n} \\ M_{y+1,1} & \dots & M_{y+1,y'-1} & M_{y+1,y'} & M_{y+1,y'+1} & \dots & M_{y+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n,1} & \dots & M_{n,y'-1} & M_{n,y'} & M_{n,y'+1} & \dots & M_{n,n} \end{vmatrix} \tag{9.9}$$

But this determinant is a weighted sum over products such that each product contains exactly one element from each row of the new matrix, and exactly one element from each column of the new matrix. In graphical terms this means one 'out' arc from each of the vertices labeled $\{1, 2, ...y-1, y+1, ..., n\}$ and one 'in' arc from each of the vertices labeled $\{1, 2, ...y'-1, y'+1, ..., n\}$. These constraints correspond to either paths from y' to y; or cycles. □

Example 4. *We give an example of Theorem 68 and Corollary 69. Let the digraph D be*



$$(9.10)$$

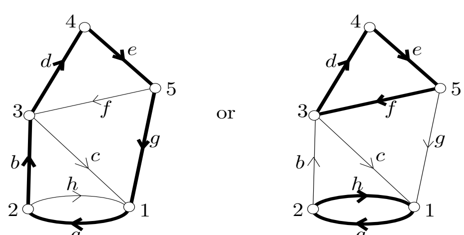
Then its adjacency matrix is

$$T = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ h & 0 & b & 0 & 0 \\ c & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \\ g & 0 & f & 0 & 0 \end{pmatrix} \tag{9.11}$$

with determinant equal to

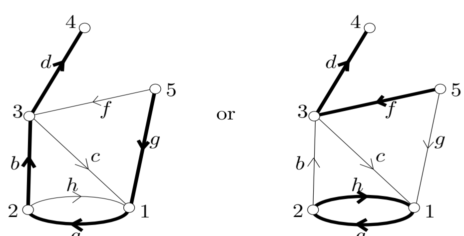
$$\det T = abdeg - adefh \tag{9.12}$$

summing over (signed) spanning cycles of the digraph.



$$(9.13)$$

Then to find the $(4,5)^{th}$ cofactor, for instance, remove arcs out of vertex 4 as well as arcs into vertex 5, and sum over spanning walks such that each walk is either a path from '5' to '4', or a cycle.



$$(9.14)$$

Hence the cofactor is

$$\mathrm{cof}_{4,5} T = abdg - adfh. \tag{9.15}$$

It is now a small modification of Theorem 68 and Corollary 69 to give determinants and cofactors for matrices of the form '$I - xT$'. The following classical corollary is of key importance in finding generating functions.

Corollary 70. *Let D be a psuedo-digraph with n vertices and arc weights $T_{i,j}$ and T its weighted adjacency matrix. Then*

$$\det(I - xT) = \sum_{c \in \mathcal{C}} (-1)^{N_e(c)} \prod_{T_{i,j} \in A(c)} (-xT_{i,j}) \tag{9.16}$$

where $\mathcal{C}$ is the set of all sets of cycles on D. Note, the set of cycles need not be a spanning set. Further, N_e is the number of even length cycles in c and $A(c)$ the arc set of $\mathcal{C}$. Also,

$$\mathrm{cof}_{y,y'}(I - xT) = \sum_{c \in \mathcal{C}'} (-1)^{N_e(c)} \prod_{T_{i,j} \in A(c)} (-xT_{i,j}), \tag{9.17}$$

where $\mathcal{C}'$ is the set of sets of walks on D such that

- *precisely one walk is a path from y to y', and*
- *all other walks are cycles.*

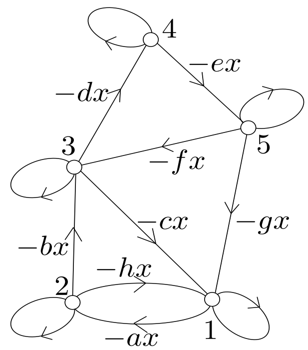
Note, the set of walks need not be a spanning set on the vertices. Further, N_e is the number of even length cycles in c and $A(c)$ the arc set of $\mathcal{C}'$.

The idea of the proof of this classic result is illustrated in Example 5.

Example 5. *[Continuation of Example 4] Now let $M = I - xT$, i.e.*

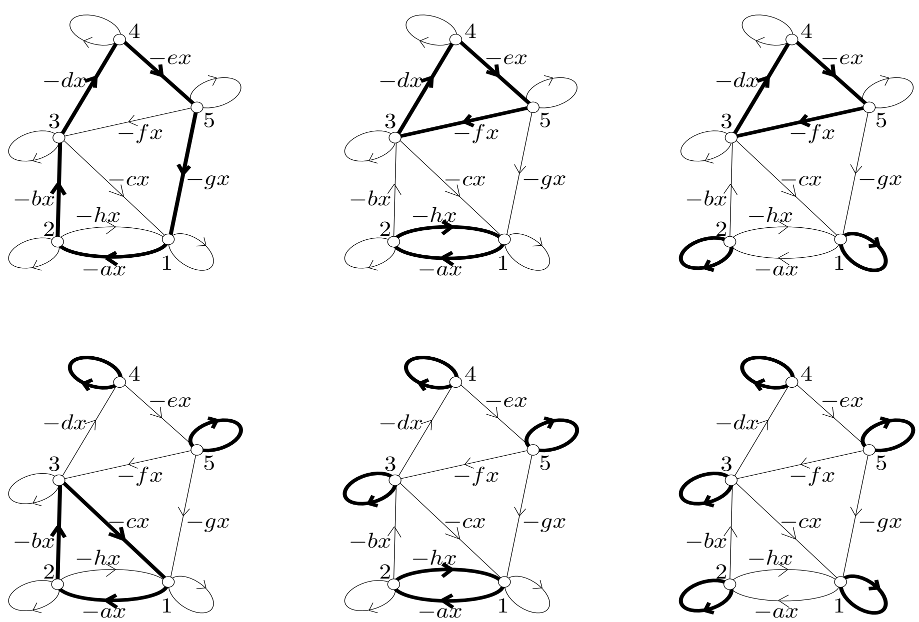
$$M = \begin{pmatrix} 1 & -ax & 0 & 0 & 0 \\ -hx & 1 & -bx & 0 & 0 \\ -cx & 0 & 1 & -dx & 0 \\ 0 & 0 & 0 & 1 & -ex \\ -gx & 0 & -fx & 0 & 1 \end{pmatrix} \tag{9.18}$$

This new matrix has a similar digraph to the original 'T', but now with a *loop at each vertex, and modified weights on the arcs, viz:*



(9.19)

The loops increase the possibilites for sets of spanning cycles. We have



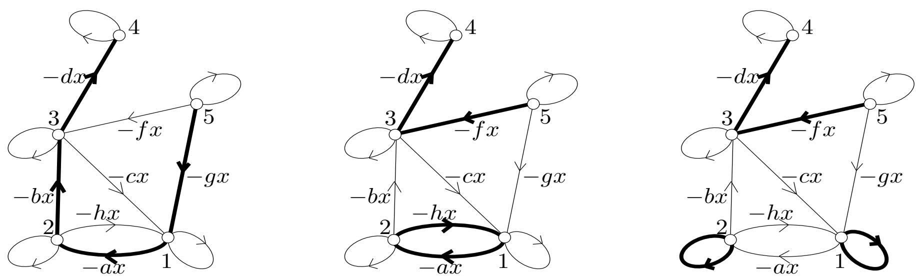
(9.20)

which gives

$$\det(I - xT) = -abdegx^5 + adefhx^5 - defx^3 - abcx^3 - ahx^2 + 1. \quad (9.21)$$

We notice that this is the same as if we had taken a sum over all *cycles on the original graph, but relaxed the constraint that the cycles be spanning.*

For the $(4,5)^{th}$ *cofactor we obtain*



(9.22)

which gives

$$\mathrm{cof}_{4,5}(I - xT) = abdgx^4 - adfhx^4 + dfx^2. \tag{9.23}$$

We notice that this is the same as if we had taken the sum over all sets of walks (not necessarily spanning) on the original graph, minus the arc from '4' to '5', such that there is exactly one walk from '5' to '4', and all other walks are cycles.

9.1.2 In terms of eigenvalues

The 'x' of our path length generating function, Equations (9.4)–(9.6), is closely related to the eigenvalues of the transfer matrix. Rewriting the determinants and cofactors in the form

$$\det(I - xT_L) = x^{L+1} \det\left(\frac{1}{x}I - T_L\right) \tag{9.24}$$

$$\mathrm{cof}_{y,y'}(I - xT_L) = x^L \mathrm{cof}_{y,y'}\left(\frac{1}{x}I - T_L\right) \tag{9.25}$$

gives the path length generating function in the form

$$G_{y',y;L}(x) = \frac{\mathrm{cof}_{y,y'}\left(\left(\frac{1}{x}\right)I - T_L\right)}{x\det\left(\left(\frac{1}{x}\right)I - T_L\right)}. \tag{9.26}$$

Thus the reciprocal of x,

$$\mu := \frac{1}{x}, \tag{9.27}$$

is an eigenvalue of T_L whenever

$$\det(\mu I - T_L) = 0. \tag{9.28}$$

It is a small modification of Corollary 70 to get determinants and cofactors directly in terms of μ. We have

Corollary 71. *Let D be a psuedo-digraph with n vertices and arc weights $T_{i,j}$ and T its weighted adjacency matrix. Then*

$$\det(\mu I - T) = \sum_{c \in \mathcal{C}} (-1)^{N_e(c)} \prod_{T_{i,j} \in A(c)} (-T_{i,j}) \mu^{|u|} \tag{9.29}$$

where $\mathcal{C}$ is the set of all sets of cycles on D. Note, the set of cycles need not be a spanning set. Further, N_e is the number of even length cycles in c, $A(c)$ the arc set of $\mathcal{C}$ and $|u|$ the number of uncovered vertices in c. Also,

$$\mathrm{cof}_{y,y'}(\mu I - T) = \sum_{c \in \mathcal{C}'} (-1)^{N_e(c)} \prod_{T_{i,j} \in A(c)} (-T_{i,j}) \mu^{|u|}, \tag{9.30}$$

where $\mathcal{C}'$ is the set of sets of walks on D such that

- *precisely one walk is a path from y to y', and*
- *all other walks are cycles.*

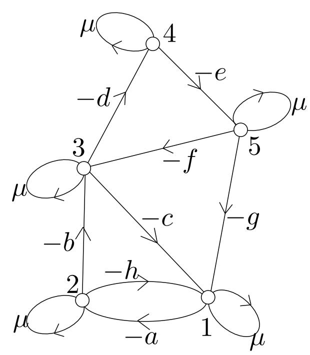
Note, the set of walks need not be a spanning set on the vertices. Further, N_e is the number of even length cycles in c, $A(c)$ the arc set of $\mathcal{C}'$ and $|u|$ the number of uncovered vertices in c.

We give an example of Corollary 71 which is a continuation of Examples 4 and 5.

Example 6 (Continuation of Examples 4 and 5). *Now let $M' = \mu I - T$, i.e.*

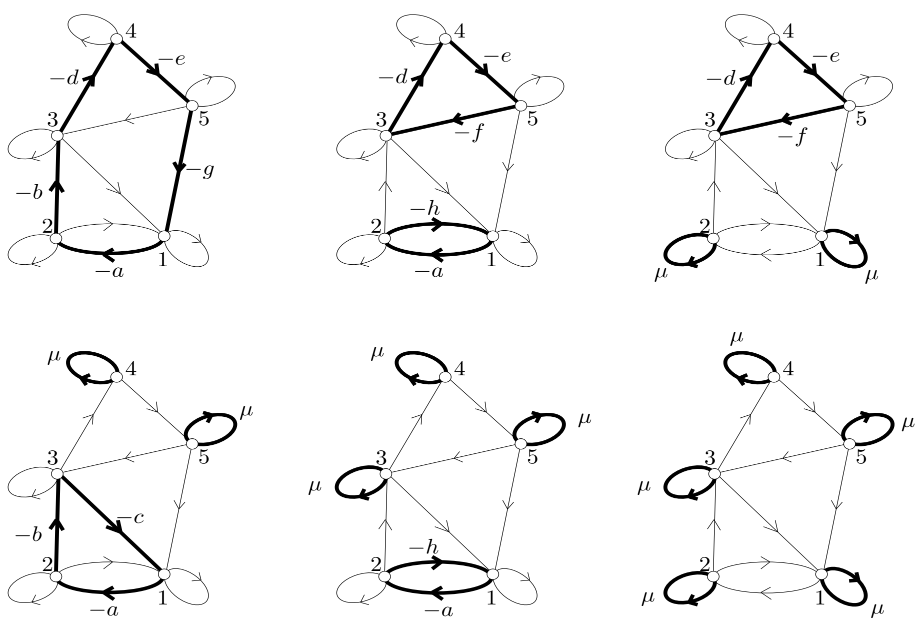
$$M' = \begin{pmatrix} \mu & -a & 0 & 0 & 0 \\ -h & \mu & -b & 0 & 0 \\ -c & 0 & \mu & -d & 0 \\ 0 & 0 & 0 & \mu & -e \\ -g & 0 & -f & 0 & \mu \end{pmatrix} \tag{9.31}$$

The corresponding adjacency digraph is similar to those of Examples 4 and 5, but now there is a loop at each vertex carrying weight μ, and arcs weights *are the negative of those in the original digraph, viz:*



(9.32)

Sets of spanning cycles are

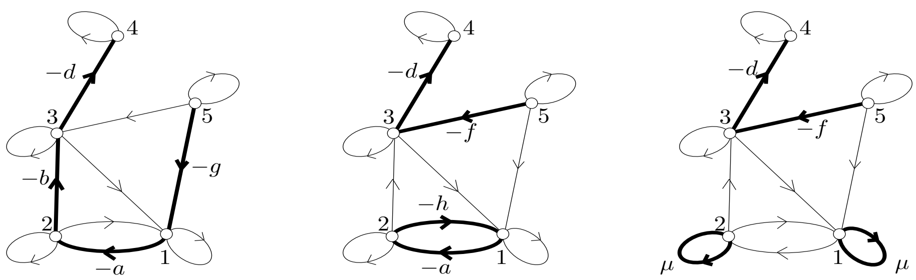


(9.33)

This gives

$$\det(\mu I - T) = \mu^5 - ah\mu^3 - abc\mu^2 - def\mu^2 + adefh - abdeg. \qquad (9.34)$$

Compare this with Equation (9.21). Also notice that the result is the same as if we had taken a sum over all *cycles on the original graph, but relaxed the constraint that the cycles be spanning and instead pick up a factor of μ for each vertex not visited. (See Figure 9.1.) For the $(4,5)^{th}$ cofactor we* *obtain*



(9.35)

This gives

$$\mathrm{cof}_{4,5}(\mu I - T) = df\mu^2 - adfh + abdg. \tag{9.36}$$

Compare this with Equation (9.23). Also notice that the result is the same as if we had taken the sum over all sets of walks (not necessarily spanning) on the original graph, minus the arc from '4' to '5', such that there is exactly one walk from '5' to '4', and all other walks are cycles; and again, picked up a factor of μ for each vertex not visited.

Finally we illustrate the determinant calculation of Equation (9.34) again, in Figure 9.1, without spanning cycles or loops.

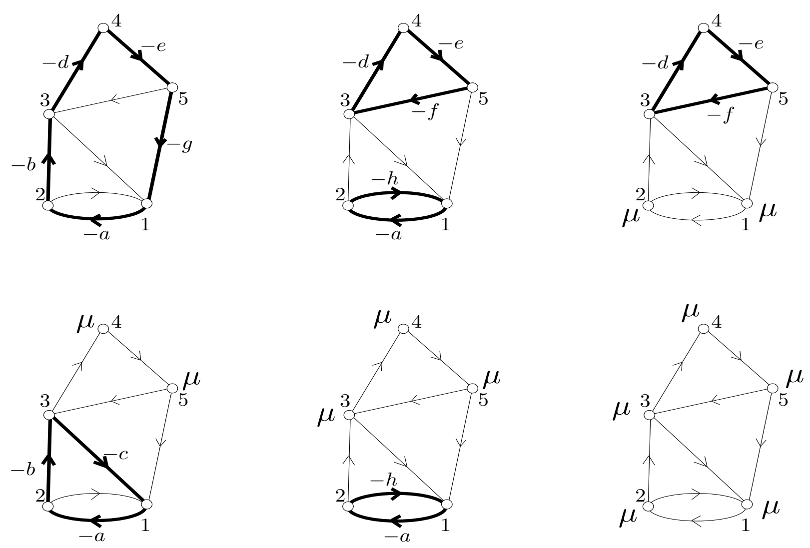


Figure 9.1: We repeat the determinant calculation of Equation (9.34), this time discarding loops, allowing non-spanning cycles and giving non-visited vertices a weight of μ.

9.1.3 Generating Functions for Ballot and Motzkin Paths

The most important consequence of Corollary 71 for specific lattice path problems of interest is the application to Ballot and Motzkin paths, given by classic Theorem 72 below. This theorem relies upon Viennot's representation of sums over cycles as sums over pavings. The practical usefulness of the theorem comes with a capacity to find closed-form expressions for the paving polynomials, which we developed in Part II, Chapters 5 and 6, building upon the observation of Richard Brak that breaking and recombining pavings gives a combinatorial route to the closed-form expressions required.

Theorem 72. *Let T_L be a finite tridiagonal matrix, of order $L+1$, of the form*

$$T_L = \begin{pmatrix} b_0 & 1 & 0 & \dots \\ \lambda_1 & b_1 & 1 & \\ 0 & \lambda_2 & b_2 & \\ \vdots & & & \ddots \end{pmatrix}, \tag{9.37}$$

with all λ_i's non-zero. Let

$$Y' = \min\{y', y\}, \qquad Y = \max\{y', y\}. \tag{9.38}$$

Then

1. $\det(I - xT_L) = x^{L+1} P_{L+1}(1/x)$
2. $cof_{y,y'}(I - xT_L) = x^L h_{y',y} P_{Y'}(1/x) P^{(Y+1)}_{L-Y}(1/x)$

so that the path length generating function

$$G_{y',y;L}(x) = \sum_{n\geq 0} (T_L^n)\big|_{y',y} x^n \tag{9.39}$$

is given by

$$G_{y',y;L}(x) = \frac{h_{y',y} P_{Y'}(1/x) P^{(Y+1)}_{L-Y}(1/x)}{x P_{L+1}(1/x)}, \tag{9.40}$$

where the factor $h_{y',y} = 1$ for $y' \leq y$, and is otherwise the product

$$h_{y',y} = \prod_{y<l\leq y'} \lambda_l. \tag{9.41}$$

The P_k's are paving polynomials, satisfying recurrence relation

$$P_{k+1}(\mu) = (\mu - b_k) P_k(\mu) - \lambda_k P_{k-1}(\mu), \tag{9.42}$$

subject to normalization

$$P_0(\mu) = 1, \qquad P_1(\mu) = \mu - b_0. \tag{9.43}$$

The $P_k^{(s)}$*'s are similar polynomials except with shifted coefficients, so that* $P_0^{(s)}(\mu) = 1$, $P_1^{(s)}(x) = \mu - b_j$ *and* $P_{k+1}^{(s)}(\mu) = (\mu - b_{k+s})P_k^{(s)}(\mu) - \lambda_{k+s}P_{k-1}^{(s)}(\mu)$. *When all* b_k*'s are zero, these are* Ballot polynomials*. Otherwise, we have* Motzkin polynomials*. Note that both Ballot and Motzkin polynomials are orthogonal polynomials.*

9.2 Constant coefficient recurrences for paths

The following well-known theorem relates rational generating functions to constant coefficient recurrences (see for instance [63] and [99]).

Theorem 73. *(i) Let* $\{b_t\}_{t\geq 0}$ *be a sequence of complex numbers with rational generating function*

$$\sum_{t\geq 0} b_t x^t = \frac{A(x)}{B(x)}, \tag{9.44}$$

with $\deg(A(x)) < \deg(B(x))$ *and*

$$B(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_d x^d. \tag{9.45}$$

Then the sequence $\{b_t\}_{t\geq 0}$ *satisfies* d^{th} *order recurrence*

$$0 = b_{t+d} + c_1 b_{t+d-1} + c_2 b_{t+d-2} + \cdots + c_d b_t, \tag{9.46}$$

for all $t \geq 0$.

(ii) Let $\{b_t\}_{t\geq 0}$ *be a sequence of complex numbers satisfying* d^{th} *order constant coefficient recurrence*

$$0 = b_{t+d} + c_1 b_{t+d-1} + c_2 b_{t+d-2} + \cdots + c_d b_t, \tag{9.47}$$

for all $t \geq 0$*. Then the sequence* $\{b_t\}_{t\geq 0}$ *has rational generating function*

$$\sum_{t\geq 0} b_t x^t = \frac{A(x)}{B(x)}, \tag{9.48}$$

with

$$A(x) = b_0 + (b_1 + c_1 b_0)x + \cdots + (b_{d-1} + c_1 b_{d-2} + \cdots + c_{d-1} b_0)x^{d-1} \tag{9.49}$$

and

$$B(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_d x^d. \tag{9.50}$$

From Part (i) of Theorem 73, together with the generating functions of the previous section, we can find constant coefficient recurrences. Instead we use a new method to find constant coefficient recurrences combinatorially, and Part (ii) of Theorem 73, to give a new combinatorial route to generating functions.

9.2.1 Ballot-like Paths

For Ballot-like paths, we have a new proof for a variant of Theorem 72 – this version (stated in terms of weight polynomials rather than transfer matrices) specifies the single variable constant coefficient recurrence on weight polynomials explicitly in a new way.

Theorem 74. *Let $B_t(y', y; \{\lambda_i\}_{i=1,..,L}; L)$ be the weight polynomial for Ballot-like paths of length t, beginning at height y', ending at height y, confined to a strip of height L; and weighted as follows:*

$$w(\textit{upstep}) = 1 \qquad (9.51)$$
$$w(\textit{downstep from height `}y\textit{' to height `}y-1\textit{'}) = \lambda_y. \qquad (9.52)$$

For any fixed y', y and L abbreviate

$$b_t := B_t(y', y; \{\lambda_i\}_{i=1,..,L}; L). \qquad (9.53)$$

Then the path length generating function for the b_t's is

$$\sum_{t=0}^{\infty} b_t x^t = \frac{A(x)}{B(x)}, \qquad (9.54)$$

where $A(x)$, $B(x)$ are polynomials in x, with $\deg(A(x)) < \deg(B(x))$. The denominator polynomial $B(x)$ is

$$B(x) = \sum_{i=0}^{L+1} c_i x^i \qquad (9.55)$$

where the coefficient c_i's are also coefficients in recurrence

$$0 = \sum_{i=0}^{L} c_i b_{t+L+1-i} \qquad (9.56)$$

and are obtainable equivalently either as

1. *coefficients in Ballot paving polynomial*

$$P_{L+1}(\mu) = \sum_{i=0}^{L+1} c_i \mu^{L+1-i}, \tag{9.57}$$

as defined as a sum over pavings/cycles in Definition 54 and satisfying recurrence

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda_k P_{k-1}(\mu) \tag{9.58}$$

with initial conditions

$$P_0(\mu) = 1, P_1(\mu) = \mu; \tag{9.59}$$

or as

2. *sums over weighted walks on a 'skewed binomial box lattice'. This lattice is* $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ *with step set*

$$S = (0, -2) \text{ (South)} \tag{9.60}$$
$$E = (1, -1) \text{ (southEast).} \tag{9.61}$$

Steps have weights

$$w(\text{'}S\text{' step from height } y \text{ to height } y-2) = -1/\lambda_y \tag{9.62}$$
$$w(\text{'}E\text{' step from height } y \text{ to height } y-1) = 1/\lambda_y \tag{9.63}$$

The weight of a walk is the product of the weights of the steps in the walk, i.e.

$$w(\alpha) = \prod_{s \in \mathcal{S}} w(s) \tag{9.64}$$

where $\mathcal{S}$ *is the set of arcs in* α*, and the* c_i*'s are*

$$c_i = \left(\prod_{i=1}^{L} \lambda_i \right) \sum_{\alpha \in A_i} w(\alpha), \tag{9.65}$$

where A_i *is the set of paths from vertex* $(t, L+1)$ *to* $(t+L+1-i, 0)$*.*

Comment 12. *Note that in the notation of Theorem 74 (and Theorem 72), the denominator polynomial* $B(x)$ *is*

$$B(x) = x^{L+1} P_{L+1}(1/x) \tag{9.66}$$

and that polynomials in x *and* μ *respectively:*

$$x^{L+1} P_{L+1}(1/x) \text{ and } P_{L+1}(\mu) \tag{9.67}$$

have the same coefficients but with their order reversed.

Before presenting the proof we give the unweighted case as a corollary, and illustrate the principle behind the proof in this simpler context.

Corollary 75. *Let notation be as in Theorem 74 and let*

$$\lambda_y = 1 \tag{9.68}$$

for all y (i.e. these are unweighted Ballot-like paths). Then the coefficients of recurrence(9.56) are signed binomial coefficients:

$$c_i = (-1)^i \binom{L+1-i}{i}. \tag{9.69}$$

For example, the first few instances of Corollary 75 are given in the following table.

L	Recurrence	Paving Polynomial
1	$0 = \binom{2}{0} b_t - \binom{1}{1} b_{t-2}$	$P_2(\mu) = \mu^2 - 1$
2	$0 = \binom{3}{0} b_t - \binom{2}{1} b_{t-2}$	$P_3(\mu) = \mu^3 - 2\mu$
3	$0 = \binom{4}{0} b_t - \binom{3}{1} b_{t-2} + \binom{2}{2} b_{t-4}$	$P_4(\mu) = \mu^4 - 3\mu^2 + 1$
4	$0 = \binom{5}{0} b_t - \binom{4}{1} b_{t-2} + \binom{3}{2} b_{t-4}$	$P_5(\mu) = \mu^5 - 4\mu^3 + 3\mu$
5	$0 = \binom{6}{0} b_t - \binom{5}{1} b_{t-2} + \binom{4}{2} b_{t-4} - \binom{3}{3} b_{t-6}$	$P_6(\mu) = \mu^6 - 5\mu^4 + 6\mu^2 - 1$

The derivation of Corollary 75 and Theorem 74 is inspired by the Euclidean algorithm, being based on a repeated back-substitution. We illustrate the principal for the un-weighted case before giving the general proof in the next subsection.

Consider Figure 9.2. Vertices are labelled by the number of paths which may reach that vertex, starting at the origin. Vertices above the upper wall are labelled '0' because paths are not allowed to cross the wall. Consider the triple of circled vertices. They exhibit the relation illustrated in the right half of the figure. So we write

$$0 = 4 - 4. \tag{9.70}$$

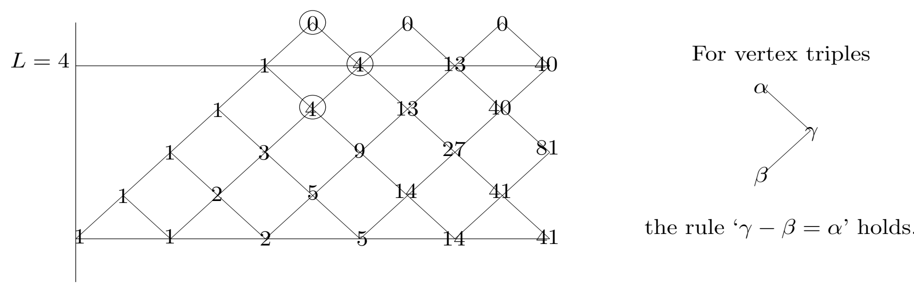


Figure 9.2: Vertex grid entries count ways from the origin to each vertex.

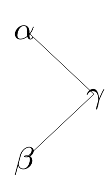
But then each of the 13’s is the uppermost vertex in similar triples. Back substituting all the way down to the base line, we get

$$0 = 4 - 4 \tag{9.71}$$
$$= (13 - 9) - (9 - 5) \tag{9.72}$$
$$= 13 - 2(9) + 5 \tag{9.73}$$
$$= (27 - 14) - 2(14 - 5) + 5 \tag{9.74}$$
$$= 27 - 3(14) + 3(5) \tag{9.75}$$
$$= (41 - 14) - 3(14) + 3(5) \tag{9.76}$$
$$= 41 - 4(14) + 3(5) \tag{9.77}$$
$$= 1(41) - 4(14) + 3(5) \tag{9.78}$$

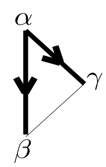
The recurrence relation obtained is the same independently of which ‘0’ was the starting point. In a strip of height $L = 4$, Ballot-like paths that start and end on the baseline satisfy recurrence

$$0 = a_t - 4a_{t-2} + 3a_{t-4}, \tag{9.79}$$

for $t \geq 6$. The general form of the recurrence coefficients can be seen by considering the process of back substitution pictorially as counting a class of walks from a zero above the upper wall down to the baseline. The algebraic rule ‘$\alpha = \gamma - \beta$’, for vertex triples



becomes in geometric terms a choice between a South step and a southEast step



with a negative sign associated with each South step. Figures 9.3 show the walks that give the coefficients to 41, 14 and 5 respectively in Equation (9.78).

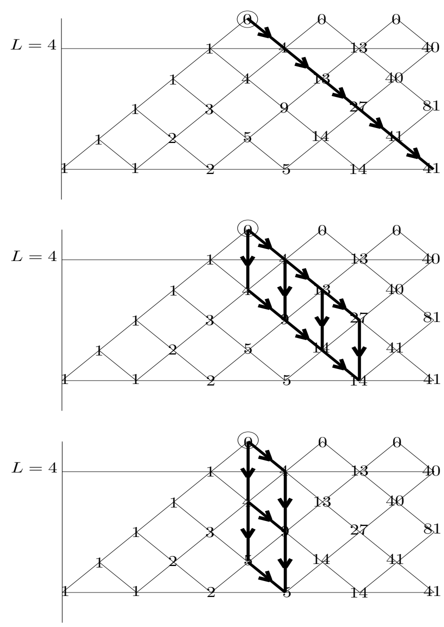


Figure 9.3: There are $\binom{5}{0}$ walks from '0' (circled) to the baseline vertex marked '41', $\binom{4}{1}$ to the baseline vertex marked '14' and $\binom{3}{2}$ to the baseline vertex marked '5'.

Combinatorial proof of the Single-variable Constant Coefficient Recurrence for decorated Ballot-like paths

Proof of Theorem 74. Let notation be as in Theorem 74. Fix L. We prove the recurrence – the generating function then follows from Theorem 73. We first prove the result for paths beginning and ending at height zero.

Assume $y' = 0$. Under this assumption, define new abbreviation

$$B_{t,y} := B_t(y', y; \{\lambda_i\}_{i=1,..,L}; L). \tag{9.80}$$

Then the $B_{t,y}$'s may be illustrated on a grid as in Figure 9.4. The defining

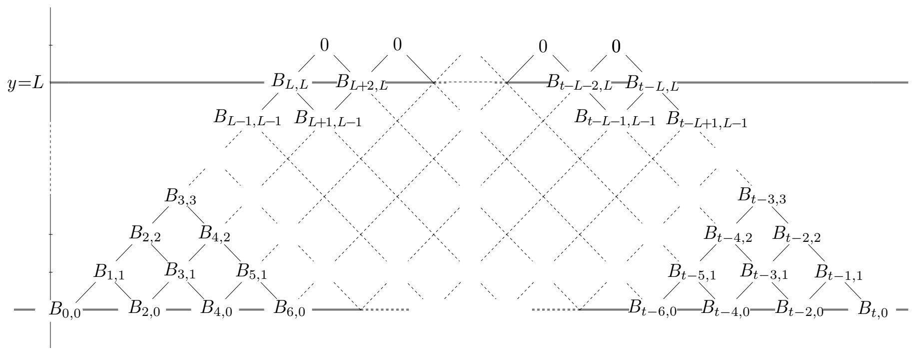


Figure 9.4: A grid showing the $B_{t,y}$ for paths which start at height $y' = 0$.

recurrence (9.1) on the $B_{t,y}$'s is illustrated geometrically in Figure 9.5. The general weight λ_k on downsteps from height k is also shown.

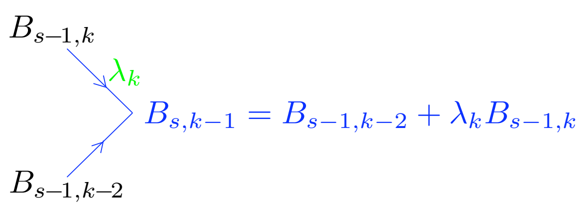


Figure 9.5: The general defining recurrence on $B_{t,y}$'s is shown.

Rearranging the defining recurrence on $B_{t,y}$'s gives Figure 9.6. Thus propagating the recurrence through decreasing heights is the same as propagating walks down through 'skewed' binomial boxes, picking up a factor of $1/\lambda_k$ for south east steps (abbreviated 'E'), and $-1/\lambda_k$ for south steps (abbreviated 'S'). Figures 9.8–9.9 of Example 7 illustrate.

Thus the coefficient of $B_{t-2j,0}$ is given by the sum of those weighted paths which start at '0' in position $(t-L-1, L+1)$ and end at the vertex labeled '$B_{t-2j,0}$' in position $(t-2j, 0)$. This process gives a recurrence of the form of Equation (9.92) in Example 7. In particular, the coefficient of '$B_{t,0}$' is $\prod_{i=1}^{L} \frac{1}{\lambda_i}$, being the weight of the sole path from $(t-L-1, L+1)$

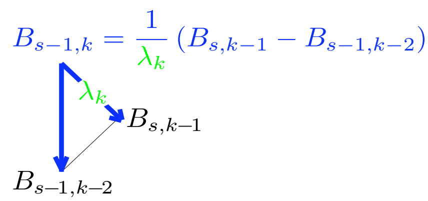


Figure 9.6: The general defining recurrence on $B_{t,y}$'s, rearranged.

to $(t, 0)$. Also, coefficients to $B_{t-2j,0}$'s are sums of products of $\frac{1}{\lambda_i}$'s, with at most $L - j$ terms in the product.

Since the recurrence obtained by counting 'skewed' binomial box paths is of the form

$$0 = \text{right hand side}, \tag{9.81}$$

multiplying by the factor

$$\prod_{i=1}^{L} \lambda_i \tag{9.82}$$

gives a new valid recurrence, which by the considerations of the previous paragraph, is rational in the λ_i's. See Example 7, Equation (9.94). This new recurrence may be obtained directly from pavings, via the following bijection with 'skewed' binomial paths. Start with paths. A path down a 'skewed' binomial box from '0' to '$B_{t-2j,0}$' consists of $L+1-j$ steps; where j of these are S steps and the rest are E steps. The path is determined by the position of the S steps in the sequence of S's and E's.

For each S step from height $k+1$ to height $k-1$, place a dimer between heights k and $k-1$ in the paving. Since we place dimers in positions corresponding to the lower half of an S step, they can never be contiguous. Hence the result is a valid paving with j pavers.

The weight of the paving is equal to

$$\prod_{i=1}^{L} \lambda_i$$

times the weight of the path, since the paving contains occurrences of λ_i for precisely those i which are not present in the path weighting. To check this notice that the weights in the path come from the upper half of S steps, and from E steps, whereas the weights in the paving come from the lower half of S steps. These sets are disjoint. Furthermore, signs match up since S steps and dimers both have a negative weight.

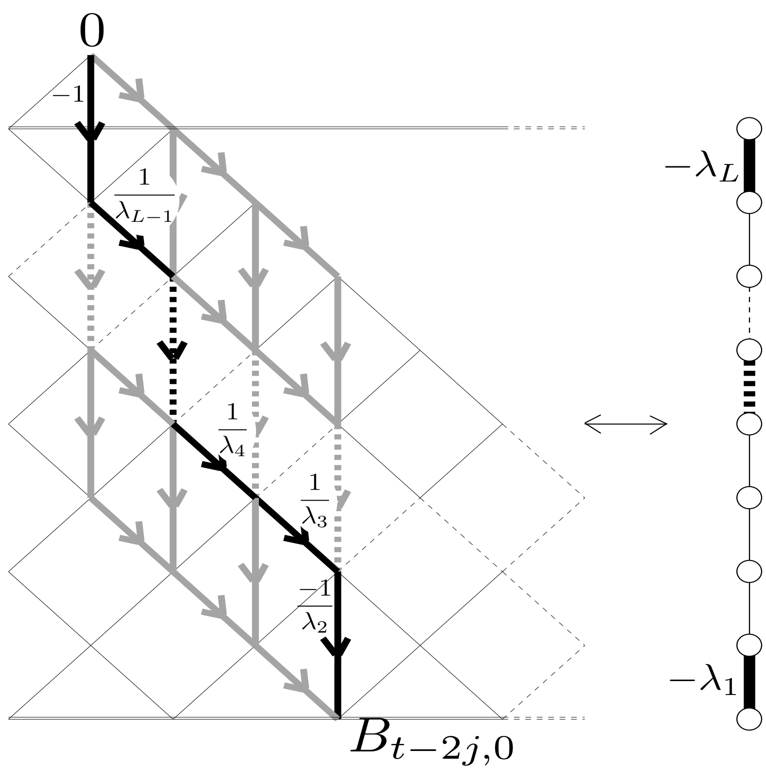


Figure 9.7: The bijection between weighted paths down a 'skewed' binomial box, and pavings.

To reverse the mapping and go from the paving to the path, construct a path as a sequence of $L-j$ occurrences of S's and E's, where j is the number of dimers present and each dimer gives an 'S' with each non-paved edge giving an 'E'. The corresponding path must have j downsteps so will be a valid path from '0' to '$B_{t-2j,0}$'. The position of the arcs determines the weighting, which by considerations as in the previous paragraph, must have weight equal to

$$\prod_{i=1}^{L} \frac{1}{\lambda_i}$$

times the weight of the paving. See Example 7, Figure (9.11).

We now need to show that the same recurrence holds for paths ending at heights greater than zero. We have already proven, for some specified c_i's, that

$$0 = \sum_{i=0}^{\lceil L/2 \rceil} c_i B_{t-2i,0}. \tag{9.83}$$

Now since

$$B_{t-1,1} = \frac{1}{\lambda_1} B_{t,0}, \tag{9.84}$$

we have that

$$0 = \sum_{i=0}^{\lceil L/2 \rceil} c_i \frac{1}{\lambda_1} B_{t-2i-1,1}, \tag{9.85}$$

Multiplying Equation (9.85) by λ_1 shows that the desired recurrence holds for paths ending at height one, also. From this the result for arbitrary heights may be shown inductively, since

$$B_{t-1,y+1} = \frac{1}{\lambda_{y+1}} \left(B_{t,y} - B_{t-1,y-1} \right) \tag{9.86}$$

and a linear combination of sequences each satisfying the same linear recurrence, together satisfies the same recurrence. Finally we note that nowhere did we use the assumption that $y' = 0$, so that the proof holds for paths with arbitrary starting and ending heights within $[0, L]$. □

Example 7. *An example of Theorem 74 with general downstep weighting is given for $L = 3$.*

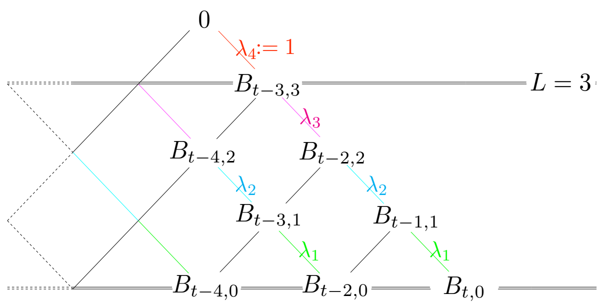


Figure 9.8: An example, for $L = 3$, of a grid for Ballot-like paths, with general downstep weighting.

$$
\begin{aligned}
0 &= B_{t-3,3} - B_{t-4,2} && (9.87)\\
&= \left(\frac{1}{\lambda_3}\right)(B_{t-2,2} - B_{t-3,1}) - \left(\frac{1}{\lambda_2}\right)(B_{t-3,1} - B_{t-4,0}) && (9.88)\\
&= \left(\frac{1}{\lambda_3}\right) B_{t-2,2} - \left(\frac{1}{\lambda_3} + \frac{1}{\lambda_2}\right) B_{t-3,1} + \left(\frac{1}{\lambda_2}\right) B_{t-4,0} && (9.89)\\
&= \left(\frac{1}{\lambda_3}\right)\left(\frac{1}{\lambda_2}\right)(B_{t-1,1} - B_{t-2,0}) - \left(\frac{1}{\lambda_3} + \frac{1}{\lambda_2}\right)\left(\frac{1}{\lambda_1}\right)(B_{t-2,0})\\
&\quad + \left(\frac{1}{\lambda_2}\right) B_{t-4,0} && (9.90)\\
&= \left(\frac{1}{\lambda_3\lambda_2}\right) B_{t-1,1} - \left(\frac{1}{\lambda_3\lambda_2} + \frac{1}{\lambda_3\lambda_1} + \frac{1}{\lambda_2\lambda_1}\right) B_{t-2,0} + \left(\frac{1}{\lambda_2}\right) B_{t-4,0} && (9.91)\\
&= \left(\frac{1}{\lambda_3\lambda_2\lambda_1}\right) B_{t,0} - \left(\frac{1}{\lambda_3\lambda_2} + \frac{1}{\lambda_3\lambda_1} + \frac{1}{\lambda_2\lambda_1}\right) B_{t-2,0} + \left(\frac{1}{\lambda_2}\right) B_{t-4,0}. && (9.92)
\end{aligned}
$$

The coefficients of each of the $B_{t-2j,0}$ come from traversing all binomial paths from '0' to '$B_{t-2j,0}$' within the binomial boxes shown in Figure 9.9, where 'E' steps have weight '$\frac{1}{\lambda_k}$', and 'S' steps have weight '$-\frac{1}{\lambda_k}$', for k the height at the top of the step, as shown. Multiplying Equation (9.92) by

$$\prod_{i=1}^{L} \lambda_i = \lambda_1\lambda_2\lambda_3 \tag{9.93}$$

gives

$$0 = B_{t,0} - (\lambda_1 + \lambda_2 + \lambda_3) B_{t-2,0} + \lambda_1\lambda_3 B_{t-4,0}. \tag{9.94}$$

This neater recurrence is given directly by pavings, as in Figure 9.10. A bijection between the weighted paths and the weighted pavings is illustrated in Figure 9.11.

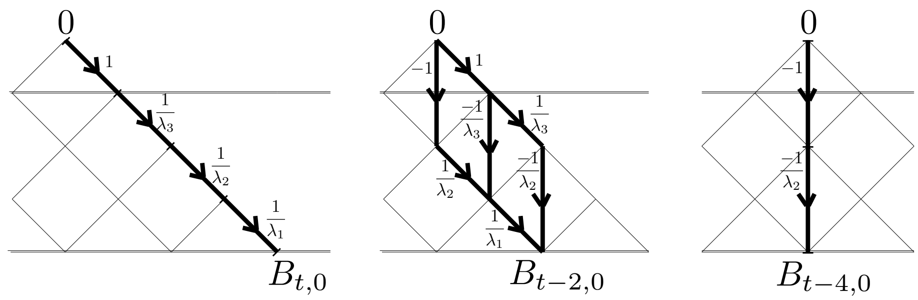


Figure 9.9: Coefficients in Equation (9.92) come from paths down binomial boxes, drawn 'skewed'.

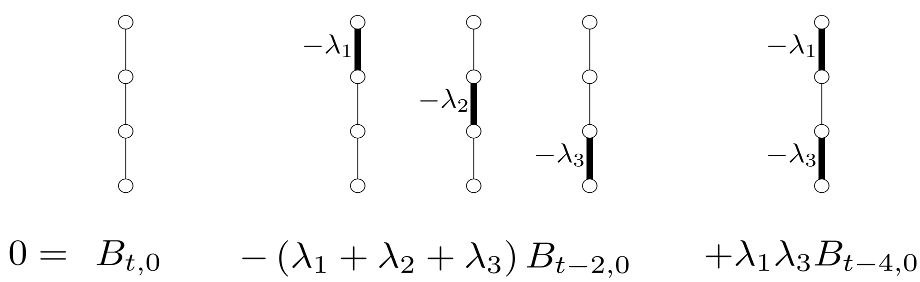


Figure 9.10: Coefficients in Equation (9.94) come from pavings.

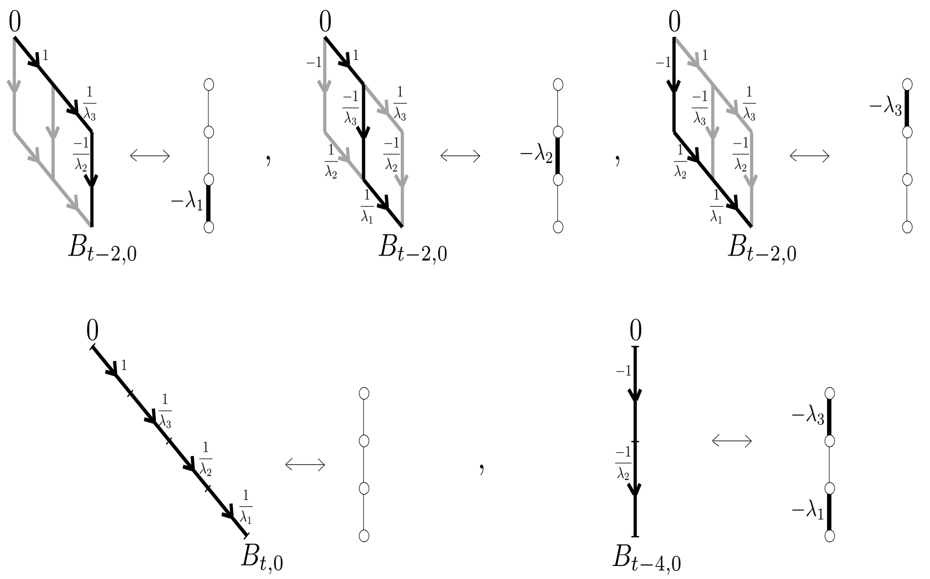


Figure 9.11: The bijection between weighted paths down a 'skewed' binomial box and ballot pavings of a path graph is shown for $L = 3$.

9.2.2 Motzkin-like Paths

Motzkin-like paths have weight polynomials which satisfy a theorem giving classical generating function and constant coefficient recurrence similar to that shown to hold for Ballot-like paths in Theorem 74, with, again, a proof based upon a new combinatorial construction and characterization of the recurrence.

Theorem 76. *Let $B_t(y', y; \{\lambda_i\}_{i=1,..,L}, \{b_j\}_{j=0,..,L}; L)$ be the weight polynomial for Motzkin-like paths of length t, beginning at height y', ending at height y, confined to a strip of height L; and weighted as follows:*

$$w(\textit{upstep}) = 1 \tag{9.95}$$

$$w(\textit{across step from height 'y' to height 'y'}) = b_y \tag{9.96}$$

$$w(\textit{downstep from height 'y' to height 'y}-1\textit{'}) = \lambda_y. \tag{9.97}$$

For any fixed y', y and L abbreviate

$$m_t := B_t(y', y; \{\lambda_i\}_{i=1,..,L}, \{b_j\}_{j=0,..,L}; L). \tag{9.98}$$

Then the path length generating function for the m_t's is

$$\sum_{t=0}^{\infty} m_t x^t = \frac{A(x)}{B(x)}, \tag{9.99}$$

where $A(x)$, $B(x)$ are polynomials in x, with $\deg(A(x)) < \deg(B(x))$. The denominator polynomial $B(x)$ is

$$B(x) = \sum_{i=0}^{L+1} c_i x^i \tag{9.100}$$

where the coefficient c_i's are also coefficients in recurrence

$$0 = \sum_{i=0}^{L} c_i m_{t+L+1-i} \tag{9.101}$$

and are obtainable equivalently either as

1. *coefficients in Motzkin paving polynomial*

$$P_{L+1}(\mu) = \sum_{i=0}^{L+1} c_i \mu^{L+1-i}, \tag{9.102}$$

as defined as a sum over pavings/cycles in Definition 55 and satisfying recurrence

$$P_{k+1}(\mu) = (\mu - b_k)P_k(\mu) - \lambda_k P_{k-1}(\mu) \tag{9.103}$$

with initial conditions

$$P_0(\mu) = 1, P_1(\mu) = \mu - b_0; \tag{9.104}$$

or as

2. *sums over weighted walks on a lattice with underlying vertex set $\mathbb{Z}\times\mathbb{Z}_{\geq 0}$ and step set*

$$\begin{aligned} H &= (0,-1) \text{ (Half south)} && (9.105)\\ S &= (0,-2) \text{ (South)} && (9.106)\\ E &= (1,-1) \text{ (southEast).} && (9.107) \end{aligned}$$

Steps have weights

$$w(\text{'}H\text{' step from height '}y\text{' to height '}y-1\text{'}) = \frac{-b_{y-1}}{\lambda_y} \quad (9.108)$$

$$w(\text{'}S\text{' step from height '}y\text{' to height '}y-2\text{'}) = \frac{-1}{\lambda_y} \quad (9.109)$$

$$w(\text{'}E\text{' step from height '}y\text{' to height '}y-1\text{'}) = \frac{1}{\lambda_y} \quad (9.110)$$

subject to the convention that $\lambda_{L+1} := 1$. *The weight of a walk is the product of the weights of the steps in the walk, i.e.*

$$w(\alpha) = \prod_{s \in \mathcal{S}} w(s) \quad (9.111)$$

where $\mathcal{S}$ *is the set of arcs in* α, *and the* c_i*'s are*

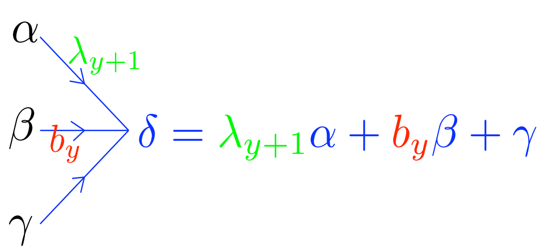
$$c_i = \left(\prod_{i=1}^{L} \lambda_i \right) \sum_{\alpha \in A_i} w(\alpha), \quad (9.112)$$

where A_i *is the set of paths from vertex* $(t, L+1)$ *to* $(t+L+1-i, 0)$.

Proof. The proof is very similar to that for Theorem 74, and we only show the key differences. We have

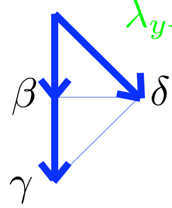
$$B_{t,y} = B_{t-1,y-1} + b_y B_{t-1,y} + \lambda_{y+1} B_{t-1,y+1}. \quad (9.113)$$

Pictorially, this is



with $\alpha = B_{t-1,y+1}$, $\beta = B_{t-1,y}$, $\gamma = B_{t-1,y-1}$ and $\delta = B_{t,y}$. Rearranging gives

$$\alpha = \frac{1}{\lambda_{y+1}} (\delta - b_y \beta - \gamma)$$



Starting at $B_{t-L-1,L+1} := 0$, and repeatedly back-substituting may be accomplished visually by walking down through a weighted 'skewed' directed graph, which has arcs and arc weights as in the Theorem. Thus we obtain coefficients, c_i, in a recurrence of the form

$$0 = c_0 B_{t,0} + c_1 B_{t-1,0} + \ldots + c_L B_{t-L,0}. \tag{9.114}$$

Multiplying said recurrence by

$$\sum_{i=1}^{L} \lambda_i \tag{9.115}$$

rationalizes the coefficients to give a new recurrence. The coefficients of this new recurrence may also be obtained directly via pavings. The connection between the paving and path derivation is given by a bijection between the path and paving pictures, as shown in Figure 9.12.

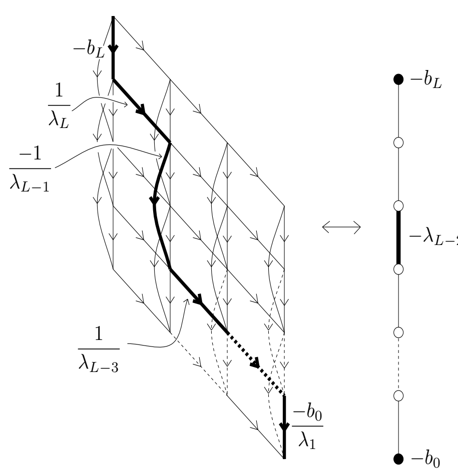


Figure 9.12: The bijection between paths and Motzkin pavings.

Finally, since each row in the grid of $B_{t,y}$'s is a linear combination of other rows, the recurrence holds for any fixed value of y in $0 \leq y \leq L$. □

Example 8. *We give an example of Theorem 76 strip height $L = 2$. A grid showing positions that a Motzkin-like path in the strip may obtain is shown in Figure 9.13. The recurrence obtained using the walks is*

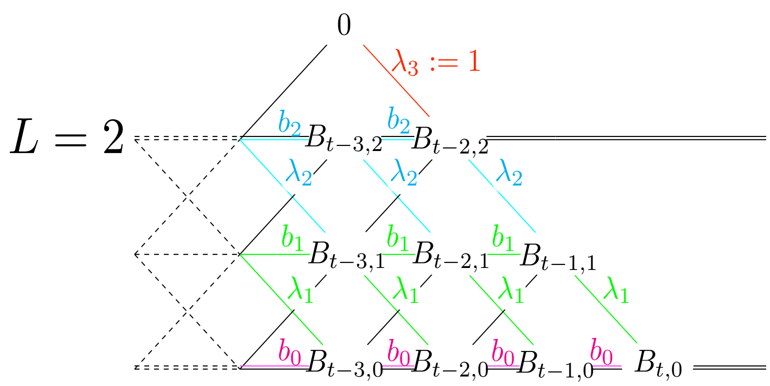


Figure 9.13: An example, for $L = 2$, of a grid for Motzkin-like paths, with general downstep weighting.

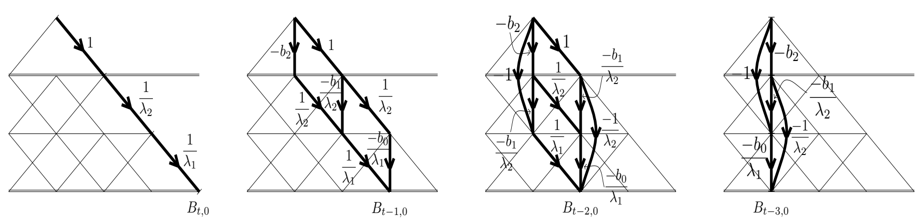


Figure 9.14: Walks down each of the four weighted digraphs give the coefficients of the $B_{t-j,0}$'s in the constant coefficient recurrence.

$$\begin{aligned} 0 &= \frac{1}{\lambda_1\lambda_2}B_{t,0} - \left(\frac{b_0}{\lambda_1\lambda_2} + \frac{b_1}{\lambda_1\lambda_2} + \frac{b_2}{\lambda_1\lambda_2}\right)B_{t-1,0} \\ &\quad + \left(\frac{b_0b_1}{\lambda_1\lambda_2} + \frac{b_0b_2}{\lambda_1\lambda_2} + \frac{b_1b_2}{\lambda_1\lambda_2} - \frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)B_{t-2,0} \\ &\quad - \left(\frac{b_0b_1b_2}{\lambda_1\lambda_2} - \frac{b_0}{\lambda_1} - \frac{b_2}{\lambda_2}\right)B_{t-3,0}. \end{aligned} \tag{9.116}$$

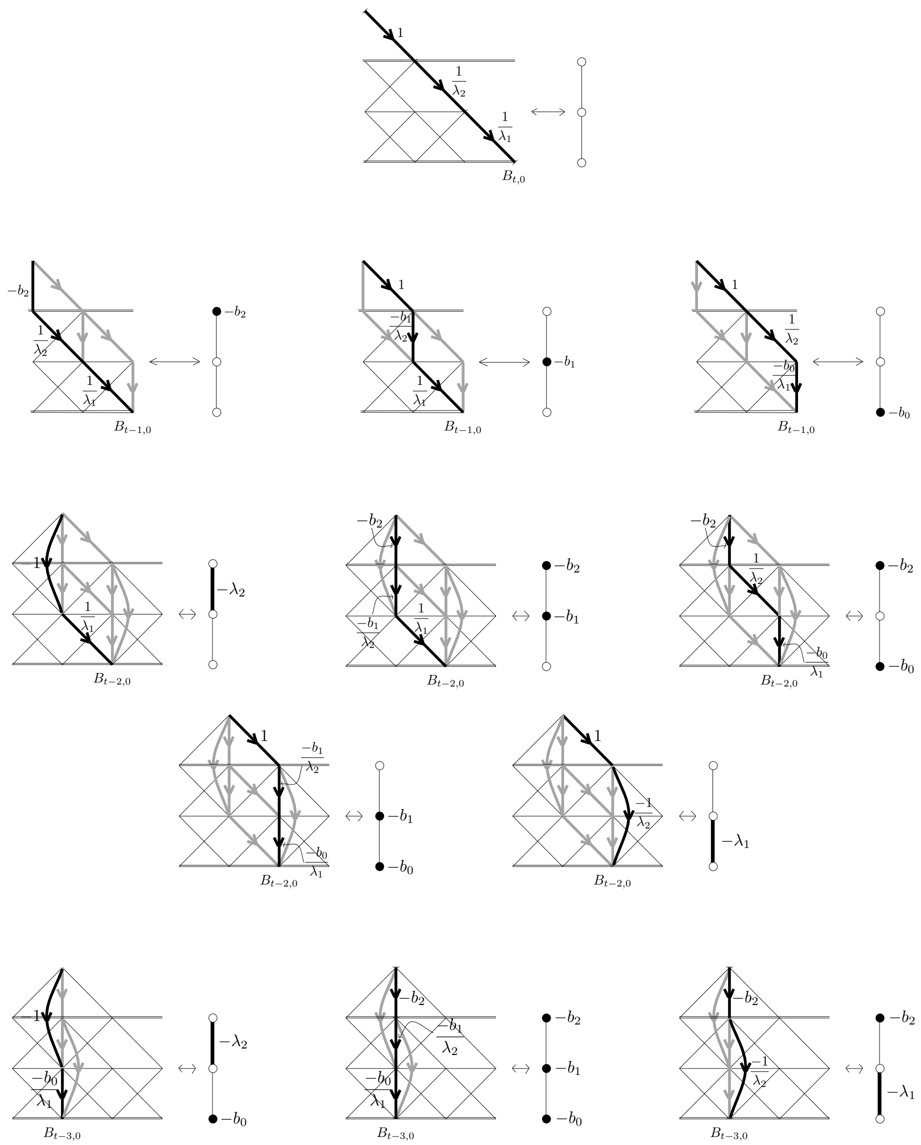


Figure 9.15: The coefficients for constant coefficient recurrences on Motzkin paths in a strip of height 2 may be obtained either via paths or via pavings, as indicated in the bijection.

Multiplying Equation (9.116) by

$$\sum_{i=1}^{L} \lambda_i = \lambda_1 \lambda_2 \tag{9.117}$$

gives the recurrence one would obtain directly using pavings, being

$$\begin{aligned} 0 \;=\; & B_{t,0} - (b_0 + b_1 + b_2)\, B_{t-1,0} + (b_0 b_1 + b_0 b_2 + b_1 b_2 - \lambda_1 - \lambda_2)\, B_{t-2,0} \\ & - (b_0 b_1 b_2 - b_0 \lambda_2 - b_2 \lambda_1)\, B_{t-3,0}. \end{aligned} \tag{9.118}$$

The term-by-term bijection is shown in Figure 9.15.

9.2.3 2-Up Paths

The combinatorial recurrence constructions of Theorem 74 for Ballot paths and Theorem 76 for Motzkin paths extend naturally to 2-up paths. We have

Theorem 77. *Let $B_t(y', y; \{\lambda_i\}_{i=1,..,L}; L)$ be the weight polynomial for 2-up paths of length t, beginning at height y', ending at height y, confined to a strip of height L; and weighted as follows:*

$$w(\textit{short upstep from height 'y' to height 'y+1'}) = 1 \tag{9.119}$$

$$w(\textit{long upstep from height 'y' to height 'y+2'}) = 1 \tag{9.120}$$

$$w(\textit{downstep from height 'y' to height 'y-1'}) = \lambda_y \tag{9.121}$$

For any fixed y', y and L abbreviate

$$u_t := B_t(y', y; \{\lambda_i\}_{i=1,..,L}; L). \tag{9.122}$$

Then the path length generating function for the u_t's is

$$\sum_{t=0}^{\infty} u_t x^t = \frac{A(x)}{B(x)}, \tag{9.123}$$

where $A(x)$, $B(x)$ are polynomials in x, with $\deg(A(x)) < \deg(B(x))$. The denominator polynomial $B(x)$ is

$$B(x) = \sum_{i=0}^{L+1} c_i x^i \tag{9.124}$$

where the coefficient c_i's are also coefficients in recurrence

$$0 = \sum_i c_i u_{t+L+1-i} \tag{9.125}$$

and are obtainable equivalently either as

1. *coefficients in 2-up paving polynomial*

$$P_{L+1}(\mu) = \sum_{i=0}^{L+1} c_i \mu^{L+1-i}, \tag{9.126}$$

as defined as a sum over pavings/cycles in Definition 58 and satisfying recurrence

$$P_{k+1}(\mu) = \mu P_k(\mu) - \lambda_k P_{k-1}(\mu) - \lambda_{k-1}\lambda_k P_{k-2}(\mu) \tag{9.127}$$

with initial conditions

$$P_0(\mu) = 1, P_1(\mu) = \mu, P_2(\mu) = \mu^2 - \lambda_1 \tag{9.128}$$

or as

2. *sums over weighted walks on a lattice with underlying vertex set* $\mathbb{Z}\times\mathbb{Z}_{\geq 0}$ *and step set*

$$S = (0,-2) \text{ (South)} \tag{9.129}$$
$$T = (0,-3) \text{ (Tall south)} \tag{9.130}$$
$$E = (1,-1) \text{ (southEast).} \tag{9.131}$$

Steps have weights

$$w(\text{`}S\text{' step from height `}y\text{' to height `}y-2\text{'}) = \frac{-1}{\lambda_y} \tag{9.132}$$
$$w(\text{`}T\text{' step from height `}y\text{' to height `}y-3\text{'}) = \frac{-1}{\lambda_y} \tag{9.133}$$
$$w(\text{`}E\text{' step from height `}y\text{' to height `}y-1\text{'}) = \frac{1}{\lambda_y} \tag{9.134}$$

subject to the convention that $\lambda_{L+1} := 1$*. The weight of a walk is the product of the weights of the steps in the walk, i.e.*

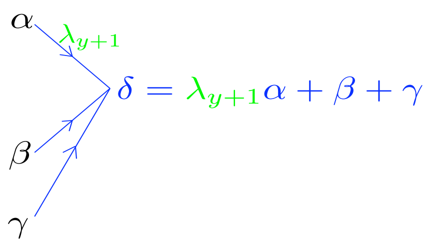
$$w(\alpha) = \prod_{s\in\mathcal{S}} w(s) \tag{9.135}$$

where $\mathcal{S}$ *is the set of arcs in* α*, and the* c_i*'s are*

$$c_i = \left(\prod_{i=1}^{L} \lambda_i\right) \sum_{\alpha\in A_i} w(\alpha), \tag{9.136}$$

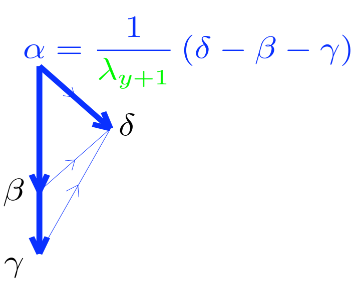
where A_i *is the set of paths from vertex* $(t, L+1)$ *to* $(t+L+1-i, 0)$*.*

Theorem 77 is derived in a similar way to Theorems 74 and 76. We illustrate with an example. As with Ballot and Motzkin paths, we begin by illustrating how paths at time t depend on paths at time $t-1$:



(9.137)

with $\alpha = B_{t-1}(y+1)$, $\beta = B_{t-1}(y)$, $\gamma = B_{t-1}(y-1)$ and $\delta = B_t(y)$. Next, we rearrange Equation (9.137) to get Equation (9.138), and a new picture illustrating this relation:



(9.138)

This new picture gives points at height $y+1$ in terms of points at height y and $y-1$, on a grid illustrated in Figure 9.16.

Take all paths from the rightmost '0' above the upper wall in Figure 9.16 to the baseline, with allowed steps of the form $(1,-1)$, $(0,-2)$ and $(0,-3)$, weighted respectively $1/\lambda_y$, $-1/\lambda_y$ and $-1/\lambda_y$ for steps down from height y. The step set and weights are given by the illustration of Equation (9.138). The resulting set of paths is displayed in Figure 9.17.

Collecting the results from the paths in Figure 9.17 gives recurrence on 2-up weight polynomials in a strip of height $L=3$ to be

$$0 = \frac{1}{\lambda_1\lambda_2\lambda_3}B_{t,0} - \left(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3}\right)B_{t-2,0} - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_3}\right)B_{t-3,0} + \frac{1}{\lambda_2}B_{t-4,0}. \tag{9.139}$$

Multiplying by

$$\sum_{i=1}^{L} \lambda_i = \lambda_1\lambda_2\lambda_3 \tag{9.140}$$

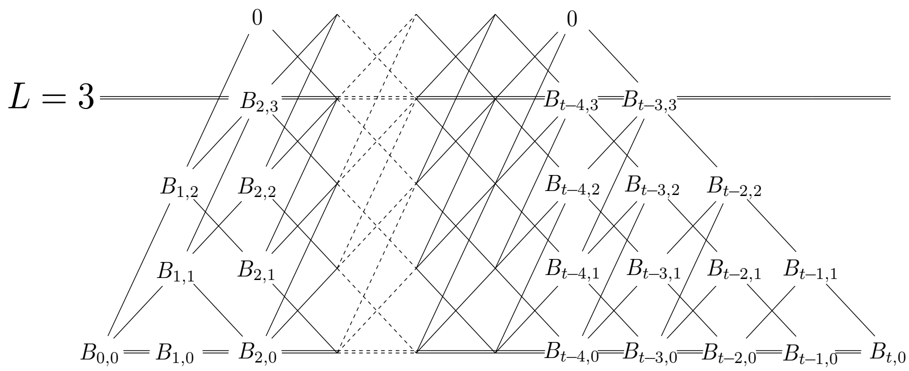


Figure 9.16: A grid for 2-up lattice paths in a strip of height $L = 3$; showing weight polynomials at each point labeled $B_{t,y}$, with lines indicating how each point is accessible via steps from the left.

gives 4-term recurrence

$$0 = B_{t,0} - (\lambda_3 + \lambda_2 + \lambda_1)\, B_{t-2,0} - (\lambda_2\lambda_3 + \lambda_1\lambda_2)\, B_{t-3,0} + \lambda_1\lambda_3 B_{t-4,0}, \tag{9.141}$$

as may be read off the pavings in Figure 9.17.

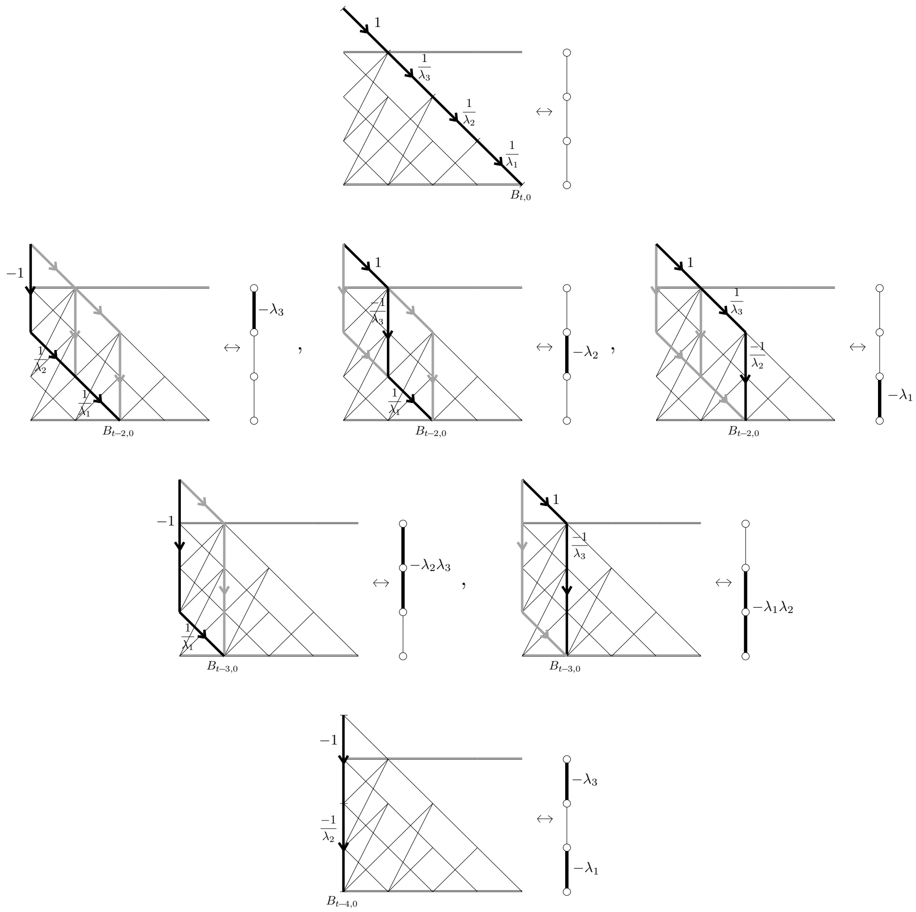


Figure 9.17: The set of all paths from a point at height $y = L+1$ to the baseline, taking steps drawn from the set defined by the illustration of Equation (9.138).

Chapter 10

A new 'Constant Term'

Now that we have generating functions, we want their coefficients, the path weight polynomials. We utilize a new twist on a very old technique. Given a generating function

$$G(x) := \sum_{i\geq 0} a_i x^i, \tag{10.1}$$

the classic and ancient form of 'Constant Term' is to note that for desired coefficient a_n,

$$a_n = \mathrm{CT}\left[\frac{G(x)}{x^n}\right], \tag{10.2}$$

where CT represents the constant term in the Laurent expansion of $G(x)$ around $x = 0$. Whilst true, Equation (10.2) is not very useful unless we can both

1. write down a closed form expression for $G(x)$; and
2. extract the constant term of the series.

This chapter presents a new way of achieving Item 2, since we have already satisfied Item 1 for many path problems of interest by the methods of Chapter 9 incorporating the earlier work on pavings in Part II.

The point of the method is that the generic alternatives, listed on page 25 of the introduction, all have drawbacks which make them difficult or impossible to apply to the weight polynomials we have evaluated in this Chapter. In particular

1. The first few terms of the series expansion of the generating function is usually quite complicated, making the method of proposing and verifying an Ansatz difficult.

2. It may not be clear how to manipulate the generating function into a form in which the Lagrange Inversion Theorem applies.

3. We are rarely lucky enough that the generating function in the original generating variable is in an explicitly rational form so that we could geometrically expand.

4. Taylor series expansion of the argument of the traditional constant term expression (10.2) involves finding n^{th} derivatives, for arbitrary n. This is likely to be intractable in practice.

5. Treating the constant term as a shifted residue, and calculating the Cauchy integral is also onerous, though more likely to succeed than Taylor expansion.

10.1 Derivation of a new CT

Consider Method 3 above, that of geometrically expanding a CT argument which is explicitly rational. What we would like to do, given a generating function which is *not explicitly rational*, is to change variables so that it is. But simply substituting one variable for another destroys the constant term of a series.

The key tool in the derivation of a new CT is a theorem which gives us the capacity to modify the argument of the traditional CT (utilizing a change of variables) in such a manner as to preserve the constant term of the Laurent expansion of the series. This key result, Theorem 78, depends upon the following lemmas.

Now the theorem which will allow a change of variables inside

$$\mathrm{CT}[\ldots]$$

is as follows.

Theorem 78. *Let $f(z)$ be a rational function and let $z(\rho) = 1/(a\rho+b+c\rho^{-1})$ be a change of variables; $a, c \neq 0$. Then*

$$Res\,[f(z), \{z, 0\}] = Res\left[f(z(\rho))\frac{dz}{d\rho}, \{\rho, 0\}\right]. \tag{10.3}$$

Proof. To prove the theorem first we expand the left hand in terms of a Laurent series which we show must exist. Then we do the same for the right hand side, to conclude that the two evaluations are equal.

Left hand side:

Since f is rational, f has only isolated singularities. Thus there exists a Laurent series $L(z)$ for $f(z)$ which converges in the annulus $0 < |z| < r$; for some $r > 0$. Let

$$L(z) = \sum_{k=1}^{m} b_k z^{-k} + \sum_{k=0}^{\infty} a_k z^k. \tag{10.4}$$

Recall that the residue of $f(z)$ at the point 0 is the coefficient of z^{-1} in the Laurent series $L(z)$. Thus the left hand side of equation (10.3) is

$$\begin{aligned} \text{Res}\left[f(z), \{z, 0\}\right] &= \text{coeff of } z^{-1} \left[\sum_{k=1}^{\infty} b_k z^{-k} + \sum_{k=0}^{m} a_k z^k\right] && (10.5)\\ &= b_1. && (10.6) \end{aligned}$$

Right hand side:

Observe that for any fixed a, b, c; by choosing sufficiently small ρ, its inverse ρ^{-1} may be made sufficiently large that

$$\left|\frac{1}{a\rho + b + c\rho^{-1}}\right| < r. \tag{10.7}$$

Thus, there exists $r' > 0$ such that within the annulus $0 < |\rho| < r'$,

$$\begin{aligned} f(z(\rho)) &= L\left(\frac{1}{a\rho + b + c\rho^{-1}}\right) && (10.8)\\ &= \sum_{k=1}^{m} b_k \left(a\rho + b + c\rho^{-1}\right)^k + \sum_{k=0}^{\infty} a_k \left(\frac{1}{a\rho + b + c\rho^{-1}}\right)^k. && (10.9) \end{aligned}$$

Hence, within the same annulus $0 < |\rho| < r'$, we also have the equality

$$f(z(\rho))\frac{dz}{d\rho} = \underbrace{\left(\sum_{k=1}^{m} b_k \left(a\rho + b + c\rho^{-1}\right)^k\right)\frac{dz}{d\rho}}_{(P1)} + \underbrace{\left(\sum_{k=0}^{\infty} a_k \left(\frac{1}{a\rho + b + c\rho^{-1}}\right)^k\right)\frac{dz}{d\rho}}_{(P2)}, \tag{10.10}$$

where

$$\frac{dz}{d\rho} = \frac{c\rho^{-2} - a}{(a\rho + b + c\rho^{-1})^2}. \tag{10.11}$$

The right hand side of Equation (10.10) as it stands specifies a function of ρ as a mixture of series and closed-form expressions. We aim to take each piece of this mixture and find a Laurent series for it.

We break the first piece into two further parts before proceeding, defining

$$(P1) := \underbrace{\left(\sum_{k=2}^{m} b_k \left(a\rho + b + c\rho^{-1}\right)^k\right)\frac{dz}{d\rho}}_{(P1a)} + \underbrace{b_1\left(a\rho + b + c\rho^{-1}\right)\frac{dz}{d\rho}}_{(P1b)} \tag{10.12}$$

Right hand side: Part (P1a)
Now, substituting for $\frac{dz}{d\rho}$ gives

$$(P1a) = \left(\sum_{k=2}^{m} b_k \left(a\rho + b + c\rho^{-1}\right)^k\right)\left(\frac{c\rho^{-2} - a}{(a\rho + b + c\rho^{-1})^2}\right) \tag{10.13}$$

$$= \sum_{k=2}^{m} b_k \left(a\rho + b + c\rho^{-1}\right)^{k-2}\left(c\rho^{-2} - a\right) \tag{10.14}$$

Since the right hand side of Equation (10.14) is a finite sum, it is the unique Laurent series for (P1a), and thus we may calculate the residue of (P1a) by picking out the coefficient of ρ^{-1}. Furthermore, since the sum is finite, we may carry out this aim for one value of k at a time, and then add them up to get the total residue for (P1a).

Thus, we want (for each $k \geq 2$) the coefficient of ρ^{-1} in

$$\left(a\rho + b + c\rho^{-1}\right)^{k-2}\left(c\rho^{-2} - a\right) \tag{10.15}$$

$$= \sum_{j=0}^{k-2} \binom{k-2}{j} b^{k-2-j}\left(a\rho + c\rho^{-1}\right)^j\left(c\rho^{-2} - a\right) \tag{10.16}$$

Again, finiteness means that it is sufficient to know (for each $0 \leq j \leq k-2$) what happens to the

$$\left(a\rho + c\rho^{-1}\right)^j\left(c\rho^{-2} - a\right) \tag{10.17}$$

For j even, all powers of ρ in the expansion of expression (10.17) will be even, hence the coefficient of ρ^{-1} is zero. So let j be odd. Then

$$\begin{aligned}
& (a\rho + c\rho^{-1})^j (c\rho^{-2} - a) && (10.18)\\
= & \left(\cdots + \binom{j}{\frac{\lfloor j \rfloor}{2}} a^{\lceil j \rceil/2} c^{\lfloor j \rfloor/2} \rho + \binom{j}{\frac{\lceil j \rceil}{2}} a^{\lfloor j \rfloor/2} c^{\lceil j \rceil/2} \rho^{-1} + \cdots \right) (c\rho^{-2} - a) && (10.19)\\
= & \cdots + \binom{j}{\frac{\lfloor j \rfloor}{2}} a^{\lceil j \rceil/2} c^{\lceil j \rceil/2} \rho^{-1} - \binom{j}{\frac{\lceil j \rceil}{2}} a^{\lceil j \rceil/2} c^{\lceil j \rceil/2} \rho^{-1} + \cdots && (10.20)\\
= & \text{ an expression with no } \rho^{-1} \text{ term,}
\end{aligned}$$

since $\binom{j}{\frac{\lfloor j \rfloor}{2}} = \binom{j}{\frac{\lceil j \rceil}{2}}$. We conclude that

$$\text{Res}\left[(P1a), \{\rho, 0\}\right] = 0. \tag{10.21}$$

Right hand side: Part (P1b)
Substituting for $\frac{dz}{d\rho}$ in (P1b) gives

$$\begin{aligned}
(P1b) &= b_1 \left(ap + b + c\rho^{-1}\right) \left(\frac{c\rho^{-2} - a}{(a\rho + b + c\rho^{-1})^2} \right) && (10.22)\\
&= \frac{b_1(c\rho^{-2} - a)}{a\rho + b + c\rho^{-1}} && (10.23)\\
&= b_1\rho^{-1} - (b/c)b_1 + (b_1(b^2 - 2ac)/c^2)\rho + \cdots, && (10.24)
\end{aligned}$$

for sufficiently small ρ. Thus we have immediately that

$$\text{Res}\left[(P1b), \{\rho, 0\}\right] = b_1. \tag{10.25}$$

Right hand side: Part (P2)

$$\begin{aligned}
(P2) &= \left(\sum_{k=0}^{\infty} a_k \left(\frac{1}{a\rho + b + c\rho^{-1}} \right)^k \right) \frac{dz}{d\rho} && (10.26)\\
&= \left(\sum_{k=0}^{\infty} a_k \left(\frac{1}{a\rho + b + c\rho^{-1}} \right)^k \right) \left(\frac{c\rho^{-2} - a}{(a\rho + b + c\rho^{-1})^2} \right) && (10.27)\\
&= (c\rho^{-2} - a) \sum_{k=0}^{\infty} a_k \left(\frac{1}{(a\rho + b + c\rho^{-1})^{k+2}} \right) && (10.28)\\
&= (c\rho^{-2} - a) \sum_{k=0}^{\infty} a_k \left(\rho^{k+2} \sum_{j=0}^{\infty} d_{j,k} \rho^j \right), && (10.29)
\end{aligned}$$

where $d_{j,k}$ is given by the finite sum

$$d_{j,k} := \sum_{i=0}^{\lfloor j/2 \rfloor} \binom{j-i+k+1}{k+1} \binom{j-i}{i} (-1)^{i+j} a^i b^{j-2i} c^{i-j-k-2}. \tag{10.30}$$

We aim to find the residue of (P2), and to this end we would like to swap the order of the sums in Equation (10.29). For this operation to be valid we require uniform convergence of the series

$$S(\rho) := \sum_{k=0}^{\infty} a_k \left(\frac{1}{(a\rho + b + c\rho^{-1})^{k+2}} \right) \tag{10.31}$$

from Equation (10.28). It turns out to be easier to show uniform convergence of a related function with the same residue:

$$\Theta(\rho) := \begin{cases} \sum_{k=0}^{\infty} a_k \left(\frac{1}{(a\rho+b+c\rho^{-1})^{k+2}} \right) & |\rho| < r'', \\ 0 & |\rho| \geq r''; \end{cases} \tag{10.32}$$

for choice of $r'' > 0$ given in Lemma 79 below. The function $\Theta(\rho)$ has the same residue at zero as does the series (10.31) because it has the same Taylor series expansion around zero. By Lemma 79, $\Theta(\rho)$ is uniformly convergent. Hence, *within the disc* $|\rho| < r''$, we may interchange the order of sums as follows:

$$\begin{aligned} \Theta(\rho) &= \sum_{k=0}^{\infty} a_k \left(\frac{1}{(a\rho + b + c\rho^{-1})^{k+2}} \right) && (10.33) \\ &= \sum_{k=0}^{\infty} a_k \left(\rho^{k+2} \sum_{j=0}^{\infty} d_{j,k} \rho^j \right) && (10.34) \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_k d_{j,k} \rho^{j+k+2}. && (10.35) \end{aligned}$$

We see that the least power of ρ occurring in expression (10.35) is ‘2’. Thus the least power of ρ occurring in the product

$$(c\rho^{-2} - a)\Theta(\rho) \tag{10.36}$$

is ‘0’. Hence expression (10.36) has no ‘ρ^{-1}’ term, and so has residue zero at zero. We have

$$\begin{aligned} \mathrm{Res}[(P2), \{\rho, 0\}] &\equiv \mathrm{Res}[(c\rho^{-2} - a)S(\rho), \{\rho, 0\}] && (10.37) \\ &= \mathrm{Res}[(c\rho^{-2} - a)\Theta(\rho), \{\rho, 0\}] && (10.38) \\ &= 0. && (10.39) \end{aligned}$$

Combining the three calculations, we have

$$\begin{aligned}
\operatorname{Res}\left[f(z(\rho))\frac{dz}{dp},\{\rho,0\}\right] &= \operatorname{Res}\left[(P1a)+(P1b)+(P2),\{\rho,0\}\right] && (10.40)\\
&= \operatorname{Res}\left[(P1a),\{\rho,0\}\right]+\operatorname{Res}\left[(P1b),\{\rho,0\}\right]\\
&\quad +\operatorname{Res}\left[(P2),\{\rho,0\}\right] && (10.41)\\
&= 0+b_1+0 && (10.42)\\
&= b_1, && (10.43)
\end{aligned}$$

which agrees with Equation (10.6). □

Lemma 79. *Let the series*

$$S(\rho)=\sum_{k=0}^{\infty}a_k\left(\frac{1}{(a\rho+b+c\rho^{-1})^{k+2}}\right), \tag{10.44}$$

as defined in the proof of Theorem 78, be convergent in a disc of the form $|p|<r'$. Then there exists a real number $r''>0$ such that the function

$$\Theta(\rho):=\begin{cases} S(\rho) & |\rho|<r'',\\ 0 & |\rho|\geq r''; \end{cases} \tag{10.45}$$

is uniformly convergent in a disc of the form $|\rho|<r''$.

Proof. It is sufficient, due to a Theorem of Weierstrauss (quoted in [109]), to show that

$$\left|a_k\left(\frac{1}{(a\rho+b+c\rho^{-1})^{k+2}}\right)\right|<M_k \tag{10.46}$$

for positive M_k independent of ρ such that

$$\sum_k M_k \tag{10.47}$$

converges. Choose $n\in\mathbb{N}$ such that $n>2$ and

$$\frac{1}{n}<r'. \tag{10.48}$$

Now let

$$|\rho|<r'':=\min\left\{r',\frac{|c|}{n+|b|+1},\frac{1}{|a|(n+|b|+1)}\right\}. \tag{10.49}$$

Then

$$|c||\rho^{-1}| > n + |b| + 1 \tag{10.50}$$

and

$$\begin{aligned} |a||\rho| &< \frac{1}{n + |b| + 1} && (10.51) \\ &< 1, && (10.52) \end{aligned}$$

since $n + |b| + 1 > 1$. Thus

$$-|a||\rho| > -1. \tag{10.53}$$

Now, by the reverse triangle inequality,

$$\begin{aligned} |a\rho + b + c\rho^{-1}| &\geq |c||\rho^{-1}| - |b| - |a||\rho| && (10.54) \\ &> n + |b| + 1 - |b| - 1 && (10.55) \\ &= n. && (10.56) \end{aligned}$$

Hence

$$\left| \frac{1}{a\rho + b + c\rho^{-1}} \right| < \frac{1}{n} \tag{10.57}$$

for $|\rho| < r''$. Write

$$M_k = \frac{|a_k|}{n^{k+2}}. \tag{10.58}$$

Then by Equation (10.57),

$$\left| a_k \left(\frac{1}{(a\rho + b + c\rho^{-1})^{k+2}} \right) \right| < M_k, \tag{10.59}$$

for all $k \geq 0$. We also know from the definition of the a_k's in Theorem 78 that

$$\sum_{k=0}^{\infty} a_k z^k \tag{10.60}$$

converges in a disc of the form $|z| < r$, hence by the ratio test

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lambda, \tag{10.61}$$

for some fixed

$$\lambda \leq 1. \tag{10.62}$$

Applying the ratio test to the sum over M_k's gives

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{M_{k+1}}{M_k}\right| &= \lim_{k\to\infty}\left|\frac{a_{k+1}n^{k+2}}{a_k n^{k+3}}\right| && (10.63)\\
&= \frac{\lambda}{n} && (10.64)\\
&< 1, && (10.65)
\end{aligned}$$

hence the series (10.47), with M_k's defined by Equation (10.58), converges by the ratio test. □

10.2 A new CT for Ballot and Motzkin Paths

The residue result Theorem 78, applied to the Ballot/Motzkin path length generating function Theorem 72, with the earlier work on pavings brought to bear, gives us a powerful new tool for tackling decorated lattice paths which have resisted enumeration for decades. We have general Ballot/Motzkin result

Theorem 80. *Let $B_t(y', y; \{\lambda_i\}_{i=1,..,L}, \{b_j\}_{j=0,1,..,L}; L)$ be the path weight polynomial for Motzkin-like paths of length t, beginning at height y', ending at height y, confined to a strip of height L; and weighted as follows:*

$$\begin{aligned}
w(\textit{upstep}) &= 1 && (10.66)\\
w(\textit{across step from height `}y\textit{' to height `}y\textit{'}) &= b_y && (10.67)\\
w(\textit{downstep from height `}y\textit{' to height `}y-1\textit{'}) &= \lambda_y; && (10.68)
\end{aligned}$$

so that the transfer matrix for the paths is an order $L+1$ matrix of the form (9.37):

$$T_L = \begin{pmatrix} b_0 & 1 & 0 & \dots \\ \lambda_1 & b_1 & 1 & \\ 0 & \lambda_2 & b_2 & \\ \vdots & & & \ddots \end{pmatrix}.$$

Note that in the special case of weightings for which $b_j = 0$ for all j, the paths are Ballot-like. Abbreviate

$$B_t(y', y; L) := B_t(y', y; \{\lambda_i\}_{i=1,..,L}, \{b_j\}_{j=0,1,..,L}; L). \tag{10.69}$$

Then

$$B_t(y', y; L) = \mathrm{CT}\left[\left(\rho + b + \lambda\rho^{-1}\right)^t \left(\frac{h_{y',y}\tilde{P}_{Y'}(\rho)\tilde{P}^{(Y+1)}_{L-Y}(\rho)}{\tilde{P}_{L+1}(\rho)}\right)\left(\lambda\rho^{-1} - \rho\right), \{\rho, 0\}\right]. \tag{10.70}$$

where CT[...] *means the constant term of the Laurent expansion of the argument around zero.*

The factor $h_{y',y} = 1$ *for* $y' \le y$*, and is otherwise the product*

$$h_{y',y} = \prod_{y<l\le y'} \lambda_l. \tag{10.71}$$

The $\tilde{P}$*'s are Laurent Motzkin (or Ballot) paving polynomials defined by*

$$\tilde{P}_k^{(j)}(\rho) := P_k^{(j)}(\mu(p)) \tag{10.72}$$

with

$$\mu(\rho) := \rho + b + \lambda\rho^{-1} \tag{10.73}$$

for constant ‘background weights’ b *and* λ*. Finally the* P_k*'s are Motzkin (or Ballot) paving polynomials, satisfying recurrence relation*

$$P_{k+1}(\mu) = (\mu - b_k)P_k(\mu) - \lambda_k P_{k-1}(\mu), \tag{10.74}$$

subject to normalization

$$P_0(\mu) = 1, \qquad P_1(\mu) = \mu - b_0. \tag{10.75}$$

The $P_k^{(s)}$*'s satisfy the same relations except with shifted coefficients, so that* $P_0^{(s)}(\mu) = 1$*,* $P_1^{(s)}(x) = \mu - b_j$ *and* $P_{k+1}^{(s)}(\mu) = (\mu - b_{k+s})P_k^{(s)}(\mu) - \lambda_{k+s}P_{k-1}^{(s)}(\mu)$*.*

Proof. We start with the classic CT of equation (10.2), and the Motzkin generating function of Theorem 72.

$$B_t(y', y; L) = \mathrm{CT}\left[\frac{h_{y',y}P_{Y'}(1/x)P_{L-Y}^{(Y+1)}(1/x)}{x^{t+1}P_{L+1}(1/x)}\right]. \tag{10.76}$$

Since a residue is the coefficient of the -1^{th} term in a Laurent series, shifting Equation (10.76) by a factor of x gives

$$B_t(y', y; L) = \mathrm{Res}\left[\frac{h_{y',y}P_{Y'}(1/x)P_{L-Y}^{(Y+1)}(1/x)}{x^{t+2}P_{L+1}(1/x)}, \{x, 0\}\right]. \tag{10.77}$$

The variable change

$$x(\rho) := \frac{1}{\rho + b + \lambda\rho^{-1}}, \tag{10.78}$$

gives derivative

$$\frac{dx}{d\rho} = \frac{\lambda p^{-2} - 1}{(\rho + b + \lambda\rho^{-1})^2} \tag{10.79}$$

so that an application of Theorem 78 yields

$$B_t(y', y; L) = \text{Res}\left[\left(\rho + b + \lambda\rho^{-1}\right)^t \left(\frac{h_{y',y}\tilde{P}_{Y'}(\rho)\tilde{P}_{L-Y}^{(Y+1)}(\rho)}{\tilde{P}_{L+1}(\rho)}\right)\left(\lambda\rho^{-2} - 1\right), \{\rho, 0\}\right]. \tag{10.80}$$

Finally, shifting the argument by a factor of ρ gives the result as a constant term. □

10.3 Decorated Path Enumerations

One of the long unsolved problems that Theorem 80 makes accessible to us is that posed by DiMazio and Rubin in 1971, in [37]. This is the problem of paths in a strip with a Ballot-like step set, with just two independent decorations: one a 'return' weight on steps from height '1' to the lower wall at height '0', and the other a 'departing' weight on steps from height L on the upper wall to height $L - 1$ below.

The problem is a natural one to consider as a model of polymers between two surfaces with which they interact. The solution to the enumeration problem has also been shown, subsequently to DiMazio and Rubin asking the question, to have application in the calculation of the stationary state of a stochastic process in traffic modeling, [41].

In the intervening years between 1971 and now, work on the problem produced a partial result by Brak, Essam and Owczarek [16]. The solution was obtained when the wall weights are not independent, but instead satisfy relation

$$\kappa + \omega = \kappa\omega. \tag{10.81}$$

This relationship implies the advantage that the orthogonal polynomial in the denominator of the generating function factorizes nicely, which makes computations more tractable. However the assumption diminishes the usefulness of the result in both the polymer modeling context and the traffic flow application.

Here we present the general solution for independent weights κ and ω. First we have the theorem stated in the form of a constant term.

Theorem 81. *Let $W_{2r}(\kappa,\omega;L)$ be the weight polynomial for Dyck paths of length $2r$ confined to a strip of height L, and with weights*

$$w(\text{downstep from height '1' to height '0'}) = \kappa, \quad (10.82)$$
$$w(\text{downstep from height '}L\text{' to height '}L-1\text{'}) = \omega; \quad (10.83)$$

and all other weights equal to '1'. Then

$$W_{2r}(\kappa,\omega;L) = \mathrm{CT}\left[(\rho+\rho^{-1})^{2r}(1-\rho^2)\frac{A\rho^L - B\rho^{-L}}{AC\rho^L - BD\rho^{-L}}\right] \quad (10.84)$$

where

$$A = \rho^2 - \hat{\omega} \quad (10.85)$$
$$B = 1 - \hat{\omega}\rho^2 \quad (10.86)$$
$$C = \rho^2 - \hat{\kappa} \quad (10.87)$$
$$D = 1 - \hat{\kappa}\rho^2 \quad (10.88)$$

for $\hat{\kappa} := \kappa - 1$ and $\hat{\omega} := \omega - 1$.

Proof. The Theorem follows from Theorem 80 and Theorem 43 with background weights $\lambda = 1$, $b = 0$. □

Next we give the explicit expansion in terms of binomials.

Theorem 82. *Let $W_{2r}(\kappa,\omega;L)$ be as in Theorem 81. Then*

$$W_{2r}(0,0;\kappa,\omega;L) = \sum_{m\geq 1}\sum_{p_1,p_2\geq 0}\sum_{s_1,s_2=0}^{m}(-1)^{s_1+s_2}\hat{\kappa}^{s_2+p_2}\hat{\omega}^{s_1+p_1}$$
$$\times\binom{m}{s_1}\binom{m+p_1-1}{p_1}\times\left[C_{r;k+1}\binom{m-1}{s_2}\binom{m+p_2-1}{p_2}\right.$$
$$\left.-C_{r;k}\binom{m}{s_2}\binom{m+p_2}{p_2}\right] \quad (10.89)$$

where $k = r + s_1 + s_2 - p_1 - p_2 - (L+2)m$, $C_{n;x} = \binom{2n}{n-x} - \binom{2n}{n-x-1}$ and $\hat{\kappa} = \kappa - 1$, $\hat{\omega} = \omega - 1$.

Proof. Expanding and rearranging gives

$$\begin{aligned}
\frac{A\rho^L - B\rho^{-L}}{AC\rho^L - BD\rho^{-L}} &= \frac{B - A\rho^{2L}}{BD - AC\rho^{2L}} && (10.90)\\
&= \left(\frac{B - A\rho^{2L}}{BD}\right) \sum_{m\geq 0} \left(\frac{AC}{BD}\right)^m \rho^{2mL} && (10.91)\\
&= \sum_{m\geq 0} \frac{A^m C^m}{B^m D^m}\rho^{2mL} - \sum_{m\geq 0} \frac{A^{m+1} C^m}{B^{m+1} D^{m+1}}\rho^{2mL+2L} && (10.92)\\
&= \frac{1}{D} + \sum_{m\geq 1} \frac{A^m C^m}{B^m D^m}\rho^{2mL} - \sum_{m\geq 1} \frac{A^m C^{m-1}}{B^m D^m}\rho^{2mL} && (10.93)
\end{aligned}$$

Expansion of A^m and C^m give finite sums over s_1 and s_2 respectively. Expansion of B^m and D^m from the denominators give infinite sums over p_1 and p_2 respectively. Together with the sum on m in Equation 10.93 this makes five sums. A sixth sum appears in the expansion

$$\left(\rho + \rho^{-1}\right)^{2r} = \sum_{i=0}^{2r} \binom{2r}{i} \rho^{2r-2i} \tag{10.94}$$

This sixth sum disappears from the final answer when we choose i to give ρ with zero exponent, i.e. the constant term. □

An extension of the DiMazio/Rubin problem is the decorated weighted path enumeration problem for which there are two weights near each wall. This may be relevant to a physical polymer model in which a single weight near each wall is too crude an approximation to a situation in which attraction between polymer and wall varies significantly over small distances near the wall.

The CT method of Theorem 80 also gives a solution to this problem. We have, in terms of the constant term,

Theorem 83. *Let $W_{2r}(\kappa_1, \kappa_2\omega_1, \omega_2; L)$ be the weight polynomial for Dyck paths of length $2r$ confined to a strip of height L, and with weights*

$$\begin{aligned}
w(\textit{downstep from height `}L\textit{' to height `}L-1\textit{'}) &= \omega_1, && (10.95)\\
w(\textit{downstep from height `}L-1\textit{' to height `}L-2\textit{'}) &= \omega_2, && (10.96)\\
w(\textit{downstep from height `2' to height `1'}) &= \kappa_2, && (10.97)\\
w(\textit{downstep from height `1' to height `0'}) &= \kappa_1, && (10.98)
\end{aligned}$$

and all other weights equal to ‘1’. Then

$$W_{2r}(\kappa_1,\kappa_2,\omega_1,\omega_2;L) = \mathrm{CT}\left[(\rho+\rho^{-1})^{2r}\left(\frac{AB\rho^L - \overline{A}\,\overline{B}\rho^{-L}}{CB\rho^L - \overline{C}\,\overline{B}\rho^{-L}}\right)(\rho^{-1}-\rho)\right] \tag{10.99}$$

where

$$A = 1-\hat{\kappa}_2\rho^{-2} \tag{10.100}$$
$$\overline{A} = 1-\hat{\kappa}_2\rho^{2} \tag{10.101}$$
$$B = \rho-(\hat{\omega}_1+\hat{\omega}_2)\rho^{-1}-\hat{\omega}_2\rho^{-3} \tag{10.102}$$
$$\overline{B} = \rho^{-1}-(\hat{\omega}_1+\hat{\omega}_2)\rho-\hat{\omega}_2\rho^{3} \tag{10.103}$$
$$C = \rho-(\hat{\kappa}_1+\hat{\kappa}_2)\rho^{-1}-\hat{\kappa}_2\rho^{-3} \tag{10.104}$$
$$\overline{C} = \rho^{-1}-(\hat{\kappa}_1+\hat{\kappa}_2)\rho-\hat{\kappa}_2\rho^{3} \tag{10.105}$$

for $\hat{\kappa} := \kappa - 1$ and $\hat{\omega} := \omega - 1$.

Proof. Start with Theorem 58 with background weight $\lambda = 1$; and two weights per wall. We have denominator Laurent polynomial

$$\begin{aligned}\left(\rho-\rho^{-1}\right)\tilde{P}_{L+1}(\rho) &= \left(\tilde{D}_3^\kappa(\rho)-\rho^{-1}\tilde{D}_2^\kappa\right)\left(\tilde{D}_3^\omega(\rho)-\rho^{-1}\tilde{D}_2^\omega\right)\rho^{L-4}\\ &\quad-\left(\tilde{D}_3^\kappa(\rho)-\rho\tilde{D}_2^\kappa\right)\left(\tilde{D}_3^\omega(\rho)-\rho\tilde{D}_2^\omega\right)\rho^{-L+4}\end{aligned} \tag{10.106}$$

which equals

$$\begin{aligned}&= \left(\rho-(\hat{\kappa}_1+\hat{\kappa}_2)\rho^{-1}-\hat{\kappa}_2\rho^{-3}\right)\left(\rho-(\hat{\omega}_1+\hat{\omega}_2)\rho^{-1}-\hat{\omega}_2\rho^{-3}\right)\rho^{L}\\ &\quad-\left(\rho^{-1}-(\hat{\kappa}_1+\hat{\kappa}_2)\rho-\hat{\kappa}_2\rho^{3}\right)\left(\rho^{-1}-(\hat{\omega}_1+\hat{\omega}_2)\rho-\hat{\omega}_2\rho^{3}\right)\rho^{-L}.\end{aligned} \tag{10.107}$$

To find the shifted numerator Laurent polynomial without having to redo the calculation from the beginning, take Equation (10.107), replace $\hat{\kappa}_2$ by zero, replace $\hat{\kappa}_1$ by $\hat{\kappa}_2$ and send $L \mapsto L-1$. Thus we have

$$\begin{aligned}\left(\rho-\rho^{-1}\right)\tilde{P}_L^{(1)}(\rho) &= \left(1-\hat{\kappa}_2\rho^{-2}\right)\left(\rho-(\hat{\omega}_1+\hat{\omega}_2)\rho^{-1}-\hat{\omega}_2\rho^{-3}\right)\rho^{L}\\ &\quad-\left(1-\hat{\kappa}_2\rho^{2}\right)\left(\rho^{-1}-(\hat{\omega}_1+\hat{\omega}_2)\rho-\hat{\omega}_2\rho^{3}\right)\rho^{-L}\end{aligned} \tag{10.108}$$

The other numerator Laurent polynomial is $\tilde{P}_0(\rho) := 1$, hence

$$\frac{\tilde{P}_0(\rho)\tilde{P}_L^{(1)}(\rho)}{\tilde{P}_{L+1}(\rho)} = \frac{AB\rho^L-\overline{A}\,\overline{B}\rho^{-L}}{CB\rho^L-\overline{C}\,\overline{B}\rho^{-L}} \tag{10.109}$$

for A, $\overline{A}$, B, $\overline{B}$, C, $\overline{C}$, as stated in the theorem. Now the result follows from Theorem 80, with background weights $\lambda = 1$, $b = 0$. □

The explicit expansion of the argument of (10.99) gives the following result.

Theorem 84. *Let $W_{2r}(\kappa_1, \kappa_2\omega_1, \omega_2; L)$ be as in Theorem 83. Then*

$$
\begin{aligned}
&W_{2r}(\kappa_1, \kappa_2, \omega_1, \omega_2; L) \\
&= \sum_{i\geq 0}\sum_{j=0}^{i}\binom{2r}{u}\binom{i}{j}(\hat\kappa_1+\hat\kappa_2)^j\hat\kappa_2^{i-j}\left(\hat\kappa_2\binom{2r}{u_0+2}-(\hat\kappa_2+1)\binom{2r}{u_0+1}+\binom{2r}{u_0}\right) \\
&+ \sum_{m\geq 1}\sum_{s_1=0}^{m}\sum_{i_1=0}^{s_1}\sum_{s_2=0}^{m}\sum_{i_2=0}^{s_2}\sum_{v_1\geq 0}\sum_{j_1=0}^{v_1}\sum_{v_2\geq 0}\sum_{j_2=0}^{v_2}\binom{s_1}{i_1}\binom{m}{s_2}\binom{s_2}{i_2}\binom{v_1}{j_1}\binom{v_2+m-1}{m-1}\binom{v_2}{j_2} \\
&\quad(-1)^{s_1+s_2+i_1+i_2}\hat\kappa_2^{i_1+j_1}(\hat\kappa_1+\hat\kappa_2)^{m+v_1-1-s_1-j_1}\hat\omega_2^{i_2+j_2}(\hat\omega_1+\hat\omega_2)^{m+v_2-s_2-j_2}\times \\
&\quad\left\{\binom{m}{s_1}\binom{v_1+m}{m}(\hat\kappa_1+\hat\kappa_2)\left(\hat\kappa_2\binom{2r}{u_1+2}-(\hat\kappa_1+1)\binom{2r}{u_1+1}+\binom{2r}{u_1}\right)\right. \\
&\quad\left.-\binom{m-1}{s_1}\binom{v_1+m-1}{m-1}\left(\hat\kappa_2\binom{2r}{u_1-1}-(\hat\kappa_1+1)\binom{2r}{u_1}+\binom{2r}{u_1+1}\right)\right\}, \qquad (10.111)
\end{aligned}
$$

for $u_0 = r + 2i - j$ and $u_1 = r + mL + v_1 + v_2 + s_1 + s_2 + j_1 + j_2 - 2i_1 - 2i_2$.

Proof.

$$
\begin{aligned}
\frac{\tilde P_0(\rho)\tilde P_L^{(1)}(\rho)}{\tilde P_{L+1}(\rho)} &= \frac{AB\rho^L-\overline{A}\,\overline{B}\rho^{-L}}{CB\rho^L-\overline{C}\,\overline{B}\rho^{-L}} && (10.112)\\
&= \frac{\overline{A}\,\overline{B}-AB\rho^{2L}}{\overline{C}\,\overline{B}-CB\rho^{2L}} && (10.113)\\
&= \frac{\overline{A}\,\overline{B}-AB\rho^{2L}}{\overline{C}\,\overline{B}}\sum_{m\geq 0}\left(\frac{CB}{\overline{C}\,\overline{B}}\right)^m\rho^{2mL} && (10.114)\\
&= \overline{A}\sum_{m\geq 0}\left(\frac{C^mB^m}{\overline{C}^{m+1}\,\overline{B}^m}\right)\rho^{2mL}-A\sum_{m\geq 0}\left(\frac{C^mB^{m+1}}{\overline{C}^{m+1}\,\overline{B}^{m+1}}\right)\rho^{2(m+1)L} && (10.115)\\
&= \underbrace{\frac{\overline{A}}{\overline{C}}}_{(T_1)}\underbrace{+\overline{A}\sum_{m\geq 1}\left(\frac{C^mB^m}{\overline{C}^{m+1}\,\overline{B}^m}\right)\rho^{2mL}}_{(T_2)}\underbrace{-A\sum_{m\geq 1}\left(\frac{C^{m-1}B^m}{\overline{C}^m\,\overline{B}^m}\right)\rho^{2mL}}_{(T_3)} && (10.116)
\end{aligned}
$$

Now, we want

$$
\mathrm{CT}\left[(\rho+\rho^{-1})^{2r}\Big((T_1)+(T_2)+(T_3)\Big)(\rho^{-1}-\rho)\right] \qquad (10.117)
$$

We split the calculation up according to the three terms.

1. Considering the first piece, we have:

$$\frac{\overline{A}}{\overline{C}} = \frac{1-\hat{\kappa}_2\rho^2}{\rho^{-1}-(\hat{\kappa}_1+\hat{\kappa}_2)\,\rho-\hat{\kappa}_2\rho^3} \tag{10.118}$$

$$= \frac{\rho-\hat{\kappa}_2\rho^3}{1-\left[(\hat{\kappa}_1+\hat{\kappa}_2)+\hat{\kappa}_2\rho^2\right]\rho^2} \tag{10.119}$$

$$= \left(\rho-\hat{\kappa}_2\rho^3\right)\sum_{i\geq 0}\left[(\hat{\kappa}_1+\hat{\kappa}_2)+\hat{\kappa}_2\rho^2\right]^i\rho^{2i} \tag{10.120}$$

$$= \left(\rho-\hat{\kappa}_2\rho^3\right)\sum_{i\geq 0}\sum_{j=0}^{i}\binom{i}{j}(\hat{\kappa}_1+\hat{\kappa}_2)^j\hat{\kappa}_2^{i-j}\rho^{4i-2j}. \tag{10.121}$$

We want the following constant term:

$$\mathrm{CT}\left[(\rho+\rho^{-1})^{2r}(\rho^{-1}-\rho)(-\hat{\kappa}_2\rho^3+\rho)\sum_{i\geq 0}\sum_{j=0}^{i}\binom{i}{j}(\hat{\kappa}_1+\hat{\kappa}_2)^j\hat{\kappa}_2^{i-j}\rho^{4i-2j}\right]$$

$$= \mathrm{CT}\left[\sum_{u=0}^{2r}\sum_{i\geq 0}\sum_{j=0}^{i}\binom{2r}{u}\binom{i}{j}(\hat{\kappa}_1+\hat{\kappa}_2)^j\hat{\kappa}_2^{i-j}\rho^{2r-2u+4i-2j}\times \left(\hat{\kappa}_2\rho^4-(\hat{\kappa}_2+1)\rho^2+1\right)\right] \tag{10.122}$$

To find the constant term, solve

- $2r-2u+4i-2j+4=0 \iff$
$$u = r+2i-j+2 \tag{10.123}$$
- $2r-2u+4i-2j+2=0 \iff$
$$u = r+2i-j+1 \tag{10.124}$$
- $2r-2u+4i-2j+0=0 \iff$
$$u = r+2i-j \tag{10.125}$$

so that

$$\begin{aligned}
&\mathrm{CT}\left[(\rho+\rho^{-1})^{2r}\Big((T_1)\Big)(\rho-\rho^{-1})\right] \\
&= \sum_{i\geq 0}\sum_{j=0}^{i}\binom{i}{j}(\hat{\kappa}_1+\hat{\kappa}_2)^j\hat{\kappa}_2^{i-j}\left(\hat{\kappa}_2\binom{2r}{u_0+2}-(\hat{\kappa}_2+1)\binom{2r}{u_0+1}+\binom{2r}{u_0}\right),
\end{aligned} \tag{10.126}$$

where $u_0 = r + 2i - j$.

2. Considering the second piece, we have:

$$\begin{aligned}
\frac{C^m B^m}{\overline{C}^{m+1}\overline{B}^m} &= \frac{\left(\rho-(\hat{\kappa}_1+\hat{\kappa}_2)\rho^{-1}-\hat{\kappa}_2\rho^{-3}\right)^m\left(\rho-(\hat{\omega}_1+\hat{\omega}_2)\rho^{-1}-\hat{\omega}_2\rho^{-3}\right)^m}{(\rho^{-1}-(\hat{\kappa}_1+\hat{\kappa}_2)\rho-\hat{\kappa}_2\rho^3)^{m+1}(\rho^{-1}-(\hat{\omega}_1+\hat{\omega}_2)\rho-\hat{\omega}_2\rho^3)^m} \\
&= \frac{\rho\left(\left(\rho^2-\hat{\kappa}_2\rho^{-2}\right)-(\hat{\kappa}_1+\hat{\kappa}_2)\right)^m\left(\left(\rho^2-\hat{\omega}_2\rho^{-2}\right)-(\hat{\omega}_1+\hat{\omega}_2)\right)^m}{(1-[(\hat{\kappa}_1+\hat{\kappa}_2)\rho^2+\hat{\kappa}_2\rho^4])^{m+1}(1-[(\hat{\omega}_1+\hat{\omega}_2)\rho^2+\hat{\omega}_2\rho^4])^m}
\end{aligned} \tag{10.127}$$

General facts

$$(a-b)^m = \sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}a^s b^{m-s}, \tag{10.128}$$

$$\frac{1}{(1-x)^m} = \sum_{v\geq 0}\binom{v+m-1}{m-1}x^v \tag{10.129}$$

imply that Equation (10.127) becomes

$$\begin{aligned}
\frac{C^m B^m}{\overline{C}^{m+1}\overline{B}^m} = {} & \rho\left(\sum_{s_1=0}^{m}\binom{m}{s_1}\left(\rho^2-\hat{\kappa}_2\rho^{-2}\right)^{s_1}(-1)^{m-s_1}(\hat{\kappa}_1+\hat{\kappa}_2)^{m-s_1}\right)\times \\
& \left(\sum_{s_2=0}^{m}\binom{m}{s_2}\left(\rho^2-\hat{\omega}_2\rho^{-2}\right)^{s_2}(-1)^{m-s_2}(\hat{\omega}_1+\hat{\omega}_2)^{m-s_2}\right)\times \\
& \left(\sum_{v_1\geq 0}\binom{v_1+m}{m}\left[(\hat{\kappa}_1+\hat{\kappa}_2)\rho^2+\hat{\kappa}_2\rho^4\right]^{v_1}\right)\times \\
& \left(\sum_{v_2\geq 0}\binom{v_2+m-1}{m-1}\left[(\hat{\omega}_1+\hat{\omega}_2)\rho^2+\hat{\omega}_2\rho^4\right]^{v_2}\right).
\end{aligned} \tag{10.130}$$

We also have

$$\left(\rho^2 - \hat{\kappa}_2\rho^{-2}\right)^{s_1} = \sum_{i_1=0}^{s_1} \binom{s_1}{i_1} (-1)^{i_1} \hat{\kappa}_2^{i_1} \rho^{2s_1-4i_1}, \quad (10.131)$$

$$\left(\rho^2 - \hat{\omega}_2\rho^{-2}\right)^{s_2} = \sum_{i_2=0}^{s_2} \binom{s_2}{i_2} (-1)^{i_2} \hat{\omega}_2^{i_2} \rho^{2s_2-4i_2}, \quad (10.132)$$

$$\left[(\hat{\kappa}_1 + \hat{\kappa}_2)\rho^2 + \hat{\kappa}_2\rho^4\right]^{v_1} = \sum_{j_1=0}^{v_1} \binom{v_1}{j_1} \hat{\kappa}_2^{j_1} (\hat{\kappa}_1 + \hat{\kappa}_2)^{v_1-j_1} \rho^{2v_1+2j_1}, \quad (10.133)$$

$$\left[(\hat{\omega}_1 + \hat{\omega}_2)\rho^2 + \hat{\omega}_2\rho^4\right]^{v_2} = \sum_{j_2=0}^{v_2} \binom{v_2}{j_2} \hat{\omega}_2^{j_2} (\hat{\omega}_1 + \hat{\omega}_2)^{v_2-j_2} \rho^{2v_2+2j_2}. \quad (10.134)$$

Thus

$$\frac{C^m B^m}{\overline{C}^{m+1}\overline{B}^m} = \sum_{s_1=0}^{m}\sum_{i_1=0}^{s_1}\sum_{s_2=0}^{m}\sum_{i_2=0}^{s_2}\sum_{v_1\geq 0}\sum_{j_1=0}^{v_1}\sum_{v_2\geq 0}\sum_{j_2=0}^{v_2} \binom{m}{s_1}\binom{s_1}{i_1}\binom{m}{s_2}\binom{s_2}{i_2} \times \binom{v_1+m}{m}\binom{v_1}{j_1}\binom{v_2+m-1}{m-1}\binom{v_2}{j_2}(-1)^{s_1+s_2+i_1+i_2} \times \hat{\kappa}_2^{i_1+j_1}\left(\hat{\kappa}_1+\hat{\kappa}_2\right)^{m+v_1-s_1-j_1}\hat{\omega}_2^{i_2+j_2}\left(\hat{\omega}_1+\hat{\omega}_2\right)^{m+v_2-s_2-j_2} \times \rho^{2v_1+2v_2+2s_1+2s_2+2j_1+2j_2-4i_1-4i_2+1}. \quad (10.135)$$

Now we want

$$\mathrm{CT}\left[(\rho+\rho^{-1})^{2r}(\rho^{-1}-\rho)\overline{A}\sum_{m\geq 1}\left(\frac{C^m B^m}{\overline{C}^{m+1}\overline{B}^m}\right)\rho^{2mL}\right] \quad (10.136)$$

which equals

$$\mathrm{CT}\left[\left(\sum_{u=0}^{2r}\binom{2r}{u}\rho^{2r-2u}\right)\left(\hat{\kappa}_2\rho^3-(\hat{\kappa}_2+1)\rho+\rho^{-1}\right)\left(\sum_{m\geq 1}\left(\frac{C^m B^m}{\overline{C}^{m+1}\overline{B}^m}\right)\rho^{2mL}\right)\right] \quad (10.137)$$

which in turn is

$$\mathrm{CT}\quad\Bigg[\Bigg(\sum_{u=0}^{2r}\sum_{m\geq 1}\sum_{s_1=0}^{m}\sum_{i_1=0}^{s_1}\sum_{s_2=0}^{m}\sum_{i_2=0}^{s_2}\sum_{v_1\geq 0}\sum_{j_1=0}^{v_1}\sum_{v_2\geq 0}\sum_{j_2=0}^{v_2}\binom{2r}{u}\binom{m}{s_1}\times$$
$$\binom{s_1}{i_1}\binom{m}{s_2}\binom{s_2}{i_2}\binom{v_1+m}{m}\binom{v_1}{j_1}\binom{v_2+m-1}{m-1}\binom{v_2}{j_2}\times$$
$$(-1)^{s_1+s_2+i_1+i_2}\times$$
$$\hat{\kappa}_2^{i_1+j_1}\,(\hat{\kappa}_1+\hat{\kappa}_2)^{m+v_1-s_1-j_1}\,\hat{\omega}_2^{i_2+j_2}\,(\hat{\omega}_1+\hat{\omega}_2)^{m+v_2-s_2-j_2}\times$$
$$\rho^{2r-2u+2mL+2v_1+2v_2+2s_1+2s_2+2j_1+2j_2-4i_1-4i_2+1}\Bigg)\times$$
$$\left(\hat{\kappa}_2\rho^3-(\hat{\kappa}_2+1)\rho+\rho^{-1}\right)\Bigg] \tag{10.138}$$

There are three pieces to this (second) term in the expansion, which correspond to the cases

- $2r-2u+2mL+2v_1+2v_2+2s_1+2s_2+2j_1+2j_2-4i_1-4i_2+1+3=0$ $\iff$
$$u=r+mL+v_1+v_2+s_1+s_2+j_1+j_2-2i_1-2i_2+2 \tag{10.139}$$
- $2r-2u+2mL+2v_1+2v_2+2s_1+2s_2+2j_1+2j_2-4i_1-4i_2+1+1=0$ $\iff$
$$u=r+mL+v_1+v_2+s_1+s_2+j_1+j_2-2i_1-2i_2+1 \tag{10.140}$$
- $2r-2u+2mL+2v_1+2v_2+2s_1+2s_2+2j_1+2j_2-4i_1-4i_2+1-1=0$ $\iff$
$$u=r+mL+v_1+v_2+s_1+s_2+j_1+j_2-2i_1-2i_2 \tag{10.141}$$

Thus we have, for the CT expansion of the second term,

$$\mathrm{CT}\left[(\rho+\rho^{-1})^{2r}\Big((T_2)\Big)(\rho-\rho^{-1})\right]$$
$$=\sum_{m\geq 1}\sum_{s_1=0}^{m}\sum_{i_1=0}^{s_1}\sum_{s_2=0}^{m}\sum_{i_2=0}^{s_2}\sum_{v_1\geq 0}\sum_{j_1=0}^{v_1}\sum_{v_2\geq 0}\sum_{j_2=0}^{v_2}\binom{m}{s_1}\binom{s_1}{i_1}\binom{m}{s_2}\binom{s_2}{i_2}\times$$
$$\binom{v_1+m}{m}\binom{v_1}{j_1}\binom{v_2+m-1}{m-1}\binom{v_2}{j_2}(-1)^{s_1+s_2+i_1+i_2}\times$$
$$\hat{\kappa}_2^{i_1+j_1}\,(\hat{\kappa}_1+\hat{\kappa}_2)^{m+v_1-s_1-j_1}\,\hat{\omega}_2^{i_2+j_2}\,(\hat{\omega}_1+\hat{\omega}_2)^{m+v_2-s_2-j_2}\times$$
$$\left(\hat{\kappa}_2\binom{2r}{u_1+2}-(\hat{\kappa}_2+1)\binom{2r}{u_1+1}+\binom{2r}{u_1}\right), \tag{10.142}$$

where $u_1 = r + mL + v_1 + v_2 + s_1 + s_2 + j_1 + j_2 - 2i_1 - 2i_2$.

3. The expansion of the third term is very similar to that of the second. We have:

$$\frac{C^{m-1}B^m}{\overline{C}^m\overline{B}^m} = \frac{\rho\left((\rho^2 - \hat{\kappa}_2\rho^{-2}) - (\hat{\kappa}_1 + \hat{\kappa}_2)\right)^{m-1}\left((\rho^2 - \hat{\omega}_2\rho^{-2}) - (\hat{\omega}_1 + \hat{\omega}_2)\right)^m}{(1 - [(\hat{\kappa}_1 + \hat{\kappa}_2)\rho^2 + \hat{\kappa}_2\rho^4])^m (1 - [(\hat{\omega}_1 + \hat{\omega}_2)\rho^2 + \hat{\omega}_2\rho^4])^m}. \tag{10.143}$$

Utilizing facts (10.128) and (10.129) again, Equation (10.143) becomes

$$\begin{aligned}\frac{C^{m-1}B^m}{\overline{C}^m\overline{B}^m} &= \rho\left(\sum_{s_1=0}^{m-1}\binom{m-1}{s_1}(\rho^2 - \hat{\kappa}_2\rho^{-2})^{s_1}(-1)^{m-1-s_1}(\hat{\kappa}_1 + \hat{\kappa}_2)^{m-1-s_1}\right)\times \\ &\left(\sum_{s_2=0}^{m}\binom{m}{s_2}(\rho^2 - \hat{\omega}_2\rho^{-2})^{s_2}(-1)^{m-s_2}(\hat{\omega}_1 + \hat{\omega}_2)^{m-s_2}\right)\times \\ &\left(\sum_{v_1\geq 0}\binom{v_1 + m - 1}{m-1}\left[(\hat{\kappa}_1 + \hat{\kappa}_2)\rho^2 + \hat{\kappa}_2\rho^4\right]^{v_1}\right)\times \\ &\left(\sum_{v_2\geq 0}\binom{v_2 + m - 1}{m-1}\left[(\hat{\omega}_1 + \hat{\omega}_2)\rho^2 + \hat{\omega}_2\rho^4\right]^{v_2}\right).\end{aligned} \tag{10.144}$$

Thus

$$\begin{aligned}\frac{C^{m-1}B^m}{\overline{C}^m\overline{B}^m} &= \sum_{s_1=0}^{m-1}\sum_{i_1=0}^{s_1}\sum_{s_2=0}^{m}\sum_{i_2=0}^{s_2}\sum_{v_1\geq 0}\sum_{j_1=0}^{v_1}\sum_{v_2\geq 0}\sum_{j_2=0}^{v_2}\binom{m-1}{s_1}\binom{s_1}{i_1}\times \\ &\binom{m}{s_2}\binom{s_2}{i_2}\binom{v_1+m-1}{m-1}\binom{v_1}{j_1}\binom{v_2+m-1}{m-1}\binom{v_2}{j_2}\times \\ &(-1)^{s_1+s_2+1+i_1+i_2}\hat{\kappa}_2^{i_1+j_1}(\hat{\kappa}_1 + \hat{\kappa}_2)^{v_1+m-1-s_1-j_1}\times \\ &\hat{\omega}_2^{i_2+j_2}(\hat{\omega}_1 + \hat{\omega}_2)^{v_2+m-s_2-j_2}\times \\ &\rho^{2v_1+2v_2+2s_1+2s_2+2j_1+2j_2-4i_1-4i_2+1}.\end{aligned} \tag{10.145}$$

Now we want

$$\mathrm{CT}\left[(\rho + \rho^{-1})^{2r}(\rho^{-1} - \rho)(-A)\sum_{m\geq 1}\left(\frac{C^{m-1}B^m}{\overline{C}^m\overline{B}^m}\right)\rho^{2mL}\right] \tag{10.146}$$

which equals

$$\mathrm{CT}\left[\left(\sum_{u=0}^{2r}\binom{2r}{u}\rho^{2r-2u}\right)\left(\hat{\kappa}_2\rho^{-3}-(\hat{\kappa}_2+1)\rho^{-1}+\rho\right)\times \left(\sum_{m\geq 1}\left(\frac{C^{m-1}B^m}{\overline{C}^m\overline{B}^m}\right)\rho^{2mL}\right)\right] \tag{10.147}$$

which in turn is

$$\begin{aligned}\mathrm{CT}\quad & \Bigg[\Bigg(\sum_{u=0}^{2r}\sum_{m\geq 1}\sum_{s_1=0}^{m-1}\sum_{i_1=0}^{s_1}\sum_{s_2=0}^{m}\sum_{i_2=0}^{s_2}\sum_{v_1\geq 0}\sum_{j_1=0}^{v_1}\sum_{v_2\geq 0}\sum_{j_2=0}^{v_2}\binom{2r}{u}\binom{m-1}{s_1}\times\\ & \binom{s_1}{i_1}\binom{m}{s_2}\binom{s_2}{i_2}\binom{v_1+m-1}{m-1}\binom{v_1}{j_1}\binom{v_2+m-1}{m-1}\binom{v_2}{j_2}\times\\ & (-1)^{s_1+s_2+1+i_1+i_2}\times\\ & \hat{\kappa}_2^{i_1+j_1}\,(\hat{\kappa}_1+\hat{\kappa}_2)^{v_1+m-1-s_1-j_1}\,\hat{\omega}_2^{i_2+j_2}\,(\hat{\omega}_1+\hat{\omega}_2)^{v_2+m-s_2-j_2}\times\\ & \rho^{2v_1+2v_2+2s_1+2s_2+2j_1+2j_2-4i_1-4i_2+1}\Bigg)\times\\ & \left(\hat{\kappa}_2\rho^{-3}-(\hat{\kappa}_2+1)\rho^{-1}+\rho\right)\Bigg]\end{aligned} \tag{10.148}$$

There are three pieces to this (third) term in the expansion.

- $2r-2u+2mL+2v_1+2v_2+2s_1+2s_2+2j_1+2j_2-4i_1-4i_2+1-3=0$ $\iff$

$$u=r+mL+v_1+v_2+s_1+s_2+j_1+j_2-2i_1-2i_2-1 \tag{10.149}$$

- $2r-2u+2mL+2v_1+2v_2+2s_1+2s_2+2j_1+2j_2-4i_1-4i_2-1=0$ $\iff$

$$u=r+mL+v_1+v_2+s_1+s_2+j_1+j_2-2i_1-2i_2+1 \tag{10.150}$$

- $2r-2u+2mL+2v_1+2v_2+2s_1+2s_2+2j_1+2j_2-4i_1-4i_2+1+1=0$ $\iff$

$$u=r+mL+v_1+v_2+s_1+s_2+j_1+j_2-2i_1-2i_2 \tag{10.151}$$

Thus we have, for the CT expansion of the third term,

$$\mathrm{CT}\left[(\rho+\rho^{-1})^{2r}\Big((T_3)\Big)(\rho-\rho^{-1})\right]$$

$$= \sum_{m\geq 1}\sum_{s_1=0}^{m-1}\sum_{i_1=0}^{s_1}\sum_{s_2=0}^{m}\sum_{i_2=0}^{s_2}\sum_{v_1\geq 0}\sum_{j_1=0}^{v_1}\sum_{v_2\geq 0}\sum_{j_2=0}^{v_2}\binom{m-1}{s_1}\binom{s_1}{i_1}\binom{m}{s_2}\times$$
$$\binom{s_2}{i_2}\binom{v_1+m-1}{m-1}\binom{v_1}{j_1}\binom{v_2+m-1}{m-1}\binom{v_2}{j_2}(-1)^{s_1+s_2+1+i_1+i_2}\times$$
$$\hat{\kappa}_2^{i_1+j_1}\,(\hat{\kappa}_1+\hat{\kappa}_2)^{m+v_1-1-s_1-j_1}\,\hat{\omega}_2^{i_2+j_2}\,(\hat{\omega}_1+\hat{\omega}_2)^{m+v_2-s_2-j_2}\times$$
$$\left(\hat{\kappa}_2\binom{2r}{u_1-1}-(\hat{\kappa}_2+1)\binom{2r}{u_1}+\binom{2r}{u_1+1}\right),\tag{10.152}$$

for u_1 defined as before to be $u_1 = r+mL+v_1+v_2+s_1+s_2+j_1+j_2-2i_1-2i_2$.

Putting the three terms back together gives

$$\mathrm{CT}\left[(\rho+\rho^{-1})^{2r}\Big((T_1)+(T_2)+(T_3)\Big)(\rho-\rho^{-1})\right]$$

$$= \sum_{i\geq 0}\sum_{j=0}^{i}\binom{2r}{u}\binom{i}{j}(\hat{\kappa}_1+\hat{\kappa}_2)^j\hat{\kappa}_2^{i-j}\left(\hat{\kappa}_2\binom{2r}{u_0+2}-(\hat{\kappa}_2+1)\binom{2r}{u_0+1}+\binom{2r}{u_0}\right)$$

$$+ \sum_{m\geq 1}\sum_{s_1=0}^{m}\sum_{i_1=0}^{s_1}\sum_{s_2=0}^{m}\sum_{i_2=0}^{s_2}\sum_{v_1\geq 0}\sum_{j_1=0}^{v_1}\sum_{v_2\geq 0}\sum_{j_2=0}^{v_2}\binom{m}{s_1}\binom{s_1}{i_1}\binom{m}{s_2}\binom{s_2}{i_2}\times$$
$$\binom{v_1+m}{m}\binom{v_1}{j_1}\binom{v_2+m-1}{m-1}\binom{v_2}{j_2}(-1)^{s_1+s_2+i_1+i_2}\times$$
$$\hat{\kappa}_2^{i_1+j_1}\,(\hat{\kappa}_1+\hat{\kappa}_2)^{m+v_1-s_1-j_1}\,\hat{\omega}_2^{i_2+j_2}\,(\hat{\omega}_1+\hat{\omega}_2)^{m+v_2-s_2-j_2}\times$$
$$\left(\hat{\kappa}_2\binom{2r}{u_1+2}-(\hat{\kappa}_2+1)\binom{2r}{u_1+1}+\binom{2r}{u_1}\right)$$

$$+ \sum_{m\geq 1}\sum_{s_1=0}^{m-1}\sum_{i_1=0}^{s_1}\sum_{s_2=0}^{m}\sum_{i_2=0}^{s_2}\sum_{v_1\geq 0}\sum_{j_1=0}^{v_1}\sum_{v_2\geq 0}\sum_{j_2=0}^{v_2}\binom{m-1}{s_1}\binom{s_1}{i_1}\binom{m}{s_2}\binom{s_2}{i_2}\times$$
$$\binom{v_1+m-1}{m-1}\binom{v_1}{j_1}\binom{v_2+m-1}{m-1}\binom{v_2}{j_2}(-1)^{s_1+s_2+1+i_1+i_2}\times$$
$$\hat{\kappa}_2^{i_1+j_1}\,(\hat{\kappa}_1+\hat{\kappa}_2)^{m+v_1-1-s_1-j_1}\,\hat{\omega}_2^{i_2+j_2}\,(\hat{\omega}_1+\hat{\omega}_2)^{m+v_2-s_2-j_2}\times$$
$$\left(\hat{\kappa}_2\binom{2r}{u_1+1}-(\hat{\kappa}_2+1)\binom{2r}{u_1}+\binom{2r}{u_1-1}\right)\tag{10.153}$$

for $u_0 = r + 2i - j$ and $u_1 = r + mL + v_1 + v_2 + s_1 + s_2 + j_1 + j_2 - 2i_1 - 2i_2$. Collecting terms gives final answer

$$\begin{aligned}
&\mathrm{CT}\left[(\rho+\rho^{-1})^{2r}\left(\frac{\tilde{P}_0(\rho)\tilde{P}_L^{(1)}(\rho)}{\tilde{P}_{L+1}(\rho)}\right)(\rho-\rho^{-1})\right] \\
= \;& \sum_{i\geq 0}\sum_{j=0}^{i}\binom{2r}{u}\binom{i}{j}(\hat{\kappa}_1+\hat{\kappa}_2)^j\hat{\kappa}_2^{i-j}\left(\hat{\kappa}_2\binom{2r}{u_0+2}-(\hat{\kappa}_2+1)\binom{2r}{u_0+1}+\binom{2r}{u_0}\right) \\
+ \;& \sum_{m\geq 1}\sum_{s_1,s_2=0}^{m}\sum_{i_1=0}^{s_1}\sum_{i_2=0}^{s_2}\sum_{v_1,v_2\geq 0}\sum_{j_1=0}^{v_1}\sum_{j_2=0}^{v_2}\binom{s_1}{i_1}\binom{m}{s_2}\binom{s_2}{i_2}\binom{v_1}{j_1}\binom{v_2+m-1}{m-1}\binom{v_2}{j_2} \\
& (-1)^{s_1+s_2+i_1+i_2}\hat{\kappa}_2^{i_1+j_1}\,(\hat{\kappa}_1+\hat{\kappa}_2)^{m+v_1-1-s_1-j_1}\,\hat{\omega}_2^{i_2+j_2}\,(\hat{\omega}_1+\hat{\omega}_2)^{m+v_2-s_2-j_2}\times \\
& \left\{\binom{m}{s_1}\binom{v_1+m}{m}(\hat{\kappa}_1+\hat{\kappa}_2)\left(\hat{\kappa}_2\binom{2r}{u_1+2}-(\hat{\kappa}_2+1)\binom{2r}{u_1+1}+\binom{2r}{u_1}\right)\right. \\
& \left.-\binom{m-1}{s_1}\binom{v_1+m-1}{m-1}\left(\hat{\kappa}_2\binom{2r}{u_1-1}-(\hat{\kappa}_2+1)\binom{2r}{u_1}+\binom{2r}{u_1+1}\right)\right\},
\end{aligned} \tag{10.154}$$

for $u_0 = r+2i-j$ and $u_1 = r+mL+v_1+v_2+s_1+s_2+j_1+j_2-2i_1-2i_2$. □

Index

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