A SHORT PROOF OF THE SOBOLEV INEQUALITY IN $\mathbb{R}^2$

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We display here a short proof of the fact that

$$\|u\|_{L^2} \leq C\|\nabla u\|_{L^1} \quad \text{for } u \in C_c^\infty(\mathbb{R}^2).$$

The argument is an adaptation of the one by Bourgain-Brezis [1], which they used to prove the stronger inequality

$$\|u\|_{L^2} \leq C\|\nabla u\|_{L^1+\dot{W}^{-1,2}}$$

on the 2-dimensional torus.

To do so, write

$$\begin{aligned} u &= (R_1^2 - R_2^2)^2 u + 4R_1^2 R_2^2 u \\ &= (R_1 - R_2)^2 (R_1 + R_2)^2 u + 4R_1^2 R_2^2 u, \end{aligned}$$

where R_1, R_2 are the Riesz transforms. It suffices to estimate $(R_1^2 R_2^2 u, u)$ and $((R_1 - R_2)^2 (R_1 + R_2)^2 u, u)$ respectively, where $(\cdot, \cdot)$ is the L^2 inner product on $\mathbb{R}^2$. By rotating the function u, the second estimate can be reduced to the first one. Now

$$(R_1^2 R_2^2 u, u) = ((-\Delta)^{-1} R_1 R_2 \partial_1 u, \partial_2 u),$$

and this is bounded in absolute value by $\|\nabla u\|_{L^1}^2$ if we can show that $(-\Delta)^{-1} R_1 R_2$ is bounded from L^1 to L^∞ on $\mathbb{R}^2$. But this is the case since we can compute the kernel $K(x)$ of this operator, and show that it is a bounded function; in fact the kernel has to be homogeneous of degree zero, so it suffices to see that the kernel is bounded on the unit sphere. Now

$$K(x) = -\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon < |\xi| < R} \frac{\xi_1 \xi_2}{|\xi|^4} e^{2\pi i x \cdot \xi} d\xi.$$

Let η be a smooth radial function with compact support on $\mathbb{R}^2$ such that η is identically 1 on the ball of radius $1/2$, and zero outside the ball of radius 2. Then $K(x) = K_0(x) + K_1(x)$, where K_0 and K_1 are the integrals with $\eta(\xi)$ and $(1 - \eta(\xi))$ inserted respectively. Now if $|x| = 1$, then

$$K_1(x) = -\lim_{R \to \infty} \int_{|\xi| < R} \frac{\xi_1 \xi_2}{|\xi|^4} (1 - \eta(\xi)) \frac{\Delta_\xi e^{2\pi i x \cdot \xi}}{-4\pi^2} d\xi,$$

and when one integrates by parts, this is bounded uniformly in x on the unit sphere. Also, if $|x| = 1$, then

$$K_0(x) = -\frac{1}{4} \lim_{\varepsilon \to 0} \int_{|\xi| > \varepsilon} \frac{\xi_1 \xi_2}{|\xi|^4} \eta(\xi) (e^{2\pi i x_1 \xi_1} - e^{-2\pi i x_1 \xi_1})(e^{2\pi i x_2 \xi_2} - e^{-2\pi i x_2 \xi_2}) d\xi,$$

since $\frac{\xi_1 \xi_2}{|\xi|^4} \eta(\xi)$ is odd in ξ_1 and ξ_2. This kills the singularity of the integrand at $\xi = 0$, and shows that $K_0(x)$ is uniformly bounded in x on the unit sphere.

References

[1] Jean and Brezis Bourgain Haïm, *On the equation* div $Y = f$ *and application to control of phases*, J. Amer. Math. Soc. **16** (2003), no. 2, 393–426.