Solution counting via incidence geometry

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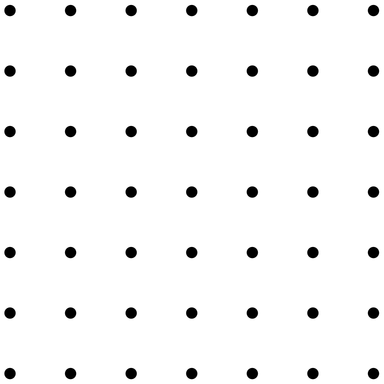
Two questions

Let S be a finite subset of $\mathbb{Z}^2$.

Main case of interest: $S = [0, N]^2 \cap \mathbb{Z}^2$.



Question 1: Diophantine equations

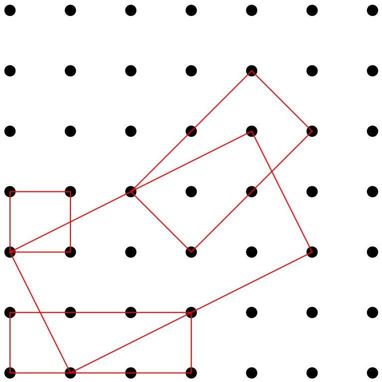
How many (integer) solutions $(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4) \in S^4$ are there, so that

$$\begin{cases} \vec{n}_1 + \vec{n}_3 = \vec{n}_2 + \vec{n}_4 \\ |\vec{n}_1|^2 + |\vec{n}_3|^2 = |\vec{n}_2|^2 + |\vec{n}_4|^2 \end{cases} \quad ?$$

Clearly $\vec{n}_1 = \vec{n}_2 \in S$ and $\vec{n}_3 = \vec{n}_4 \in S$ gives $(\#S)^2$ solutions. Can there be many more?

Question 2: Counting rectangles

How many rectangles can you form with vertices in S?



Let's try...

Rewrite the system as

$$\begin{cases} \vec{n}_1 - \vec{n}_2 = \vec{n}_4 - \vec{n}_3 \\ |\vec{n}_1|^2 - |\vec{n}_2|^2 = |\vec{n}_4|^2 - |\vec{n}_3|^2. \end{cases}$$

The first equation says $(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4)$ form a parallelogram.

The second equation can be rewritten as

$$(\vec{n}_1 - \vec{n}_2) \cdot (\vec{n}_1 + \vec{n}_2) = (\vec{n}_4 - \vec{n}_3) \cdot (\vec{n}_4 + \vec{n}_3)$$

which under the first equation is the same as

$$(\vec{n}_1 - \vec{n}_2) \cdot (\vec{n}_1 + \vec{n}_2 - \vec{n}_3 - \vec{n}_4) = 0$$

i.e.

$$(\vec{n}_1 - \vec{n}_2) \cdot 2(\vec{n}_2 - \vec{n}_3) = 0.$$

Hence the second equation says the parallelogram $(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4)$ has a right angle at $\vec{n}_2$, i.e. it is a rectangle.

Summary: The conditions

$$\begin{cases} \vec{n}_1 - \vec{n}_2 = \vec{n}_4 - \vec{n}_3 \\ |\vec{n}_1|^2 - |\vec{n}_2|^2 = |\vec{n}_4|^2 - |\vec{n}_3|^2. \end{cases}$$

is the same as saying that $(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4)$ forms a rectangle!

Questions 1 and 2 are equivalent.

We now investigate Question 2.

For most of the discussion, we can relax our assumption from $S \subset \mathbb{Z}^2$ to $S \subset \mathbb{R}^2$.

Counting rectangles in $\mathbb{R}^2$: Trivial bounds

Let S be a set of m points in $\mathbb{R}^2$.

The number of rectangles in S is $\leq m^4$. (Of course.)

The number of rectangles in S is $\leq m^3$. (Three vertices of a rectangle determine the fourth.)

One should be able to do better since once the first two vertices of a rectangle are picked, the third point is not arbitrary.

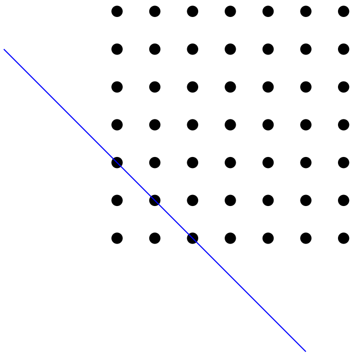
But Euclidean geometry must be brought into play: in a toy model over a finite field, a set of m points can form as many as $m^{2.5}$ rectangles!

In other words, need to exploit that $\mathbb{R}^2$ is not $\mathbb{F}_q^2$...

Counting r-rich lines

Let S be a finite subset of $\mathbb{R}^2$, and $r \geq 2$.

An r-rich line through S is a line in $\mathbb{R}^2$ that passes through at least r points from S.

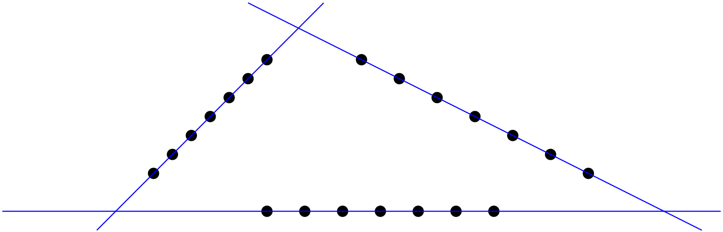


How many r-rich lines are there?

If the points of S are in general position, then no 3-rich lines through S.

So the question is: can we have a lot of r-rich lines through a set S of m points?

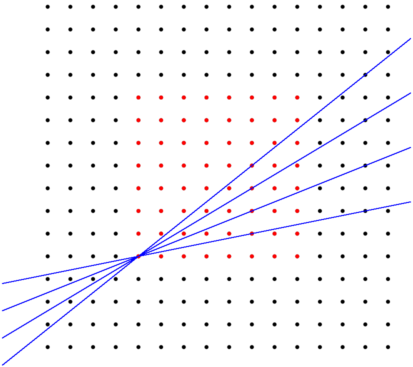
It is possible to have $\frac{m}{r}$ many r-rich lines through S: e.g. Each of these lines has exactly r points from S.



$r = 7$, $m = \#S$

$\frac{m}{r}$ many r-rich lines

If $r \ll \sqrt{m}$ we can arrange the m points from S in an $\sqrt{m} \times \sqrt{m}$ lattice grid to get lots of r-rich lines:



$m = 16^2$, $r = 3$

lines through a red dot of slope a/b where $1 \leq a \leq b \leq 5$ are 3-rich.

Generally speaking, lines through most points through a $\sqrt{m} \times \sqrt{m}$ square lattice with slope a/b where $1 \leq a \leq b \leq \frac{\sqrt{m}}{r}$ are r-rich.

There are m base points and $\simeq (\frac{\sqrt{m}}{r})^2$ slopes. But two base points can give the same line. If a line passes through exactly r points from the lattice, then it is counted r times. One can find out how much double counting there is for each line.

It turns out there are $\sim m(\frac{\sqrt{m}}{r})^2 \frac{1}{r} = \frac{m^2}{r^3}$ many r-rich lines through an $\sqrt{m} \times \sqrt{m}$ lattice grid.

Summary: For a set of m points S in the plane, there can be as many as $\frac{m}{r}$ many r-rich lines through S. If $r \ll \sqrt{m}$ there can even be as many as $\frac{m^2}{r^3}$ many r-rich lines through S (note $\frac{m^2}{r^3} > \frac{m}{r}$ if $r < \sqrt{m}$).

The Szemeredi-Trotter theorem

The above examples capture the extreme scenarios in the plane $\mathbb{R}^2$.

Theorem (Szemeredi-Trotter 1983): For any set of m points $S \subset \mathbb{R}^2$, if $r \geq 2$, there are at most $\ll \frac{m}{r} + \frac{m^2}{r^3}$ many r-rich lines through S.

The theorem is false if $\mathbb{R}^2$ is replaced by $\mathbb{F}_q^2$, where $\mathbb{F}_q$ is a finite field. You can have more r-rich lines in $\mathbb{F}_q^2$.

The proof of the theorem uses crucially the topology of $\mathbb{R}^2$.

One proof uses Euler's theorem: $V - E + F = 2$. Another uses the polynomial ham sandwich theorem: Given N open subsets of $\mathbb{R}^2$ with finite volume, there exist a polynomial $p \in \mathbb{R}[x, y]$ with degree $\ll \sqrt{N}$ whose zero set bisects all N open subsets. (The case $N = 2$ is a fun exercise.) The topology of $\mathbb{R}^2$ enters.

Back to counting rectangles

How many rectangles can you form with vertices among a set of m points in $\mathbb{R}^2$?

Theorem (Pach-Sharir 1992): For any set of m points $S \subset \mathbb{R}^2$, there are at most $\ll m^2 \log m$ many rectangles with vertices in S.

This is a smaller bound than the case of $\mathbb{F}_q^2$ (one can have an example with $\gg m^{2.5}$ many rectangles).

The proof must use something about $\mathbb{R}^2$: in this case, the Szemeredi-Trotter theorem! (So ultimately it is topology of $\mathbb{R}^2$ at work.)

Herr and Kwak has given a new proof of this result of Pach-Sharir, and their proof gives something more general.

Below I would like to illustrate some beautiful ideas from Herr and Kwak.

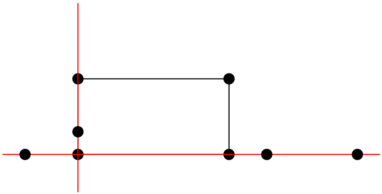
The mass of a vertex in a rectangle

Fix a finite set $S \subset \mathbb{R}^2$.

A non-degenerate rectange in S is one that has four distinct vertices in S.

If R is a non-degenerate rectangle in S and $\vec{n}$ is a vertex of R, we compute the *mass* of $\vec{n}$ in R by:

- First extend the two sides of R that intersect at $\vec{n}$ to form two (infinite) lines ℓ and ℓ'.
- Then the mass of $\vec{n}$ in R is defined to be $\max\{\#(S \cap \ell), \#(S \cap \ell')\}$.



The mass of any vertex in a non-degenerate rectangle is $\in [2, \#S]$.

Special case 1

Suppose $S \subset \mathbb{R}^2$ is a finite subset of m points, and $\mathcal{Q} = \mathcal{Q}(S)$ be the set of non-degenerate rectangles in S.

For $a \in \mathbb{N}$ and $2^a \leq m$, let $\mathcal{Q}_a$ be the set of all rectangles in S for which the heaviest vertex has mass $\in [2^a, 2^{a+1})$.

Then $\mathcal{Q} = \bigsqcup_a \mathcal{Q}_a = \Big(\bigsqcup_{2^a \ll \sqrt{m}} \mathcal{Q}_a \Big) \bigcup \Big(\bigsqcup_{\sqrt{m} \ll 2^a \ll m} \mathcal{Q}_a \Big)$.

Lemma 1: For $2^a \ll \sqrt{m}$, we have $\#\mathcal{Q}_a \ll m^2$.

Since there are $\ll \log m$ many such a's, this gives

$$\#\Big(\bigsqcup_{2^a \ll \sqrt{m}} \mathcal{Q}_a \Big) \ll m^2 \log m.$$

Proof of Lemma 1.

Let $2^a \ll \sqrt{m}$. Let R be a rectangle in $\mathcal{Q}_a$.

This means the heaviest vertex of R (say $\vec{n}_1$) has mass $\in [2^a, 2^{a+1})$.

$\vec{n}_1$ lies on a 2^a-rich line of S.

Since $2^a \ll \sqrt{m}$, Szemeredi-Trotter theorem says that the number of 2^a-rich lines in S is $\ll \frac{m}{2^a} + \frac{m^2}{2^{3a}} \simeq \frac{m^2}{2^{3a}}$.

Once such a line has been fixed, there are $< 2^{a+1}$ choices for $\vec{n}_1$ along that line.

To determine a rectangle R in $\mathcal{Q}_a$ we need to pick two more vertices from S so that their mass in R is $< 2^{a+1}$.

There are $< 2^{a+1}$ choices for each of these two vertices.

Altogether, $\#\mathcal{Q}_a \ll \frac{m^2}{2^{3a}} 2^{a+1} 2^{a+1} 2^{a+1} \ll m^2$, as desired.
(Argument fails without red condition.)

Special case 2

Suppose $S \subset \mathbb{R}^2$ is a finite subset of m points. A line is said to be very heavy if it is $C\sqrt{m}$-rich line where C is a large absolute constant.

Lemma 2: Suppose additionally that through every point of S there is at most one very heavy line. Then #rectangles in $S \ll m^2 \log m$.

Proof. Using Lemma 1, we only need to count the number of very heavy rectangles, namely those whose heaviest vertex has mass $\geq C\sqrt{m}$.

Let R be such a very heavy rectangle. The number of possible choices of the heaviest vertex of R is (trivially) $\leq m$.

Once the heaviest vertex $\vec{n}_1$ is fixed, since there is only one very heavy line through this vertex, the orientation of R is fixed.

Hence R is determined by the choice of the opposite vertex $\vec{n}_3$ to the heaviest vertex $\vec{n}_1$, for which there are $\leq m$ many choices.

The number of very heavy rectangles is thus $\leq m^2$.

General case

By choosing a sufficiently large absolute constant C one can prove:

Lemma 3: Let $S \subset \mathbb{R}^2$ be a finite set of points. Then there exists a subset S_1 of S, such that through every point of S_1 there is at most one line that contains $\geq C\sqrt{\#S_1}$ many points of S_1, and $\#(S \setminus S_1) \leq \frac{\#S}{2}$.

In other words, we only need to remove less than half of the points of S, to obtain a set $S_1 \subset S$ satisfying the additional hypothesis in Lemma 2.

Proof of Lemma 3: Let S' be the set of points in S that lie on at least two distinct lines with $\geq C\sqrt{\frac{\#S}{2}}$ points of S.

Then

$$\#S' \leq \#(C\sqrt{\frac{\#S}{2}}\text{-rich lines in } S)^2 \ll \Big(\frac{\#S}{C\sqrt{\frac{\#S}{2}}}\Big)^2 \leq \frac{\#S}{2}$$

where we used Szemeredi-Trotter in the second inequality and C sufficiently large for the last.

Let $S_1 := S \setminus S'$. Then $\#S_1 \geq \frac{\#S}{2}$, and through any $\vec{n} \in S_1$, there is at most one line that contains $\geq C\sqrt{\frac{\#S}{2}}$ points of S.

In particular, since $\#S_1 \geq \frac{\#S}{2}$, through any $\vec{n} \in S_1$, there is at most one line that contains $\geq C\sqrt{\#S_1}$ points of S_1.

Finally $\#(S \setminus S_1) = \#S' \leq \frac{\#S}{2}$ as desired.

Iterating Lemma 3

Given $S \subset \mathbb{R}^2$, Lemma 3 gives a subset $S_1 \subset S$ that satisfies the additional hypothesis in Lemma 2, with $\#(S \setminus S_1) \leq \frac{\#S}{2}$.

Now apply Lemma 3 again to $S \setminus S_1$. This gives a subset $S_2 \subset S \setminus S_1$, such that S_2 satisfies the additional hypothesis in Lemma 2, with

$$\#(S \setminus (S_1 \sqcup S_2)) \leq \frac{\#(S \setminus S_1)}{2} \leq \frac{\#S}{4}.$$

Iterating this process, we obtain:

Lemma 3'. For any finite set of m points $S \subset \mathbb{R}^2$, there exists a decomposition $S = \bigsqcup_{j \geq 1} S_j$ so that each S_j satisfies the hypothesis in Lemma 2, and

$$\#S_j \leq \#(S \setminus (S_1 \sqcup \cdots \sqcup S_{j-1})) \leq 2^{1-j} \#S \quad \text{for all } j.$$

We are now ready to prove the theorem of Pach and Sharir.

Theorem. For any set of m points $S \subset \mathbb{R}^2$, there are at most $\ll m^2 \log m$ many rectangles with vertices in S.

Proof. For $\vec{n} \in \mathbb{R}^2$, let $\varphi(\vec{n}) = (\vec{n}, |\vec{n}|^2)$. The notation is convenient because four points $\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4$ form a rectangle in $\mathbb{R}^2$ if and only if

$$\varphi(\vec{n}_1) + \varphi(\vec{n}_3) = \varphi(\vec{n}_2) + \varphi(\vec{n}_4). \tag{*}$$

For $E_1, E_2, E_3, E_4 \subset \mathbb{R}^2$, let $\mathcal{Q}(E_1, E_2, E_3, E_4)$ be the number of tuples $(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4)$ so that (*) is satisfied and $\vec{n}_i \in E_i$ for all i.

Then from Lemma 3', we have $S = \bigsqcup_{j \geq 1} S_j$, so

$$\#\mathcal{Q}(S) = \sum_{j_1, j_2, j_3, j_4 \geq 1} \#\mathcal{Q}(S_{j_1}, S_{j_2}, S_{j_3}, S_{j_4}).$$

Claim: It is well-known that for any set $E_1, E_2, E_3, E_4 \subset \mathbb{R}^2$,

$$\#\mathcal{Q}(E_1, E_2, E_3, E_4) \leq \prod_{i=1}^{4} \#\mathcal{Q}(E_i)^{1/4}.$$

From

$$\#\mathcal{Q}(S) = \sum_{j_1, j_2, j_3, j_4 \geq 1} \#\mathcal{Q}(S_{j_1}, S_{j_2}, S_{j_3}, S_{j_4}).$$

and

$$\#\mathcal{Q}(S_{j_1}, S_{j_2}, S_{j_3}, S_{j_4}) \leq \prod_{i=1}^{4} \#\mathcal{Q}(S_{j_i})^{1/4},$$

we obtain

$$\#\mathcal{Q}(S) \leq \Big(\sum_{j \geq 1} \#\mathcal{Q}(S_j)^{1/4} \Big)^4.$$

Lemma 2 implies

$$\#\mathcal{Q}(S_j) \ll (\#S_j)^2 \log(\#S_j) \ll 2^{2(1-j)} (\#S)^2 \log(\#S)$$

for each j.

The two bounds together give the Pach-Sharir bound

$$\#\mathcal{Q}(S) \ll (\#S)^2 \log(\#S).$$

Finally we come back to an easy proof of the claim (I learned this from Bryce Kerr).

For any set $E_1, E_2, E_3, E_4 \subset \mathbb{R}^2$, by Cauchy-Schwarz,

$$\begin{aligned} &\#\mathcal{Q}(E_1, E_2, E_3, E_4) \\ &= \sum_{\vec{w} \in \mathbb{R}^3} \#\{(\vec{n}_1, \vec{n}_3) \in E_1 \times E_3 \colon \varphi(\vec{n}_1) + \varphi(\vec{n}_3) = \vec{w}\} \\ &\qquad\qquad \cdot \#\{(\vec{n}_2, \vec{n}_4) \in E_2 \times E_4 \colon \varphi(\vec{n}_2) + \varphi(\vec{n}_4) = \vec{w}\} \\ &\leq \mathcal{Q}(E_1, E_1, E_3, E_3)^{1/2} \mathcal{Q}(E_2, E_2, E_4, E_4)^{1/2}. \end{aligned}$$

Similarly, by rewriting the system (*) as $\varphi(\vec{n}_1) - \varphi(\vec{n}_2) = \varphi(\vec{n}_4) - \varphi(\vec{n}_3)$,

$$\#\mathcal{Q}(E_1, E_2, E_3, E_4) \leq \mathcal{Q}(E_1, E_2, E_2, E_1)^{1/2} \mathcal{Q}(E_3, E_4, E_4, E_3)^{1/2}.$$

Together we get $\#\mathcal{Q}(E_1, E_2, E_3, E_4) \leq \prod_{i=1}^{4} \#\mathcal{Q}(E_i)^{1/4}$.

More general exponential sum estimates

Herr and Kwak actually proved a stronger result:

Theorem" (Herr-Kwak 2024). Let $S \subset \mathbb{Z}^2$ be a finite set. For every choice of coefficients $\{b_{\vec{n}}\}_{\vec{n}\in S}$, and every interval $J \subset [0,1]$ of length $\frac{1}{\log(\#S)}$, one has

$$\Big\| \sum_{\vec{n}\in S} b_{\vec{n}} e(\vec{n}\cdot x + |\vec{n}|^2 t) \Big\|_{L^4([0,1]^2\times J, dxdt)} \ll \|b_{\vec{n}}\|_{\ell^2}.$$

In particular, by breaking $[0,1]$ into the union of $\log(\#S)$ many intervals of length $\frac{1}{\log(\#S)}$, we obtain a sharp discrete Strichartz estimate (power $1/4$ cannot be lowered):

$$\Big\| \sum_{\vec{n}\in S} b_{\vec{n}} e(\vec{n}\cdot x + |\vec{n}|^2 t) \Big\|_{L^4([0,1]^3, dxdt)} \ll [\log(\#S)]^{1/4} \|b_{\vec{n}}\|_{\ell^2}.$$

Decoupling only gives a bound with constant $(\#S)^{\varepsilon}$ for any $\varepsilon > 0$.