THE BOURGAIN-GUTH ITERATION FOR PROVING RESTRICTION ESTIMATES

PO-LAM YUNG

The following is a more detailed exposition of the iteration that Bourgain and Guth [2] used to prove the Fourier extension estimate at exponent $q = 10/3$ for the paraboloid in $\mathbb{R}^3$, written for my own benefit. In particular, care will be taken to make precise and justify all use of the uncertainty principle, but not much heuristics will be explained here (the original article [2] contains a very good exposition of the heuristics already).

Let Q be the cube $[-1,1]^2$ in $\mathbb{R}^2$, and $\Phi\colon Q \to \mathbb{R}^3$ be the parametrization of a compact part of the paraboloid in $\mathbb{R}^3$ given by $\Phi(\xi) = (\xi, |\xi|^2)$. Let $E\colon L^\infty(Q) \to L^\infty(\mathbb{R}^3)$ be the extension operator associated to the paraboloid in $\mathbb{R}^3$, i.e.

$$Ef(x) = \int_Q f(\xi) e^{2\pi i x\cdot\Phi(\xi)} d\xi, \quad x \in \mathbb{R}^3.$$

Let $q = 10/3$, and for each $R \geq 1$, let $A(R)$ be the smallest constant for which

$$\|Ef\|_{L^q(B_R)} \leq A(R)\|f\|_{L^\infty(Q)}$$

for all $f \in L^\infty(Q)$ and all cubes $B_R \subset \mathbb{R}^3$ of side length R. Following Bourgain and Guth [2], we will show that for any $\varepsilon > 0$, there exists a finite constant A_ε such that $A(R) \leq A_\varepsilon R^\varepsilon$ for all R.

Suppose $\varepsilon > 0$. We will determine two positive integers K_1 and K, both depending only on ε, with $K \geq K_1$; as we will see shortly, it will be convenient to choose K to be a large multiple of K_1, so let's choose K as such.

From now on, we fix a cube $B_R \subset \mathbb{R}^3$ of side length $R \geq 1$. Without loss of generality we assume that R is an integer multiple of K. We partition B_R into essentially disjoint cubes of side lengths K, and call the collection of resulting cubes $\mathcal{B}_K$. A cube from $\mathcal{B}_K$ is typically denoted by μ; since K is a multiple of K_1, every such μ can be further partitioned into a disjoint union of cubes of side lengths K_1. Call the collection of these resulting cubes $\mathcal{B}_{K_1}(\mu)$; then we have $\mu = \bigcup_{\nu\in\mathcal{B}_{K_1}(\mu)} \nu$ for every $\mu \in \mathcal{B}_K$.

We also partition Q into essentially disjoint cubes of side lengths K^{-1} and K_1^{-1} respectively, and call the collection of resulting cubes $\mathcal{P}_{K^{-1}}$ and $\mathcal{P}_{K_1^{-1}}$. A cube in $\mathcal{P}_{K^{-1}}$ is usually denoted α; a cube in $\mathcal{P}_{K_1^{-1}}$ is usually denoted β. Since K is a multiple of K_1, if $\alpha \in \mathcal{P}_{K^{-1}}$ and $\beta \in P_{K_1^{-1}}$, then either α and β are essentially disjoint, or $\alpha \subset \beta$. The distance between two sets will be denoted by d; for instance, if $\alpha_1, \alpha_2 \in \mathcal{P}_{K^{-1}}$, then $d(\alpha_1, \alpha_2)$ denotes the distance between α_1 and α_2. Three cubes $\alpha_1, \alpha_2, \alpha_3$ in $\mathcal{P}_{K^{-1}}$ will be said to be transverse, if for every $\xi_1 \in \alpha_1$, $\xi_2 \in \alpha_2$ and $\xi_3 \in \alpha_3$, the unit normal vectors to the paraboloid at $(\xi_1, |\xi_1|^2)$, $(\xi_2, |\xi_2|^2)$

and $(\xi_3, |\xi_3|^2)$ form a determinant whose absolute value is $\gtrsim K^{-2}$ (which holds if and only if the area of the triangle whose vertices are ξ_1, ξ_2, ξ_3 has area $\gtrsim K^{-2}$). Hence $\alpha_1, \alpha_2, \alpha_3$ would be transverse, if for instance $\max_{1 \leq i < j \leq 3} d(\alpha_i, \alpha_j) = d(\alpha_1, \alpha_2) \geq 10/K$ and the distance of α_3 to the line joining the centers of α_1 and α_2 is $\geq 10/K$.

Now fix $f \in L^\infty(Q)$ with $\|f\|_{L^\infty(Q)} = 1$. For each cube $\alpha \in \mathcal{P}_{K^{-1}}$, let $f_\alpha = f\chi_\alpha$ where χ_α is the characteristic function of α. Then

$$Ef = \sum_{\alpha \in \mathcal{P}_{K-1}} Ef_\alpha.$$

Informally, the uncertainty principle asserts that if $\alpha \in \mathcal{P}_{K^{-1}}$, then Ef_α is locally constant on any cube μ of side length K. To make this precise, let $w(x) = \frac{1}{(1+|x|)^{30}}$ and $w_K(x) = \frac{1}{K^3} w(\frac{x}{K})$. For any cube μ of side length K and any $\alpha \in \mathcal{P}_{K^{-1}}$, let

$$c_{\mu,\alpha} = \int_{\mathbb{R}^3} |Ef_\alpha(x)| w_K(z_\mu - x) dx$$

where z_μ is the center of μ. Then

$$\|Ef_\alpha\|_{L^\infty(\mu)} \lesssim c_{\mu,\alpha};$$

indeed

$$\|Ef_\alpha\|_{L^\infty(\mu')} \lesssim \left(1 + \frac{d(\mu,\mu')}{K}\right)^{30} c_{\mu',\alpha}$$

for every cubes μ, μ' of side lengths K.

Now for $\mu \in \mathcal{B}_K$, let

$$c_{\mu,*} = \max_{\alpha \in \mathcal{P}_{K-1}} c_{\mu,\alpha}$$

We consider two conditions on μ:

Condition 1. There exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{P}_{K^{-1}}$, transverse to each other, with $c_{\mu,\alpha_j} > K^{-2} c_{\mu,*}$ for $j = 1, 2, 3$.

Condition 2. There exists a line $L \subset \mathbb{R}^2$, such that if $\alpha \in \mathcal{P}_{K^{-1}}$ satisfy $c_{\mu,\alpha} > K^{-2} c_{\mu,*}$, then $d(\alpha, L) < 10/K$.

One can check that every $\mu \in \mathcal{B}_K$ satisfies at least one of these two conditions: indeed, let's call a cube $\alpha \in \mathcal{P}_{K^{-1}}$ important if $c_{\mu,\alpha} > K^{-2} c_{\mu,*}$. Among all the important cubes, choose two that are as far apart as possible (if there are two), and see whether every other important cube is at a distance at most $10/K$ from the line joining the centers of the two important cubes chosen just now. If yes, then μ satisfies condition 2; if not, then μ satisfies condition 1.

Let's write $\mu \in \mathfrak{C}_i$ if $\mu \in \mathcal{B}_K$ satisfies condition i, for $i = 1, 2$. The $\alpha_1, \alpha_2, \alpha_3$ in condition 1, and the line L in condition 2, will generally depend on μ; we write $\alpha_i(\mu)$ and $L(\mu)$ if there is a need to carry the dependence of such on μ.

Our goal is to estimate $\|Ef\|_{L^q(B_R)}$. Since

$$\|Ef\|_{L^q(B_R)}^q = \sum_{\mu \in \mathfrak{C}_1} \|Ef\|_{L^q(\mu)}^q + \sum_{\mu \in \mathfrak{C}_2} \|Ef\|_{L^q(\mu)}^q,$$

we will estimate the two terms on the right hand side separately.

If $\mu \in \mathfrak{C}_1$, then
$$\|Ef\|_{L^\infty(\mu)} \lesssim K^2 c_{\mu,*} \le K^4 \operatorname*{geom}_{1\le j\le 3} c_{\mu,\alpha_j(\mu)}$$
where $\operatorname{geom}_{1\le j\le 3} a_j$ is the geometric mean of a_1, a_2, a_3. Hence
$$\begin{aligned}\|Ef\|_{L^q(\mu)}^q &\le \|Ef\|_{L^3(\mu)}^3 \\ &\lesssim K^{12} \iiint w_K(x_1)w_K(x_2)w_K(x_3) \int_\mu \operatorname*{geom}_{1\le j\le 3} |Ef_{\alpha_j(\mu)}(z-x_j)|^3 dz\, dx_1 dx_2 dx_3\end{aligned}$$
which is
$$\lesssim K^{12} \sum_{\substack{\alpha_1,\alpha_2,\alpha_3\\ \text{transverse}}} \iiint w_K(x_1)w_K(x_2)w_K(x_3) \int_\mu \operatorname*{geom}_{1\le j\le 3} |Ef_{\alpha_j}(z-x_j)|^3 dz\, dx_1 dx_2 dx_3.$$
Summing over $\mu \in \mathfrak{C}_1$, we have
$$\begin{aligned}\sum_{\mu\in\mathfrak{C}_1} \|Ef\|_{L^q(\mu)}^q &\lesssim K^{12} \sum_{\substack{\alpha_1,\alpha_2,\alpha_3\\ \text{transverse}}} \iiint w_K(x_1)w_K(x_2)w_K(x_3) \int_{B_R} \operatorname*{geom}_{1\le j\le 3} |Ef_{\alpha_j}(z-x_j)|^3 dz\, dx_1 dx_2 dx_3 \\ &\lesssim_{K,\varepsilon} R^{\varepsilon q} \operatorname*{geom}_{1\le j\le 3} \|f_{\alpha_j}\|_{L^2}^3 \\ &\lesssim_{K,\varepsilon} R^{\varepsilon q}.\end{aligned}$$
We used the trilinear restriction estimate of Bennett, Carbery and Tao [1] in the second inequality above.

Next we estimate $\sum_{\mu\in\mathfrak{C}_2} \|Ef\|_{L^q(\mu)}^q$. Suppose $\mu \in \mathfrak{C}_2$. Let $L(\mu)$ be the line in the statement of condition 2, and $S(\mu)$ for the strip given by the $10/K$ neighborhood of $L(\mu)$. Let
$$f_{S(\mu)} = \sum_{\substack{\alpha\in\mathcal{P}_{K^{-1}}\\ \alpha\cap S(\mu)\neq\emptyset}} f_\alpha,$$
and for each $\beta \in \mathcal{P}_{K_1^{-1}}$, let
$$f_{S(\mu),\beta} = \sum_{\substack{\alpha\in\mathcal{P}_{K^{-1}},\, \alpha\subset\beta\\ \alpha\cap S(\mu)\neq\emptyset}} f_\alpha$$
so that
$$f_{S(\mu)} = \sum_{\beta\in\mathcal{P}_{K_1}^{-1}} f_{S(\mu),\beta}.$$
Then
$$|Ef(z)| \le c_{\mu,*} + |Ef_{S(\mu)}(z)|$$
for all $z \in \mu$, and hence
$$\|Ef\|_{L^q(\mu)}^q \lesssim |\mu| c_{\mu,*}^q + \|Ef_{S(\mu)}\|_{L^q(\mu)}^q. \tag{1}$$
This is true if $\mu \in \mathfrak{C}_2$. We want to sum over all $\mu \in \mathfrak{C}_2$. The first term on the right hand side of (1) can be estimated easily by the following parabolic rescaling lemma:

Lemma 1. *We have*

$$\sum_{\mu\in\mathcal{B}_K}|\mu|\sum_{\alpha\in\mathcal{P}_{K^{-1}}}c_{\mu,\alpha}^q\lesssim K^{6-2q}A(R/K)^q.$$

Indeed then

$$\sum_{\mu\in\mathfrak{C}_2}|\mu|c_{\mu,*}^q\lesssim K^{6-2q}A(R/K)^q. \tag{2}$$

To control the second term on the right hand side of (1), we introduce some further notation. Still suppose $\mu\in\mathfrak{C}_2$. Let $W(x)=\frac{1}{(1+|x|)^{3000}}$, and $W_{K_1}(x)=\frac{1}{K_1^3}W(\frac{x}{K_1})$. For $\nu\in\mathcal{B}_{K_1}(\mu)$, let

$$c_{\nu,\beta}=\int_{\mathbb{R}^3}|Ef_{S(\mu),\beta}(x)|W_{K_1}(z_\nu-x)dx$$

for all $\beta\in\mathcal{P}_{K_1^{-1}}$, where z_ν is the center of ν. Let

$$c_{\nu,*}=\max_{\beta\in\mathcal{P}_{K_1^{-1}}}c_{\nu,\beta}.$$

If $\beta_1,\beta_2\in\mathcal{P}_{K_1^{-1}}$, we say they are transverse, if $d(\beta_1,\beta_2)\geq 10/K_1$. For each $\nu\in\mathcal{B}_{K_1}(\mu)$, we consider two conditions:
Condition 2a. There exist $\beta_1,\beta_2\in\mathcal{P}_{K_1^{-1}}$ transverse, such that $c_{\nu,\beta_j}>K_1^{-1}c_{\nu,*}$ for $j=1,2$. (Note such β_1,β_2 must intersect $S(\mu)$, for otherwise $c_{\nu,\beta_j}=0$.)
Condition 2b. There exists a point $P\subset\mathbb{R}^2$, such that if $\beta\in\mathcal{P}_{K_1^{-1}}$ satisfy $c_{\nu,\beta}>K_1^{-1}c_{\nu,*}$, then the distance between β and P is $\leq 100/K_1$.
Clearly every $\nu\in\mathcal{B}_{K_1}(\mu)$ satisfies at least one of these two conditions. Let's write $\nu\in\mathfrak{C}_{2j}(\mu)$ if it satisfies condition $2j$, for $j=a,b$. Also write $\beta_1(\nu)$ and $\beta_2(\nu)$ for the cubes arising in condition 2a, if $\nu\in\mathfrak{C}_{2a}(\mu)$. Recall our goal was to control $\sum_{\mu\in\mathfrak{C}_2}\|Ef_{S(\mu)}\|_{L^q(\mu)}^q$, which is given by

$$\sum_{\mu\in\mathfrak{C}_2}\|Ef_{S(\mu)}\|_{L^q(\mu)}^q=\sum_{\mu\in\mathfrak{C}_2}\sum_{\nu\in\mathfrak{C}_{2a}(\mu)}\|Ef_{S(\mu)}\|_{L^q(\mu)}^q+\sum_{\mu\in\mathfrak{C}_2}\sum_{\nu\in\mathfrak{C}_{2b}(\mu)}\|Ef_{S(\mu)}\|_{L^q(\mu)}^q.$$

Hence we will estimate each of the two terms on the right hand side one by one.

Now suppose $\nu\in\mathfrak{C}_{2a}(\mu)$. Then

$$\|Ef_{S(\mu)}\|_{L^\infty(\nu)}\leq K_1c_{\nu,*}\leq K_1^2\operatorname*{geom}_{j=1,2}c_{\nu,\beta_j(\nu)}.$$

Hence

$$\|Ef_{S(\mu)}\|_{L^q(\nu)}^q\leq K_1^{2q}|\nu|\operatorname*{geom}_{j=1,2}c_{\nu,\beta_j(\nu)}^q\leq K_1^{2q}\sum_{\substack{\beta_1,\beta_2\\ \text{transverse}}}|\nu|\operatorname*{geom}_{j=1,2}c_{\nu,\beta_j}^q,$$

which is bounded by

$$\lesssim K_1^{2q}\iint W_{K_1}(x_1)W_{K_1}(x_2)\sum_{\substack{\beta_1,\beta_2\\ \text{transverse}}}\int_\nu\operatorname*{geom}_{j=1,2}|Ef_{S(\mu),\beta_j}(z-x_j)|^q dz\,dx_1dx_2.$$

Summing over $\nu \in \mathfrak{C}_{2a}(\mu)$, we get

$$\sum_{\nu \in \mathfrak{C}_{2a}(\mu)} \|Ef_{S(\mu)}\|_{L^q(\nu)}^q \lesssim K_1^{2q} \iint W_{K_1}(x_1)W_{K_1}(x_2) \sum_{\substack{\beta_1,\beta_2 \\ \text{transverse}}} \int_\mu \operatorname*{geom}_{j=1,2} |Ef_{S(\mu),\beta_j}(z-x_j)|^q dz\, dx_1 dx_2.$$

Applying Hölder's inequality in the integral over μ, we bound this by

$$K_1^{2q}|\mu|^{1-\frac{q}{4}} \iint W_{K_1}(x_1)W_{K_1}(x_2) \sum_{\substack{\beta_1,\beta_2 \\ \text{transverse}}} \left(\int_\mu \operatorname*{geom}_{j=1,2} |Ef_{S(\mu),\beta_j}(z-x_j)|^4 dz \right)^{q/4} dx_1 dx_2. \quad (3)$$

We estimate this using the following lemma, which is a consequence of the bilinear restriction theorem in 2 dimensions:

Lemma 2. *Fix two constants K, K_1 with $K >> K_1$. Suppose L is a line in $\mathbb{R}^2$, C is the curve $\Phi(L \cap Q)$, and N is the $100/K$ neighborhood of C in $\mathbb{R}^3$. Let U_1 and U_2 be two balls in $\mathbb{R}^3$ of radius $10K_1^{-1}$, that are at a distance $\geq 100K_1^{-1}$ from each other. Suppose F_1 and F_2 are two functions on $\mathbb{R}^3$, so that the support of $\widehat{F}_j$ is in $U_j \cap N$, for $j = 1, 2$. Then for any cube $\mu \subset \mathbb{R}^3$ of side length K, we have*

$$\int_\mu \operatorname*{geom}_{j=1,2} |F_j(z)|^4 dz \lesssim_{K_1} |\mu|^{-1} \operatorname*{geom}_{j=1,2} \|F_j\|_{L^2(\mathbb{R}^3)}^4.$$

Proof. This is a consequence of bilinear restriction in $\mathbb{R}^2$. Indeed, rotating the coordinate system, we may assume that $L = \{\xi_1 = c\}$ where $c \in [-1, 1]$ is a constant. For $j = 1, 2$, let $\tilde{F}_j(\xi_1, z_2, z_3)$ be the partial Fourier transform of F_j in the first variable. Then whenever $|\xi_1 - c| \geq 100/K$, we have

$$\tilde{F}_j(\xi_1, z_2, z_3) = \int_{\mathbb{R}^2} \widehat{F}_j(\xi) e^{2\pi i(z_2\xi_2+z_3\xi_3)} d\xi_2 d\xi_3 = 0$$

for all $(z_2, z_3) \in \mathbb{R}^2$. Thus if $\omega(y) = \frac{1}{1+y^2}$ and $\omega_K(y) = \frac{1}{K}\omega(\frac{y}{K})$ for $y \in \mathbb{R}$, then for all $z \in \mathbb{R}^3$ we have

$$|F_j(z)| \lesssim \int_{y_j \in \mathbb{R}} |F_j(y_j, z_2, z_3)| \omega_K(z_1 - y_j) dy_j, \quad j = 1, 2.$$

Thus

$$\int_\mu \operatorname*{geom}_{j=1,2} |F_j(z)|^4 dz$$
$$\lesssim \iint_{y_1,y_2 \in \mathbb{R}} \int_{z_1 \in \mathbb{R}} \omega_K(z_1 - y_1)\omega_K(z_1 - y_2) \int_{(z_2,z_3) \in \pi_1(\mu)} \operatorname*{geom}_{j=1,2} |F_j(y_j, z_2, z_3)|^4 dz_2 dz_3\; dz_1 dy_1 dy_2$$

where π_1 denotes the coordinate projection from $\mathbb{R}^3$ onto the plane that forgets the first coordinates. Now for each $y_1 \in \mathbb{R}$, let $\check{F}_j(y_1, \xi_2, \xi_3)$ be the partial Fourier transform of F_j in the last two variables. Then whenever $(\xi_2, \xi_3) \notin \pi_1(U_j \cap N)$, we have

$$\check{F}_j(y_1, \xi_2, \xi_3) = \int_{\mathbb{R}} \widehat{F}_j(\xi) e^{2\pi i y_1 \xi_1} d\xi_1 = 0 \quad \text{for all } y_1 \in \mathbb{R}.$$

Thus for each fixed $y_1 \in \mathbb{R}$, the functions $(z_2, z_3) \mapsto F_j(y_1, z_2, z_3)$, $j = 1, 2$, satisfy the hypothesis of the bilinear restriction theorem on the plane; indeed $\pi_1(U_1 \cap N)$ and $\pi_1(U_2 \cap N)$

are $100/K$ neighborhoods of two arcs of length $\simeq K_1^{-1}$, that are at a distance $\geq 100K_1^{-1}$. It follows that

$$\int_{(z_2,z_3)\in\pi_1(\nu)} \operatorname*{geom}_{j=1,2} |F_j(y_j,z_2,z_3)|^4 dz_2dz_3 \lesssim_{K_1} K^{-2} \prod_{j=1}^{2} \int_{\mathbb{R}^2} |F_j(y_j,z_2,z_3)|^2 dz_2dz_3.$$

Plugging this back, we get

$$\int_\mu \operatorname*{geom}_{j=1,2} |F_j(z)|^4 dz$$

$$\lesssim_{K_1} K^{-2} \iint_{y_1,y_2\in\mathbb{R}} \int_{z_1\in\mathbb{R}} \prod_{j=1}^{2} \omega_K(z_1-y_j)\|F_j(y_j,\cdot)\|_{L^2(\mathbb{R}^2)}^2 dz_1dy_1dy_2.$$

Applying Cauchy-Schwarz in the z_1 integral, we bound this by

$$\lesssim_{K_1} K^{-3} \prod_{j=1}^{2} \|F_j\|_{L^2(\mathbb{R}^3)}^2,$$

since $\|\omega_K\|_{L^2(\mathbb{R})} \lesssim K^{-1/2}$. This completes the proof of the lemma. $\square$

Now we claim that (3) is bounded by

$$\lesssim_{K_1} |\mu| \sum_{\substack{\beta_1,\beta_2\\ \text{transverse}}} \operatorname*{geom}_{j=1,2} \left(\sum_{\substack{\alpha\in\mathcal{P}_{K^{-1}}\\ \alpha\subset\beta_j}} c_{\mu,\alpha}^2 \right)^{q/2}. \tag{4}$$

This is because we can take a Schwartz function η on $\mathbb{R}^3$, whose Fourier support is in a unit ball, and such that $|\eta(z)| \geq 1$ for $|z| \leq 1$. Let $\eta_K(z) = \eta(\frac{z}{K})$ for $z \in \mathbb{R}^3$. Given $x_1, x_2 \in \mathbb{R}^n$, and $\beta_1, \beta_2 \in \mathcal{P}_{K_1^{-1}}$ that are transverse, let $F_j(z) = Ef_{S(\mu),\beta_j}(z-x_j)\eta_K(z-z_\mu)$ for $j = 1,2$; again z_μ is the center of μ. Then

$$\int_\mu \operatorname*{geom}_{j=1,2} |Ef_{S(\mu),\beta_j}(z-x_j)|^4 dz \leq \int_\mu \operatorname*{geom}_{j=1,2} |F_j(z)|^4 dz.$$

Since β_1, β_2 are transverse, F_1 and F_2 verify the hypothesis of Lemma 2, unless F_1F_2 is identically zero in which case there is nothing to prove. Hence by Lemma 2, the above is bounded by

$$\lesssim_{K_1} |\mu|^{-1} \operatorname*{geom}_{j=1,2} \|F_j\|_{L^2(\mathbb{R}^3)}^4.$$

But by orthogonality, for $j = 1,2$,

$$\|F_j\|_{L^2(\mathbb{R}^3)}^2 \lesssim \sum_{\substack{\alpha\in\mathcal{P}_{K^{-1}},\, \alpha\subset\beta_j\\ \alpha\cap S(\mu)\neq\emptyset}} \int_{\mathbb{R}^2} |Ef_\alpha(z-x_j)|^2 |\eta_K(z-z_\mu)|^2 dz$$

and

$$\int_{\mathbb{R}^2} |Ef_\alpha(z-x_j)|^2 |\eta_K(z-z_\mu)|^2 dz \lesssim \sum_{\mu'} |\mu'| \left\| Ef_\alpha(z) \frac{1}{1+|z+x_j-z_\mu|^{300}} \right\|_{L^\infty(\mu')}^2$$

where the sum is over all μ' in a partition of $\mathbb{R}^3$ into cubes of side lengths K. Also

$$\|Ef_\alpha\|_{L^\infty(\mu')} \lesssim \left(1+\frac{d(\mu,\mu')}{K}\right)^{30} c_{\mu,\alpha},$$

and

$$\left\|\frac{1}{1+|z+x_j-z_\mu|^{300}}\right\|_{L^\infty(\mu')} \left(1+\frac{d(\mu,\mu')}{K}\right)^{-300} \left(1+\frac{|x_j|}{K}\right)^{300}.$$

Thus altogether, we have

$$\begin{aligned}\|F_j\|^2_{L^2(\mathbb{R}^3)} &\lesssim \sum_{\substack{\alpha\in\mathcal{P}_{K^{-1}},\,\alpha\subset\beta_j\\ \alpha\cap S(\mu)\neq\emptyset}} |\mu| \sum_{\mu'} \left(1+\frac{d(\mu,\mu')}{K}\right)^{60} c^2_{\mu,\alpha} \left(1+\frac{d(\mu,\mu')}{K}\right)^{-600} \left(1+\frac{|x_j|}{K}\right)^{600} \\ &\lesssim \sum_{\substack{\alpha\in\mathcal{P}_{K^{-1}},\,\alpha\subset\beta_j\\ \alpha\cap S(\mu)\neq\emptyset}} |\mu| c^2_{\mu,\alpha} \left(1+\frac{|x_j|}{K}\right)^{600},\end{aligned}$$

which in turn gives

$$\int_\mu \operatorname*{geom}_{j=1,2} |Ef_{S(\mu),\beta_j}(z-x_j)|^4 dz \lesssim_{K_1} |\mu| \prod_{j=1}^2 \sum_{\substack{\alpha\in\mathcal{P}_{K^{-1}},\,\alpha\subset\beta_j\\ \alpha\cap S(\mu)\neq\emptyset}} c^2_{\mu,\alpha} \left(1+\frac{|x_j|}{K}\right)^{600}.$$

Plugging this back into (3), we see that (3) is bounded by

$$\lesssim_{K_1} |\mu|^{1-\frac{q}{4}} |\mu|^{\frac{q}{4}} \sum_{\substack{\beta_1,\beta_2\\ \text{transverse}}} \prod_{j=1}^2 \left(\sum_{\substack{\alpha\in\mathcal{P}_{K^{-1}}\\ \alpha\subset\beta_j}} c^2_{\mu,\alpha} \right)^{q/4}$$

which is (4). Now (4) is bounded by

$$\lesssim_{K_1} |\mu| K^{\frac{q}{2}-1} \sum_{\alpha\in\mathcal{P}_{K^{-1}}} c^q_{\mu,\alpha},$$

i.e.

$$\sum_{\nu\in\mathfrak{C}_{2a}(\mu)} \|Ef_{S(\mu)}\|^q_{L^q(\nu)} \lesssim_{K_1} |\mu| K^{\frac{q}{2}-1} \sum_{\alpha\in\mathcal{P}_{K^{-1}}} c^q_{\mu,\alpha}.$$

Summing over all $\mu\in\mathfrak{C}_2$, we get

$$\sum_{\mu\in\mathfrak{C}_2} \sum_{\nu\in\mathfrak{C}_{2a}(\mu)} \|Ef_{S(\mu)}\|^q_{L^q(\nu)} \lesssim_{K_1} K^{\frac{q}{2}-1} \sum_{\mu\in\mathcal{B}_K} |\mu| \sum_{\alpha\in\mathcal{P}_{K^{-1}}} c^q_{\mu,\alpha} \lesssim_{K_1} K^{\frac{q}{2}-1} K^{6-2q} A(R/K), \tag{5}$$

the last inequality following from Lemma 1.

On the other hand, if $\nu\in\mathfrak{C}_{2b}(\mu)$, then

$$\|Ef_{S(\mu)}\|_{L^\infty(\nu)} \lesssim c_{\nu,*},$$

so

$$\|Ef_{S(\mu)}\|^q_{L^q(\nu)} \lesssim |\nu| c^q_{\nu,*} \lesssim |\nu| \sum_{\beta\in\mathcal{P}_{K_1^{-1}}} c^q_{\nu,\beta}.$$

Summing over $\nu \in \mathfrak{C}_{2b}(\mu)$ and then over $\mu \in \mathfrak{C}_2$, we get

$$\sum_{\mu\in\mathfrak{C}_2}\sum_{\nu\in\mathfrak{C}_{2b}(\mu)} \|Ef_{S(\mu)}\|^q_{L^q(\nu)} \lesssim \sum_{\mu\in\mathcal{B}_K}\sum_{\nu\in B_{K_1}(\mu)} |\nu| \sum_{\beta\in\mathcal{P}^{-1}_{K_1}} c^q_{\nu,\beta}.$$

Similar to Lemma 1, we have

Lemma 3.

$$\sum_{\mu\in\mathcal{B}_K}\sum_{\nu\in B_{K_1}(\mu)} |\nu| \sum_{\beta\in\mathcal{P}^{-1}_{K_1}} c^q_{\nu,\beta} \lesssim K_1^{6-2q} A(R/K_1)^q.$$

Thus

$$\sum_{\mu\in\mathfrak{C}_2}\sum_{\nu\in\mathfrak{C}_{2b}(\mu)} \|Ef_{S(\mu)}\|^q_{L^q(\nu)} \lesssim K_1^{6-2q} A(R/K_1)^q. \tag{6}$$

From (5) and (6), we get

$$\sum_{\mu\in\mathfrak{C}_2} \|Ef_{S(\mu)}\|^q_{L^q(\mu)} \leq C_{K_1} K^{\frac{q}{2}-1} K^{6-2q} A(R/K)^q + C K_1^{6-2q} A(R/K_1)^q. \tag{7}$$

where C_{K_1} is a constant depending on K_1. From (2) and (7), we see that

$$\sum_{\mu\in\mathfrak{C}_2} \|Ef\|^q_{L^q(\mu)} \leq C_{K_1} K^{\frac{q}{2}-1} K^{6-2q} A(R/K)^q + C K_1^{6-2q} A(R/K_1)^q.$$

As a result,

$$\begin{aligned}\|Ef\|^q_{L^q(B_R)} &\leq \sum_{\mu\in\mathfrak{C}_1} \|Ef\|^q_{L^q(\mu)} + \sum_{\mu\in\mathfrak{C}_2} \|Ef\|^q_{L^q(\mu)} \\ &\leq C_{K,\varepsilon} R^{\varepsilon q} + C_{K_1} K^{\frac{q}{2}-1} K^{6-2q} A(R/K)^q + C K_1^{6-2q} A(R/K_1)^q.\end{aligned}$$

Since $q = 10/3$, the power of K in front of $A(R/K)^q$ is zero. This shows

$$A(R) \leq C_{K,\varepsilon} R^\varepsilon + C_{K_1} A(R/K) + C K_1^{\frac{6}{q}-2} A(R/K_1).$$

By first choosing K_1 to be sufficiently large, so that $CK_1^{\frac{6}{q}-2} \leq 1$, then K to be sufficiently large, so that $C_{K_1} K^{-\varepsilon/2} \leq 1$, we get

$$A(R) \leq C_{K,\varepsilon} R^\varepsilon + K^{\varepsilon/2} A(R/K) + A(R/K_1),$$

so iterating, we get

$$A(R) \leq C_{K,\varepsilon} R^\varepsilon \sum_{j=0}^{\infty} (K^{-j\varepsilon/2} + K_1^{-j\varepsilon}) \lesssim_\varepsilon R^\varepsilon,$$

as desired.

We remark that a small refinement of the above argument also shows that

$$\|Ef\|_{L^q(B_R)} \lesssim_\varepsilon R^\varepsilon \|f\|_{L^q(Q)}$$

for all $f \in L^q(Q)$.

References

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