ON BOURGAIN'S COUNTEREXAMPLE FOR THE SCHRÖDINGER MAXIMAL FUNCTION

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Below I write up, for my own benefit, Bourgain's counterexample [1] for the Schrödinger maximal function. The exposition follows closely the excellent survey article [3] by Pierce.[1]

For every $n \geq 2$ and every $s < \frac{1}{2} - \frac{1}{2(n+1)}$, Bourgain [1] constructed a family of functions $F_R \in H^s(\mathbb{R}^n)$, such that

$$\lim_{R\to\infty} \frac{\| \sup_{0<t<1/R} |e^{-it\Delta} F_R(x)| \|_{L^1(B(0,1))}}{\|F_R\|_{H^s(\mathbb{R}^n)}} = \infty. \tag{1}$$

Here $B(0,1)$ is the unit ball in $\mathbb{R}^n$ centered at 0, and $\|F_R\|_{H^s(\mathbb{R}^n)}$ is the Sobolev norm given by $\|(I-\Delta)^{s/2} F_R\|_{L^2(\mathbb{R}^n)}$. We will repeat this construction, with a bit more detail than in [1].

Notation. Throughout we will fix a Schwartz function φ on $\mathbb{R}$ so that $\widehat{\varphi}$ is supported in the unit interval $[-1,1]$, and so that $\varphi(0) = 1$, and write

$$e(x) := e^{ix}.$$

We will allow all implicit constants to depend on φ as necessary.

1. Counterexample in dimension $n = 1$

First, it helps to understand what happens in dimension $n = 1$. Let $R \gg 1$. We will choose a parameter $S = S(R)$ such that $1 \ll S \ll R$, and set

$$f_R(x) = e(Rx)\varphi(Sx), \quad x \in \mathbb{R}. \tag{2}$$

It follows that $\widehat{f_R}$ is contained in the set where $|\xi| \simeq R$, and

$$\|f_R\|_{H^s(\mathbb{R})} \simeq R^s S^{-1/2}. \tag{3}$$

Now let

$$u_R(x,t) = e^{-it\partial_x^2} f_R(x).$$

Then

$$u_R(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\xi) e((S\xi + R)x + (S\xi + R)^2 t) d\xi$$

so

$$u_R(x,t) = e(Rx + R^2 t) \int_{\mathbb{R}} \widehat{\varphi}(\xi) e(\xi(Sx + 2SRt)) e(S^2\xi^2 t) d\xi.$$

We write $e(S^2\xi^2 t) = 1 + O(S^2\xi^2|t|)$ and apply Fourier inversion for the main term. We then obtain

$$u_R(x,t) = e(Rx + R^2 t)\varphi(Sx + 2SRt) + O(S^2 t). \tag{4}$$

[1] I am very grateful to Lillian B. Pierce and Keith Rogers for discussions about the proof presented here. It is worth pointing out that Lucà and Rogers [2] gave another proof of (1) using ergodic considerations in lieu of number theoretic estimates.

If $|x| \le c$ for some sufficiently small constant $c > 0$ and

$$t := -\frac{1}{2R}x,$$

then as long as

$$S^2 \le R, \tag{5}$$

we have

$$|u_R(x,t)| \ge |\varphi(0)| - \frac{1}{2} = \frac{1}{2}.$$

We fix such c for the remainder of this section. Then under assumption (5),

$$\sup_{0<t<1/R} |u_R(x,t)| \ge \frac{1}{2}$$

for every x satisfying $|x| \le c$, which implies

$$\left\| \sup_{0<t<1/R} |u_R(x,t)| \right\|_{L^1(B(0,1))} \ge \frac{1}{2}c.$$

Recall the H^s norm of f_R in (3). As a result, under assumption (5),

$$\frac{\| \sup_{0<t<1/R} |u_R(x,t)| \|_{L^1(B(0,1))}}{\|f_R\|_{H^s(\mathbb{R}^n)}} \ge \frac{1}{2}cR^{-s}S^{1/2}.$$

We now take $S = R^{1/2}$, so that the lower bound above is maximized under assumption (5). This gives

$$\frac{\| \sup_{0<t<1/R} |u_R(x,t)| \|_{L^1(B(0,1))}}{\|f_R\|_{H^s(\mathbb{R}^n)}} \ge \frac{1}{2}cR^{\frac{1}{4}-s},$$

which tends to $+\infty$ as $R \to +\infty$ if $s < 1/4$.

2. Another construction in dimension 1

To prove Bourgain's theorem in dimensions $n \ge 2$, it is perhaps natural to try tensor product of the 1-dimensional example with something else. Bourgain's insight is that you can tensor with essentially sums of well-spaced Dirac delta's in the frequency space and use bounds for exponential sums from number theory. At its root we should consider the following 1-dimensional construction.

Again let $R \gg 1$. We will choose an integer $D = D(R)$ such that $1 \ll D \ll R$, and write Λ as a shorthand for the integer part of $R/2D$. Let

$$g_R(x) = \varphi(x) \sum_{\Lambda < \ell \le 2\Lambda} e(D\ell x), \quad x \in \mathbb{R}. \tag{6}$$

Then the support of $\widehat{g_R}$ is contained in the set $|\xi| \simeq R$, and

$$\|g_R\|_{H^s(\mathbb{R})} \simeq R^s \Lambda^{1/2}.$$

Moreover, let

$$v_R(x,t) = e^{-it\partial_x^2} g_R(x).$$

Then

$$v_R(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\xi) \sum_{\Lambda < \ell \le 2\Lambda} e((\xi + D\ell)x + (x + D\ell)^2 t) d\xi$$

so

$$v_R(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi}(\xi) \sum_{\Lambda<\ell\le 2\Lambda} e(\xi(x+2D\ell t))e(D\ell x + D^2\ell^2 t)e(\xi^2 t)d\xi.$$

We write $e(\xi^2 t) = 1 + O(\xi^2 t)$ and obtain

$$\begin{aligned} v_R(x,t) =& \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi}(\xi) \sum_{\Lambda<\ell\le 2\Lambda} e(\xi(x+2D\ell t))e(D\ell x + D^2\ell^2 t)d\xi \\ &+ O\left(|t| \sup_{z\in[0,2\pi]} \left|\sum_{\Lambda<\ell\le 2\Lambda} e(\ell z + D^2\ell^2 t)\right|\right). \end{aligned}$$

Then we apply the summation by parts formula

$$\sum_{\Lambda<\ell\le 2\Lambda} a_\ell b_\ell = a_{2\Lambda} \sum_{\Lambda<\ell\le 2\Lambda} b_\ell + \sum_{\Lambda<L<2\Lambda} (a_{L+1}-a_L) \sum_{\Lambda<\ell\le L} b_\ell \tag{7}$$

with $a_\ell = e(\xi(x+2D\ell t))$ and $b_\ell = e(D\ell x + D^2\ell^2 t)$, and then integrate in ξ using the Fourier inversion formula. We obtain

$$\begin{aligned} v_R(x,t) =& \varphi(x+4\Lambda D t) \sum_{\Lambda<\ell\le 2\Lambda} e(D\ell x + D^2\ell^2 t) + O\left(|t| \sup_{z\in[0,2\pi]} \left|\sum_{\Lambda<\ell\le 2\Lambda} e(\ell z + D^2\ell^2 t)\right|\right) \\ &+ O\left(\Lambda D|t| \sup_{\Lambda<L\le 2\Lambda} \left|\sum_{\Lambda<\ell\le L} e(D\ell x + D^2\ell^2 t)\right|\right). \end{aligned} \tag{8}$$

At this point we can bring in exponential sum estimates from number theory.

Lemma 1. *If there exist integers $q, a, b \in \mathbb{N}$ with q odd and $(a,q)=1$ such that $D^2 t = 2\pi a/q$ and $Dx = 2\pi b/q$, then for $\Lambda < L \le 2\Lambda$,*

$$\sum_{\Lambda<\ell\le L} e(D\ell x + D^2\ell^2 t) = \left\lfloor \frac{L-\Lambda}{q} \right\rfloor q^{1/2} + O(q^{1/2}(\log q)^{1/2}).$$

Proof. This is because we can split the sum on the left hand side into $\left\lfloor \frac{L-\Lambda}{q} \right\rfloor$ many complete Gauss sums modulo q, each of which is equal to $q^{1/2}$, and the remaining incomplete Gauss sum can be bounded by an estimate of Weyl. See Lemma 3.1(1) and Lemma 3.2 of [3]. □

Furthermore, we may perturb x a bit and the error would still be under control:

Lemma 2. *If there exist integers $q, a, b \in \mathbb{N}$ with q odd and $(a,q)=1$ such that*

$$D^2 t = \frac{2\pi a}{q} \quad \text{and} \quad \left| Dx - \frac{2\pi b}{q} \right| < \delta,$$

then

$$\begin{aligned} \sum_{\Lambda<\ell\le L} e(D\ell x + D^2\ell^2 t) =& e\left(L\left(Dx - \frac{b}{q}\right)\right) \left\lfloor \frac{L-\Lambda}{q} \right\rfloor q^{1/2} + O(q^{1/2}(\log q)^{1/2}) \\ &+ O\left((L-\Lambda)\delta\left(\left\lfloor \frac{L-\Lambda}{q} \right\rfloor q^{1/2} + O(q^{1/2}(\log q)^{1/2})\right)\right). \end{aligned}$$

Proof. Indeed, we write the sum on the left hand side as

$$\sum_{\Lambda<\ell\leq L} e\left(\ell\left(Dx-\frac{2\pi b}{q}\right)\right) e\left(\ell\frac{2\pi b}{q}+\ell^2\frac{2\pi a}{q}\right)$$

and apply summation by parts (7) with $a_\ell = e\left(\ell\left(Dx-\frac{2\pi b}{q}\right)\right)$, $b_\ell = e\left(2\pi\left(\ell\frac{b}{q}+\ell^2\frac{a}{q}\right)\right)$; we obtain

$$\begin{aligned}\sum_{\Lambda<\ell\leq L} e(D\ell x+D^2\ell^2 t) =& e\left(L\left(Dx-\frac{b}{q}\right)\right)\sum_{\Lambda<\ell\leq L} e\left(2\pi\left(\ell\frac{b}{q}+\ell^2\frac{a}{q}\right)\right)\\ &+O\left((L-\Lambda)\delta\sup_{\Lambda<L\leq 2\Lambda}\left|\sum_{\Lambda<\ell\leq L} e\left(2\pi\left(\ell\frac{b}{q}+\ell^2\frac{a}{q}\right)\right)\right|\right).\end{aligned}$$

The desired estimate follows via the same argument in the proof of Lemma 1, where one splits into complete Gauss sums and an incomplete one. □

The following summarizes what we need below:

Lemma 3. *If there exist integers $q, a, b \in \mathbb{N}$ with q odd and $(a, q) = 1$ such that*

$$D^2 t = \frac{2\pi a}{q} \quad \text{and} \quad \left|Dx-\frac{2\pi b}{q}\right| < \delta,$$

then the following three estimates hold:

$$\left|\sum_{\Lambda<\ell\leq 2\Lambda} e(D\ell x+D^2\ell^2 t)\right| = (1+O(\Lambda\delta))\left(\left\lfloor\frac{\Lambda}{q}\right\rfloor q^{1/2}+O(q^{1/2}(\log q)^{1/2})\right),$$

$$\sup_{\Lambda<L\leq 2\Lambda}\left|\sum_{\Lambda<\ell\leq L} e(D\ell x+D^2\ell^2 t)\right| \leq (1+O(\Lambda\delta))\left(\left\lfloor\frac{\Lambda}{q}\right\rfloor q^{1/2}+O(q^{1/2}(\log q)^{1/2})\right),$$

and

$$\sup_{z\in[0,2\pi]}\left|\sum_{\Lambda<\ell\leq 2\Lambda} e(\ell z+D^2\ell^2 t)\right| = O\left(\left(\frac{\Lambda}{q}q^{1/2}+q^{1/2}\right)(\log q)^{1/2}\right).$$

In fact, the first two estimates follow directly from Lemma 2, whereas the third one follows from another Weyl sum estimate (see again Lemma 3.2 of [3]).

With these preparations, we are ready for the construction of Bourgain's counterexample in general dimensions $n \geq 2$.

3. Counterexample in dimensions $n \geq 2$

Let $R \gg 1$. As in the last section, $D = D(R)$ is an integer to be chosen, such that $1 \ll D \ll R$. Λ will be a shorthand for the integer part of $R/2D$. We define two one-variable functions f_R and g_R as in (2) and (6) in the previous sections, where we take $S = R^{1/2}$ in the definition of f_R.

Let F_R be the function on $\mathbb{R}^n$ defined by

$$F_R(x) = f_R(x_1)g_R(x_2)\dots g_R(x_n), \quad x = (x_1,\dots,x_n)\in\mathbb{R}^n.$$

Then the support of $\widehat{F_R}$ is contained in the set $|\xi| \simeq R$, and

$$\|F_R\|_{H^s(\mathbb{R}^n)} \simeq R^{s-\frac{1}{4}} \Lambda^{\frac{n-1}{2}} \simeq R^{s-\frac{1}{4}} \left(\frac{R}{D}\right)^{\frac{n-1}{2}}. \tag{9}$$

Let

$$U_R(x,t) := e^{-it\Delta} F_R(x) = u_R(x_1,t) v_R(x_2,t) \dots v_R(x_n,t)$$

where u_R and v_R are as in the previous sections. We will now construct a set $\Omega = \Omega_R \subset B(0,1)$ in $\mathbb{R}^n$, for which $\sup_{0<t<1/R} |U_R(x,t)|$ is large for $x \in \Omega$.

The construction of Ω involves the choice of another integer $Q = Q(R)$ with $1 \ll Q \ll R$, whose value we will only pick at the close. Denote by $I(r,d)$ the interval $(r-d, r+d)$. For odd primes $q \le Q$, let

$$\Omega_{1,q} := \bigcup_{\substack{1 \le a \le c\frac{qD^2}{R} \\ (a,q)=1}} I\left(-\frac{4\pi a R}{qD^2}, \frac{c}{R^{1/2}}\right) \quad \text{and} \quad \Omega_{2,q} := \bigcup_{1 \le b \le cqD} I\left(\frac{2\pi b}{qD}, \frac{c}{\Lambda D}\right)$$

where $c > 0$ is a sufficiently small constant to be determined. Then let $\Omega \subset B(0,1)$ be given by

$$\Omega = \bigcup_{\substack{q \le Q \\ q \text{ odd prime}}} \Omega_{1,q} \times \Omega_{2,q}^{n-1}$$

Suppose now $x \in \Omega$. Then there exists an odd prime $q \le Q$, and integers $a, b_2, \dots, b_n$ with $1 \le a \le c\frac{qD^2}{R}$, $(a,q) = 1$, and $1 \le b_i \le cqD$ for $2 \le i \le n$, such that

$$\left| x_1 + \frac{4\pi a R}{qD^2} \right| \le \frac{c}{R^{1/2}}, \quad \text{and} \quad \left| x_i - \frac{2\pi b_i}{qD} \right| \le \frac{c}{\Lambda D} \quad \text{for every } 2 \le i \le n.$$

Setting

$$t := \frac{2\pi a}{qD^2},$$

we have

$$|R^{1/2}(x_1 + 2Rt)| \le c,$$

which implies

$$|t| \le \left| t + \frac{x_1}{2R} \right| + \frac{|x_1|}{R} \le \frac{c}{R^{3/2}} + \frac{1}{R}\left(4\pi c + \frac{c}{R^{1/2}}\right) \le \frac{4\pi c + o(1)}{R}; \tag{10}$$

from (4) with $S^2 = R$, we get

$$|u_R(x_1,t)| \ge \frac{1}{2}|\varphi(0)| = \frac{1}{2} \tag{11}$$

if c is chosen small enough. Also note that for $2 \le i \le n$,

$$|x_i| \le \left| x_i - \frac{2\pi b_i}{qD} \right| + \frac{2\pi b_i}{qD} \le \frac{c}{\Lambda D} + 2\pi c = 2\pi c + o(1),$$

which in light of $\Lambda D \simeq R$ and (10) shows that $|x_i + 4\Lambda D t| = O(c)$ for our previous choice of t. In addition, we have

$$D^2 t = \frac{2\pi a}{q} \quad \text{with} \quad \left| D x_i - \frac{2\pi b_i}{q} \right| \le \frac{c}{\Lambda} \quad \text{for every } 2 \le i \le n,$$

which allows us to apply Lemma 3 with $\delta = \frac{c}{\Lambda}$. If c is chosen small enough, then from (8) and the previous application of Lemma 3, it follows that

$$|v_R(x_i,t)| \geq \frac{1}{2}\left(\left\lfloor\frac{\Lambda}{q}\right\rfloor q^{1/2} + O(q^{1/2}(\log q)^{1/2})\right). \tag{12}$$

We now fix, once and for all, a small c so that both (11) and (12) holds. We thus have, for $x \in \Omega$, that

$$\sup_{0<t<1/R} |U_R(x,t)| \gtrsim \left(\frac{R}{Q^{1/2}D}\right)^{n-1}, \tag{13}$$

provided that $Q = Q(R)$ and $D = D(R)$ are chosen so that

$$\frac{R}{D} \gg Q(\log Q)^{1/2} \tag{14}$$

(recall $\Lambda \simeq R/D$).

We will now show that the measure of Ω is large, so that $\|\sup_{0<t<1/R}|U_R(x,t)|\|_{L^1(B(0,1))}$ has a favorable lower bound. To do so, we should choose $D = D(R)$ and $Q = Q(R)$ so that

$$\frac{2c}{R^{1/2}} \leq \frac{4\pi R}{QD^2}; \tag{15}$$

this will ensure that the intervals in the definition of $\Omega_{1,q}$ are disjoint for every prime $q \leq Q$. We will also choose $D = D(R)$ such that

$$D^2 \gg R; \tag{16}$$

this ensures that the union in the definition of $\Omega_{1,q}$ runs over many complete residue classes modulo q, and also ensures that

$$\frac{2c}{\Lambda} \leq \frac{2\pi}{Q}$$

(recall $\Lambda \simeq R/D$ so (15) and (16) ensures that $\frac{2\pi}{Q} \geq \frac{cD^2}{R^{3/2}} \simeq \frac{c}{\Lambda}\frac{D}{R^{1/2}} \gg \frac{c}{\Lambda}$). The latter ensures that the intervals in the definition of $\Omega_{2,q}$ are disjoint for every prime $q \leq Q$. If we were so fortunate that the sets $\Omega_{1,q} \times \Omega_{2,q}^{n-1}$ in the definition of Ω are disjoint as q varies over all odd primes $\leq Q$, then the measure of Ω can now be easily computed: we would then have

$$|\Omega| = \sum_{\substack{q \leq Q \\ q \text{ odd prime}}} |\Omega_{1,q} \times \Omega_{2,q}^{n-1}| \geq \sum_{\substack{q \leq Q \\ q \text{ odd prime}}} \left\lfloor c\frac{D^2}{R}\right\rfloor (q-1)\frac{2c}{R^{1/2}} \cdot \left(\lfloor cqD \rfloor \frac{c}{\Lambda D}\right)^{n-1},$$

because there are at least $\left\lfloor c\frac{qD^2}{R}\frac{1}{q}\right\rfloor (q-1)$ many disjoint intervals of lengths $\frac{2c}{R^{1/2}}$ in the set $\Omega_{1,q}$, which shows

$$|\Omega_{1,q}| \geq \left\lfloor c\frac{D^2}{R}\right\rfloor (q-1)\frac{2c}{R^{1/2}}, \tag{17}$$

and there are $\lfloor cqD \rfloor$ disjoint intervals of length $\frac{c}{\Lambda D}$ in $\Omega_{2,q}$, which shows

$$|\Omega_{2,q}| \geq \lfloor cqD \rfloor \frac{c}{\Lambda D}. \tag{18}$$

We make the earlier sum smaller by summing only over those q that satisfies additionally $q > Q/2$. Indeed, the prime number theorem allows us to count the number of primes q satisfying $Q/2 < q \leq Q$: for every $\varepsilon > 0$, when $Q \gg 1$,

$$\pi(Q) - \pi(Q/2) \geq (1-\varepsilon)\frac{Q}{\log Q} - (1+\varepsilon)\frac{Q/2}{\log(Q/2)} \geq \left(\frac{1}{2} - 2\varepsilon\right)\frac{Q}{\log Q}, \tag{19}$$

where $\pi(Q)$ is the number of primes $\leq Q$. Hence for $Q \gg 1$, we have

$$|\Omega| \geq \frac{1}{4}\frac{Q}{\log Q}\left(c\frac{D^2}{R}\frac{Q}{2}\frac{2c}{R^{1/2}}\right)\cdot\left(c\frac{Q}{2}D\frac{c}{\Lambda D}\right)^{n-1} = \frac{c^{2n}}{2^{n+1}\log Q}\frac{QD^2}{R^{3/2}}\frac{Q^n D^{n-1}}{R^{n-1}}. \tag{20}$$

(We used the assumption (16) to bound $\left\lfloor c\frac{D^2}{R}\right\rfloor \geq (1-\varepsilon)c\frac{D^2}{R}$ for every $\varepsilon > 0$, and $Q \gg 1$ to bound $\lfloor cqD \rfloor \geq (1-\varepsilon)c\frac{Q}{2}D$ whenever $q > Q/2$. If ε is sufficiently small, we can make the coefficient on the right hand side of the first inequality in the earlier display equation $> 1/4$.) This calculation turns out to give essentially the correct answer when Q and $1/\Lambda \simeq D/R$ are small, but is too good to be true when Q or D/R is too big; the correct lower bound for $|\Omega|$ is given in (25) below, whose rigorous derivation we now give.

To begin with the rigorous argument, note that the measure of Ω can be computed using Fubini's theorem, by first integrating the charateristic function $\mathbf{1}_\Omega$ of Ω in the x_1 variable, before integrating in the x' variables. The projection of Ω to the x' space is

$$\bigcup_{\substack{q\leq Q\\ q \text{ odd prime}}} \Omega_{2,q}^{n-1}$$

and for each x' in this projection, we may choose a prime $q \leq Q$ so that $x' \in \Omega_{2,q}^{n-1}$; it then follows that

$$\int_{\mathbb{R}} \Omega(x_1,x')dx_1 \geq |\Omega_{1,q}| \geq \left\lfloor c\frac{D^2}{R}\right\rfloor (q-1)\frac{2c}{R^{1/2}}$$

where we used (17). This shows that

$$\int_{\mathbb{R}} \Omega(x_1,x')dx_1 \geq \left\lfloor c\frac{D^2}{R}\right\rfloor \frac{Q}{2}\frac{2c}{R^{1/2}} \geq \frac{c^2}{2}\frac{QD^2}{R^{3/2}} \quad \text{whenever} \quad x' \in \Omega' := \bigcup_{\substack{Q/2<q\leq Q\\ q \text{ odd prime}}} \Omega_{2,q}^{n-1}.$$

(We used the assumption (16) to bound $\left\lfloor c\frac{D^2}{R}\right\rfloor \geq \frac{c}{2}\frac{D^2}{R}$.) In particular,

$$|\Omega| \geq |\Omega'|\frac{c^2}{2}\frac{QD^2}{R^{3/2}}. \tag{21}$$

But Ω' is the support of the function

$$\eta(x') := \sum_{\substack{Q/2<q\leq Q\\ q \text{ odd prime}}} \mathbf{1}_{\Omega_{2,q}^{n-1}}(x'),$$

and for $Q \gg 1$,

$$\|\eta\|_{L^1(\mathbb{R}^{n-1})} = \sum_{\substack{Q/2<q\leq Q\\ q \text{ odd prime}}} |\Omega_{2,q}|^{n-1} \geq \frac{1}{4}\frac{Q}{\log Q}\left(\frac{1}{2}cQD\frac{c}{\Lambda D}\right)^{n-1} = \frac{c^{2(n-1)}}{2^{n+1}\log Q}\frac{Q^n}{\Lambda^{n-1}} \tag{22}$$

where we used (18) (together with the bound $\lfloor cqD \rfloor \geq (\frac{1}{2}-\varepsilon)cQD$ when $q > Q/2 \gg 1$) and (19). On the other hand,

$$\|\eta\|^2_{L^2(\mathbb{R}^{n-1})} = \sum_{\substack{Q/2<q,q'\leq Q\\ q,q' \text{ odd prime}}} |\Omega_{2,q}\cap\Omega_{2,q'}|^{n-1} = \|\eta\|_{L^1} + \sum_{\substack{Q/2<q,q'\leq Q\\ q,q' \text{ distinct odd prime}}} |\Omega_{2,q}\cap\Omega_{2,q'}|^{n-1} \tag{23}$$

For distinct primes q, q' with $Q/2 < q, q' \leq Q$, we count the number of pairs (b, b') with $1 \leq b \leq cqD$ and $1 \leq b' \leq cq'D$ so that the intervals $I(\frac{2\pi b}{qD}, \frac{c}{\Lambda D})$ and $I(\frac{2\pi b'}{q'D}, \frac{c}{\Lambda D})$ intersect. This happens precisely when

$$\left|\frac{2\pi b}{qD} - \frac{2\pi b'}{q'D}\right| < \frac{2c}{\Lambda D},$$

i.e.

$$|bq' - b'q| < \frac{c}{\pi\Lambda} qq'.$$

But since q, q' are relatively prime, for all integers m and k, the number of integer solutions (b, b') to the equation $bq' - b'q = m$ with $kq < b \leq (k+1)q$ is at most 1. (If (b, b') and $(\bar{b}, \bar{b}')$ are both such solutions, then $(b - \bar{b})q' = (b' - \bar{b}')q$ is divisible by q, so q divides $b - \bar{b}$, which implies $b = \bar{b}$ and hence $b' = \bar{b}'$.) As a result, the number of pairs (b, b') with $1 \leq b \leq cqD$ and $1 \leq b' \leq cq'D$ so that $|bq' - b'q| \leq \frac{c}{\pi\Lambda} qq'$ is at most

$$\lceil cD \rceil \left(2\frac{c}{\pi\Lambda} qq' + 1\right) \leq C\frac{Q^2 D}{\Lambda}$$

provided that $Q = Q(R)$, $D = D(R)$ are chosen so that

$$Q^2 \gg \frac{R}{D} \tag{24}$$

(recall $\Lambda \simeq R/D$ so (24) impiles $\lceil cD \rceil \cdot 1 \leq c\frac{Q^2 D}{\Lambda}$). It follows that for distinct primes q, q' with $Q/2 < q, q' \leq Q$,

$$|\Omega_{2,q} \cap \Omega_{2,q'}| \leq \sum_{1 \leq b \leq cqD} \sum_{1 \leq b' \leq cq'D} \left| I\left(\frac{2\pi b}{qD}, \frac{c}{D\Lambda}\right) \cap I\left(\frac{2\pi b'}{q'D}, \frac{c}{D\Lambda}\right)\right| \leq C\frac{Q^2 D}{\Lambda}\frac{c}{D\Lambda} = C\frac{Q^2}{\Lambda^2},$$

so using (19) again, we obtain

$$\sum_{\substack{Q/2 < q, q' \leq Q \\ q, q' \text{ distinct odd prime}}} |\Omega_{2,q} \cap \Omega_{2,q'}|^{n-1} \leq C\left(\frac{Q}{\log Q}\right)^2 \left(\frac{Q^2}{\Lambda^2}\right)^{n-1} = \frac{C}{\log Q}\left(\frac{Q^n}{\Lambda^{n-1}}\right)^2.$$

Plugging back into (23),

$$\|\eta\|^2_{L^2(\mathbb{R}^{n-1})} \leq 2\max\left\{\|\eta\|_{L^1}, \frac{C}{\log Q}\left(\frac{Q^n}{\Lambda^{n-1}}\right)^2\right\}.$$

But then from Cauchy-Schwarz, we obtain

$$|\Omega'| \geq \frac{\|\eta\|^2_{L^1(\mathbb{R}^{n-1})}}{\|\eta\|^2_{L^2(\mathbb{R}^{n-1})}} \geq \frac{1}{2}\min\left\{\|\eta\|_{L^1(\mathbb{R}^{n-1})}, \|\eta\|^2_{L^1(\mathbb{R}^{n-1})}\left(\frac{C}{\log Q}\left(\frac{Q^n}{\Lambda^{n-1}}\right)^2\right)^{-1}\right\},$$

so the lower bound (22) for $\|\eta\|_{L^1(\mathbb{R}^{n-1})}$ gives

$$|\Omega'| \geq \frac{1}{2}\min\left\{\frac{c^{2(n-1)}}{2^{n+1}\log Q}\frac{Q^n}{\Lambda^{n-1}}, \frac{c^{4(n-1)}}{2^{2(n+1)} C \log Q}\right\}$$

Using (21), we then obtain our final bound for $|\Omega|$, namely

$$|\Omega| \gtrsim \frac{1}{\log Q}\frac{QD^2}{R^{3/2}}\min\left\{\frac{Q^n D^{n-1}}{R^{n-1}}, 1\right\} \tag{25}$$

(note that this is only worse than the naive guess in (20) when $\frac{Q^n D^{n-1}}{R^{n-1}} \le 1$). From (13), it follows that

$$\left\| \sup_{0<t<1/R} |U_R(x,t)| \right\|_{L^1(B(0,1))} \gtrsim \left(\frac{R}{Q^{1/2} D} \right)^{n-1} \frac{1}{\log Q} \frac{QD^2}{R^{3/2}} \min \left\{ \frac{Q^n D^{n-1}}{R^{n-1}}, 1 \right\}.$$

Hence in view of (9), we obtain

$$\frac{\left\| \sup_{0<t<1/R} |U_R(x,t)| \right\|_{L^1(B(0,1))}}{\|F_R\|_{H^s(\mathbb{R}^n)}} \gtrsim R^{\frac{1}{4}-s} \left(\frac{R}{QD} \right)^{\frac{n-1}{2}} \frac{1}{\log Q} \frac{QD^2}{R^{3/2}} \min \left\{ \frac{Q^n D^{n-1}}{R^{n-1}}, 1 \right\} \tag{26}$$

under the assumptions on $D = D(R)$ and $Q = Q(R)$ we have made so far, namely (14), (15), (16) and (24).

As we will see shortly, to maximize this lower bound under the assumptions made, it is natural to take $D = D(R)$ and $Q = Q(R)$ so that

$$\frac{QD^2}{R^{3/2}} \simeq 1 \quad \text{and} \quad \frac{Q^n D^{n-1}}{R^{n-1}} \simeq 1;$$

this is so that $\frac{QD^2}{R^{3/2}}$ achieves the maximum allowed value in (16), and so that the minimum in (26) equals 1. In other words, we are taking $D = D(R)$ and $Q = Q(R)$ so that

$$D \simeq R^{\frac{n+2}{2(n+1)}} \quad \text{and} \quad Q \simeq R^{\frac{n-1}{2(n+1)}}. \tag{27}$$

This is the choice made by Bourgain, and it is easy to check that the assumptions (14), (16) and (24) are all satisfied. The right hand side of (26) is now bounded below by

$$R^{\frac{1}{4}-s} R^{(1-\frac{n-1}{2(n+1)}-\frac{n+2}{2(n+1)})\frac{n-1}{2}-o(1)} = R^{\frac{1}{4}+\frac{n-1}{4(n+1)}-o(1)-s} = R^{\frac{n}{2(n+1)}-o(1)-s}$$

which tends to $+\infty$ as $R \to +\infty$ if $s < \frac{n}{2(n+1)}$.

4. Optimality of Bourgain's counterexample

A simple linear programming exercise shows that under assumptions (14), (15), (16) and (24) on D and Q, the choice (27) of $D = D(R)$ and $Q = Q(R)$ maximizes the right hand side of (26), thereby giving the optimal lower bound for the left hand side of (26).

Indeed, assumptions (15) and (16) almost implies the condition (14), because (15) says $QD^2 \lesssim R^{3/2}$, and (16) says $D \gg R^{1/2}$. Together they imply

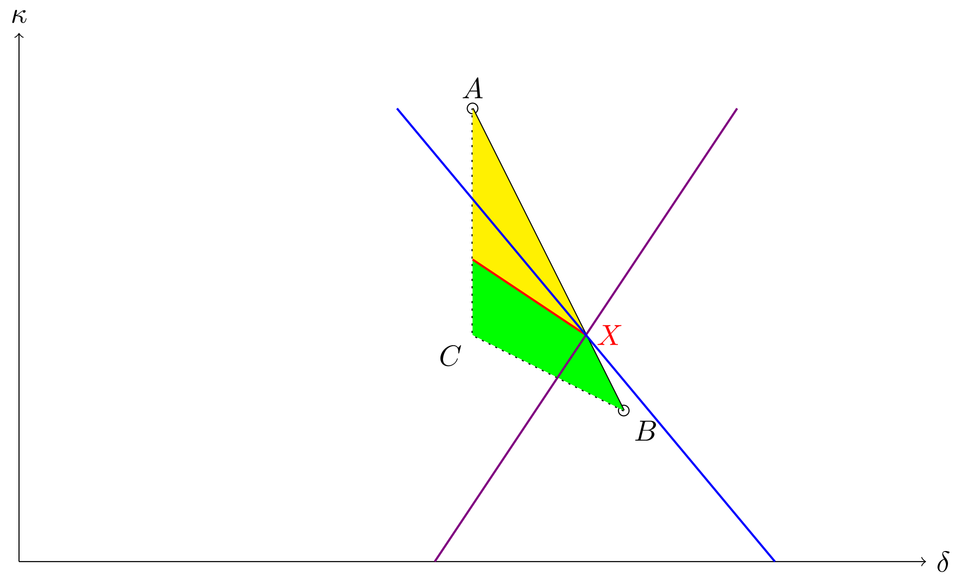
$$Q \lesssim \frac{R^{3/2}}{D^2} = \frac{R}{D} \cdot \frac{R^{1/2}}{D} \ll \frac{R}{D}$$

which is (14) up to a factor of $(\log Q)^{1/2}$. So when we carry out linear programming, we may as well just focus on the assumptions (15), (16) and (24) on Q and D and check that the final solution does satisfy (14). Writing $D = R^\delta$ and $Q = R^\kappa$, we see that the assumptions (15), (16) and (24) translate into

$$2\delta + \kappa \le \frac{3}{2}, \quad \delta > \frac{1}{2}, \quad \text{and} \quad \delta + 2\kappa > 1.$$

These inequalities describe, in the (δ, κ) plane, the union of the open triangle ABC, together with the open edge (AB), where

$$A = (1/2, 1/2), \quad B = (2/3, 1/6), \quad \text{and} \quad C = (1/2, 1/4).$$



To maximize the right hand side of (26), we need to maximize

$$L(\delta,\kappa) := 2\delta + \frac{3-n}{2}\kappa + \min\{(n-1)\delta + n\kappa - (n-1), 0\}.$$

The line $(n-1)\delta + n\kappa - (n-1) = 0$ (pictured in red above) cuts the triangle ABC into two parts, one where $(n-1)\delta + n\kappa - (n-1) \leq 0$ and

$$L(\delta,\kappa) = L_1(\delta,\kappa) := (n+1)\delta + \frac{n+3}{2}\kappa - (n-1)$$

(let's call this region P_1; it is shaded in green above); and another where $(n-1)\delta + n\kappa - (n-1) > 0$ and

$$L(\delta,\kappa) = L_2(\delta,\kappa) := 2\delta + \frac{3-n}{2}\kappa$$

(let's call this region P_2; it is shaded in yellow above). Let

$$X = \left(\frac{n+2}{2(n+1)}, \frac{n-1}{2(n+1)}\right)$$

be the intersection of the red line with the line segment AB. The slope of the level sets of L_1 (namely $-\frac{2(n+1)}{n+3}$, as for the blue line in the picture) is sandwiched between the slope of the red line (namely $-\frac{n}{n-1}$) and the slope of XB (namely -2), and when $n \geq 2$, the level sets of L_2 (such as the purple line in the picture) makes an angle $\arctan(\frac{4}{n-3})$ with the positive δ axis, which is smaller than the angle that AX makes with the positive δ axis (namely $\arctan(-2)$; here arctan is defined to have range $[0,\pi)$). As a result, the maximum of $L(\delta,\kappa)$ on both P_1 and P_2 occurs at the point $(\delta,\kappa) = X$. This corresponds precisely to the choice of $D = D(R)$ and $Q = Q(R)$ in (27).

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