# A new twist of the Carleson operator

Po-Lam Yung

University of Oxford

July 16, 2014

# Introduction

- Joint work with Lillian Pierce
- Our main theorem concerns a variant of the Carleson operator, which was first studied in relation to pointwise a.e. convergence of Fourier series.
- Part 1: Introduction, and statement of our main theorem
- Part 2: A taste of time-frequency analysis
- Part 3: Some aspects of the proof of our main theorem

# Part 1: Introduction, and statement of our main theorem

# Motivation: Pointwise a.e. convergence of Fourier series

- Given an integrable function $f$ on $[0,1]$, we associate to $f$ its Fourier series
$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x},$$
where
$$\widehat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x} dx.$$
- Fourier had the insight that perhaps "every" function can be expanded as its Fourier series.
- It took mathematicians quite some time to clarify and make precise this claim of Fourier.

- Let

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x}.$$

- If $f$ is $C^\alpha$ for some $\alpha > 0$, then

$$S_N f(x) \to f(x) \quad \text{for every } x \in [0,1].$$

  The corresponding result is false if $f$ is merely continuous.
- If $f$ is in $L^p$, $1 < p < \infty$, then

$$S_N f \to f \quad \text{in } L^p \text{ norm}.$$

  The corresponding result is false if $p = 1$.
- Things can go really wrong with $L^1$: there exist $L^1$ functions whose Fourier series diverges everywhere.
- Question: what if one is only interested in almost everywhere convergence of Fourier series?

- Theorem (Carleson 1966, Hunt 1967):
  If $f$ is in $L^p$, $1 < p < \infty$, then

$$S_N f(x) \to f(x) \quad \text{for almost every } x \in [0,1].$$

- The proof proceeds via approximating a function in $L^p$ by smooth functions.
- The key then is to control a certain maximal operator, which has since been called the Carleson operator:

$$f(x) \mapsto \sup_{N \in \mathbb{N}} |S_N f(x)|.$$

  This operator maps $L^p$ boundedly into $L^p$, for all $1 < p < \infty$.
  ($p = 2$: Carleson's theorem; other values of $p$: result of Hunt)
- Later C. Fefferman (1973) and Lacey-Thiele (2000) gave very interesting alternative proofs. The techniques developed have since evolved into a field called *time-frequency analysis*.

# A variant on $\mathbb{R}^n$

- If we are given instead an $L^p$ function $f$ on $\mathbb{R}$, we could also ask whether
$$\int_{-N}^{N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi$$
converges to $f(x)$ for almost every $x \in \mathbb{R}$.
- This would require one to study the operator
$$f(x) \mapsto \sup_{N>0} \left| \int_{-N}^{N} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \right|.$$
- Modulo some trivial operators, one has then to bound the operator
$$f(x) \mapsto \sup_{\lambda \in \mathbb{R}} \left| \text{p.v.} \int_{\mathbb{R}} f(x-y) \frac{1}{y} e^{2\pi i \lambda y} dy \right|.$$
- Note p.v. $\frac{1}{y}$ is the simplest Calderon-Zygmund kernel on $\mathbb{R}$.

- More precisely, by a Calderon-Zygmund kernel on $\mathbb{R}^n$, we mean a distribution $K$ that agrees with a smooth function $K_0(y)$ outside the origin, with

$$|K_0(y)| \leq C|y|^{-n}, \quad |\nabla^\alpha K_0(y)| \leq C|y|^{-(n+|\alpha|)}, \quad \widehat{K} \in L^\infty.$$

- e.g. $K = \text{p.v.}\dfrac{y_j}{|y|^{n+1}}$ on $\mathbb{R}^n$.
- Sjölin formulated a version of Carleson's operator on $\mathbb{R}^n$. It is given by

$$Cf(x) := \sup_{\lambda \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)K(y)e^{i\lambda \cdot y}\,dy \right|,$$

where $K(y)$ is a fixed Calderon-Zygmund kernel on $\mathbb{R}^n$.
- Theorem (Sjölin): $C : L^p \to L^p$, for all $1 < p < \infty$.

# A variant with polynomial phases

- Stein-Wainger (2001) initiated the study of a variant of the Carleson operator, where the phase $\lambda \cdot y$ in the exponential $e^{i\lambda \cdot y}$ is replaced by a real polynomial of higher degree in $y$.
- More precisely, let $\mathcal{P}_d$ be the set of all real polynomials of degrees $\leq d$ on $\mathbb{R}^n$, and $\mathcal{P}'_d$ be the set of all polynomials in $\mathcal{P}_d$ that vanishes at the origin to order $\geq 2$. Define

$$C_d f(x) = \sup_{P \in \mathcal{P}'_d} \left| \int_{\mathbb{R}^n} f(x-y) K(y) e^{iP(y)} dy \right| .$$

  Then Stein-Wainger proved that $C_d \colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and all $d$, using stationary phases.
- This does not cover the Carleson's Theorem stated above; later, using time-frequency analysis, Lie (2009) improved Stein-Wainger's result when $n = 1$ by replacing $\mathcal{P}'_d$ by the bigger class $\mathcal{P}_d$, thereby obtaining a true generalization of Carleson's Theorem.

# A variant with Radon transform

- In joint work with Lillian Pierce, we study a variant of the theorem of Stein-Wainger, where an additional Radon transform is involved. For concreteness, we will work with the Radon transform along the paraboloid in $\mathbb{R}^3$.
- Recall that if $K(y)$ is a Calderon-Zygmund kernel on $\mathbb{R}^2$, then given a function $f(x,t)$ on $\mathbb{R}^3$, where $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$, the singular Radon transform of $f$ along the paraboloid is given by

$$Rf(x,t) := \int_{\mathbb{R}^2} f(x-y, t-|y|^2) K(y) dy.$$

- The operator we study takes the form

$$\mathfrak{C}_d f(x,t) := \sup_{P \in \mathcal{Q}'_d} \left| \int_{\mathbb{R}^2} f(x-y, t-|y|^2) K(y) e^{iP(y)} dy \right|,$$

where $\mathcal{Q}'_d$ is a class of polynomials of degree $\leq d$ on $\mathbb{R}^2$ to be specified.

- More precisely, fix a positive integer $d \geq 2$.

  For $2 \leq j \leq d$, fix some polynomial $p_j(y)$ on $\mathbb{R}^2$, that is homogeneous of degree $j$, and that has real coefficients.

  Let $\mathcal{Q}'_d = \left\{ \sum_{j=2}^{d} \lambda_j p_j(y) \colon \lambda_j \in \mathbb{R} \text{ for all } j \right\} \subset \mathcal{P}'_d$

  $K(y) =$ a Calderon-Zygmund kernel on $\mathbb{R}^2$

## Theorem 1 (Pierce-Y.)

*If $p_2(y) \not\equiv C|y|^2$ for any non-zero constant $C$, then the operator*

$$\mathfrak{C}_d f(x,t) := \sup_{P \in \mathcal{Q}'_d} \left| \int_{\mathbb{R}^2} f(x-y, t-|y|^2) K(y) e^{iP(y)} \, dy \right|$$

*is bounded on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$ and all $d$.*

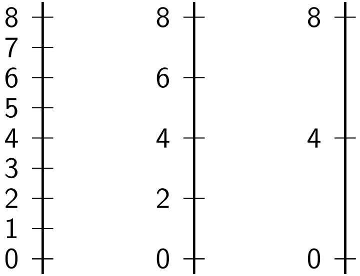
# Part 2: A taste of time-frequency analysis

# Heuristics of the Proof of Carleson's theorem

- Let
$$H_N f(x) = \sum_{n>N} \widehat{f}(n) e^{2\pi i n x},$$
$$\mathcal{C}f(x) = \sup_{N\geq 0} |H_N f(x)|.$$
- Then the original theorem of Carleson is equivalent to the statement that
$$\mathcal{C} : L^2 \to \text{weak-}L^2.$$
- Let's first understand $H_N f$.
- Step 1: an orthogonal decomposition in frequency space
- Step 2: an orthogonal decomposition in the physical space.

# A first decomposition: decomposition in frequency space

- A dyadic interval is an interval of the form $[(m-1)2^k, m2^k)$ for some integers $m$ and $k$.
- For each fixed $k$, the dyadic intervals of length $2^k$ tile the real axis.

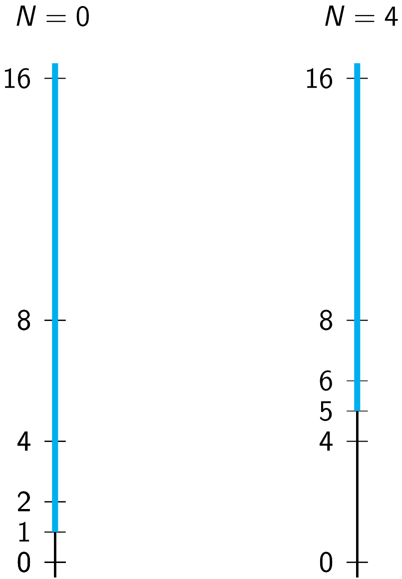


- Each dyadic interval $\omega$ has a parent $\omega^*$, which is the unique dyadic interval that contains $\omega$, and has twice the length of $\omega$.
- Given any integer $N \geq 0$, one can decompose $[N+1, \infty)$ into a disjoint union of dyadic intervals:

$$[N+1, \infty) = \bigcup_{k=0}^{\infty} \omega_{k,N}$$

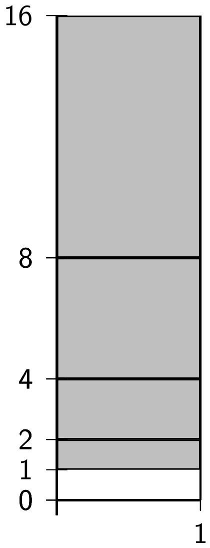
  where each $\omega_{k,N}$ is either a empty set, or a dyadic interval of length $2^k$ with $\omega_{k,N}^* \not\subset [N+1, \infty)$.
- Examples

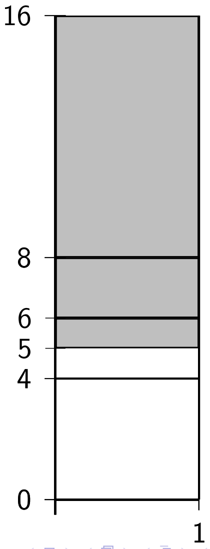
# Decomposition of $[N+1, \infty)$



- Let
$$\Pi_{\omega_{k,N}} f(x) = \sum_{n \in \omega_{k,N}} \widehat{f}(n) e^{2\pi i n x}.$$
- Then
$$H_N f = \sum_{k=0}^{\infty} \Pi_{\omega_{k,N}} f.$$
- It is now helpful to draw a picture of the *time-frequency* plane (or the $x$-$n$ plane), to illustrate where (morally speaking) each term is supported.

$$H_0 f = \sum_{k=0}^{\infty} \Pi_{\omega_{k,0}} f, \qquad H_4 f = \sum_{k=0}^{\infty} \Pi_{\omega_{k,4}} f.$$



# A second decomposition: decomposition in physical space

- For $k \geq 0$, $\ell = 1, \ldots, 2^k$, let $I_{k,\ell} = [(\ell - 1)2^{-k}, \ell 2^{-k})$, so that

$$[0,1) = \bigcup_{\ell=1}^{2^k} I_{k,\ell}.$$

Also let

$$\psi_{k,\ell} = 2^{k/2} \chi_{I_{k,\ell}}(x).$$

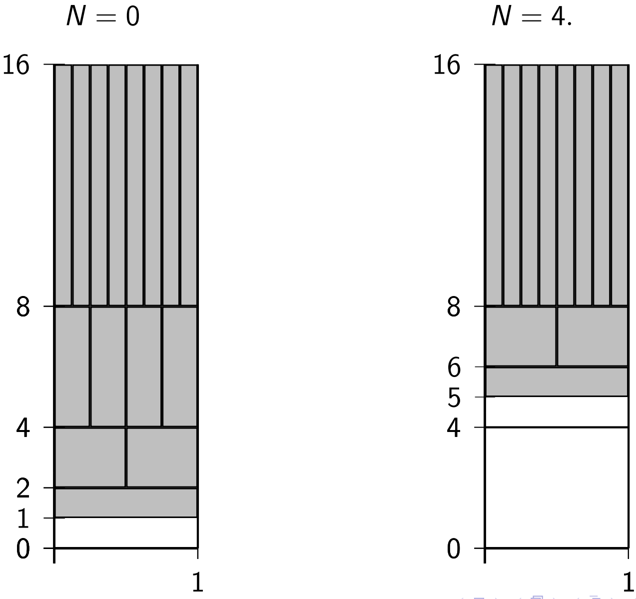
Then since $\Pi_{\omega_{k,N}} f$ is roughly constant on a physical scale $2^{-k}$,

$$\Pi_{\omega_{k,N}} f \simeq \sum_{\ell=1}^{2^k} \langle \Pi_{\omega_{k,N}} f, \psi_{k,\ell} \rangle \psi_{k,\ell} \simeq \sum_{\ell=1}^{2^k} \langle f, \Pi_{\omega_{k,N}} \psi_{k,\ell} \rangle \Pi_{\omega_{k,N}} \psi_{k,\ell},$$

so that

$$H_N f \simeq \sum_{k=0}^{\infty} \sum_{\ell=1}^{2^k} \langle f, \Pi_{\omega_{k,N}} \psi_{k,\ell} \rangle \Pi_{\omega_{k,N}} \psi_{k,\ell}.$$

$$H_N f \simeq \sum_{k=0}^{\infty} \sum_{\ell=1}^{2^k} \langle f, \Pi_{\omega_{k,N}} \psi_{k,\ell} \rangle \Pi_{\omega_{k,N}} \psi_{k,\ell}.$$



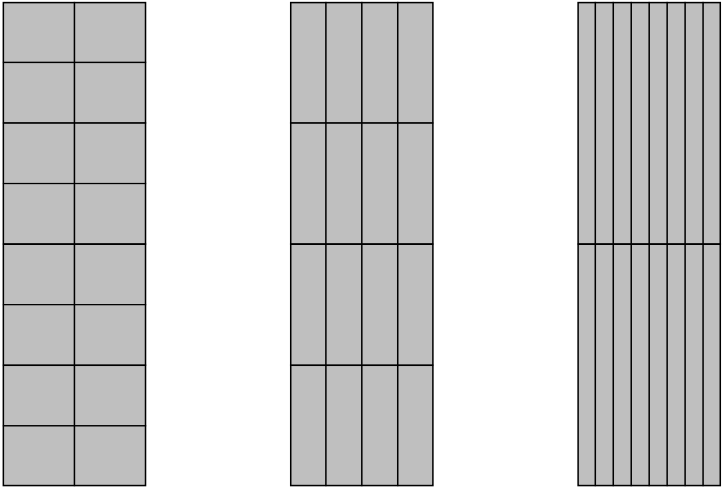
- A tile is a dyadic rectangle (i.e. a product of two dyadic intervals) in the time-frequency plane with area 1.
- We just saw that $H_N f$ is essentially a sum over tiles; essentially any tile could arise in the decomposition of $H_N f$ for a suitable $N$.
- Therefore analyzing the Carleson operator

$$\mathcal{C}f(x) = \sup_{N \geq 0} |H_N f(x)|$$

  forces one to consider essentially the set of all tiles in the time-frequency plane.
- But that is a lot of tiles!
- For each $k \in \mathbb{N}$, the set of tiles whose dimensions are $2^k \times 2^{-k}$ tile the time-frequency plane:

# Tiling with tiles of sizes $2^{-k} \times 2^{k}$

$k = 1$ $k = 2$ $k = 3$



- Fortunately, not all tiles contribute, and one can organize the ones that contribute into unions of trees.
- The Carleson operator is then a sum over trees, and each tree would give rise to an operator that is like the Hilbert transform, that is bounded on any $L^p$.
- The key is then to exhibit orthogonality between operators corresponding to different trees, so as to prove that the full Carleson operator is weak-type $(p, p)$.
- Carleson, Fefferman, and Lacey and Thiele each has a different way of exhibiting this crucial orthogonality; we will not enter into details here.

- On the other hand, Stein and Wainger's approach to their theorem does not require the use of all this machinery from time-frequency analysis.
- Instead, it is based on a simple, but very clever observation, that allows them to exploit the additional oscillations they have in the polynomial phases they consider.
- Our approach to our main theorem is a refinement of that, to take into account the presence of a Radon transform along the paraboloid.

Part 3: Some aspects of proof of our main theorem

# Proof of our main Theorem 1

- Recall our polynomial Carleson operator along the paraboloid:

$$\mathfrak{C}_d f(x,t) := \sup_{P \in \mathcal{Q}'_d} \left| \int_{\mathbb{R}^2} f(x-y, t-|y|^2) K(y) e^{iP(y)} dy \right|.$$

- We want to prove that it is bounded on $L^2(\mathbb{R}^3)$.
- Proof uses the method of stationary phases (rather than time-frequency analysis).
- In fact, when $|y|$ is small, $e^{iP(y)} \simeq 1$. This suggests that one should decompose $K(y)$ dyadically:

$$K(y) = \sum_{k=-\infty}^{\infty} 2^{-2k} \eta^{(k)}(2^{-k} y),$$

and estimate the terms with small $k$ (say $k \leq k(P)$) with a maximal truncated singular Radon transform.

- Thus we are reduced to the terms with $k$ large:

$$\begin{aligned} k \text{ large} \quad &\to \quad \text{Integrating over large values of } |y| \\ &\to \quad \text{The phase } P(y) \text{ oscillates rapidly} \\ &\to \quad \text{Decay in } k \end{aligned}$$

  This then allows us to sum over large $k$ in the previous slide, and prove our main theorem.

- Notation: If $P$ is a polynomial in $y$, say

$$P(y) = \sum_{0 \le |\alpha| \le d} \lambda_\alpha y^\alpha,$$

  then the isotropic norm of $P$ is defined by

$$\|P\| = \sum_{|\alpha| \ge 1} |\lambda_\alpha|.$$

## Theorem 2 (Pierce-Y.)

*Let $\eta$ be a $C^1$ function supported in the unit ball, and $d \in \mathbb{N}$. For each polynomial $P \in \mathcal{Q}'_d$ and each $k \in \mathbb{Z}$, define*

$$\mathcal{I}_k^P f(x,t) = \int_{\mathbb{R}^2} f(x-y, t-|y|^2) 2^{-2k} \eta(2^{-k}y) e^{iP(2^{-k}y)} dy.$$

*Then there is some $\delta_0 > 0$ such that for all $r \geq 1$, the operator*

$$M_r f(x,t) := \sup_{k \in \mathbb{Z}} \sup_{\substack{P \in \mathcal{Q}'_d \\ \|P\| \simeq r}} |\mathcal{I}_k^P f(x,t)|$$

*is bounded on $L^2(\mathbb{R}^3)$ with norm $\lesssim r^{-\delta_0}$.*

- Since $M_r$ is pointwisely bounded by the maximal Radon transform of $f$, it is certainly bounded on $L^2$. The key here is the decay in $r$ of the norm of the operator when $r$ is large.

# A toy model for Theorem 2

- To illustrate our method and to understand some of the difficulties involved, we consider the following toy problem.
- Let's fix $k = 0$ and consider

$$M_{r,0}f(x,t) := \sup_{\substack{P \in \mathcal{Q}'_d \\ \|P\| \simeq r}} |\mathcal{I}_0^P f(x,t)|.$$

## Proposition (Pierce-Y.)

*There exists $\delta > 0$ such that for all $r \geq 1$, $M_{r,0}$ is bounded on $L^2(\mathbb{R}^3)$, with norm $\lesssim r^{-\delta}$.*

- Again the key is to gain the decay in $r$ of the norm of the operator. To do so we use stopping times and $TT^*$.

- Recall $M_{r,0}f(x,t) = \sup_P |\mathcal{I}_0^P f(x,t)|$.
- At each point $(x,t)$, this supremum is almost attained at a possibly different polynomial $P$, say $P_{(x,t)}$.
- Consider the linear operator

$$Tf(x,t) := \mathcal{I}_0^{P_{(x,t)}} f(x,t).$$

  It suffices to show that $T$ is bounded on $L^2$ with norm $\lesssim r^{-\delta}$.
- Since $T$ is linear, it suffices to prove that $TT^*$ is bounded on $L^2$, with norm $\lesssim r^{-2\delta}$. This is desirable since the kernel of $TT^*$ exhibits more cancellation than that of $T$:

- In fact,

$$Tf(x,t) = \int_{\mathbb{R}^2} f(x-y, t-|y|^2)\eta(y)e^{iP_{(x,t)}(y)}dy$$

$$T^*f(x,t) = \int_{\mathbb{R}^2} f(x+z, t+|z|^2)\eta(z)e^{iP_{(x+z,t+|z|^2)}(z)}dz$$

so $TT^*f(x,t)$ is given as an integral over 4-dimensions.

- One can then (almost) write $TT^*f(x,t)$ as a 3-dimensional convolution against a convolution kernel; the kernel will then be a 1-dimensional (oscillatory) integral.
- In fact,

$$TT^*f(x,t) = \int_{\mathbb{R}^2\times\mathbb{R}} f(x-u, t-|u|^2-2|u|\tau)K^\sharp_{P_1,P_2}(u,\tau)dud\tau,$$

for a suitable kernel $K^\sharp_{P_1,P_2}$, where

$$P_1 := P_{(x,t)}, \quad P_2 := P_{(x-u,t-|u|^2-2|u|\tau)}.$$

- The kernel $K^{\sharp}_{P_1,P_2}$ is defined in terms of an oscillatory integral:

$$K^{\sharp}_{P_1,P_2}(u,\tau) := \int_{\mathbb{R}} e^{iP_1(u+z)-iP_2(z)}\eta(u+z)\eta(z)d\sigma,$$

where $z = (z_1, z_2)$ is defined by $z_1 = \frac{u_1\tau+u_2\sigma}{|u|}$, $z_2 = \frac{u_2\tau-u_1\sigma}{|u|}$.

- Note that while $Tf(x,t)$ is given as an integral over 2 dimensions, the kernel $K^{\sharp}_{P_1,P_2}$ is only a 1-dimensional integral.

  $\rightarrow$ Less oscillations in the integral defining $K^{\sharp}_{P_1,P_2}$

  $\rightarrow$ so method of $TT^*$ is less effective in the Radon case.

- But we still hope to gain something non-trivial from the oscillatory nature of the integral defining $K^{\sharp}_{P_1,P_2}$.

- First we have a trivial bound:

$$|K^{\sharp}_{P_1,P_2}(u,\tau)| \lesssim 1, \text{ and is supported where } |u|, |\tau| \lesssim 1.$$

- As a result,

$$|TT^*f(x,t)| \lesssim \int_{\mathbb{R}^2\times\mathbb{R}} |f|(x-u, t-|u|^2-2|u|\tau)\chi_{B_1}(u)\chi_{B_1}(\tau)dud\tau$$

  which is bounded on $L^2$ with norm $\lesssim 1$; we need to do better.
- Using the method of stationary phases, we improve the above trivial bound on $K^{\sharp}_{P_1,P_2}$.
  (It is only here that we use our assumption that our polynomial phases $P(y)$ are from our specific class $\mathcal{Q}'_d$.)

### Lemma

*If $P_1$, $P_2 \in \mathcal{Q}'_d$ with $\|P_1\|, \|P_2\| \simeq r$, then there exists*

- *some $\delta > 0$ depending only on $d$,*
- *a set $E(P_1)$ depending only on $P_1$, and*
- *a family of sets $F(P_1, u)$ depending only on $P_1$ and $u$,*

*such that*

$$|K^\sharp_{P_1,P_2}(u,\tau)| \lesssim r^{-2\delta}\chi_{B_1}(u)\chi_{B_1}(\tau) + \chi_{E(P_1)}(u)\chi_{B_1}(\tau) + \chi_{B_1}(u)\chi_{F(P_1,u)}(\tau).$$

*Furthermore, the sets $E(P_1)$ and $F(P_1, u)$ has small measures:*

$$|E(P_1)| \lesssim r^{-8\delta}, \quad \text{and} \quad |F(P_1,u)| \lesssim r^{-8\delta} \quad \text{for all } u.$$

- Then we have the following bound for $|TT^*f(x,t)|$:

$$\int_{\mathbb{R}^2\times\mathbb{R}} |f|(x-u, t-|u|^2-2|u|\tau)\left[r^{-2\delta}\chi_{B_1}(u)\chi_{B_1}(\tau) + \chi_{E(P_{(x,t)})}(u)\chi_{B_1}(\tau) + \chi_{B_1}(u)\chi_{F(P_{(x,t)},u)}(\tau)\right] du d\tau.$$

- This is a sum of three terms, and they are all bounded on $L^2$ with a small norm $\lesssim r^{-2\delta}$ (e.g. by interpolation between a good $L^\infty \to L^\infty$ bound, and a trivial $L^1 \to L^1$ bound).
- It is important here that we get small exceptional sets $E(P_1)$ and $F(P_1, u)$ that are independent of $P_2$: otherwise we then need to estimate things like

$$\int_{\mathbb{R}^2\times\mathbb{R}} |f|(x-u, t-|u|^2-2|u|\tau)\chi_{E(P_{(x,t)}, P_{(x-u, t-|u|^2-2|u|\tau)})}(u)\chi_{B_1}(\tau) du d\tau,$$

which we cannot quite estimate.

# Back to Theorem 2: bounding a square function

- At this point, we observe that if one wants to apply the same argument to prove the desired bound for our original operator

$$M_r f(x,t) = \sup_k \sup_P |\mathcal{I}_k^P f(x,t)|,$$

one would naively also adopt a stopping time in $k$, and do a $TT^*$, because otherwise one does not have a linear operator $T$, and linearity is crucial for the application of $TT^*$.

- Unfortunately, stopping times and the method of $TT^*$ are not good for bounding the supremum in $k$, as is known when people tried to bound the (ordinary) maximal Radon transform along the paraboloid.
- So we proceed differently, by introducing a smoother variant of our maximal operator, and estimating a square function.

- Recall that our operator $M_r$ is given by

$$M_r f(x,t) = \sup_k \sup_P \left| \mathcal{I}_k^P f(x,t) \right|,$$

$$\mathcal{I}_k^P f(x,t) = \int_{\mathbb{R}^2} f(x-y, t-|y|^2) e^{iP(\frac{y}{2^k})} \eta(\frac{y}{2^k}) \frac{dyds}{2^{2k}}.$$

- The key is to compare $\mathcal{I}_k^P f(x,t)$ to a smoother variant, which we call $\mathcal{J}_k^P f(x,t)$.
- Let now $\zeta \in C_c^\infty[-1,1]$ be fixed, with $\int_{\mathbb{R}} \zeta(s)ds = 1$. Define

$$\mathcal{J}_k^P f(x,t) := \int \int_{\mathbb{R}^2 \times \mathbb{R}} f(x-y, t-s) e^{iP(\frac{y}{2^k})} \eta(\frac{y}{2^k}) \zeta(\frac{s}{2^{2k}}) \frac{dyds}{2^{4k}}.$$

- Note an important cancellation property between $\mathcal{I}_k^P$ and $\mathcal{J}_k^P$:

$$\int_{\mathbb{R}} (\mathcal{I}_k^P - \mathcal{J}_k^P) f(x,t) dt = 0 \quad \text{for every } x.$$

- Now

$$\begin{aligned} M_r f &= \sup_k \sup_P |\mathcal{I}_k^P f| \\ &\leq \sup_k \sup_P |\mathcal{J}_k^P f| + \sup_k \sup_P |(\mathcal{I}_k^P - \mathcal{J}_k^P) f| \\ &\leq \sup_k \sup_P |\mathcal{J}_k^P f| + \left( \sum_{k \in \mathbb{Z}} \sup_P |(\mathcal{I}_k^P - \mathcal{J}_k^P) f|^2 \right)^{1/2} . \end{aligned}$$

- The first term is known to be bounded on $L^2$ with norm $\lesssim r^{-2\delta}$, by an easy modification of Stein and Wainger's argument. (No Radon behavior for $\mathcal{J}_k^P$!)
- Thus to show that $M_r$ is bounded on $L^2$ with the desired norm, it suffices to prove the same for the second term:

$$S_r f(x,t) := \left( \sum_{k \in \mathbb{Z}} \sup_{\substack{P \in \mathcal{Q}_d' \\ \|P\| \simeq r}} |(\mathcal{I}_k^P - \mathcal{J}_k^P) f(x,t)|^2 \right)^{1/2} .$$

- Square function again:

$$S_r f(x,t) := \left( \sum_{k\in\mathbb{Z}} \sup_P |(\mathcal{I}_k^P - \mathcal{J}_k^P) f(x,t)|^2 \right)^{1/2}.$$

- Recall that $\mathcal{I}_k^P - \mathcal{J}_k^P$ satisfies a cancellation property
$\rightarrow$ Morally speaking, $\mathcal{I}_k^P - \mathcal{J}_k^P$ should act only on the part of $f$ with 'frequency' $\simeq 2^{-k}$.
- So by decomposing $f$ in the 'frequency' space, one can hope to carry out the sum in $k$ in the definition of $S_r$.
- More precisely, we hope to be able to find Littlewood-Paley projections $\Delta_j$, such that one can decompose any $f \in L^2(\mathbb{R}^3)$ as

$$f = \sum_{j\in\mathbb{Z}} \Delta_j F_j, \qquad \sum_{j\in\mathbb{Z}} \|F_j\|_{L^2}^2 \lesssim \|f\|_{L^2}^2,$$

and $\Delta_j f$ has 'frequency' $2^{-j}$.

- More importantly, we hope to choose Littlewood-Paley projections $\Delta_j$, such that the following holds:

### Theorem 3 (Pierce-Y.)

*There exists $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for all $r \geq 1$ and $j, k \in \mathbb{Z}$,*

$$M_{r,j,k}f(x,t) := \sup_{\substack{P \in \mathcal{Q}'_d \\ \|P\| \simeq r}} |(\mathcal{I}_k^P - \mathcal{J}_k^P)\Delta_j f(x,t)|$$

*is bounded on $L^2(\mathbb{R}^3)$, with norm $\lesssim 2^{-\varepsilon_0|j-k|} r^{-\delta_0}$.*

- Note the additional decay when $|j-k|$ is large.
- When $2^{-j} \leq 2^{-k}$, use the cancellation property of $\mathcal{I}_k^P - \mathcal{J}_k^P$;
- When $2^{-j} > 2^{-k}$, one needs a cancellation property from $\Delta_j$.

- The question now is then two-fold, namely:

(1) What cancellation property do we require of $\Delta_j$?
(2) How can we exploit that cancellation property?

- The answer to the first question has a nice simple form: We just take $\Delta_j$ to be a Littlewood-Paley projection in the last variable:

$$\Delta_j f(x,t) = \int_{\mathbb{R}} f(x, t-s) \frac{1}{2^{2j}} \Delta\left(\frac{s}{2^{2j}}\right) ds.$$

- But this is a bit tricky to use, in the presence of the Radon transform:

- e.g. when $0 = j < k$, let's try to gain some cancellation between $j$ and $k$ from $\mathcal{I}_k^P \Delta_j f$. Note

$$\mathcal{I}_k^P \Delta_0 f(x,t) = \int_{\mathbb{R}^3} f(x-y, t-s) \frac{1}{2^{2k}} \eta(\frac{y}{2^k}) e^{iP(\frac{y}{2^k})} \Delta(s-|y|^2) dyds.$$

- Even if say $\int_{\mathbb{R}} \Delta(s) ds = 0$, or $\Delta(s) = \tilde{\Delta}'(s)$ for some function $\tilde{\Delta}$, it is not immediate how one can gain $2^{-k}$ by integrating by parts.
- This difficulty only arises from the presence of Radon behaviour in $\mathcal{I}_k^P$!
- Fortunately, one can succeed by using yet another $TT^*$ argument here.