A formula for Sobolev seminorms involving weak-L^p

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Introduction

- Fix $n \geq 1$ and $1 \leq p < \infty$.
- Goal: Compute the $\dot{W}^{1,p}$ (semi)norm for $u \in C_c^\infty(\mathbb{R}^n)$, i.e.

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |\nabla u|^p dx \Big)^{1/p}.$$

- In joint work with Haïm Brezis and Jean Van Schaftingen, we established a new formula for these (semi)norms that involves only difference quotients and no gradients.
- The formula was motivated by a weak-L^p estimate on the product space $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, which allowed us to fix up certain Gagliardo-Nirenberg interpolations involving $\dot{W}^{1,1}$.
- In joint work with Qingsong Gu, we proved a similar formula for the L^p norm of any function $u \in L^p(\mathbb{R}^n)$.
- Finally, with Andreas Seeger, Brian Street and Jean Van Schaftingen, we studied similar questions regarding fractional order Besov spaces, which allowed us to clarify the nature of the aforementioned fix to Gagliardo-Nirenberg interpolations.

L^p versus weak-L^p

- For $1 \leq p < \infty$, if $f \in L^p(\mathbb{R}^n)$, then for every $\lambda > 0$,

$$\|f\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |f(x)|^p dx \geq \lambda^p |\{x \in \mathbb{R}^n \colon |f(x)| > \lambda\}|.$$

  In particular, if $f \in L^p(\mathbb{R}^n)$, then

$$\sup_{\lambda > 0} \left(\lambda |\{x \in \mathbb{R}^n \colon |f(x)| > \lambda\}|^{1/p} \right) < \infty$$

  but the converse is not necessarily true.
- If f is measurable on $\mathbb{R}^n$ and the supremum above is finite, then f is said to be in weak-L^p. Its weak-L^p norm is defined as the above supremum, and denoted by $\|f\|_{L^{p,\infty}(\mathbb{R}^n)}$.
- Example: $f(x) = |x|^{-n/p}$ is in weak-L^p on $\mathbb{R}^n$, because

$$|\{x \in \mathbb{R}^n \colon |x|^{-n/p} > \lambda\}| = |B(0, \lambda^{-p/n})| = \lambda^{-p} |B(0,1)|.$$

  It is not in $L^p(\mathbb{R}^n)$, because $\int_{\mathbb{R}^n} |f|^p dx = \int_{\mathbb{R}^n} |x|^{-n} dx = +\infty$.

A formula for $\|\nabla u\|_{L^p(\mathbb{R}^n)}$

Theorem (Brezis, Van Schaftingen, Yung)

Let $n \geq 1$, $1 \leq p < \infty$ and $u \in C_c^\infty(\mathbb{R}^n)$. Define the modified difference quotient for u on $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ by

$$Qu(x,y) := \frac{|u(y)-u(x)|}{|y-x|} \frac{1}{|y-x|^{\frac{n}{p}}} = \frac{|u(y)-u(x)|}{|y-x|^{1+\frac{n}{p}}}.$$

For $\lambda > 0$, define the superlevel set of Qu by

$$E_\lambda := \Big\{(x,y) \in \mathbb{R}^{2n} \colon Qu(x,y) > \lambda\Big\}.$$

Then

$$\lim_{\lambda \to \infty} \left(\lambda |E_\lambda|^{1/p}\right) = \left(\frac{k(p,n)}{n}\right)^{1/p} \|\nabla u\|_{L^p(\mathbb{R}^n)},$$

where $k(p,n) := \int_{\mathbb{S}^{n-1}} |e \cdot \omega|^p d\omega$ and $e \in \mathbb{S}^{n-1}$.

- Below we discuss a heuristic proof of this theorem, and more importantly how this formula came by.

A heuristic proof (in 1 dimension)

- Let $u \in C_c^\infty(\mathbb{R})$. We want to see why one can compute $\|u'\|_{L^p(\mathbb{R})}$ by understanding $\lim_{\lambda\to\infty}\left(\lambda|E_\lambda|^{1/p}\right)$ where

$$E_\lambda := \left\{(x,y) \in \mathbb{R}^2 \colon \frac{|u(y)-u(x)|}{|y-x|^{1+\frac{1}{p}}} > \lambda\right\}.$$

- Note that for λ large, $(x,y) \notin E_\lambda$ unless $|y-x|$ is small.
- When $|y-x|$ is small,

$$\frac{|u(y)-u(x)|}{|y-x|^{1+\frac{1}{p}}} \simeq \frac{|u'(x)|}{|y-x|^{\frac{1}{p}}},$$

so may hope $E_\lambda \simeq \tilde{E}_\lambda := \left\{(x,y) \in \mathbb{R}^2 \colon \frac{|u'(x)|}{|y-x|^{\frac{1}{p}}} > \lambda\right\}$; but

$$|\tilde{E}_\lambda| = \int_{\mathbb{R}} \left|\left\{y \in \mathbb{R} \colon |y-x| < \frac{|u'(x)|^p}{\lambda^p}\right\}\right| dx = 2\int_{\mathbb{R}} \frac{|u'(x)|^p}{\lambda^p} dx$$

$$\implies \qquad \lambda|\tilde{E}_\lambda|^{1/p} = 2^{1/p}\|u'\|_{L^p(\mathbb{R})} \quad \text{for all } \lambda > 0.$$

How did this modified difference quotient first arise?

- If we believe that

$$|\nabla u(x)| \simeq \frac{|u(y)-u(x)|}{|y-x|},$$

then to express $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ using a difference quotient instead of a gradient, a naive guess might be to try

$$\iint_{\mathbb{R}^{2n}} \frac{|u(y)-u(x)|^p}{|y-x|^p} dydx \quad \text{in place of} \quad \int_{\mathbb{R}^n} |\nabla u(x)|^p dx.$$

- Not working, because it doesn't scale upon $u(x) \mapsto u(tx)$.
- A proper scaling will be achieved if we consider

$$\iint_{\mathbb{R}^{2n}} \frac{|u(y)-u(x)|^p}{|y-x|^p} \frac{dydx}{|y-x|^n}$$

instead, which is $\iint_{\mathbb{R}^{2n}} Qu(x,y)^p dydx$ if

$$Qu(x,y) := \frac{|u(y)-u(x)|}{|y-x|} \frac{1}{|y-x|^{\frac{n}{p}}}.$$

Interplay between L^p and weak-L^p

- This is how one was first led to consider the modified difference quotient
$$Qu(x,y) := \frac{|u(y)-u(x)|}{|y-x|}\frac{1}{|y-x|^{\frac{n}{p}}}.$$
- But should one compute $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ via $\|Qu\|_{L^p(\mathbb{R}^{2n})}$?
- For fixed x, the factor $|y-x|^{-\frac{n}{p}}$ is only in weak-$L^p(dy)$ but not in $L^p(dy)$.
- Indeed, for this reason, if $u \in C_c^\infty(\mathbb{R}^n)$, unless u is identically zero, the L^p norm of $Qu(x,y)$ on $\mathbb{R}^{2n}$ is *always* infinite!
- Hence impossible to compute $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ via $\|Qu\|_{L^p(\mathbb{R}^{2n})}$.
- But what if we take the weak-L^p norm of Qu on $\mathbb{R}^{2n}$ instead, i.e.
$$\|Qu\|_{L^{p,\infty}(\mathbb{R}^{2n})} = \sup_{\lambda>0}\left(\lambda |E_\lambda|^{1/p}\right)$$
where as before $E_\lambda := \{(x,y) \in \mathbb{R}^{2n} \colon Qu(x,y) > \lambda\}$?

A weak-L^p estimate for the modified difference quotient

- By the previous theorem,

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \simeq_{p,n} \lim_{\lambda\to\infty} \left(\lambda |E_\lambda|^{1/p}\right) \leq \sup_{\lambda>0} \left(\lambda |E_\lambda|^{1/p}\right),$$

  so $\|\nabla u\|_{L^p(\mathbb{R}^n)} \lesssim_{p,n} \|Qu\|_{L^{p,\infty}(\mathbb{R}^{2n})}$.
- Is the reversed inequality true?

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  so $\|\nabla u\|_{L^p(\mathbb{R}^n)} \lesssim_{p,n} \|Qu\|_{L^{p,\infty}(\mathbb{R}^{2n})}$.
- Is the reversed inequality true? Yes by the following theorem.

Theorem (Brezis, Van Schaftingen, Yung)

Let $n \geq 1$ *and* $1 \leq p < \infty$. *For* $u \in C_c^\infty(\mathbb{R}^n)$, *define as before*

$$Qu(x,y) := \frac{|u(y)-u(x)|}{|y-x|} \frac{1}{|y-x|^{\frac{n}{p}}}.$$

Then

$$\|Qu(x,y)\|_{L^{p,\infty}(\mathbb{R}^{2n})} \simeq_{p,n} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

- Towards the end we will prove this theorem when $n = p = 1$.

Three applications involving $\dot{W}^{1,1}$

- The issue is that $\|\nabla u\|_{L^1(\mathbb{R}^n)}$ computes an L^1 norm, and L^1 is borderline for many purposes in harmonic analysis.
- First we introduce fractional order Besov (semi)norms on $\mathbb{R}^n$.
- Suppose $u \in C_c^\infty(\mathbb{R}^n)$ (or a Schwartz function on $\mathbb{R}^n$).
- For $s \in (0,1)$ and $1 < p < \infty$, define the Besov (semi)norm by

$$[u]_{\dot{B}^s_p(\mathbb{R}^n)} := \left(\sum_{j \in \mathbb{Z}} \|2^{js} \Delta_j u\|^p_{L^p(\mathbb{R}^n)} \right)^{1/p}$$

  where $\{\Delta_j\}_{j \in \mathbb{Z}}$ is an appropriate family of Littlewood-Paley projections on $\mathbb{R}^n$.
- It is known that for such s and p,

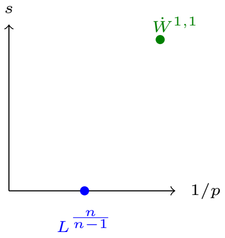
$$[u]_{\dot{B}^s_p(\mathbb{R}^n)} \simeq_{s,p,n} \left\| \frac{|u(y) - u(x)|}{|y - x|^s} \frac{1}{|y - x|^{\frac{n}{p}}} \right\|_{L^p(\mathbb{R}^{2n})}.$$

- So incidentally, the idea of taking the L^p norm of a modified difference quotient on $\mathbb{R}^{2n}$ works for fractional Besov spaces!

Application 1: Fixing a 1-d fractional Sobolev embedding

- In $\mathbb{R}^n$, Sobolev embedding for $\dot{W}^{1,1}$ says

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \lesssim_n \|\nabla u\|_{L^1(\mathbb{R}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n).$$



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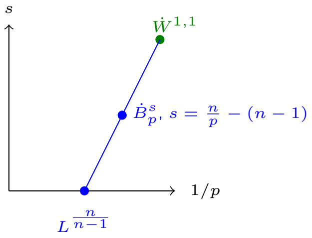
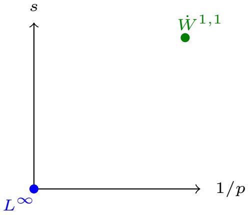
$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \lesssim_n \|\nabla u\|_{L^1(\mathbb{R}^n)} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n).$$
- What about embeddings into fractional Besov spaces?
- Scaling considerations suggest that
$$[u]_{\dot{B}^s_p(\mathbb{R}^n)} \lesssim_{s,n} \|\nabla u\|_{L^1(\mathbb{R}^n)} \quad \text{for } \frac{1}{p} = 1 - \frac{1-s}{n},\ 0 < s < 1$$
and this is true if $n \geq 2$ (Solonnikov / Van Schaftingen).
- Situation changes in 1 dimension.



- In 1 dimension we (still) have the inequality

$$\|u\|_{L^\infty(\mathbb{R})} \leq \|u'\|_{L^1(\mathbb{R})} \quad \text{for all } u \in C_c^\infty(\mathbb{R}).$$



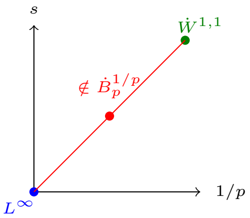
- In 1 dimension we (still) have the inequality

$$\|u\|_{L^\infty(\mathbb{R})} \leq \|u'\|_{L^1(\mathbb{R})} \quad \text{for all } u \in C_c^\infty(\mathbb{R}).$$

- Scaling considerations suggest that perhaps

$$[u]_{\dot{B}_p^{1/p}(\mathbb{R})} \lesssim_p \|u'\|_{L^1(\mathbb{R})} \quad \text{for } 1 < p < \infty$$

  but this was known to be false for all $1 < p < \infty$.



$$[u]_{\dot{B}_p^{1/p}(\mathbb{R})} \lesssim_p \|u'\|_{L^1(\mathbb{R})} \quad \text{false for } 1 < p < \infty$$

- Recall

$$[u]_{\dot{B}_p^s(\mathbb{R})} \simeq_{s,p} \left\| \frac{u(y) - u(x)}{|y - x|^{s+\frac{1}{p}}} \right\|_{L^p(\mathbb{R}^2)}.$$

$$[u]_{\dot{B}^{1/p}_p(\mathbb{R})} \lesssim_p \|u'\|_{L^1(\mathbb{R})} \quad \text{false for } 1 < p < \infty$$

- Recall

$$[u]_{\dot{B}^{1/p}_p(\mathbb{R})} \simeq_p \left\| \frac{u(y) - u(x)}{|y - x|^{\frac{2}{p}}} \right\|_{L^p(\mathbb{R}^2)}.$$

- From our earlier weak-L^p estimate for the modified difference quotient, we obtain the following substitute:

Corollary (Brezis, Van Schaftingen, Yung)

There exists an absolute constant C such that if $1 < p < \infty$ and $u \in C^\infty_c(\mathbb{R})$, then

$$\left\| \frac{u(y) - u(x)}{|y - x|^{\frac{2}{p}}} \right\|_{L^{p,\infty}(\mathbb{R}^2)} \leq C\|u'\|_{L^1(\mathbb{R})}.$$

- The case $p = 2$ was originally due to Greco and Schiattarella (2020), which inspired our current work.

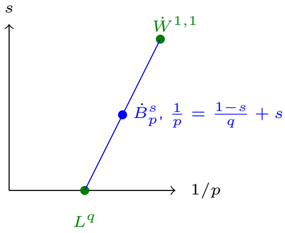
Application 2: Fixing a Gagliardo-Nirenberg interpolation

- In $\mathbb{R}^n$, if $1 \leq q < \infty$, then for $0 < s < 1$, Gagliardo-Nirenberg interpolation between $\dot{W}^{1,1}$ and L^q gives

$$[u]_{\dot{B}^s_p(\mathbb{R}^n)} \lesssim_{s,q,n} \|u\|^{1-s}_{L^q(\mathbb{R}^n)} \|\nabla u\|^s_{L^1(\mathbb{R}^n)} \quad \text{for } \frac{1}{p} = \frac{1-s}{q} + s$$

if $u \in C^\infty_c(\mathbb{R}^n)$.

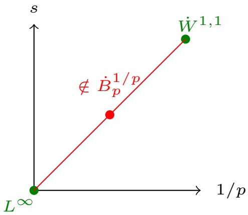
- Situation changes when $q = \infty$.



- When $q = \infty$, the anticipated inequality

$$[u]_{\dot{B}^s_p(\mathbb{R}^n)} \lesssim_{s,n} \|u\|^{1-s}_{L^q(\mathbb{R}^n)} \|\nabla u\|^s_{L^1(\mathbb{R}^n)} \quad \text{for } s = \frac{1}{p}$$

fails for every $1 < p < \infty$.



$$[u]_{\dot{B}^{1/p}_p(\mathbb{R}^n)} \lesssim_{p,n} \|u\|^{1-\frac{1}{p}}_{L^\infty(\mathbb{R}^n)} \|\nabla u\|^{\frac{1}{p}}_{L^1(\mathbb{R}^n)} \quad \text{false for } 1 < p < \infty$$

- Recall

$$[u]_{\dot{B}^{1/p}_p(\mathbb{R}^n)} \simeq_{p,n} \left\| \frac{u(y)-u(x)}{|y-x|^{\frac{1+n}{p}}} \right\|_{L^p(\mathbb{R}^{2n})}.$$

- Again from our earlier weak-L^p estimate for the modified difference quotient, we obtain the following substitute:

Corollary (Brezis, Van Schaftingen, Yung)

There exists a dimensional constant $C = C(n)$ such that if $1 < p < \infty$ and $u \in C^\infty_c(\mathbb{R}^n)$, then

$$\left\| \frac{u(y)-u(x)}{|y-x|^{\frac{1+n}{p}}} \right\|_{L^{p,\infty}(\mathbb{R}^{2n})} \leq C \|u\|^{1-\frac{1}{p}}_{L^\infty(\mathbb{R}^n)} \|\nabla u\|^{\frac{1}{p}}_{L^1(\mathbb{R}^n)}.$$

- When $n = 1$, $\|u\|_{L^\infty(\mathbb{R})} \leq \|u'\|_{L^1(\mathbb{R})}$, so we recover our previous corollary about fractional Sobolev embedding.

Proof of Corollary in Application 2

- Let $1 \leq p < \infty$. To prove

$$\left\| \frac{u(y) - u(x)}{|y - x|^{\frac{1+n}{p}}} \right\|_{L^{p,\infty}(\mathbb{R}^{2n})} \leq C \|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{1}{p}} \|\nabla u\|_{L^1(\mathbb{R}^n)}^{\frac{1}{p}},$$

  remember our weak-L^1 estimate for the modified difference quotient says

$$\left| \left\{ \frac{|u(y) - u(x)|}{|y - x|^{1+n}} > \Lambda \right\} \right| \lesssim \frac{1}{\Lambda} \|\nabla u\|_{L^1(\mathbb{R}^n)} \quad \text{for all } \Lambda > 0.$$

- But for any $\lambda > 0$,

$$\left\{ \frac{|u(y) - u(x)|}{|y - x|^{\frac{1+n}{p}}} > \lambda \right\} \subseteq \left\{ \frac{|u(y) - u(x)|}{|y - x|^{1+n}} > \frac{\lambda^p}{(2\|u\|_{L^\infty})^{p-1}} \right\}.$$

- The Lebesgue measure of the latter set is

$$\lesssim \frac{(2\|u\|_{L^\infty})^{p-1}}{\lambda^p} \|\nabla u\|_{L^1(\mathbb{R}^n)}.$$

  Rearranging this inequality gives the desired conclusion.

Application 3: Another Gagliardo-Nirenberg interpolation

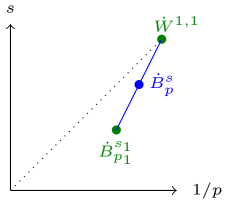
- In $\mathbb{R}^n$, we might consider Gagliardo-Nirenberg interpolation between $\dot{W}^{1,1}$ and $\dot{B}^{s_1}_{p_1}$ where $0 < s_1 < 1$ and $1 < p_1 < \infty$.
- The anticipated inequality

$$[u]_{\dot{B}^s_p(\mathbb{R}^n)} \lesssim_{\theta,s_1,p_1,n} [u]^{1-\theta}_{\dot{B}^{s_1}_{p_1}(\mathbb{R}^n)} \|\nabla u\|^{\theta}_{L^1(\mathbb{R}^n)}$$

where

$$(\frac{1}{p}, s) = (1-\theta)(\frac{1}{p_1}, s_1) + \theta(1,1)$$

holds for all $\theta \in (0,1)$ if $s_1 < \frac{1}{p_1}$
(by Cohen, Dahmen, Daubechies, DeVore) but
fails for all $\theta \in (0,1)$ if $s_1 \geq \frac{1}{p_1}$ (Brezis, Mironescu).



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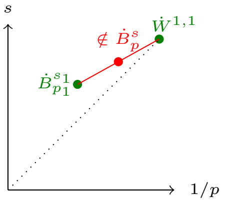

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- Recall
$$[u]_{\dot{B}^s_p(\mathbb{R}^n)} \simeq_{s,p,n} \left\| \frac{u(y)-u(x)}{|y-x|^{s+\frac{n}{p}}} \right\|_{L^p(\mathbb{R}^{2n})}.$$
- From our earlier weak-L^p estimate for the modified difference quotient, we obtain the following substitute:

Corollary (Brezis, Van Schaftingen, Yung)

There exists a dimensional constant $C = C(n)$ such that for any $0 < s_1 < 1$, $1 < p_1 < \infty$ with $s_1 \geq \frac{1}{p_1}$, and any $\theta \in (0,1)$, one has

$$\left\| \frac{u(y)-u(x)}{|y-x|^{s+\frac{n}{p}}} \right\|_{L^{p,\infty}(\mathbb{R}^{2n})} \leq C \left\| \frac{u(y)-u(x)}{|y-x|^{s_1+\frac{n}{p_1}}} \right\|^{1-\theta}_{L^{p_1}(\mathbb{R}^{2n})} \|\nabla u\|^{\theta}_{L^1(\mathbb{R}^n)}$$

if $u \in C^\infty_c(\mathbb{R}^n)$ and $(\frac{1}{p}, s) = (1-\theta)(\frac{1}{p_1}, s_1) + \theta(1,1)$.

Interlude: A formula for L^p norm

- Recall our main results: if $Qu(x,y) := \frac{|u(y)-u(x)|}{|y-x|^{1+\frac{n}{p}}}$, then

$$\|Qu(x,y)\|_{L^{p,\infty}(\mathbb{R}^n)} \simeq \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

  and

$$\lim_{\lambda\to\infty} \Big(\lambda|E_\lambda|\Big)^{1/p} = \Big(\frac{k(p,n)}{n}\Big)^{1/p} \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

  where

$$E_\lambda := \Big\{(x,y) \in \mathbb{R}^{2n} \colon Qu(x,y) > \lambda\Big\}.$$

- What if we consider $\|u\|_{L^p(\mathbb{R}^n)}$ in place of $\|\nabla u\|_{L^p(\mathbb{R}^n)}$?
- One possible difference quotient to look at would then be

$$\frac{|u(y)-u(x)|}{|y-x|^{\frac{n}{p}}}.$$

- Dominguez and Milman observed the existence of a dimensional constant $C = C(n)$ so that

$$\left\| \frac{u(y)-u(x)}{|y-x|^{\frac{n}{p}}} \right\|_{L^{p,\infty}(\mathbb{R}^{2n})} \leq C^{1/p} \|u\|_{L^p(\mathbb{R}^n)}$$

for all $1 \leq p < \infty$ and $u \in L^p(\mathbb{R}^n)$.
- Reversed inequality established in work with Qingsong Gu:

Theorem (Gu, Yung)

If $1 \leq p < \infty$ *and* $u \in L^p(\mathbb{R}^n)$*, then*

$$\lim_{\lambda \to 0^+} \left(\lambda |\mathcal{E}_\lambda|^{1/p} \right) = (2V_n)^{1/p} \|u\|_{L^p(\mathbb{R}^n)}$$

where

$$\mathcal{E}_\lambda := \left\{ (x,y) \in \mathbb{R}^{2n} : \frac{|u(y)-u(x)|}{|y-x|^{\frac{n}{p}}} > \lambda \right\}$$

and V_n *is the volume of the unit ball in* $\mathbb{R}^n$*.*

- Note that the hypothesis $u \in L^p(\mathbb{R}^n)$ is necessary, for the left hand side is zero and the right hand side is infinite if $u \equiv 1$.

Besov meets weak-L^p

- The above considerations led us to consider
$$\left\| \frac{u(y)-u(x)}{|y-x|^{s+\frac{n}{p}}} \right\|_{L^{p,\infty}(\mathbb{R}^{2n})}$$
if $0 < s < 1$ and $1 < p < \infty$.
- This quantity is in general smaller than $[u]_{\dot{B}^s_p(\mathbb{R}^n)}$.
- It also arises in various substitutes when fractional Sobolev embeddings / Gagliardo-Nirenberg interpolations fail.
- But what else can we say about it?
- In joint work with Andreas Seeger, Brian Street and Jean Van Schaftingen, we clarify the role of this quantity.
- The first result is a Fourier analytic characterization.

- It relies on the observation that the Besov (semi)norm

$$[u]_{\dot{B}^s_p(\mathbb{R}^n)} = \left(\sum_{j \in \mathbb{Z}} \|2^{js} \Delta_j u\|^p_{L^p(\mathbb{R}^n)} \right)^{1/p}$$

can also be rewritten as

$$\|2^{j(s+\frac{n}{p})} \Delta_j u(x)\|_{L^p(\mu)}$$

where μ is the measure on $\mathbb{R}^n \times \mathbb{Z}$ given by

$$\mu(E \times \{j\}) = 2^{-jn}|E| \quad \text{for all measurable } E \subset \mathbb{R}^n.$$

- Indeed,

$$\begin{aligned}
&\|2^{j(s+\frac{n}{p})} \Delta_j u(x)\|^p_{L^p(\mu)} \\
&= \int_0^\infty \lambda^p \mu\{(x,j) \colon 2^{j(s+\frac{n}{p})} |\Delta_j u(x)| > \lambda\} \frac{d\lambda}{\lambda} \\
&= \sum_{j \in \mathbb{Z}} \int_0^\infty 2^{-jn} \lambda^p |\{x \colon 2^{js} |\Delta_j u(x)| > 2^{-j\frac{n}{p}} \lambda\}| \frac{d\lambda}{\lambda} \\
&= \sum_{j \in \mathbb{Z}} \|2^{js} \Delta_j u\|^p_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Theorem (Seeger, Street, Van Schaftingen, Yung)

If $u \in C_c^\infty(\mathbb{R}^n)$ *(or a Schwartz function on* $\mathbb{R}^n$*), then*

$$\left\| \frac{u(y) - u(x)}{|y - x|^{s+\frac{n}{p}}} \right\|_{L^{p,\infty}(\mathbb{R}^{2n})} \simeq \|2^{j(s+\frac{n}{p})} \Delta_j u(x)\|_{L^{p,\infty}(\mu)}$$

whenever $s \in (0,1)$, $1 < p < \infty$, *and* μ *is the measure on* $\mathbb{R}^n \times \mathbb{Z}$ *given by*

$$\mu(E \times \{j\}) = 2^{-jn}|E| \quad \text{for all measurable } E \subset \mathbb{R}^n,\ j \in \mathbb{Z}.$$

More explicitly, the norm on the right hand side is

$$\sup_{\lambda > 0} \left(\sum_{j \in \mathbb{Z}} 2^{-jn} \lambda^p |\{x \in \mathbb{R}^n \colon |2^{js} \Delta_j u(x)| > 2^{-j\frac{n}{p}} \lambda\}| \right)^{1/p}.$$

- Such distribution of weight into the measure has appeared also in work on radial Fourier multipliers, and Fourier restriction theorems with affine arclength measure on curves.

- The second result is a characterization via real interpolation, that follows from the first theorem.

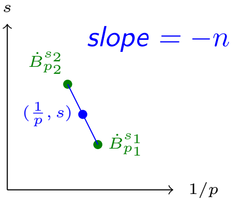
Theorem (Seeger, Street, Van Schaftingen, Yung)

If $u \in C_c^\infty(\mathbb{R}^n)$ (or a Schwartz function on $\mathbb{R}^n$), then

$$\left\| \frac{u(y)-u(x)}{|y-x|^{s+\frac{n}{p}}} \right\|_{L^{p,\infty}(\mathbb{R}^{2n})} \simeq \|u\|_{[\dot{B}^{s_1}_{p_1}, \dot{B}^{s_2}_{p_2}]_{\theta,\infty}}$$

whenever $s_1, s_2 \in (0,1)$, $p_1, p_2 \in (1,\infty)$, $\theta \in (0,1)$,

$$(\frac{1}{p}, s) = (1-\theta)(\frac{1}{p_1}, s_1) + \theta(\frac{1}{p_2}, s_2) \quad \text{and} \quad \frac{s_1 - s_2}{\frac{1}{p_1} - \frac{1}{p_2}} = -n.$$



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(The norm on the right hand side is

$$\sup_{t>0} \left(t^{-\theta} \inf \left\{ \|u_1\|_{\dot{B}^{s_1}_{p_1}} + t\|u_2\|_{\dot{B}^{s_2}_{p_2}} : u = u_1 + u_2 \right\} \right)$$

that arises from the real method of interpolation.)

Back to the beginning (with Brezis and Van Schaftingen)

- Recall while the L^p norm of the modified difference quotient

$$Qu(x,y) := \frac{|u(y)-u(x)|}{|y-x|^{1+\frac{n}{p}}}$$

  on $\mathbb{R}^{2n}$ is usually infinite, its weak-L^p norm on $\mathbb{R}^{2n}$ is indeed comparable to $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ for $1 \leq p < \infty$.
- Let's prove this when $n = p = 1$, i.e.

$$\left\| \frac{u(y)-u(x)}{|y-x|^2} \right\|_{L^{p,\infty}(\mathbb{R}^2)} \lesssim \|u'\|_{L^1(\mathbb{R})}.$$

  (Passage to higher dimensions possible via the method of rotation.)
- The proof relies on the Vitali covering lemma in 1-dimension: If X is a collection of intervals on $\mathbb{R}$ with $\sup_{I \in X} |I| < \infty$, then there exists a subcollection $Y \subset X$ such that all intervals from Y are pairwise disjoint up to end-points, and every $I \in X$ is contained in $5J$ for some $J \in Y$.

- Goal: Show that for $u \in C_c^\infty(\mathbb{R})$ and $\lambda > 0$,

$$|E_\lambda| \lesssim \frac{1}{\lambda}\|u'\|_{L^1(\mathbb{R})}$$

where $E_\lambda := \left\{(x,y) \in \mathbb{R}^2 \colon \frac{|u(y)-u(x)|}{|y-x|^2} > \lambda\right\}$.
- Let X be the collection of intervals $[x,y]$ where $(x,y) \in E_\lambda$.
- Vitali covering lemma applies to X because for every $I \in X$,

$$|I| < \left(\frac{1}{\lambda}\int_I |u'|\right)^{1/2} \le \left(\frac{1}{\lambda}\|u'\|_{L^1(\mathbb{R})}\right)^{1/2}.$$

- We obtain a subcollection $Y \subset X$ such that all intervals from Y are pairwise disjoint up to end-points, and every $I \in X$ is contained in $5J$ for some $J \in Y$.
- As a result, $E_\lambda \subset \bigcup_{I \in X} I \times I \subset \bigcup_{J \in Y} (5J) \times (5J)$, and

$$|E_\lambda| \le \sum_{J \in Y} |5J|^2 \le 25 \sum_{J \in Y} \frac{1}{\lambda}\int_J |u'| \le \frac{25}{\lambda}\|u'\|_{L^1(\mathbb{R})}.$$

Another limiting equality for $\|\nabla u\|_{L^p(\mathbb{R}^n)}$

- Now recall for $u \in C_c^\infty(\mathbb{R}^n)$, if

$$E_\lambda := \left\{(x,y) \in \mathbb{R}^{2n} \colon \frac{|u(y)-u(x)|}{|y-x|^{1+\frac{n}{p}}} > \lambda\right\},$$

then for $1 \leq p < \infty$,

$$\lim_{\lambda \to \infty} \left(\lambda |E_\lambda|^{1/p}\right) = \left(\frac{k(p,n)}{n}\right)^{1/p} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

- This is strikingly similar to a consequence of the BBM formula of Bourgain, Brezis and Mironescu (2001), which says

$$\lim_{s \to 1^-} (1-s)^{1/p} \left\| \frac{u(y)-u(x)}{|y-x|^{s+\frac{n}{p}}} \right\|_{L^p(\mathbb{R}^{2n})} = \left(\frac{k(p,n)}{p}\right)^{1/p} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

(thanks to Armin Schikorra who showed me this consequence; incidentally, this shows that $\left\|\frac{u(y)-u(x)}{|y-x|^{s+\frac{n}{p}}}\right\|_{L^p(\mathbb{R}^{2n})}$ blows up like $(1-s)^{-1/p}$ as $s \to 1^-$, unless u is a constant).

Closing remarks

- Recently, Oscar Dominguez and Mario Milman have been able to put some of the above results in an abstract framework.
- They proved that if X is a σ-finite measure space, $1 \leq p < \infty$ and $\{T_t\}_{t>0}$ is a family of sublinear operators on $L^p(X)$, then for all $f \in L^p(X)$ satisfying

$$\|T_t f - f\|_{L^\infty(X)} \lesssim_f t^{1/p} \quad \text{for all } t > 0,$$

  we have

$$\lim_{\lambda \to \infty} \left(\lambda |E_\lambda|^{1/p} \right) = \|f\|_{L^p(X)},$$

  where

$$E_\lambda := \left\{ (x,t) \in X \times (0,\infty) \colon \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\}.$$

- They found an impressive list of applications, from a characterization of $\|\Delta u\|_{L^p(\mathbb{R}^n)}$ and $\|\partial_{x_1}\partial_{x_2} u\|_{L^p(\mathbb{R}^2)}$, to relations between $\|f\|_{L^p(\mathbb{R}^n)}$ with level set estimates for spherical averages of f for $p > \frac{n}{n-1}$, to ergodic theory, etc.