Yau's Gradient Estimate

Theorem 1. *If M is a complete Riemannian manifold of dimension $n \geq 2$ with $Ric \geq -(n-1)K$ for some $K \geq 0$, then any positive harmonic function on B_r satisfies*

$$\sup_{B_{r/2}} \frac{|\nabla u|}{u} \leq C_n \left(\frac{1}{r} + \sqrt{K} \right),$$

where $B_r = \exp B_r$ denotes any geodesic ball of radius r.

Proof. Let x_0 be the center of B_r and $\rho(x)$ denote the distance between x and x_0. Then the key is to consider the function

$$F(x) = (r^2 - \rho^2) \frac{|\nabla u|}{u}$$

on B_r. It suffices to show that F is bounded on B_r by $C_n r^2 \left(\frac{1}{r} + \sqrt{K} \right)$, for if we restrict to $B_{r/2}$ then $r^2 - \rho^2$ is bounded below by a multiple of r^2. To estimate F, we argue at a point where F assumes its maximum: suppose F attains its maximum at a point y, which has to be in the interior of B_r since $F = 0$ on the boundary. Then at y, unless y is a cut point of x, we have F smooth near y and

$$\nabla F(y) = 0,$$

$$\Delta F(y) \leq 0.$$

This says at y,

$$(r^2 - \rho^2) \nabla \phi - \phi \nabla \rho^2 = 0 \tag{1}$$

$$(r^2 - \rho^2) \Delta \phi - \phi \Delta \rho^2 - 2 \nabla \rho^2 \cdot \nabla \phi \leq 0 \tag{2}$$

where

$$\phi = \frac{|\nabla u|}{u}.$$

Substitute (1) into (2), we get

$$(r^2 - \rho^2) \Delta \phi - \phi \Delta \rho^2 - 8 \rho^2 \frac{\phi}{r^2 - \rho^2} \leq 0 \tag{3}$$

since $|\nabla \rho^2| = |2\rho \nabla \rho| = 2\rho$. Now $\Delta \rho^2$ can be controlled rather easily by the geometry of M: by Laplacian comparison theorem,

$$\Delta \rho^2 = 2\rho \Delta \rho + 2|\nabla \rho|^2 = 2\rho \Delta \rho + 2 \leq C_n (1 + \sqrt{K} \rho).$$

It turns out that one can estimate $\Delta \phi$ from below by high powers of ϕ; it would then follow from (3) that a high power of $\phi(y)$ is controlled by a small power of $\phi(y)$, and from that one can obtain a bound of $|F(y)|$. We proceed as follows:

First,

$$|\nabla u| = \phi u,$$

so

$$\Delta |\nabla u| = u \Delta \phi + \phi \Delta u + 2 \nabla u \cdot \nabla \phi,$$

i.e.

$$\Delta \phi = \frac{\Delta |\nabla u|}{u} - \frac{2 \nabla u \cdot \nabla \phi}{u}$$

since $\Delta u = 0$. To estimate $\Delta|\nabla u|$, we look at $\Delta|\nabla u|^2$: On one hand

$$\Delta|\nabla u|^2 = 2|\nabla u|\Delta|\nabla u| + 2|\nabla|\nabla u||^2 = 2|\nabla u|\Delta|\nabla u| + 2\left|u_{ij}\frac{u_j}{|\nabla u|}\right|^2$$

and on the other

$$\begin{aligned}\Delta|\nabla u|^2 &= 2u_{ij}^2 + 2u_{ijj}u_i = 2u_{ij}^2 + 2u_{jji}u_i + 2\mathrm{Ric}_{ij}u_iu_j \\ &\geq 2u_{ij}^2 - 2(n-1)K|\nabla u|^2\end{aligned}$$

because $\Delta u = 0$. Note that $U = (u_{ij})$ is a trace-free symmetric matrix. Hence its any eigenvalue is bounded by $(n-1)/n$ times the trace of U^*U. Indeed let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of U. Then

$$\lambda_1^2 = (\lambda_2 + \cdots + \lambda_n)^2 \leq (n-1)(\lambda_2^2 + \cdots + \lambda_n^2)$$

so

$$n\lambda_1^2 \leq (n-1)(\lambda_1^2 + \cdots + \lambda_n^2).$$

(Geometrically, this says if λ is a point on the n-dimensional unit sphere that lies in the hyperplane $\lambda_1 + \cdots + \lambda_n = 0$ then $|\lambda_1|$ cannot be as big as 1; it is at most $(n-1)/n$. This trick is useful as long as the trace of U is very small, say bounded by some ε_n, in absolute value.) As a result, combining the above, if we write the unit vector $\frac{u_j}{|\nabla u|} = \sum a_je_j$ in an orthonormal eigenbasis e_j of (u_{ij}) (so $\sum a_j^2 = 1$), then

$$\begin{aligned}|\nabla u|\Delta|\nabla u| &\geq u_{ij}^2 - (n-1)K|\nabla u|^2 - \left|u_{ij}\frac{u_j}{|\nabla u|}\right|^2 \\ &= u_{ij}^2 - (n-1)K|\nabla u|^2 - \lambda_j^2a_j^2 \\ &\geq \frac{n}{n-1}\lambda_1^2 - (n-1)K|\nabla u|^2 - \lambda_j^2a_j^2 \\ &\geq \frac{n}{n-1}\lambda_j^2a_j^2 - (n-1)K|\nabla u|^2 - \lambda_j^2a_j^2 \\ &= \frac{1}{n-1}\lambda_j^2a_j^2 - (n-1)K|\nabla u|^2 \\ &= \frac{|\nabla|\nabla u||^2}{n-1} - (n-1)K|\nabla u|^2.\end{aligned}$$

Hence

$$\begin{aligned}\Delta\phi &\geq \frac{1}{|\nabla u|u}\left(\frac{|\nabla|\nabla u||^2}{n-1} - (n-1)K|\nabla u|^2\right) - \frac{2\nabla u\cdot\nabla\phi}{u} \\ &= \frac{|\nabla|\nabla u||^2}{(n-1)|\nabla u|u} - (n-1)K\phi - \frac{2\nabla u\cdot\nabla\phi}{u}.\end{aligned}$$

Notice how we squeezed out the first positive term on the right hand side using

$\Delta u = 0$. This term will cancel with part of the third term as follows:

$$\begin{aligned}\frac{\nabla u \cdot \nabla \phi}{u} &= \frac{\nabla |\nabla u| \cdot \nabla u}{u} - \frac{|\nabla u|^3}{u^3} \\ &\leq \frac{|\nabla |\nabla u|| |\nabla u|}{u} - \phi^3 \\ &= \frac{|\nabla |\nabla u||}{(|\nabla u| u)^{\frac{1}{2}}} \frac{|\nabla u|^{\frac{3}{2}}}{u^{\frac{3}{2}}} - \phi^3 \\ &\leq \frac{1}{2}\left(\frac{|\nabla |\nabla u||^2}{|\nabla u| u} + \phi^3\right) - \phi^3 \\ &= \frac{|\nabla |\nabla u||^2}{2|\nabla u| u} - \frac{1}{2}\phi^3.\end{aligned}$$

Hence

$$\Delta \phi \geq -(n-1)K\phi - \left(2 - \frac{2}{n-1}\right)\frac{\nabla u \cdot \nabla \phi}{u} + \frac{1}{n-1}\phi^3$$

and this is how $\Delta \phi$ is bounded below by a high power of ϕ. Now the second term is at least $-C_n \phi |\nabla \phi|$, which at y is

$$-C_n \frac{\phi^2 |\nabla \rho^2|}{r^2 - \rho^2} = -C_n \frac{\phi^2 \rho}{r^2 - \rho^2}$$

by (1). Hence at y we have

$$\Delta \phi \geq -(n-1)K\phi - C_n \frac{\phi^2 \rho}{r^2 - \rho^2} + \frac{1}{n-1}\phi^3$$

and from (3)

$$\begin{aligned}&(r^2 - \rho^2)^2 \left(-(n-1)K\phi - C_n \frac{\phi^2 \rho}{r^2 - \rho^2} + \frac{1}{n-1}\phi^3\right) \\ \leq &C_n \phi (1 + \sqrt{K}\rho)(r^2 - \rho^2) + 8\rho^2 \phi.\end{aligned}$$

Dividing through by ϕ, and collecting terms,

$$\begin{aligned}\frac{1}{n-1}F^2 &\leq C_n r F + r^2 \left((n-1)K r^2 + C_n (1 + \sqrt{K} r) + 8\right) \\ &\leq C_n \left(r F + r^2 (1 + \sqrt{K} r)^2\right)\end{aligned}$$

from which one gets

$$|F(y)| \leq C_n r^2 (1 + \sqrt{K} r).$$

Hence we are done in the case when y is not a cut point of x_0. Note how geometry of M (bound of Ricci curvature) enters only via the Laplacian comparison theorem and the Ricci formula for u_{ijj}.

If y is a cut point of x_0, then take x_1 to be a point along a minimizing geodesic from x_0 to y that is close to x_1. Then y is not a cut point of x_1, and we can run the above argument for the support function

$$F_1(x) = \left(r^2 - (d(x, x_1) + d(x_1, x_0))^2\right) \phi(x)$$

instead. Note $F_1(x) \leq F(x)$ and $F_1(y) = F(y)$, so y is still a maximum point of F_1, and F_1 is smooth near y. This completes the proof in general. □