SHARP DISCRETE STRICHARTZ INEQUALITY IN $2+1$ DIMENSIONS: AN EXPOSITION

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In a beautiful paper [3], Herr and Kwak proved sharp discrete Strichartz inequality for the Schrödinger equation on $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$:

Theorem 1 (Herr-Kwak). *There is a universal constant c so that for any finite subsets $S \subset \mathbb{Z}^2$ and any initial data F on $\mathbb{T}^2$ with Fourier support in S, we have*[1]

$$\|e^{it\Delta}F\|_{L^4([0,2\pi]\times\mathbb{T}^2)} \leq c(\log \#S)^{1/4}\|F\|_{L^2(\mathbb{T}^2)}. \tag{1}$$

The power of $\log \#S$ is the best possible, as seen by Bourgain's example [1] where $S = [-M,M]^2 \cap \mathbb{Z}^2$ and $F(x) = \sum_{\xi\in S} e^{ix\cdot\xi}$; for a nice detailed explanation see Chapter 13.2 of Demeter [2].

For general finite set $S \subset \mathbb{Z}^2$, if $F(x) = \sum_{\xi\in S} e^{ix\cdot\xi}$, then

$$\begin{aligned} \|e^{it\Delta}F\|^4_{L^4([0,2\pi]\times\mathbb{T}^2)} &= \sum_{(\xi_1,\xi_2,\xi_3,\xi_4)\in S^4} \int_0^{2\pi}\int_{\mathbb{T}^2} e^{ix\cdot(\xi_1-\xi_2+\xi_3-\xi_4)} e^{it(|\xi_1|^2-|\xi_2|^2+|\xi_3|^2-|\xi_4|^2)} dx dt \\ &= (2\pi)^3 N(S) \end{aligned} \tag{2}$$

where $N(S)$ is the number of solutions $(\xi_1,\xi_2,\xi_3,\xi_4) \in S^4$ to the equations

$$\xi_1 + \xi_3 = \xi_2 + \xi_4$$
$$|\xi_1|^2 + |\xi_3|^2 = |\xi_2|^2 + |\xi_4|^2.$$

The first (vector) equation says $\xi_1 - \xi_2 = \xi_4 - \xi_3$, i.e. $(\xi_1,\xi_2,\xi_3,\xi_4)$ are vertices of a parallelogram. Under this first equation, the second equation can be rearranged to say

$$2(\xi_1 - \xi_2)\cdot(\xi_1 - \xi_4) = 0$$

so one of the angles of the parallelogram $(\xi_1,\xi_2,\xi_3,\xi_4)$ is a right angle, i.e. $(\xi_1,\xi_2,\xi_3,\xi_4)$ is a rectangle. Thus $N(S)$ counts the number of rectangles in S (henceforth by a rectangle in S we mean a quadruple $(\xi_1,\xi_2,\xi_3,\xi_4) \in S^4$ that forms a (possibly degenerate) rectangle). Pach and Sharir [5] proved that $N(S) \lesssim \#S^2 \log \#S$; in fact, using the Szemeredi-Trotter theorem for counting incidences of lines in the plane, they showed that for any finite set S in the plane, there are at most $O(\#S^2 \log \#S)$ many right angled triangles whose vertices all lie in S. This establishes (1) when $F(x) = \sum_{\xi\in S} e^{ix\cdot\xi}$; the issue is that we do not know F extremizes the inequality (1) (the proof of Herr and Kwak does not proceed this way).

Herr and Kwak actually proved a lot more than Theorem 1: the following is interesting even for the example F mentioned above.

Theorem 2 (Herr-Kwak). *There is a universal constant c so that for any finite subsets $S \subset \mathbb{Z}^2$ and any initial data F on $\mathbb{T}^2$ supported on S, we have*

$$\|e^{it\Delta}F\|_{L^4([0,T_0]\times\mathbb{T}^2)} \leq c\|F\|_{L^2(\mathbb{T}^2)} \quad \textit{for} \quad T_0 := \frac{1}{\log \#S}. \tag{3}$$

Below we explain their proof of Theorem 2 (slightly reorganized to highlight some key ideas). In fact, we show that $\#S$ in Theorem 2, and hence Theorem 1, can be sharpened to $\max_{k\in\mathbb{Z}} \#S_k$ where

$$S_k := \{\xi \in \mathbb{Z}^2 \colon 2^k \leq |\widehat{F}(\xi)| < 2^{k+1}\}.$$

[1] Throughout this note $\log x$ means $\max\{1, \log_e x\}$.

1. Initial reductions

The argument of Herr and Kwak is entirely Fourier analytic and takes advantage of the fact that the exponent 4 is an even integer, so that one can access Fourier expansions of $\|e^{it\Delta}F\|^4_{L^4([0,T_0]\times\mathbb{T}^2)}$ similar to (2). Once they passed to the Fourier side, they rely on geometric ideas surrounding the Szemeredi-Trotter theorem from incidence geometry[2]. Hence it is useful to introduce some shorthands: we denote by $\mathcal{Q}$ the set of all (possibly degenerate) parallelograms in $\mathbb{Z}^2$, i.e. the set of all quadruples $(\xi_1,\xi_2,\xi_3,\xi_4)\in(\mathbb{Z}^2)^4$ with $\xi_1+\xi_3=\xi_2+\xi_4$. For any integer τ, let $\mathcal{Q}^\tau$ be the set of all parallelograms $Q\in\mathcal{Q}$ with

$$\tau = |\xi_1|^2-|\xi_2|^2+|\xi_3|^2-|\xi_4|^2 = 2(\xi_1-\xi_2)\cdot(\xi_1-\xi_4).$$

In particular, $\mathcal{Q}^0$ is the set of all rectangles. Additionally, for $Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}$ and $f\colon\mathbb{Z}^2\to\mathbb{C}$ let $f(Q)=f(\xi_1)f(\xi_2)f(\xi_3)f(\xi_4)$.

The first reduction of Herr and Kwak is in proving the following proposition.

Lemma 1. *For any initial data F on $\mathbb{T}^2$ Fourier supported on a finite set, and any $0<T_0\le 2\pi$,*

$$\|e^{it\Delta}F\|^4_{L^4([0,T_0]\times\mathbb{T}^2)} \lesssim T_0\sum_{Q\in\mathcal{Q}^0}|f(Q)| + \sup_{M\in 2^{\mathbb{N}}}\frac{1}{M}\sum_{|\tau|\simeq M}\sum_{Q\in\mathcal{Q}^\tau}|f(Q)|. \tag{4}$$

where $f(\xi):=\widehat{F}(\xi)=\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}F(x)e^{-ix\cdot\xi}dx$.

Proof. To prove the lemma, motivated by (2), one might be tempted to write

$$\|e^{it\Delta}F\|^4_{L^4([0,T_0]\times\mathbb{T}^2)} = (2\pi)^2\sum_{\xi_1+\xi_3=\xi_2+\xi_4} f(\xi_1)\overline{f(\xi_2)}f(\xi_3)\overline{f(\xi_4)}\int_0^{T_0} e^{it(|\xi_1|^2-|\xi_2|^2+|\xi_3|^2-|\xi_4|^2)}dt,$$

and then bound the right hand side above by the right hand side of (4). But this is not what Herr and Kwak did; instead, they pulled up a very nice trick, and estimated

$$\|e^{it\Delta}F\|^4_{L^4([0,T_0]\times\mathbb{T}^2)} \le \frac{1}{T_0}\int_0^{2T_0}\|e^{it\Delta}F\|^4_{L^4([0,T]\times\mathbb{T}^2)}dT$$

(the inequality holds because

$$\frac{1}{T_0}\int_0^{2T_0}\int_0^T H(t)dt\,dT = \int_0^{2T_0}\frac{2T_0-t}{T_0}H(t)dt \ge \int_0^{T_0}H(t)dt$$

for any nonnegative function $H(t)$). This gives

$$\|e^{it\Delta}F\|^4_{L^4([0,T_0]\times\mathbb{T}^2)} \le (2\pi)^2\sum_{\xi_1+\xi_3=\xi_2+\xi_4} f(\xi_1)\overline{f(\xi_2)}f(\xi_3)\overline{f(\xi_4)}\frac{1}{T_0}\int_0^{2T_0}\int_0^T e^{it(|\xi_1|^2-|\xi_2|^2+|\xi_3|^2-|\xi_4|^2)}dt\,dT$$

Additionally, observe that in proving (4), without loss of generality one may assume that f is real, because one can express f in terms of its real and imaginary parts. Under such hypothesis, one can take the real parts of the above inequality, and obtain, with our earlier shorthands, that[3]

$$\|e^{it\Delta}F\|^4_{L^4([0,T_0]\times\mathbb{T}^2)} \lesssim T_0\sum_{Q\in\mathcal{Q}^0}f(Q) + \sum_{\tau\in\mathbb{Z}\setminus\{0\}}\frac{1-\cos(2T_0\tau)}{T_0\tau^2}\sum_{Q\in\mathcal{Q}^\tau}f(Q).$$

(4) then follows from the inequality $\left|\frac{1-\cos(2T_0\tau)}{T_0\tau^2}\right|\lesssim\min\{T_0,\frac{1}{T_0\tau^2}\}$, together with

$$\sum_{\tau\in\mathbb{Z}\setminus\{0\}}\min\{T_0,\frac{1}{T_0\tau^2}\}\sum_{Q\in\mathcal{Q}^\tau}|f(Q)| \lesssim \sum_{M\in 2^{\mathbb{N}}}\min\{T_0M,\frac{1}{T_0M}\}\frac{1}{M}\sum_{|\tau|\simeq M}\sum_{Q\in\mathcal{Q}^\tau}|f(Q)|. \tag{5}$$

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[2] It is worth noting that Herr and Kwak did not use induction on scales at all (contrary to proofs of decoupling for the paraboloid in $\mathbb{R}^3$). On the other hand, they did use a pruning argument, as we will soon see.

[3] The key here is that via this additional averaging in time and reduction to real parts, we obtained an additional factor of $T_0\tau$ in the denominator of the second term, which we will use in (5). This gain is only possible by applying both tricks!

It is actually possible to strengthen Lemma 1, and prove without resorting to real-valued f that

$$\|e^{it\Delta}F(x)\|^4_{L^4([0,T_0]\times\mathbb{T}^2)} \lesssim T_0 \sum_{Q\in\mathcal{Q}^0} |f(Q)| + \sup_{\substack{M\in 2^{\mathbb{N}}\\ M\le 1/T_0}} \frac{1}{M} \sum_{|\tau|\simeq M} \sum_{Q\in\mathcal{Q}^\tau} |f(Q)|. \tag{4'}$$

Indeed, let $\eta(t)$ be a non-negative Schwartz function on $\mathbb{R}$ with Fourier support in $[-1,1]$, and satisfies $\eta(t)\gtrsim 1$ on $[0,1]$. For $T_0>0$ let $\eta_{T_0}(t) := \sum_{n\in\mathbb{Z}} \eta(\frac{t+2\pi n}{T_0})$. η_{T_0} is a smooth 2π-periodic function of t, and has Fourier support in $\mathbb{Z}\cap[-T_0^{-1},T_0^{-1}]$ (as one can verify using the Poisson summation formula). Writing $u(x,t)=e^{it\Delta}F(x)$, we have

$$\begin{aligned}
\|u\|^4_{L^4([0,T_0]\times\mathbb{T}^2)} &\lesssim \int_{\mathbb{T}^3} |u|^4 \eta_{T_0} = \widehat{u}*\widehat{\overline{u}}*\widehat{u}*\widehat{\overline{u}}*\widehat{\eta_{T_0}}(0)\\
&= \sum_{\tau\in\mathbb{Z}} \widehat{u}*\widehat{\overline{u}}*\widehat{u}*\widehat{\overline{u}}(0,\tau)\widehat{\eta_{T_0}}(-\tau)\\
&\lesssim \sum_{\substack{\tau\in\mathbb{Z}\\ |\tau|\le 1/T_0}} |\widehat{u}*\widehat{\overline{u}}*\widehat{u}*\widehat{\overline{u}}(0,\tau)| T_0
\end{aligned}$$

where we used $\|\widehat{\eta_{T_0}}\|_{l^\infty(\mathbb{Z})} \le \|\eta_{T_0}\|_{L^1(\mathbb{T})} = T_0\|\eta\|_{L^1(\mathbb{R})}$ and the Fourier support of $\widehat{\eta_{T_0}}$. Using further that

$$\left|\widehat{u}*\widehat{\overline{u}}*\widehat{u}*\widehat{\overline{u}}(0,\tau)\right| = \Big|\sum_{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^\tau} f(\xi_1)\overline{f(\xi_2)}f(\xi_3)\overline{f(\xi_4)}\Big| \le \sum_{Q\in\mathcal{Q}^\tau} |f(Q)|,$$

we have

$$\|u\|^4_{L^4([0,T_0]\times\mathbb{T}^2)} \lesssim T_0 \sum_{Q\in\mathcal{Q}^0} |f(Q)| + T_0 \sum_{\substack{M\in 2^{\mathbb{N}}\\ M\le 1/T_0}} \sum_{|\tau|\simeq M} \sum_{Q\in\mathcal{Q}^\tau} |f(Q)|$$

which proves (4') since $T_0 \sum_{\substack{M\in 2^{\mathbb{N}}\\ M\le 1/T_0}} M \lesssim 1$. □

2. From parallelograms to rectangles

Lemma 1 reduces the proof of Theorem 2 to bounding the right hand side of (4), which has two terms. Estimating the first term involves counting rectangles, which seems manageable. Estimating the second term seems to involve counting parallelograms, which seems harder. Magically, Herr and Kwak found a way of estimating the second term by counting rectangles only: Let $\mathcal{Q}^0_{\text{nondeg}}$ be the set of non-degenerate rectangles, i.e. rectangles with four distinct vertices. For any vector $\xi\in\mathbb{Z}^2\setminus\{0\}$, we denote by by $\gcd(\xi)$ the greatest common divisor of the two components of ξ. Herr and Kwak proved:

Lemma 2. *For any $f\in\ell^2(\mathbb{Z}^2)$,*

$$\sup_{M\in 2^{\mathbb{N}}} \frac{1}{M} \sum_{|\tau|\simeq M} \sum_{Q\in\mathcal{Q}^\tau} |f(Q)| \lesssim \sum_{Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0_{nondeg}} \frac{1}{\gcd(\xi_1-\xi_4)} |f(Q)| + \|f\|^4_{\ell^2(\mathbb{Z}^2)}. \tag{6}$$

Proof. Indeed, note that $Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^\tau$ means $2(\xi_1-\xi_2)\cdot(\xi_1-\xi_4)=\tau$. So $\gcd(\xi_1-\xi_4)|\tau$ whenever $\xi_1-\xi_4\neq 0$ and $Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^\tau$. This shows

$$\begin{aligned}
\sum_{Q\in\mathcal{Q}^\tau} |f(Q)| &\le \sum_{\xi\in\mathbb{Z}^2\setminus\{0\}} \sum_{\substack{Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^\tau\\ \xi_1-\xi_4=\xi}} |f(Q)| + \sum_{\xi_1,\xi_2\in\mathbb{Z}^2} |f(\xi_1)|^2|f(\xi_2)|^2\\
&= \sum_{\substack{\xi\in\mathbb{Z}^2\setminus\{0\}\\ \gcd(\xi)|\tau}} \sum_{\substack{Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^\tau\\ \xi_1-\xi_4=\xi}} |f(Q)| + \|f\|^4_{\ell^2(\mathbb{Z}^2)}.
\end{aligned}$$

But Cauchy-Schwarz gives

$$\sum_{\substack{Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^\tau\\ \xi_1-\xi_4=\xi}}|f(Q)| = \sum_{\sigma\in\mathbb{Z}}\sum_{\substack{\xi_1-\xi_4=\xi\\ \xi_1\cdot\xi=\sigma}}|f(\xi_1)f(\xi_4)|\sum_{\substack{\xi_2-\xi_3=\xi\\ \xi_2\cdot\xi=\sigma+\tau/2}}|f(\xi_2)f(\xi_3)| \tag{7}$$
$$\leq \sum_{\sigma\in\mathbb{Z}}\Big(\sum_{\substack{\xi_1-\xi_4=\xi\\ \xi_1\cdot\xi=\sigma}}|f(\xi_1)f(\xi_4)|\Big)^2$$

which is equal to

$$\sum_{\sigma\in\mathbb{Z}}\sum_{\substack{\xi_1-\xi_4=\xi\\ \xi_1\cdot\xi=\sigma}}\sum_{\substack{\xi_2-\xi_3=\xi\\ \xi_2\cdot\xi=\sigma}}|f(\xi_1)f(\xi_2)f(\xi_3)f(\xi_4)| = \sum_{\substack{Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0\\ \xi_1-\xi_4=\xi}}|f(Q)|$$
$$\leq \sum_{\substack{Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0\\ \xi_1-\xi_4=\xi,\,\xi_1-\xi_2\neq 0}}|f(Q)| + \sum_{\xi_1-\xi_4=\xi}|f(\xi_1)|^2|f(\xi_4)|^2.$$

This shows that for $\tau\in\mathbb{Z}$, one has

$$\sum_{Q\in\mathcal{Q}^\tau}|f(Q)| \leq \sum_{\substack{Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0_{\text{nondeg}}\\ \gcd(\xi_1-\xi_4)|\tau}}|f(Q)| + 2\|f\|^4_{\ell^2(\mathbb{Z}^2)}.$$

Hence

$$\sup_{M\in 2^{\mathbb{N}}}\frac{1}{M}\sum_{|\tau|\simeq M}\sum_{Q\in\mathcal{Q}^\tau}|f(Q)| \lesssim \sup_{M\in 2^{\mathbb{N}}}\frac{1}{M}\sum_{|\tau|\simeq M}\Big(\sum_{\substack{Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0_{\text{nondeg}}\\ \gcd(\xi_1-\xi_4)|\tau}}|f(Q)| + \|f\|^4_{\ell^2(\mathbb{Z}^2)}\Big)$$
$$\lesssim \sum_{Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0_{\text{nondeg}}}\frac{1}{\gcd(\xi_1-\xi_4)}|f(Q)| + \|f\|^4_{\ell^2(\mathbb{Z}^2)}.$$

□

3. Pruning f to remove all but one 'heavy' line through each point

In light of (4), (6) and Parseval which asserts that $2\pi\|f\|_{\ell^2(\mathbb{Z}^2)} = \|F\|_{L^2(\mathbb{T}^2)}$ when $f=\widehat{F}$, Theorem 2 would follow if we could prove for all $f\in\ell^2(\mathbb{Z}^2)$ that

$$\sum_{Q\in\mathcal{Q}^0}|f(Q)| \lesssim \frac{1}{T_0}\|f\|^4_{\ell^2(\mathbb{Z}^2)} \quad\text{and}\quad \sum_{Q=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0_{\text{nondeg}}}\frac{1}{\gcd(\xi_1-\xi_4)}|f(Q)| \lesssim \|f\|^4_{\ell^2(\mathbb{Z}^2)} \tag{8}$$

with $\frac{1}{T_0} = \log\max_{k\in\mathbb{Z}}\#\{\xi\in\mathbb{Z}^2\colon 2^k\leq|f(\xi)|<2^{k+1}\}$. At this point Herr and Kwak took a shortcut: they observed that it actually suffices to prove (8) only for those f that satisfies an additional structural hypothesis (although with some careful bookkeeping, their methods actually proves (8) in full without such additional hypothesis on f).

First, given a finite set $S\subset\mathbb{R}^2$, a line ℓ in the plane is said to be heavy with respect to S, if $\#(\ell\cap S)\geq C(\#S)^{1/2}$ where $C\geq 2$ is a fixed universal constant to be specified. It turns out that to be sufficient to prove:

Proposition 1. *Let $f\in\ell^2(\mathbb{Z}^2)$ and $S_k := \{\xi\in\mathbb{Z}^2\colon 2^k\leq|f(\xi)|<2^{k+1}\}$ for $k\in\mathbb{Z}$ so that S_k are disjoint finite subsets of $\mathbb{Z}^2$. Assume that for each $k\in\mathbb{Z}$ and each $\xi\in S_k$, at most one line through ξ is heavy with respect to S_k. Then (8) holds with $\frac{1}{T_0} = \log\max_{k\in\mathbb{Z}}\#S_k$.*

To see why Proposition 1 implies Theorem 2, we apply a pruning process to a general $f\in\ell^2(\mathbb{Z}^2)$. For $k\in\mathbb{Z}$, let $E_{k,0} = \{\xi\in\mathbb{Z}^2\colon 2^k\leq|f(\xi)|<2^{k+1}\}$. If $E_{k,n}$ has been defined, let $E_{k,n+1}$ be the set of all $\xi \in E_{k,n}$ that lie at the intersection of at least 2 lines that are heavy with respect to $E_{k,n}$. For $n = 0, 1, 2, \ldots$, define

$$g_n := \sum_{k\in\mathbb{Z}} f 1_{S_{k,n}}, \quad S_{k,n} := E_{k,n} \setminus E_{k,n+1}.$$

Then $S_{k,n} = \{\xi \in \mathbb{Z}^2 \colon 2^k \le |g_n(\xi)| < 2^{k+1}\}$, so each g_n satisfies the structural hypothesis of Proposition 1 by construction. Furthermore, since $C(\#E_{k,n})^{1/2} \ge 2$ whenever $E_{k,n}$ is non-empty, Szemeredi-Trotter implies

$$\begin{aligned}\sqrt{\#E_{k,n+1}} &\le \text{number of } C(\#E_{k,n})^{1/2} \text{ rich lines through } E_{k,n} \\ &\lesssim \frac{(\#E_{k,n})^2}{(C(\#E_{k,n})^{1/2})^3} + \frac{\#E_{k,n}}{C(\#E_{k,n})^{1/2}} \le \frac{1}{2}\sqrt{\#E_{k,n}}.\end{aligned}$$

if the absolute constant C were chosen sufficiently large. Thus $\sqrt{\#E_{k,N+1}} \le 2^{-(N+1)}\sqrt{\#E_{k,0}}$, from which we have

$$\Big\| f - \sum_{n=0}^{N} g_n \Big\|_{\ell^2} = \Big\| \sum_{k\in\mathbb{Z}} f 1_{E_{k,N+1}} \Big\|_{\ell^2} \le 2 \Big\| \sum_{k\in\mathbb{Z}} 2^k 1_{E_{k,N+1}} \Big\|_{\ell^2} \le 2^{-N} \Big\| \sum_{k\in\mathbb{Z}} 2^k 1_{E_{k,0}} \Big\|_{\ell^2} \le 2^{-N}\|f\|_{\ell^2} \to 0$$

as $N \to \infty$. This gives a decomposition

$$f = \sum_{n=0}^{\infty} g_n. \tag{9}$$

By a similar token, we also have

$$\|g_n\|_{\ell^2} \le \Big\| \sum_{k\in\mathbb{Z}} f 1_{E_{k,n}} \Big\|_{\ell^2} \le 2^{1-n}\|f\|_{\ell^2}. \tag{10}$$

Now let $T_0 := (\log \max_{k\in\mathbb{Z}} \#S_k)^{-1}$ and write $\mathcal{F}^{-1}$ for the inverse Fourier transform on $\mathbb{Z}^2$. By (9) and the Minkowski inequality,

$$\|e^{it\Delta}\mathcal{F}^{-1} f\|_{L^4([0,T_0]\times\mathbb{T}^2)} \le \sum_{n=0}^{\infty} \|e^{it\Delta}\mathcal{F}^{-1} g_n\|_{L^4([0,T_0]\times\mathbb{T}^2)}, \tag{11}$$

and if (8) holds for each g_n in place of f, then the above is

$$\lesssim \sum_{n=0}^{\infty} \Big(T_0 \log \max_{k\in\mathbb{Z}} (\#S_{k,n}) + 1 \Big)^{1/4} \|g_n\|_{\ell^2} \lesssim \|f\|_{\ell^2}$$

by Lemma 1, Lemma 2, the trivial bound $\#S_{k,n} \le \#S_k$ for all n, and (10). This yields the desired bound (3) for $F = \mathcal{F}^{-1} f$ with our claimed improvement for T_0.

4. Geometric lemmas

The previous section reduces our goal to proving Proposition 1. Interestingly, Herr and Kwak managed to do so by proving something apparently weaker, as we see below.

First we introduce some key concepts. A set of two perpendicular lines $\{\ell_1, \ell_2\}$ in the plane will be called a cross. To each cross one associates a crossing, which is the point where the two lines in the cross intersect. The mass of a cross $\{\ell_1, \ell_2\}$ with respect to a finite set $S \subset \mathbb{R}^2$ is defined to be

$$m_S(\{\ell_1, \ell_2\}) := \max\{\#(\ell_1 \cap S), \#(\ell_2 \cap S)\}.$$

If the crossing of a cross $\{\ell_1, \ell_2\}$ lies in a set S, then the cross is said to be of

$$\begin{aligned} &\text{type 1 with respect to } S \quad \text{if } m_S(\{\ell_1, \ell_2\}) \ge C(\#S)^{1/2} \\ &\text{type 2 with respect to } S \quad \text{if } 2 \le m_S(\{\ell_1, \ell_2\}) < C(\#S)^{1/2} \\ &\text{type 3 with respect to } S \quad \text{if } m_S(\{\ell_1, \ell_2\}) = 1. \end{aligned}$$

Here C is the absolute constant chosen during the pruning process in the last section. If $Q \in \mathcal{Q}^0_{\text{nondeg}}$ is a non-degenerate rectangle and ξ is a vertex of Q, then $C_{\xi,Q}$ is the cross obtained by extending the two sides of Q that meet at ξ into indefinite lines. If S_1, S_2, S_3, S_4 are subsets of $\mathbb{R}^2$, and $Q = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{Q}^0_{\text{nondeg}} \cap (S_1 \times S_2 \times S_3 \times S_4)$ is a rectangle whose vertices $\xi_1, \xi_2, \xi_3, \xi_4$ are in the sets S_1, S_2, S_3, S_4 respectively, then Q is said to be of type (α, β) if $C_{\xi_j,Q}$ is type α with respect to S_j for $j = 1, 2$, and type β with respect to S_j for $j = 3, 4$. The set of all $Q \in \mathcal{Q}^0_{\text{nondeg}} \cap (S_1 \times S_2 \times S_3 \times S_4)$ of type (α, β) is then denoted $\mathcal{Q}^0_{\alpha,\beta}(S_1, S_2, S_3, S_4)$.

Proposition 1 follows from the following proposition, which is apparently weaker[4]:

Proposition 2. *Let $\{S_k\}_{k\in\mathbb{Z}}$ be finite pairwise disjoint subsets of $\mathbb{R}^2$. Suppose for each $k \in \mathbb{Z}$ and each $\xi \in S_k$, at most one line through ξ is heavy with respect to S_k. Then for $(\alpha, \beta) \neq (2, 2)$,*

$$\sum_{(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4} 2^{k_1+k_2+k_3+k_4} \# \mathcal{Q}^0_{\alpha,\beta}(S_{k_1}, S_{k_2}, S_{k_3}, S_{k_4}) \lesssim \Big(\sum_{k\in\mathbb{Z}} 2^{2k} \# S_k\Big)^2 \tag{12}$$

For $(\alpha, \beta) = (2, 2)$,

$$\sum_{(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4} 2^{k_1+k_2+k_3+k_4} \# \mathcal{Q}^0_{\alpha,\beta}(S_{k_1}, S_{k_2}, S_{k_3}, S_{k_4}) \lesssim \log(\max_{k\in\mathbb{Z}} \# S_k) \Big(\sum_{k\in\mathbb{Z}} 2^{2k} \# S_k\Big)^2 \tag{13}$$

and

$$\sum_{(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4} 2^{k_1+k_2+k_3+k_4} \sum_{Q\in\mathcal{Q}^0_{\alpha,\beta}(S_{k_1},S_{k_2},S_{k_3},S_{k_4})} \frac{1}{\gcd(\xi_1 - \xi_4)} \lesssim \Big(\sum_{k\in\mathbb{Z}} 2^{2k} \# S_k\Big)^2. \tag{14}$$

Proof of Proposition 1. Let f, S_k and T_0 be as in the statement of Proposition 1. Following the proof of Lemma 2 up to (7), we have

$$T_0 \sum_{Q\in\mathcal{Q}^0} |f(Q)| + \sup_{M\in 2^{\mathbb{N}}} \frac{1}{M} \sum_{|\tau|\simeq M} \sum_{Q\in\mathcal{Q}^\tau} |f(Q)| \lesssim T_0 G(0) + \sup_{M\in 2^{\mathbb{N}}} \frac{1}{M} \sum_{|\tau|\simeq M} G(\tau) + \|f\|^4_{\ell^2(\mathbb{Z}^2)} \tag{15}$$

where

$$G(\tau) := \sum_{\substack{\xi\in\mathbb{Z}^2\setminus\{0\} \\ \gcd(\xi)|\tau}} \sum_{\sigma\in\mathbb{Z}} \Big(\sum_{\substack{\xi_1-\xi_4=\xi \\ \xi_1\cdot\xi=\sigma}} |f(\xi_1)f(\xi_4)|\Big)^2.$$

We will see that Proposition 2 implies

$$G(0) \lesssim \log(\max_{k\in\mathbb{Z}} \# S_k) \|f\|^4_{\ell^2(\mathbb{Z}^2)} \quad \text{and} \quad \sup_{M\in 2^{\mathbb{N}}} \frac{1}{M} \sum_{|\tau|\simeq M} G(\tau) \lesssim \|f\|^4_{\ell^2(\mathbb{Z}^2)}. \tag{16}$$

Since $T_0 = (\log \max_{k\in\mathbb{Z}} \# S_k)^{-1}$, this will conclude the proof of Proposition 1.

To prove (16), we write $C(\xi_j, \xi)$ for the cross given by $\xi_j + \xi\mathbb{R}$ and $\xi_j + \xi^\perp\mathbb{R}$, for $j = 1, 4$. Then

$$\sum_{\substack{\xi_1-\xi_4=\xi \\ \xi_1\cdot\xi=\sigma}} |f(\xi_1)f(\xi_4)| \lesssim \sum_{1\leq\alpha,\beta\leq 3} \sum_{(k_1,k_4)\in\mathbb{Z}^2} 2^{k_1+k_4} \sum_{\substack{\xi_1-\xi_4=\xi \\ \xi_1\cdot\xi=\sigma}} 1_{S^{(\xi,\alpha)}_{k_1}}(\xi_1) 1_{S^{(\xi,\beta)}_{k_4}}(\xi_4)$$

where $S^{(\xi,\alpha)}_{k_1}$ is the set of all $\xi_1 \in S_{k_1}$ so that $C(\xi_1, \xi)$ is of type α with respect to S_{k_1}, and similarly for $S^{(\xi,\beta)}_{k_4}$. Then

$$\Big(\sum_{\substack{\xi_1-\xi_4=\xi \\ \xi_1\cdot\xi=\sigma}} |f(\xi_1)f(\xi_4)|\Big)^2 \lesssim \sum_{1\leq\alpha,\beta\leq 3} \sum_{(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4} 2^{k_1+k_2+k_3+k_4} \sum_{\substack{\xi_1-\xi_4=\xi \\ \xi_1\cdot\xi=\sigma}} \sum_{\substack{\xi_2-\xi_3=\xi \\ \xi_2\cdot\xi=\sigma}} 1_{S^{(\xi,\alpha)}_{k_1}}(\xi_1) 1_{S^{(\xi,\alpha)}_{k_2}}(\xi_2) 1_{S^{(\xi,\beta)}_{k_3}}(\xi_3) 1_{S^{(\xi,\beta)}_{k_4}}(\xi_4)$$

[4]Weaker because (8) requires one to sum, e.g., $\sum_{(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4} 2^{k_1+k_2+k_3+k_4} \#(\mathcal{Q}^0_{\text{nondeg}} \cap (S_{k_1} \times S_{k_2} \times S_{k_3} \times S_{k_4}))$ which includes rectangles where say one vertex is type 1, another vertex is type 2, and the remaining vertices are type 3; those do not appear explicitly in the Proposition 2.

so summing over σ,

$$\sum_{\sigma\in\mathbb{Z}}\Big(\sum_{\substack{\xi_1-\xi_4=\xi\\ \xi_1\cdot\xi=\sigma}}|f(\xi_1)f(\xi_4)|\Big)^2\lesssim\sum_{1\le\alpha,\beta\le3}\sum_{(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4}2^{k_1+k_2+k_3+k_4}\sum_{\substack{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0\\ \xi_1-\xi_4=\xi}}1_{S_{k_1}^{(\xi,\alpha)}}(\xi_1)1_{S_{k_2}^{(\xi,\alpha)}}(\xi_2)1_{S_{k_3}^{(\xi,\beta)}}(\xi_3)1_{S_{k_4}^{(\xi,\beta)}}(\xi_4).$$

To proceed further we classify $(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0$ according to whether $\xi_1-\xi_2=0$. If $\xi_1=\xi_2$ then $1_{S_{k_1}^{(\xi,\alpha)}}(\xi_1)1_{S_{k_2}^{(\xi,\alpha)}}(\xi_2)$ is non-zero only if $k_1=k_2$, in which case it is $\le 1_{S_{k_1}}(\xi_1)$; similarly $1_{S_{k_3}^{(\xi,\beta)}}(\xi_3)1_{S_{k_4}^{(\xi,\beta)}}(\xi_4)$ is non-zero only if $k_3=k_4$, in which case it is $\le 1_{S_{k_4}}(\xi_4)$. On the other hand, if $(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0$, $\xi_1-\xi_4=\xi\neq0$, and $\xi_1-\xi_2\neq0$, then $(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0_{\text{nondeg}}$, and $1_{S_{k_1}^{(\xi,\alpha)}}(\xi_1)1_{S_{k_2}^{(\xi,\alpha)}}(\xi_2)1_{S_{k_3}^{(\xi,\beta)}}(\xi_3)1_{S_{k_4}^{(\xi,\beta)}}(\xi_4)=1$ if and only if $(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0_{\alpha,\beta}(S_{k_1},S_{k_2},S_{k_3},S_{k_4})$, and equals 0 otherwise. Thus for $\xi\neq0$,

$$\sum_{\sigma\in\mathbb{Z}}\Big(\sum_{\substack{\xi_1-\xi_4=\xi\\ \xi_1\cdot\xi=\sigma}}|f(\xi_1)f(\xi_4)|\Big)^2\lesssim\sum_{1\le\alpha,\beta\le3}\sum_{\vec{k}\in\mathbb{Z}^4}2^{\vec{k}}\sum_{\substack{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0_{\alpha,\beta}(S_{\vec{k}})\\ \xi_1-\xi_4=\xi}}1+\sum_{(k_1,k_4)\in\mathbb{Z}^2}2^{2k_1+2k_4}\sum_{\substack{(\xi_1,\xi_4)\in(\mathbb{Z}^2)^2\\ \xi_1-\xi_4=\xi}}1_{S_{k_1}}(\xi_1)1_{S_{k_4}}(\xi_4)$$

where we wrote $2^{\vec{k}}$ for $2^{k_1+k_2+k_3+k_4}$ and $S_{\vec{k}}=(S_{k_1},S_{k_2},S_{k_3},S_{k_4})$ if $\vec{k}=(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4$. As a result,

$$G(\tau)=\sum_{\substack{\xi\in\mathbb{Z}^2\setminus\{0\}\\ \gcd(\xi)|\tau}}\sum_{\sigma\in\mathbb{Z}}\Big(\sum_{\substack{\xi_1-\xi_4=\xi\\ \xi_1\cdot\xi=\sigma}}|f(\xi_1)f(\xi_4)|\Big)^2\lesssim\sum_{1\le\alpha,\beta\le3}\sum_{\vec{k}\in\mathbb{Z}^4}2^{\vec{k}}\sum_{\substack{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0_{\alpha,\beta}(S_{\vec{k}})\\ \gcd(\xi_1-\xi_4)|\tau}}1+\Big(\sum_{k\in\mathbb{Z}}2^{2k}\#S_k\Big)^2$$

Now $\sum_{k\in\mathbb{Z}}2^{2k}\#S_k\lesssim\|f\|^2_{\ell^2(\mathbb{Z}^2)}$. Since the condition $\gcd(\xi_1-\xi_4)|\tau$ is vacuous when $\tau=0$, this shows

$$G(0)\lesssim\sum_{1\le\alpha,\beta\le3}\sum_{\vec{k}\in\mathbb{Z}^4}2^{\vec{k}}\#\mathcal{Q}^0_{\alpha,\beta}(S_{\vec{k}})+\|f\|^4_{\ell^2(\mathbb{Z}^2)}.$$

We also see that for $M\in2^{\mathbb{N}}$

$$\frac{1}{M}\sum_{|\tau|\simeq M}G(\tau)\lesssim\sum_{1\le\alpha,\beta\le3}\sum_{\vec{k}\in\mathbb{Z}^4}2^{\vec{k}}\sum_{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0_{\alpha,\beta}(S_{\vec{k}})}\frac{1}{\gcd(\xi_1-\xi_4)}+\|f\|^4_{\ell^2(\mathbb{Z}^2)}.$$

As a result, (16) follows from Proposition 2. ☐

The geometric ingredients needed in proving Proposition 2 are captured in the following lemmas.

Lemma 3. *Let S_1 be a finite subset of $\mathbb{R}^2$, so that through any point of S_1 there passes at most one line that is heavy with respect to S_1. Then for every $\xi_1\in S_1$ and every $\xi_3\in\mathbb{R}^2$, there exists at most two choices of $(\xi_2,\xi_4)\in\mathbb{R}^2$ such that $Q:=(\xi_1,\xi_2,\xi_3,\xi_4)$ is in $\mathcal{Q}^0_{nondeg}$ and the cross $C(\xi_1,Q)$ is of type 1 with respect to S_1.*

Lemma 4. *Let S_1,S_2,S_3,S_4 be finite subsets of $\mathbb{R}^2$, so that for $j=1,2,3,4$, through any point of S_j there passes at most one line that is heavy with respect to S_j. Then for all $(\theta_1,\theta_2,\theta_3,\theta_4)$ in some open neighborhood U of the point $(\frac12,\frac12,\frac12,\frac12)$ in the 2 dimensional plane $\theta_1+\theta_3=\theta_2+\theta_4=1$, the following is true. If $(\alpha,\beta)=(3,3)$, $(2,3)$ or $(3,2)$, then*

$$\#\mathcal{Q}^0_{\alpha,\beta}(S_1,S_2,S_3,S_4)\lesssim(\#S_1)^{\theta_1}(\#S_2)^{\theta_2}(\#S_3)^{\theta_3}(\#S_4)^{\theta_4}\tag{17}$$

If $(\alpha,\beta)=(2,2)$, then

$$\#\mathcal{Q}^0_{\alpha,\beta}(S_1,S_2,S_3,S_4)\lesssim\log(\max_{j=1,2,3,4}\#S_j)(\#S_1)^{\theta_1}(\#S_2)^{\theta_2}(\#S_3)^{\theta_3}(\#S_4)^{\theta_4}\tag{18}$$

and

$$\sum_{Q\in\mathcal{Q}^0_{\alpha,\beta}(S_1,S_2,S_3,S_4)}\frac{1}{\gcd(\xi_1-\xi_4)}\lesssim(\#S_1)^{\theta_1}(\#S_2)^{\theta_2}(\#S_3)^{\theta_3}(\#S_4)^{\theta_4}.\tag{19}$$

In the next two sections, we first deduce Proposition 2 from Lemma 3 and Lemma 4, and then prove the lemmas. We note that the proof of Proposition 2 via Lemma 4 resembles somewhat an interpolation argument of Keel and Tao [4].

Proof of Proposition 2. First consider the case $\alpha = 1$. Write $S = \bigcup_{k \in \mathbb{Z}} S_k$. Then by Lemma 3, there exists a function $K(\xi_1, \xi_2, \xi_3, \xi_4)$ defined on S^4, such that for any $(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4$,

$$\sum_{1 \leq \beta \leq 3} 1_{\mathcal{Q}^0_{1,\beta}(S_{k_1}, S_{k_2}, S_{k_3}, S_{k_4})}(\xi_1, \xi_2, \xi_3, \xi_4) \leq 1_{S_{k_1}}(\xi_1) 1_{S_{k_2}}(\xi_2) 1_{S_{k_3}}(\xi_3) 1_{S_{k_4}}(\xi_4) K(\xi_1, \xi_2, \xi_3, \xi_4), \tag{20}$$

with

$$\sup_{\xi_1, \xi_3 \in S} \sum_{\xi_2, \xi_4 \in S} K(\xi_1, \xi_2, \xi_3, \xi_4) \leq 2 \quad \text{and} \quad \sup_{\xi_2, \xi_4 \in S} \sum_{\xi_1, \xi_3 \in S} K(\xi_1, \xi_2, \xi_3, \xi_4) \leq 2. \tag{21}$$

In fact, given $(\xi_1, \xi_3) \in S^2$, first choose $k_1 \in \mathbb{Z}$ so that $\xi_1 \in S_{k_1}$. Define $K_1(\xi_1, \xi_2, \xi_3, \xi_4) = 1$ if $Q := (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{Q}^0_{\text{nondeg}}$ and the cross $C(\xi_1, Q)$ is of type 1 with respect to S_{k_1}, and define $K_1(\xi_1, \xi_2, \xi_3, \xi_4) = 0$ otherwise. Similarly, given $(\xi_2, \xi_4) \in S^2$, let $k_2 \in \mathbb{Z}$ so that $\xi_2 \in S_{k_2}$. Define $K_2(\xi_1, \xi_2, \xi_3, \xi_4) = 1$ if $Q := (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{Q}^0_{\text{nondeg}}$ and the cross $C(\xi_2, Q)$ is of type 1 with respect to S_{k_2}, and define $K_2(\xi_1, \xi_2, \xi_3, \xi_4) = 0$ otherwise. Then $K := K_1 K_2$ satisfies (20) by construction, and (21) holds by Lemma 3. Now using (20),

$$\begin{aligned}
&\sum_{(k_1,k_2,k_3,k_4) \in \mathbb{Z}^4} 2^{k_1+k_2+k_3+k_4} \# \mathcal{Q}^0_{\alpha,\beta}(S_{k_1}, S_{k_2}, S_{k_3}, S_{k_4}) \\
&\leq \sum_{(\xi_1,\xi_3) \in S^2} \sum_{(\xi_2,\xi_4) \in S^2} \Big(\sum_{(k_1,k_3) \in \mathbb{Z}^2} 2^{k_1+k_3} 1_{S_{k_1}}(\xi_1) 1_{S_{k_3}}(\xi_3) \Big) \Big(\sum_{(k_2,k_4) \in \mathbb{Z}^2} 2^{k_2+k_4} 1_{S_{k_2}}(\xi_2) 1_{S_{k_4}}(\xi_4) \Big) K(\xi_1, \xi_2, \xi_3, \xi_4)
\end{aligned}$$

which by (21) and Schur's test on $\ell^2(S^2)$ is

$$\leq 2 \sum_{(\xi_1,\xi_3) \in S^2} \Big(\sum_{(k_1,k_3) \in \mathbb{Z}^2} 2^{k_1+k_3} 1_{S_{k_1}}(\xi_1) 1_{S_{k_3}}(\xi_3) \Big)^2 = 2 \Big(\sum_{k \in \mathbb{Z}} 2^{2k} \# S_k \Big)^2.$$

This completes the proof of Proposition 2 when $\alpha = 1$, and the case $\beta = 1$ is similar.

Next, for $(\alpha, \beta) = (3,3)$, $(2,3)$ or $(3,2)$, we use (17) and estimate

$$\sum_{(k_1,k_2,k_3,k_4) \in \mathbb{Z}^4} 2^{k_1+k_2+k_3+k_4} \# \mathcal{Q}^0_{\alpha,\beta}(S_{k_1}, S_{k_2}, S_{k_3}, S_{k_4}) \lesssim \sum_{(k_1,k_2,k_3,k_4) \in \mathbb{Z}^4} 2^{k_1+k_2+k_3+k_4} \inf_{\theta \in U} \# S_{k_1}^{\theta_1} \# S_{k_2}^{\theta_2} \# S_{k_3}^{\theta_3} \# S_{k_4}^{\theta_4}.$$

By shrinking U, we may assume that U is symmetric under permutation of the ξ_1 and ξ_3 coordinates, as well as permutation of the ξ_2 and ξ_4 coordinates. So the above is

$$\begin{aligned}
&\leq 4 \sum_{\substack{(k_1,k_2,k_3,k_4) \in \mathbb{Z}^4 \\ k_1 \leq k_3, k_2 \leq k_4}} 2^{k_1+k_2+k_3+k_4} \# S_{k_1}^{\frac{1}{2}-\delta} \# S_{k_2}^{\frac{1}{2}-\delta} \# S_{k_3}^{\frac{1}{2}+\delta} \# S_{k_4}^{\frac{1}{2}+\delta} \\
&= 4 \Big(\sum_{\substack{(k_1,k_3) \in \mathbb{Z}^2 \\ k_1 \leq k_3}} 2^{k_1+k_3} \# S_{k_1}^{\frac{1}{2}-\delta} \# S_{k_3}^{\frac{1}{2}+\delta} \Big)^2 \\
&= 4 \Big(\sum_{\substack{k_1, k_3 \in \mathbb{Z} \\ k_1 \leq k_3}} 2^{2\delta(k_1-k_3)} (2^{2k_1} \# S_{k_1})^{\frac{1}{2}-\delta} (2^{2k_3} \# S_{k_3})^{\frac{1}{2}+\delta} \Big)^2 \qquad (22)
\end{aligned}$$

for some $\delta \in (0, 1/2]$. Now Schur's test gives

$$\sum_{\substack{k_1, k_3 \in \mathbb{Z} \\ k_1 \leq k_3}} 2^{2\delta(k_1-k_3)} a_{k_1} b_{k_3} \leq \Big(\sum_{k_1 \in \mathbb{Z}} a_{k_1}^p \Big)^{1/p} \Big(\sum_{k_3 \in \mathbb{Z}} b_{k_3}^q \Big)^{1/q}$$

if $\delta > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Applying this with $\frac{1}{p} = \frac{1}{2} - \delta$, $\frac{1}{q} = \frac{1}{2} + \delta$, we have (22) $\leq 4 \Big(\sum_{k \in \mathbb{Z}} 2^{2k} \# S_k \Big)^2$, as desired. Finally, the same proof above establishes (13) and (14) when $(\alpha, \beta) = (2,2)$; we only need to use (18) and (19) in place of (17). $\square$

Proof of Lemma 3. Since through any point of S_1 there passes at most one line that is heavy with respect to S_1, once we pick $\xi_1 \in S_1$, then any rectangle $Q := (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{Q}^0_{\text{nondeg}}$ with $C_{\xi_1,Q}$ being of type 1 with respect to S_1 has a fixed orientation. Thus once ξ_1 and ξ_3 are fixed, (ξ_2, ξ_4) are determined up to permutation, and this establishes Lemma 3. □

Proof of Lemma 4. Fix S_1, S_2, S_3, S_4 as in the lemma, and abbreviate $\mathcal{Q}^0_{\alpha,\beta}(S_1, S_2, S_3, S_4)$ by $Q^0_{\alpha,\beta}$.

First we count $\#\mathcal{Q}^0_{3,3}$: we have

$$\#\mathcal{Q}^0_{3,3} \leq \min\{\#S_1\#S_2, \#S_2\#S_3, \#S_3\#S_4, \#S_4\#S_1\}.$$

This is because once two consecutive vertices of $Q \in \mathcal{Q}^0_{3,3}$ are chosen, the orientation of Q is fixed, and there can only be 1 choice for each of the remaining two vertices (otherwise that vertex would not be type 3). Since $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is in the interior[5] of the convex hull of $(1, 1, 0, 0)$, $(0, 1, 1, 0)$, $(0, 0, 1, 1)$ and $(1, 0, 0, 1)$, and the four vectors affine span the plane $\theta_1 + \theta_3 = \theta_2 + \theta_4 = 1$, this proves the lemma when $(\alpha, \beta) = (3, 3)$.

Next, for $\vec{a} \in \mathbb{Z}^4_{\geq 0}$ let $\mathcal{Q}^0(\vec{a})$ be the set of all $Q = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{Q}^0_{\text{nondeg}} \cap S_1 \times S_2 \times S_3 \times S_4$ so that $2^{a_j} \leq m_{S_j}(C_{\xi_j,Q}) < 2^{a_j+1}$ for $j = 1, 2, 3, 4$. The key fact is that when $2 \leq 2^{a_1} \leq C(\#S_1)^{1/2}$,

$$\#\mathcal{Q}^0(\vec{a}) \lesssim (\#S_1)^2 2^{-2a_1+a_2} \min\{2^{a_3}, 2^{a_4}\}. \tag{23}$$

This is because Szemeredi-Trotter[6] implies that there are $\lesssim (\#S_1)^2/(2^{a_1})^3 + \#S_1/2^{a_1} \lesssim (\#S_1)^2/(2^{a_1})^3$ lines[7] that contain $\geq 2^{a_1}$ points from S_1, and on each such line there are $\leq 2^{a_1+1}$ choices for ξ_1. Once ξ_1 and the 2^{a_1} rich line through ξ_1 are chosen, the orientation of the rectangle in $\mathcal{Q}^0(\vec{a})$ is fixed, and the rectangle is determined by ξ_2 and ξ_3 ($\lesssim 2^{a_2+a_3}$ choices), or ξ_2 and ξ_4 ($\lesssim 2^{a_2+a_4}$ choices). Thus (23) follows.

By the same argument we used to count $\#Q^0_{3,3}$, we also have

$$\#Q^0(\vec{a}) \lesssim \#S_j\#S_{j+1} \min\{2^{a_{j+2}}, 2^{a_{j+3}}\} \tag{24}$$

where we used cyclic notation and identify indices that are congruent mod 4.

When $2 \leq 2^{a_j} \leq C(\#S_j)^{1/2}$ for $j = 1, 2$ and $a_3 = a_4 = 0$, (23) and (24) gives

$$\#\mathcal{Q}^0(\vec{a}) \lesssim \min\{(\#S_1)^2 2^{-2a_1}, (\#S_2)^2 2^{-2a_2}, \#S_3\#S_4 2^{a_1}, \#S_3\#S_4 2^{a_2}\}.$$

Interpolating,

$$\#\mathcal{Q}^0(\vec{a}) \lesssim ((\#S_1)^2 2^{-2a_1})^{\frac{1}{6}+\delta}((\#S_2)^2 2^{-2a_2})^{\frac{1}{6}+\delta}(\#S_3\#S_4 2^{a_1})^{\frac{1}{3}-\delta}(\#S_3\#S_4 2^{a_2})^{\frac{1}{3}-\delta}$$

for all sufficiently small $\delta > 0$, so summing over $a_1, a_2 \geq 1$ we obtain

$$\mathcal{Q}^0_{2,3} \lesssim (\#S_1)^{\frac{1}{3}+2\delta}(\#S_2)^{\frac{1}{3}+2\delta}(\#S_3)^{\frac{2}{3}-2\delta}(\#S_4)^{\frac{2}{3}-2\delta}$$

for all sufficiently small $\delta > 0$. We also have

$$\mathcal{Q}^0_{2,3} \lesssim \min\{\#S_1\#S_2, \#S_2\#S_3, \#S_4\#S_1\}$$

by the argument used to count $\#Q^0_{3,3}$. Since $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is in the interior[8] of the convex hull of $(1, 1, 0, 0)$, $(0, 1, 1, 0)$, $(1, 0, 0, 1)$ and $(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, and the four vectors affine span the plane $\theta_1 + \theta_3 = \theta_2 + \theta_4 = 1$, by continuity the same is true when the last point is replaced by $(\frac{1}{3} + 2\delta, \frac{1}{3} + 2\delta, \frac{2}{3} - 2\delta, \frac{2}{3} - 2\delta)$ whenever δ is sufficiently close to 0. In other words, the convex hull of $(1, 1, 0, 0)$, $(0, 1, 1, 0)$, $(1, 0, 0, 1)$ and $(\frac{1}{3} + 2\delta, \frac{1}{3} + 2\delta, \frac{2}{3} - 2\delta, \frac{2}{3} - 2\delta)$ contains an open neighbourhood of the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in the plane

[5] e.g. $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{4}(1, 1, 0, 0) + \frac{1}{4}(0, 1, 1, 0) + \frac{1}{4}(0, 0, 1, 1) + \frac{1}{4}(1, 0, 0, 1)$

[6] We used $2^{a_1} \geq 2$ here.

[7] The last inequality used $2^{a_1} \leq C(\#S_1)^{1/2}$.

[8] e.g. $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{8}(1, 1, 0, 0) + \frac{1}{4}(0, 1, 1, 0) + \frac{1}{4}(1, 0, 0, 1) + \frac{3}{8}(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

$\theta_1+\theta_3=\theta_2+\theta_4=1$. This proves the lemma when $(\alpha,\beta)=(2,3)$, and the argument when $(\alpha,\beta)=(3,2)$ is similar.

When $2\le 2^{a_j}\le C(\#S_j)^{1/2}$ for $j=1,2,3,4$, (23) gives

$$\#\mathcal{Q}^0(\vec{a})\lesssim \min_{1\le j\le 4}\min_{l=1,2}(\#S_j)^2 2^{-2a_j+a_{j+1}+a_{j+l}}. \tag{25}$$

Since the vectors

$$\begin{aligned} v_1&:=(2,0,0,0,-2,1,1,0), & v_2&:=(2,0,0,0,-2,1,0,1),\\ v_3&:=(0,2,0,0,0,-2,1,1), & v_4&:=(0,2,0,0,1,-2,1,0)\\ v_5&:=(0,0,2,0,1,0,-2,1), & v_6&:=(0,0,2,0,0,1,-2,1),\\ v_7&:=(0,0,0,2,1,1,0,-2), & v_8&:=(0,0,0,2,1,0,1,-2)\end{aligned}$$

span the 6 dimensional plane $\{\theta_1+\theta_2+\theta_3+\theta_4=2 \text{ and } \phi_1+\phi_2+\phi_3+\phi_4=0\}$, and $(\frac12,\frac12,\frac12,\frac12,0,0,0,0)$ is in the interior[9] of the convex hull of $v_1,\dots,v_8$, for any $(\theta_1,\theta_2,\theta_3,\theta_4)$ in a sufficiently small neighbourhood $\tilde U$ of $(\frac12,\frac12,\frac12,\frac12)$ in the 3 dimensional plane $\theta_1+\theta_2+\theta_3+\theta_4=2$, and any $(\phi_1,\phi_2,\phi_3,\phi_4)$ in a neighbourhood V of $(0,0,0,0)$ in the 3 dimensional plane $\phi_1+\phi_2+\phi_3+\phi_4=0$, one has

$$\#Q^0(\vec{a})\lesssim(\#S_1)^{\theta_1}(\#S_2)^{\theta_2}(\#S_3)^{\theta_3}(\#S_4)^{\theta_4}2^{a_1\phi_1+a_2\phi_2+a_3\phi_3+a_4\phi_4}.$$

Now for any $(\theta_1,\theta_2,\theta_3,\theta_4)\in\tilde U$,

$$\begin{aligned}\#\mathcal{Q}^0_{2,2}&\lesssim\sum_{2\le 2^{a_j}\le C(\#S_j)^{1/2}}\#Q^0(\vec{a})\\ &\lesssim(\#S_1)^{\theta_1}(\#S_2)^{\theta_2}(\#S_3)^{\theta_3}(\#S_4)^{\theta_4}\sum_{2\le 2^{a_j}\le C(\#S_j)^{1/2}}\inf_{(\phi_1,\phi_2,\phi_3,\phi_4)\in V}2^{a_1\phi_1+a_2\phi_2+a_3\phi_3+a_4\phi_4},\end{aligned}$$

and for sufficiently small $\delta>0$, V contains the point $(-\delta,0,0,\delta)$, as well as its images under any permutation of its four coordinates. By symmetry, the summation in the last display is

$$\le 16\sum_{\log(\max_j\#S_j)\gtrsim a_1\ge a_2\ge a_3\ge a_4\ge 0}2^{-\delta(a_1-a_4)}\lesssim\log(\max_{j=1,2,3,4}\#S_j),$$

establishing (18) (in fact, we showed a little more than required, since $\tilde U$ is a neighbourhood of $(\frac12,\frac12,\frac12,\frac12)$ in the 3 dimensional plane $\theta_1+\theta_2+\theta_3+\theta_4=2$).

Finally, when $2\le 2^{a_j}\le C(\#S_j)^{1/2}$ for $j=1,2,3,4$,

$$\sum_{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0(\vec{a})}\frac{1}{\gcd(\xi_1-\xi_4)}\lesssim(\#S_1)^2 2^{-2a_1+a_2+a_4/2}. \tag{26}$$

This is because when ξ_1,ξ_2 are fixed ($(\#S_1)^2 2^{-2a_1+a_2}$ choices), then ξ_3 is determined by ξ_4. Furthermore, the set of possible ξ_4 is a subset of an arithmetic progression and has at most 2^{a_4} elements. So summing over (ξ_3,ξ_4) and using

$$\max_{\substack{S\subset\mathbb{N}\\ \#S\le 2^{a_4}}}\sum_{m\in S}\frac1m\lesssim a_4\lesssim 2^{a_4/2}$$

gives (26). One can also bound the left hand side of (26) by $\#\mathcal{Q}^0(\vec{a})$, which we bounded by (25).

Let $v_9=(2,0,0,0,-2,1,0,1/2)$. Then $v_1,\dots,v_9$ span the 7 dimensional plane $\theta_1+\theta_2+\theta_3+\theta_4=2$ in $\mathbb{R}^8$, and $(\frac12,\frac12,\frac12,\frac12,0,0,0,-\varepsilon)$ is in the interior[10] of the convex hull of $v_1,\dots,v_9$ for all sufficiently small $\varepsilon>0$. Fix one such ε. Thus for any $(\theta_1,\theta_2,\theta_3,\theta_4)$ in a sufficiently small neighbourhood $\tilde U$ of $(\frac12,\frac12,\frac12,\frac12)$ in the plane $\theta_1+\theta_2+\theta_3+\theta_4=2$, and any $(\phi_1,\phi_2,\phi_3,\phi_4)$ in a neighbourhood V_ε of $(0,0,0,-\varepsilon)$ in $\mathbb{R}^4$, one has

$$\sum_{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathcal{Q}^0(\vec{a})}\frac{1}{\gcd(\xi_1-\xi_4)}\lesssim(\#S_1)^{\theta_1}(\#S_2)^{\theta_2}(\#S_3)^{\theta_3}(\#S_4)^{\theta_4}2^{a_1\phi_1+a_2\phi_2+a_3\phi_3+a_4\phi_4}.$$

[9] e.g. $(\frac12,\frac12,\frac12,\frac12,0,0,0,0)=\frac18\sum_{i=1}^8 v_i$

[10] $(\frac12,\frac12,\frac12,\frac12,0,0,0,-\varepsilon)=\frac18\sum_{i=1}^8 v_i+\frac{\varepsilon}{2}(v_9-v_8)$

Hence for sufficiently small $\delta > 0$,

$$\begin{aligned}\sum_{Q\in\mathcal{Q}^0_{2,2}} \frac{1}{\gcd(\xi_1-\xi_4)} &\lesssim \sum_{2\le 2^{a_j}\le C(\#S_j)^{1/2}} \sum_{Q\in\mathcal{Q}^0(\vec{a})} \frac{1}{\gcd(\xi_1-\xi_4)} \\ &\lesssim (\#S_1)^{\theta_1}(\#S_2)^{\theta_2}(\#S_3)^{\theta_3}(\#S_4)^{\theta_4} \sum_{a_1,a_2,a_3,a_4\ge 1} \inf_{(\phi_1,\phi_2,\phi_3,\phi_4)\in V_\varepsilon} 2^{a_1\phi_1+a_2\phi_2+a_3\phi_3+a_4\phi_4} \\ &\lesssim (\#S_1)^{\theta_1}(\#S_2)^{\theta_2}(\#S_3)^{\theta_3}(\#S_4)^{\theta_4} \sum_{a_1\ge a_2\ge a_3\ge a_4\ge 1} 2^{-\delta(a_1-a_4)-\varepsilon a_4} \\ &\lesssim (\#S_1)^{\theta_1}(\#S_2)^{\theta_2}(\#S_3)^{\theta_3}(\#S_4)^{\theta_4}.\end{aligned}$$

This completes the proof of (19). $\square$

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