Reflections on Hodge decomposition

Let M be a compact Riemannian manifold without boundary, d and d^* be the densely defined closed operators on $L^2(\Lambda^p TM^*)$, so that domain of d is the set of all forms u with L^2 coefficients such that du (defined in the sense of distributions) has L^2 coefficients, etc.

Theorem 1 (Hodge decomposition on L^2)**.** *There is an orthogonal decomposition*

$$L^2(\Lambda^p TM^*) = \mathcal{H} \oplus Im(d) \oplus Im(d^*)$$

into closed subspaces, where $\mathcal{H} = ker(d) \cap ker(d^*)$. Also $\mathcal{H}$ is finite dimensional.

Proof. It is immediate that the three subspaces are orthogonal to each other; indeed

$$(\mathrm{Im}(d) \oplus \mathrm{Im}(d^*))^\perp = \mathcal{H}.$$

(If $u \perp \mathrm{Im}(d^*)$, then $\langle u, d^* v\rangle = 0$ for all smooth forms v, so $du = 0$ in distribution, and $u \in \mathrm{Dom}(d)$ with $du = 0$.) It follows that

$$L^2(\Lambda^p TM^*) = \mathcal{H} \oplus \overline{\mathrm{Im}(d) \oplus \mathrm{Im}(d^*)}.$$

It remains to prove that $\mathrm{Im}(d)$ and $\mathrm{Im}(d^*)$ are both closed subspaces of L^2, and that $\mathcal{H}$ is finite dimensional. We need two lemmas:

Lemma 1 (Rellich)**.** *$H^1(\Lambda^p TM^*)$ embeds compactly into $L^2(\Lambda^p TM^*)$.*

Lemma 2 (Elliptic estimate)**.** *For any form $u \in Dom(d) \cap Dom(d^*)$,*

$$\|u\|_{H^1} \le C(\|du\|_{L^2} + \|d^* u\|_{L^2} + \|u\|_{L^2}).$$

One then argues, using compactness arguments, that

Lemma 3 (Special elliptic estimate)**.** *For $u \in Dom(d)$ orthogonal to the kernel of d, we have*

$$\|u\|_{H^1} \le C\|du\|_{L^2}.$$

Indeed we need only the weaker estimate

$$\|u\|_{L^2} \le C\|du\|_{L^2}$$

that holds for all u that satisfies the same conditions. By functional analytic arguments, this already implies that the closed operator d has closed range in L^2. (If $du_i \to v$ in L^2, then taking u_i to be orthogonal to the kernel of d (which is possible since the kernel of d is a closed subspace of L^2), we have $\|u_i - u_j\|_{L^2} \le C\|du_i - du_j\|_{L^2}$ so $u_i \to u$ for some $u \in L^2$, and by closedness of d we have $u \in \mathrm{Dom}(d)$ with $du = v$.) Same for $\mathrm{Im}(d^*)$.

Finally, on $\mathcal{H}$ we have, by elliptic estimate, that $\|u\|_{H^1} \simeq \|u\|_{L^2}$. Thus the unit ball of $\mathcal{H}$ in L^2, which is contained in some ball in H^1, is compact in the L^2 topology by Rellich. This proves that $\mathcal{H}$ is finite dimensional. □

Proof of Lemma 3. Suppose the inequality does not hold for any C. Then there is a sequence u_i, all in $\mathrm{Dom}(d)$ and orthogonal to the kernel of d, with

$$\|u_i\|_{H^1} = 1, \qquad \|du_i\|_{L^2} \to 0.$$

Then by Rellich, we may assume $u_i \to u$ in L^2 for some $u \in L^2$. But then $du_i \to du$ in distribution, while $du_i \to 0$ in L^2. Thus $u \in \mathrm{Dom}(d)$ with $du = 0$. Also $u \perp \ker(d)$, because each u_i does and $u_i \to u$ in L^2. It follows that $u = 0$, and that $u_i \to 0$ in L^2. Note also $u_i \in \mathrm{Dom}(d^*)$ with

$$d^* u_i = 0,$$

since for all smooth forms v, $\langle u_i, dv \rangle = 0$ by orthogonality of u to the kernel of d (here $d(dv) = 0$). Hence by elliptic estimate,

$$\|u_i\|_{H^1} \leq C(\|du_i\|_{L^2} + \|u_i\|_{L^2}) \to 0$$

while this is impossible since all $\|u_i\|_{H^1} = 1$. □

It is clear from the theorem that the kernel of d is $\mathcal{H} \oplus \mathrm{Im}(d)$, and that the kernel of d^* is $\mathcal{H} \oplus \mathrm{Im}(d^*)$. One has the corresponding solution operators for d and d^*:

Corollary 1 (Solving d and d^*)**.** *If $f \in L^2$, $f \perp \mathcal{H}$ and $df = 0$, then there exists $u \in L^2$ such that*

$$du = f.$$

Moreover, u can be chosen to be orthogonal to the kernel of d; this determines u uniquely, and such u satisfies the special elliptic estimate

$$\|u\|_{H^1} \leq C\|du\|_{L^2}.$$

We call this u the canonical solution to the equation. Same for d^*.

By techniques in elliptic PDEs, one can also show that $\mathcal{H}$ consists of smooth forms, and that there is a corresponding Hodge decomposition for the Sobolev spaces and the C^∞ spaces.

Next, let Δ be the densely defined closed operator on $L^2(\Lambda^p TM^*)$ defined by $\Delta = dd^* + d^*d$ so that the domain of Δ consists of all u with $u \in \mathrm{Dom}(d) \cap \mathrm{Dom}(d^*)$, $du \in \mathrm{Dom}(d^*)$ and $d^*u \in \mathrm{Dom}(d)$. Then

Corollary 2 (Hodge decomposition for L^2)**.** *$Ker(\Delta) = \mathcal{H}$, and there exists an orthogonal decomposition*

$$L^2(\Lambda^p TM^*) = \mathcal{H} \oplus Im(\Delta)$$

into closed subspaces.

Proof. $\mathrm{Ker}(\Delta) = \mathcal{H}$ is clear: if $u \in \mathrm{Ker}(\Delta)$, then

$$\langle du, du \rangle + \langle d^*u, d^*u \rangle = \langle u, \Delta u \rangle = 0,$$

which implies $du = d^*u = 0$. The converse is even easier. Also

$$\mathrm{Im}(\Delta)^\perp = \mathcal{H} :$$

indeed if $u \in \mathrm{Im}(\Delta)^\perp$, then $\Delta u = 0$ in distribution, so u is smooth and $u \in \mathcal{H}$. It remains to see that $\mathrm{Im}(\Delta)$ is a closed subspace of L^2. The argument is analogous to the above; indeed the elliptic estimate

$$\|u\|_{H^2} \leq C(\|du\|_{H^1} + \|d^*u\|_{H^1} + \|u\|_{H^1})$$

implies

$$\|u\|_{H^2} \leq C(\|\Delta u\|_{L^2} + \|u\|_{H^1}),$$

because

$$\begin{aligned}
\|u\|_{H^2}^2 &\leq C(\|du\|_{H^1}^2 + \|d^*u\|_{H^1}^2 + \|u\|_{H^1}^2) \\
&= C(\langle \Delta u, u\rangle_{H^1} + \|u\|_{H^1}^2) \\
&\leq C(\|\Delta u\|_{L^2}\|u\|_{H^2} + \|u\|_{H^1}^2) \\
&\leq \epsilon\|u\|_{H^2}^2 + C(\|\Delta u\|_{L^2}^2 + \|u\|_{H^1}^2).
\end{aligned}$$

Thus we can play the same compactness argument as before, proving that

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}$$

for $u \in \mathrm{Dom}(\Delta)$ orthogonal to kernel of Δ. The rest is a direct analogue of the case for d and d^*. □

There is also a corresponding solution operator for Δ.

Corollary 3 (Solving Δ)**.** *If $f \in L^2$ and $f \perp \mathcal{H}$, then there exists $u \in L^2$ such that*

$$\Delta u = f.$$

Moreover, u can be chosen to be orthogonal to the kernel of Δ; this determines u uniquely, and such u satisfies the special elliptic estimate

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2}.$$

We call this u the canonical solution to the equation, and write $u = Nf$.

For $f \perp \mathcal{H}$ with $df = 0$, $u = d^*Nf$ is the canonical solution to $du = f$. Thus

$$\|d^*Nf\|_{H^1} \leq C\|f\|_{L^2}.$$

This estimate is true without the condition that $df = 0$: (oh but this can be seen as the corollary of that N maps L^2 to H^2)

Proposition 1. *For $f \in L^2$, $f \perp \mathcal{H}$,*

$$\|d^*Nf\|_{H^1} \leq C\|f\|_{L^2}.$$

Same for d.

Proof. This is because d^*Nf is orthogonal to the kernel of d, and the special elliptic estimate applies:

$$\|d^*Nf\|_{H^1} \leq C\|dd^*Nf\|_{L^2}.$$

Now

$$\begin{aligned}
\|dd^*Nf\|_{L^2}^2 &= \langle d^*dd^*Nf, d^*Nf\rangle = \langle d^*\Delta Nf, d^*Nf\rangle \\
&= \langle d^*f, d^*Nf\rangle = \langle f, dd^*Nf\rangle \\
&\leq \|f\|_{L^2}\|dd^*Nf\|_{L^2}
\end{aligned}$$

which implies the desired estimate. □

Finally, note that as long as some subelliptic estimates hold in place of the elliptic estimate, we can get some versions of all the above theorems (with suitable modifications in regularities of the solution operators).