New bounds for large values of Dirichlet polynomials

Joint with James Maynard

Estimates for large values of Dirichlet polynomials appear in analytic number theory, in connection with bounds for the Riemann zeta function and the distribution of prime numbers in short intervals.

We prove some new estimates for Dirichlet polynomials, which lead to small improvements in bounds about zeta and primes.

Trying to understand large values of Dirichlet polynomials is also interesting from the point of view of harmonic analysis.

Applications: Riemann zeta function

$\zeta(s)$ is the Riemann zeta function.

$N(\sigma, T) := \#\{s : \zeta(s) = 0, \Re(s) > \sigma, |\Im(s)| \leq T\}$.

Previous bounds for $N(\sigma, T)$:

- Ingham

- Halasz-Montgomery-Huxley: improvements for $\sigma > 3/4$.

We prove new bounds for $7/10 < \sigma \leq 3/4 +$ little bit.

Example:

- $N(3/4, T) \lessapprox T^{\frac{6}{10}}$ (Ingham)
- $N(3/4, T) \lessapprox T^{\frac{5}{9}}$ (G-Maynard)

Applications: primes in short intervals

Prime number theorem: as $x \to \infty$,

Number of primes in $[x, 2x] = (1 + o(1)) \int_x^{2x} \frac{du}{\log u}$.

Question: Suppose $x \to \infty$. When do we have

Number of primes in $[x, x + y] = (1 + o(1)) \int_x^{x+y} \frac{du}{\log u}$?

Conjecture: True if $y \geq c_\epsilon x^\epsilon$ for any $\epsilon > 0$.

Riemann hypothesis implies: True if $y \geq c_\epsilon x^{\frac{1}{2}+\epsilon}$.

Previous bound (Huxley): True if $y \geq c_\epsilon x^{\frac{7}{12}+\epsilon}$.

New bound (G-Maynard): True if $y \geq c_\epsilon x^{\frac{17}{30}+\epsilon}$.

Setup

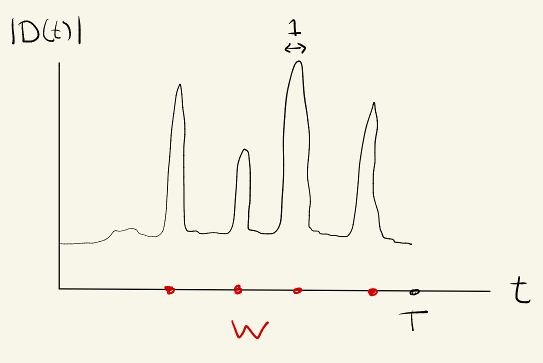
- $D(t) = \sum_{n=N}^{2N} b_n e^{it \log n}$
- $|b_n| \leq 1$.

The goal is to understand the superlevel set where $|D(t)| > N^{\sigma}$.

Setup

- $D(t) = \sum_{n=N}^{2N} b_n e^{it \log n}$
- $|b_n| \leq 1$.
- $|D(t)| > N^\sigma$ for $t \in W$.
- $W \subset [0, T]$ is 1-separated set.

Large value problem: Estimate $|W|$ in terms of N, T, σ.



Remark: $|D(t)|$ is morally $\approx$ constant on unit intervals.

Known bounds

Recall $D(t) = \sum_{n=N}^{2N} b_n e^{it \log n}$

Basic orthogonality bound. If $W \subset [0, T]$ 1-separated, and $T \geq N$, then

$\sum_{t \in W} |D(t)|^2 \lesssim T \sum_n |b_n|^2$.

Proof sketch.

$\sum_{t \in W} |D(t)|^2 \lesssim \int_0^T |D(t)|^2 dt \lesssim T \sum_n |b_n|^2$.

Known bounds

Setup

- $D(t) = \sum_{n=N}^{2N} b_n e^{it \log n}$
- $|b_n| \leq 1$.
- $|D(t)| > N^{\sigma}$ for $t \in W$.
- $W \subset [0, T]$ is 1-separated set, $T \geq N$.

Orthogonality gives

$|W| N^{2\sigma} \leq \sum_{t \in W} |D(t)|^2 \lesssim T \sum_n |b_n|^2 \lesssim TN.$

Conjecture (Montgomery) If Setup and $\sigma > 1/2$, then

$|W| N^{2\sigma} \lessapprox N^2.$

Orthogonality gives sharp bounds for $T = N$.

Known bounds beyond orthogonality

Power trick.

- D^2, D^3, etc. are Dirichlet polynomials.
- Can apply orthogonality bound to $D^2, D^3, \ldots$
- Gives strong bounds for $T = N^2, N^3, \ldots$

Halasz-Montgomery method.

- Shows conjecture true for large σ.
- But if $\sigma \leq 3/4$, gives no information.
- We will describe in detail.

If $N \leq T \leq N^{3/2}$ and $\sigma \leq 3/4$, orthogonality was the best known bound.

The new estimate

Setup

- $D(t) = \sum_{n=N}^{2N} b_n e^{it \log n}$
- $|b_n| \leq 1$.
- $|D(t)| > N^\sigma$ for $t \in W$.
- $W \subset [0, T]$ is 1-separated set, $T \geq N$.

Theorem (G - Maynard) Suppose Setup and $N^{6/5} \leq T$ and $\frac{7}{10} < \sigma$. Then

$$|W| N^{2\sigma} \lessapprox N^{-\epsilon(\sigma)} TN,$$

where $\epsilon(\sigma) > 0$ (explicit but a little messy).

Example $\epsilon(3/4) = 1/10$.

General harmonic analysis setup

Φ set of frequencies. (For us, $\Phi = \{\log n\}_{n=N}^{2N}$)

$D(t) = \sum_{\xi \in \Phi} b_\xi e^{i\xi t}$.

Want to study $D(t)$ on a finite set $W \subset \mathbb{R}$.

General harmonic analysis setup

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Want to study $D(t)$ on a finite set $W \subset \mathbb{R}$.

$M = M_{\Phi, W}$ matrix
Input: the coefficients b_ξ
Output: $D(t)$ for $t \in W$.

Coefficients of the matrix M are:
$M_{t,\xi} = e^{i\xi t}$.
(Rows indexed by $t \in W$ and columns are indexed by $\xi \in \Phi$.)

Sanity check:

$(M\vec{b})_t = \sum_{\xi \in \Phi} M_{t,\xi} b_\xi = \sum_{\xi \in \Phi} b_\xi e^{i\xi t} = D(t)$.

General harmonic analysis setup

Φ set of frequencies. (For us, $\{\log n\}_{n=N}^{2N}$)

$D(t) = \sum_{\xi \in \Phi} b_\xi e^{i\xi t}$.

To study $D(t)$ on a set $W \subset \mathbb{R}$.

$M = M_{\Phi,W}$ matrix
Input: the coefficients b_ξ
Output: $D(t)$ for $t \in W$.

The singular values of $M_{\Phi,W}$ control how $D(t)$ can behave on W.
Singular values of M are $s_1(M) \geq s_2(M) \geq \ldots$

$\sum_{t \in W} |D(t)|^2 = \|M_{\Phi,W} \vec{b}\|^2 \leq s_1(M_{\Phi,W})^2 \sum_n |b_n|^2$.

The problem boils down to estimating $s_1(M_{\Phi,W})$ for all possible W.

The TT^* method

To understand $s_1(M)$, it helps to look at MM^* or M^*M.

We have

$$s_1(M)^2 = \lambda_1(MM^*) = \lambda_1(M^*M).$$

We will apply this to our matrix $M_{\Phi,W}$.

The TT^* method

$s_1(M)^2 = \lambda_1(MM^*) = \lambda_1(M^*M)$.

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(Rows indexed by $t \in W$ and columns are indexed by $\xi \in \Phi$.)

Can compute MM^* and M^*M, and they have nice formulas for the coefficients.

The formulas are in terms of

$\hat{W}(\xi) := \sum_{t \in W} e^{-it\xi}$;
$\Phi^\vee := \sum_{\xi \in \Phi} e^{it\xi}$.

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Coefficients of the matrix M are:
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$\hat{W}(\xi) := \sum_{t \in W} e^{-it\xi}$;
$\Phi^\vee := \sum_{\xi \in \Phi} e^{it\xi}$.

MM^* has rows and columns indexed by $t \in W$.
$(MM^*)_{t_1,t_2} = \Phi^\vee(t_1 - t_2)$.

M^*M has rows and columns indexed by $\xi \in \Phi$.
$(M^*M)_{\xi_1,\xi_2} = \hat{W}(\xi_1 - \xi_2)$.

The Halasz-Montgomery method

$M = M_{\Phi,W}$ matrix, $\Phi^\vee := \sum_{\xi \in \Phi} e^{it\xi}$

MM^* has rows and columns indexed by $t \in W$.
$(MM^*)_{t_1,t_2} = \Phi^\vee(t_1 - t_2)$.

Bound entries of MM^*. $\Phi^\vee(0) = N$.

Conjecture A. $|\Phi^\vee(t)| \lessapprox N^{1/2}$ for $1 \le |t| \le N^{O(1)}$.
(Know $|\Phi^\vee(t)| \lessapprox |t|^{1/2}$ for ")

Conjecture A implies $s_1(M_{\Phi,W})^2 = \lambda_1(MM^*) \lessapprox N + N^{1/2}|W|$.

In our Setup, this gives
$|W|N^{2\sigma} \lessapprox N^2 + N^{3/2}|W|$.

For $\sigma > 3/4$, this implies the Montgomery conjecture.
For $\sigma \le 3/4$, this gives no information on W.

Pause to reflect

If $N \leq T \leq N^{3/2}$ and $\sigma \leq 3/4$, then basic orthogonality was the best known bound.

I'm quite struck by how difficult it is to improve on this simple argument.

Pause to reflect

The TT^* method appears in many places in harmonic analysis.

- Kolmogorov-Seliverstov-Plessner (convergence of Fourier series 1920s)
- Halasz-Montgomery (late 1960s)
- Tomas-Stein and Strichartz (restriction theory 1970s)
- Mattila (geometric measure theory 1980s)
- Bourgain (periodic version of Strichartz 1990s)

In the last two applications, the results of TT^* are only sharp above a threshold, analogous to $\sigma > 3/4$.
Sharp estimates below the threshold were proven by Wolff and Bourgain-Demeter using different methods, like wave packets.

Fu-G-Maldague tried to study Dirichlet polynomials with wave packets, no progress on main question.

Almost counterexample

In our problem, it is hard to improve on orthogonality because there is a cousin of our problem where orthogonality is actually sharp.

This almost counterexample helped motivate some of our approach.

Almost counterexample

$\Phi_C = \{\sqrt{\frac{n}{N}}\}_{n=N}^{2N}$. (Many properties similar to $\Phi = \{\log n\}_{n=N}^{2N}$.)

$D(t) := \sum_{n=N}^{2N} b_n e^{it\sqrt{\frac{n}{N}}}$.
Normalize: $\sum_n |b_n|^2 = N$.
$|D(t)| > N^{\sigma}$ on $W \subset [0, T]$ 1-separated.

Orthogonality gives $|W| N^{2\sigma} \lesssim TN$.

Conjecture B. $|\Phi_C^{\vee}(t)| \lessapprox N^{1/2}$ for $1 \leq |t| \leq N^{O(1)}$.
HM Method: Conj B implies that $|W| N^{2\sigma} \lessapprox N^2$ for $\sigma > 3/4$.

For every $\sigma \leq 3/4$, the bound $|W| N^{2\sigma} \lesssim TN$ is sharp!

We will write down the example for $\sigma = 3/4$.

Almost counterexample

$\Phi_C = \{\sqrt{\frac{n}{N}}\}_{n=N}^{2N}$. (Many properties similar to $\Phi = \{\log n\}_{n=N}^{2N}$.)

$D(t) := \sum_{n=N}^{2N} b_n e^{it\sqrt{\frac{n}{N}}}$.
Normalize: $\sum_n |b_n|^2 = N$.
$|D(t)| \gtrsim N^{3/4}$ on $W \subset [0, T]$ 1-separated.

The key fact is that Φ_C contains an arithmetic progression of length $\sim N^{1/2}$.

The arithmetic progression comes from n of the form m^2.

$\sqrt{\frac{m^2}{N}} = \frac{m}{\sqrt{N}}$.

Almost counterexample

$D(t) := \sum_{n=N}^{2N} b_n e^{it\sqrt{\frac{n}{N}}}$.
Normalize: $\sum_n |b_n|^2 = N$.
$|D(t)| \gtrsim N^{3/4}$ on $W \subset [0, T]$ 1-separated.

$$b_n = \begin{cases} N^{1/4} & \text{if } n = m^2 \\ 0 & \text{else} \end{cases}.$$

$$D(t) = N^{1/4} \sum_{m \sim N^{1/2}} e^{it\frac{m}{\sqrt{N}}}.$$

$D(0) \sim N^{3/4}$. And $D(t)$ is $\sqrt{N}$-periodic.
$W = \sqrt{N}\mathbb{Z} \cap [0, T]$.

Almost counterexample

Setup of almost counterexample
$D(t) := \sum_{n=N}^{2N} b_n e^{it\sqrt{\frac{n}{N}}}$.
Normalize: $\sum_n |b_n|^2 = N$.
$|D(t)| \gtrsim N^{3/4}$ on $W \subset [0, T]$ 1-separated.

This setup is different from our actual setup in two ways:

- Frequencies are $\sqrt{\frac{n}{N}}$ instead of $\log n$.
- $\sum_n |b_n|^2 = N$ instead of $|b_n| \leq 1$ for all n.

Some methods are not sensitive to these differences and so cannot help us.

Our new method will use both. In particular, it will distinguish $\log n$ from $\sqrt{\frac{n}{N}}$.

Singular values of matrices again

For any matrix M we have

$$s_1(M)^2 = \lambda_1(MM^*) = \lambda_1(M^*M),$$

and so for any integer $r \geq 1$,

$$\sum_i s_i(M)^{2r} = \mathrm{Trace}((MM^*)^r) = \mathrm{Trace}((M^*M)^r).$$

For $M = M_{\Phi,W}$ we can compute these.
Recall $\Phi^\vee := \sum_{\xi \in \Phi} e^{it\xi}$; $\hat{W}(\xi) := \sum_{t \in W} e^{-it\xi}$.

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$$\mathrm{Trace}((MM^*)^r) = \sum_{t_1,\ldots,t_r \in W} \Phi^\vee(t_1 - t_2)\Phi^\vee(t_2 - t_3)\ldots\Phi^\vee(t_r - t_1).$$

$$\mathrm{Trace}((M^*M)^r) = \sum_{\xi_1,\ldots,\xi_r \in \Phi} \hat{W}(\xi_1 - \xi_2)\hat{W}(\xi_2 - \xi_3)\ldots\hat{W}(\xi_r - \xi_1).$$

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If $r = 2$, closely related to Halasz-Montgomery method.
So $r = 2$ gives no information if $\sigma \leq 3/4$.
We use $r = 3$.

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If $r = 2$, closely related to Halasz-Montgomery method.
So $r = 2$ gives no information if $\sigma \leq 3/4$.
We use $r = 3$.

Remark. If $r = 2$ all terms are positive.
If $r \geq 3$, they are not positive, and we cannot afford to use the triangle inequality.

Cannot afford the triangle inequality

$\sum_i s_i(M)^{2r} = \mathrm{Trace}((M^* M)^r) =$

$\sum_{\xi_1,\ldots,\xi_r \in \Phi} \hat{W}(\xi_1 - \xi_2)\hat{W}(\xi_2 - \xi_3)\ldots\hat{W}(\xi_r - \xi_1).$

$\leq \sum_{\xi_1,\ldots,\xi_r \in \Phi} \left|\hat{W}(\xi_1 - \xi_2)\hat{W}(\xi_2 - \xi_3)\ldots\hat{W}(\xi_r - \xi_1)\right| = (*).$

Recall $\hat{W}(\xi) = \sum_{t \in W} e^{-it\xi}$.
Best hope is $|\hat{W}(\xi)| \lessapprox |W|^{1/2}$. (True for random W)

Recall $|\Phi| = N$.

Best hope for $|(*)| \sim N^r |W|^{r/2}$. Too big for our application.

So we need to prove some cancellation.
Tricky because we need to do it for every set W.

Special features of our frequency set

For Dirichlet polynomials, $\Phi = \{\log n\}_{n \sim N}$.
Plugging into our last formula, we get

$\sum_i s_i(M_{\Phi,W})^6 =$

$\sum_{n_1,n_2,n_3 \sim N} \hat{W}(\log n_1 - \log n_2)\hat{W}(\log n_2 - \log n_3)\hat{W}(\log n_3 - \log n_1)$

$= \sum_{n_1,n_2,n_3 \sim N} \hat{W}\left(\log(\frac{n_1}{n_2})\right) \hat{W}\left(\log(\frac{n_2}{n_3})\right) \hat{W}\left(\log(\frac{n_3}{n_1})\right)$

Since we sum over integers n, it helps to use Poisson summation.

Special features of our frequency set

$$\sum_i s_i(M_{\Phi,W})^6 =$$
$$\sum_{n_1,n_2,n_3\sim N} \hat{W}\left(\log(\tfrac{n_1}{n_2})\right) \hat{W}\left(\log(\tfrac{n_2}{n_3})\right) \hat{W}\left(\log(\tfrac{n_3}{n_1})\right)$$

$$= \sum_{n_1,n_2,n_3\in\mathbb{Z}} \psi\left(\tfrac{n}{N}\right) \hat{W}\left(\log(\tfrac{n_1}{n_2})\right) \hat{W}\left(\log(\tfrac{n_2}{n_3})\right) \hat{W}\left(\log(\tfrac{n_3}{n_1})\right),$$

where $n = (n_1, n_2, n_3)$ and ψ is a cutoff function.

Now we can do Poisson summation.

Special features of our frequency set

$\sum_i s_i(M_{\Phi,W})^6 =$

$$= \sum_{n_1,n_2,n_3\in\mathbb{Z}} \underbrace{\psi\left(\frac{n}{N}\right)\hat{W}\left(\log(\frac{n_1}{n_2})\right)\hat{W}\left(\log(\frac{n_2}{n_3})\right)\hat{W}\left(\log(\frac{n_3}{n_1})\right)}_{G(n)}$$

$$= \sum_{m\in\mathbb{Z}^3}\hat{G}(m).$$

Special features of our frequency set

$\sum_i s_i(M_{\Phi,W})^6 =$

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$$= \sum_{m \in \mathbb{Z}^3} \hat{G}(m).$$

The frequency $m = 0$ is special.
It can be computed precisely. It only depends on $|W|$.
It describes the average behavior of a set W of given cardinality.

So we have to estimate $\hat{G}(m)$ for $m \neq 0$.
We will write it out carefully and find a cancellation that depends on special structure of $\log n$.

Special features of our frequency set

$\sum_i s_i(M_{\Phi,W})^6 =$

$$= \sum_{n_1,n_2,n_3 \in \mathbb{Z}} \underbrace{\psi\left(\frac{n}{N}\right) \hat{W}\left(\log(\frac{n_1}{n_2})\right) \hat{W}\left(\log(\frac{n_2}{n_3})\right) \hat{W}\left(\log(\frac{n_3}{n_1})\right)}_{G(n)}$$

$\hat{G}(m) = \int_{\mathbb{R}^3} e^{imx} G(x) dx =$
$\int_{\mathbb{R}^3} e^{imx} \psi\left(\frac{x}{N}\right) \hat{W}\left(\log(\frac{x_1}{x_2})\right) \hat{W}\left(\log(\frac{x_2}{x_3})\right) \hat{W}\left(\log(\frac{x_3}{x_1})\right) dx.$

First observation: $\hat{W}$ factors only really depend on two variables.

$v_1 = \frac{x_1}{x_3}$ and $v_2 = \frac{x_2}{x_3}$, so

$\hat{W}\left(\log(\frac{x_1}{x_2})\right) \hat{W}\left(\log(\frac{x_2}{x_3})\right) \hat{W}\left(\log(\frac{x_3}{x_1})\right) =$
$\hat{W}\left(\log(\frac{v_1}{v_2})\right) \hat{W}\left(\log(v_2)\right) \hat{W}\left(\log(\frac{1}{v_1})\right)$

Special features of our frequency set

$$\hat{G}(m) = \int_{\mathbb{R}^3} e^{imx} \psi\left(\frac{x}{N}\right) \hat{W}\left(\log(\tfrac{x_1}{x_2})\right) \hat{W}\left(\log(\tfrac{x_2}{x_3})\right) \hat{W}\left(\log(\tfrac{x_3}{x_1})\right) dx.$$

New variables: $v_1 = \frac{x_1}{x_3}$ and $v_2 = \frac{x_2}{x_3}$ and x_3.

$$\int_{\mathbb{R}^3} e^{imx} \psi \, \hat{W}\left(\log(\tfrac{v_1}{v_2})\right) \hat{W}\left(\log(v_2)\right) \hat{W}\left(\log(\tfrac{1}{v_1})\right) \mathrm{Jac}\, dv_1 dv_2 dx_3.$$

Which integral should we do first?

Special features of our frequency set

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New variables: $v_1 = \frac{x_1}{x_3}$ and $v_2 = \frac{x_2}{x_3}$ and x_3.

$$\int_{\mathbb{R}^3} e^{imx} \psi \, \hat{W}\left(\log\left(\frac{v_1}{v_2}\right)\right) \hat{W}\left(\log(v_2)\right) \hat{W}\left(\log\left(\frac{1}{v_1}\right)\right) \, \mathrm{Jac} \, dv_1 dv_2 dx_3.$$

$$= \int \left(\int e^{imx} \psi \mathrm{Jac} dx_3\right) \hat{W}\left(\log\left(\frac{v_1}{v_2}\right)\right) \hat{W}\left(\log(v_2)\right) \hat{W}\left(\log\left(\frac{1}{v_1}\right)\right) dv.$$

The red integral doesn't depend on W. It has cancellation.

Special features of our frequency set

$$\hat{G}(m) = \int_{\mathbb{R}^3} e^{imx}\psi\left(\frac{x}{N}\right)\hat{W}\left(\log(\frac{x_1}{x_2})\right)\hat{W}\left(\log(\frac{x_2}{x_3})\right)\hat{W}\left(\log(\frac{x_3}{x_1})\right)dx.$$

New variables: $v_1 = \frac{x_1}{x_3}$ and $v_2 = \frac{x_2}{x_3}$ and x_3.

$$= \int \left(\int e^{imx}\psi \mathrm{Jac}\, dx_3\right)\hat{W}\left(\log(\frac{v_1}{v_2})\right)\hat{W}\left(\log(v_2)\right)\hat{W}\left(\log(\frac{1}{v_1})\right)dv.$$

The function ψJac is a nice function of x_3: picture a smooth bump on $[N, 2N]$.

Now v_1, v_2 are fixed and x_3 is changing.

$e^{imx} = e^{i(m_1x_1+m_2x_2+m_3x_3)}$.

We have to write it in terms of v_1, v_2, x_3. It works out in a special way.

Special features of our frequency set

$$\hat{G}(m) = \int_{\mathbb{R}^3} e^{imx}\psi\left(\frac{x}{N}\right)\hat{W}\left(\log(\frac{x_1}{x_2})\right)\hat{W}\left(\log(\frac{x_2}{x_3})\right)\hat{W}\left(\log(\frac{x_3}{x_1})\right)dx.$$

New variables: $v_1 = \frac{x_1}{x_3}$ and $v_2 = \frac{x_2}{x_3}$ and x_3.

$$\int \left(\int e^{imx}\psi \mathrm{Jac}\, dx_3\right)\hat{W}\left(\log(\frac{v_1}{v_2})\right)\hat{W}\left(\log(v_2)\right)\hat{W}\left(\log(\frac{1}{v_1})\right)dv_1 dv_2.$$

The function ψJac is a smooth bump on $[N, 2N]$.

Note $x_1 = v_1 x_3$ and $x_2 = v_2 x_3$.

$$e^{imx} = e^{i(m_1x_1+m_2x_2+m_3x_3)} = e^{i(m_1v_1+m_2v_2+m_3)x_3}.$$

So the red integral is TINY unless $|m_1v_1 + m_2v_2 + m_3| \lessapprox \frac{1}{N}$!

Special features of our frequency set

$$\hat{G}(m) = \int_{\mathbb{R}^3} e^{imx}\psi\left(\frac{x}{N}\right)\hat{W}\left(\log(\frac{x_1}{x_2})\right)\hat{W}\left(\log(\frac{x_2}{x_3})\right)\hat{W}\left(\log(\frac{x_3}{x_1})\right)dx.$$

New variables: $v_1 = \frac{x_1}{x_3}$ and $v_2 = \frac{x_2}{x_3}$ and x_3.

$$\int \left(\int e^{imx}\psi \mathrm{Jac}\, dx_3\right)\hat{W}\left(\log(\frac{v_1}{v_2})\right)\hat{W}\left(\log(v_2)\right)\hat{W}\left(\log(\frac{1}{v_1})\right)dv_1 dv_2.$$

The function ψJac is a smooth bump on $[N, 2N]$.

$$e^{imx} = e^{i(m_1x_1+m_2x_2+m_3x_3)} = e^{i(m_1v_1+m_2v_2+m_3)x_3}.$$

So the red integral is TINY unless $|m_1v_1 + m_2v_2 + m_3| \lessapprox \frac{1}{N}$!

This key fact is a special feature of $\log n$.
If we had frequencies $\sqrt{\frac{n}{N}}$, then we would have different formulas for v_1, v_2, and $e^{imx} = e^{ig_{m,v_1,v_2}(x_3)}$ with $g_{m,v_1,v_2}(x_3)$ nonlinear!

Special features of our frequency set

$$\hat{G}(m) = \int_{\mathbb{R}^3} e^{imx}\psi\left(\frac{x}{N}\right)\hat{W}\left(\log(\frac{x_1}{x_2})\right)\hat{W}\left(\log(\frac{x_2}{x_3})\right)\hat{W}\left(\log(\frac{x_3}{x_1})\right)dx.$$

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$$\int \left(\int e^{imx}\psi \mathrm{Jac}\, dx_3\right)\hat{W}\left(\log(\frac{v_1}{v_2})\right)\hat{W}(\log(v_2))\,\hat{W}\left(\log(\frac{1}{v_1})\right)dv_1 dv_2.$$

The red integral is TINY unless $|m_1 v_1 + m_2 v_2 + m_3| \lessapprox \frac{1}{N}$!
Triangle inequality: $\left|\int e^{imx}\psi \mathrm{Jac}\, dx_3\right| \lesssim N^3$.

So $|\hat{G}(m)| \lesssim$
$$N^3 \int_{|m_1 v_1 + m_2 v_2 + m_3| \lessapprox \frac{1}{N}} \left|\hat{W}\left(\log(\frac{v_1}{v_2})\right)\hat{W}(\log(v_2))\,\hat{W}\left(\log(\frac{1}{v_1})\right)\right| dv_1 dv_2.$$

At this point, we do have to deal with $|\hat{W}|$.

Special features of our frequency set

$$\hat{G}(m) = \int_{\mathbb{R}^3} e^{imx} \psi\left(\frac{x}{N}\right) \hat{W}\left(\log(\tfrac{x_1}{x_2})\right) \hat{W}\left(\log(\tfrac{x_2}{x_3})\right) \hat{W}\left(\log(\tfrac{x_3}{x_1})\right) dx.$$

New variables: $v_1 = \frac{x_1}{x_3}$ and $v_2 = \frac{x_2}{x_3}$ and x_3.

$$= \int \left(\int e^{imx} \psi \mathrm{Jac}\, dx_3\right) \hat{W}(v_1) \hat{W}(v_2/v_1) \hat{W}(1/v_2) dv_1 dv_2.$$

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Special case $|\hat{W}(\xi)| \lessapprox |W|^{1/2}$ for all ξ (away from 0).
Plug in and we get an improved bound for our main problem.

Additive energy

$$E(W) := \#\{t_1, t_2, t_3, t_4 \in W : |t_1 + t_2 - t_3 - t_4| \leq 1\}.$$

Classical fact from additive combinatorics: $E(W)$ is approximately

$\int_{|\xi|\leq 1} |\hat{W}(\xi)|^4 d\xi.$

Small energy case

- $|\hat{W}(\xi)|$ is small most of the time.
- Plug that in to bound $|\hat{W}|$ factors in our bound for $|\hat{G}(m)|$.

Large energy case

- The large number of additive quadruples helps us.
- Connected to work of Heath-Brown from the 70s.

Why does additive structure help?

Theorem (Heath-Brown):
If $D(t) = \sum_{n \sim N} b_n e^{it \log n}$ and $|b_n| \leq 1$,
and $\mathcal{T} \subset [0, T]$ 1-separated,
and $N \leq T \leq N^{4/3}$
then $\sum_{t_1, t_2 \in \mathcal{T}} |D(t_1 - t_2)|^2 \lessapprox |\mathcal{T}|^2 N + N^2 |\mathcal{T}|$.

Bound is sharp.
We have $|\mathcal{T}|^2$ terms with $|D(t)| \sim N^{1/2}$.
If $b_n = 1$ for all n, then we have $|\mathcal{T}|$ terms with $t_1 = t_2$ and $|D(t_1 - t_2)| = N$.

This gives a sharp bound for W of the form $\mathcal{T} - \mathcal{T}$.

And it leads to good bounds whenever W has large energy.

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Heath-Brown's proof has several cool ideas. We will describe the first step.
Recall $D(t) = \sum_{n \sim N} b_n e^{it \log n}$ and let $D_0(t) = \sum_{n \sim N} e^{it \log n}$.

By Plancherel

$$\sum_{t_1, t_2 \in \mathcal{T}} |D(t_1 - t_2)|^2 = \sum_{n_1, n_2 \sim N} b_{n_1} \bar{b}_{n_2} |\hat{W}(\log n_1 - \log n_2)|^2$$

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$$\leq \sum_{n_1, n_2 \sim N} |\hat{W}(\log n_1 - \log n_2)|^2 = \sum_{t_1, t_2 \in \mathcal{T}} |D_0(t_1 - t_2)|^2.$$

So it suffices to study D_0!