A few simple perspectives on Fourier uncertainty

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Fourier Uncertainty Principle

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- One of our key points of emphasis today is connections between Fourier uncertainty and exact signal recovery.

Restriction Conjecture

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(Restriction conjecture) The restriction conjecture says that if S is the unit sphere, then

$$\left(\int_S |\widehat{f}(\xi)|^r d\sigma_S(\xi)\right)^{\frac{1}{r}} \leq C_{p,r}\left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{\frac{1}{p}}$$

whenever

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- This conjecture is solved in two dimensions and in spite of a lot of brilliant work by Bourgain, Guth, Ou, Stein, Tao, Tomas, Wang and many others, the problem is still open in higher dimensions.

A signal recovery perspective on restriction

- Suppose that A is a compact set in $\mathbb{R}^d$, $d \geq 2$, $|A| > 0$, and $\widehat{\chi}_A(\xi)$ is known except for $\xi \in S^\delta$, the annulus of radius 1 and thickness δ (small). Can we recover $\chi_A(x)$ exactly?

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- $$= \int_{\xi \notin S^\delta} + \int_{\xi \in S^\delta} = I(x) + II(x).$$
- By assumption, we have no information about $II(x)$, so we must estimate it and hope for the best.

Applying the conjectured restriction inequality

- By Holder, if the restriction theorem holds with exponents (p, r), then

$$|II(x)| \leq |S^\delta| \cdot \left(\frac{1}{|S^\delta|} \int_{S^\delta} |\widehat{\chi}_A(\xi)|^r d\xi \right)^{\frac{1}{r}} \leq C_{p,r} \cdot |S^\delta| \cdot |A|^{\frac{1}{p}}.$$

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- If the right hand side is $< \frac{1}{2}$, i.e if $|A| \lesssim \delta^{-p}$ with suitable constants, then we can take the modulus of $I(x)$ and round it up to 1, or down to 0, whichever is closer, and thus recover $\chi_A(x)$ is exactly.

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- For any r, the restriction theorem always holds for $p = 1$, but according to the restriction conjecture, it holds for any

$$p < \frac{2d}{d+1},$$

which gives us a much less stringent recovery condition.

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- A natural question is whether the exponent $\frac{2d}{k}$ is sharp, and what does it have to with restriction theory?

- After all, if $k = d - 1$ and S is the unit sphere, $\frac{2d}{d-1}$ is the sharp conjectured exponent for the dual of the restriction conjecture.

Space curves

Theorem

(S. Guo, A. Iosevich, R. Zhang, and P. Zorich-Kranich (2023)) Let $d \geq 2$ be a positive integer and suppose that $1 \leq p < \frac{d^2+d+2}{2}$. If $f \in L^p(\mathbb{R}^d)$ and $\widehat{f}$ is supported on

$$\{(t, t^2, \ldots, t^d) : t \in (0,1)\},$$

then $f \equiv 0$. The exponent $\frac{d^2+d+2}{2}$ is best possible, up to the endpoint. Moreover, the conclusion is still valid for small perturbations of this curve.

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- Note that the Agranovsky-Narayanan theorem yields the same conclusion for $p < 2d$ in this case.
- We also note that $\frac{d^2+d+2}{2}$ is the optimal extension exponent.

Finite Signals and Discrete Fourier transform

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- Suppose that f is transmitted via its Fourier transforms, with

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- Fourier Inversion says that we can reconstruct (or recover) the signal completely by using the Fourier inversion:

$$f(x) = \sum \chi(x \cdot m) \widehat{f}(m).$$

Exact recovery problem

- The basic question is, can we *still* recover f **exactly** from its discrete Fourier transforms if

$$\left\{ \widehat{f}(m) : m \in S \right\}$$

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- The answer turns out to be $\boxed{YES}$ if f is supported in $E \subset \mathbb{Z}_N^d$, and

$$|E| \cdot |S| < \frac{N^d}{2},$$

with the main tool being the Fourier Uncertainty Principle (FUP).

An elementary point of view: setup

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An elementary point of view: Cauchy-Schwarz

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$$|II(x)| \leq |S|^{\frac{1}{2}} \cdot \left(\sum_{m \in S} \left| \widehat{E}(m) \right|^2 \right)^{\frac{1}{2}}.$$

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- Extending the sum in S over the sum in $\mathbb{Z}_N^d$ and applying Plancherel, we see that this expression is bounded by

$$|S|^{\frac{1}{2}} \cdot N^{-\frac{d}{2}} \cdot |E|^{\frac{1}{2}}.$$

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- But what happens if we consider general signals?

Donoho-Stark point of view

- Suppose that $h : \mathbb{Z}_N \to \mathbb{C}$ has N_t non-zero values, and its Fourier transform $\widehat{h}$ has N_w non-zero entries. Then the classical Uncertainty Principle says that

$$|supp(h)| \cdot |supp(\hat{h})| = N_t \cdot N_w \geq N.$$

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- Suppose that $f : \mathbb{Z}_N \to \mathbb{C}$ is supported in $E \subset \mathbb{Z}_N$, with the frequencies in $S \subset \mathbb{Z}_N$ unobserved.

- If f cannot be recovered uniquely, then there exists a signal $g : \mathbb{Z}_N \to \mathbb{C}$ such that g also has N_t non-zero entries,

$$\widehat{f}(m) = \widehat{g}(m) \text{ for } m \notin S,$$

and f is not identically equal to g.

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- Therefore, if

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we must have $h = 0$, and hence the recovery is *unique*.

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- By Cauchy-Schwarz, Plancherel, and the fact that f is supported on E,

$$|f(x)|^2 \leq |S| \cdot \sum_{m \in S} \left|\widehat{f}(m)\right|^2$$

$$= |S| \cdot \sum_{m \in \mathbb{Z}_N^d} \left|\widehat{f}(m)\right|^2 = |S| \cdot N^{-d} \cdot \sum_{x \in E} |f(x)|^2.$$

Conclusion of the proof of FUP

- Summing both sides over E and dividing by $\sum_{x \in E} |f(x)|^2$, we get

$$|E| \cdot |S| \geq N^d, \quad \text{(the classical Uncertainty Principle).}$$

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- Some stronger uncertainty principles that depend on the arithmetic properties of N have been obtained by Tao and Meshulam. We shall briefly discuss those in a moment.

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- Since $|S| \cdot |S^{\perp}| = N^d$, the FUP is sharp.

- However, there are very few situations of this type, and it is possible to classify them, though we will not do it here.

- We will see that in most cases, we can do much better, and the key mechanism we are going to utilize is **restriction theory**.

Restriction theory enters the picture

- We say that $S \subset \mathbb{Z}_N^d$ satisfies the (p,q) restriction estimate $(1 \leq p \leq q)$ with uniform constant $C_{p,q} > 0$ if for any function $f : \mathbb{Z}_N^d \to \mathbb{C}$,

$$\left(\frac{1}{|S|} \sum_{m \in S} \left| \widehat{f}(m) \right|^q \right)^{\frac{1}{q}} \leq C_{p,q} N^{-d} \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}}.$$

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Theorem (Uncertainty Principle via Restriction Theory – A.I. & A.Mayeli, 2023)

Suppose that $f, \widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$, with f supported in $E \subset \mathbb{Z}_N^d$, and $\widehat{f}$ supported in $S \subset \mathbb{Z}_N^d$. Suppose S satisfies the (p,q) restriction estimate with norm $C_{p,q}$. Then

$$|E|^{\frac{1}{p}} \cdot |S| \geq \frac{N^d}{C_{p,q}}.$$

A stronger (usually) restriction mechanism

Theorem (Uncertainty Principle via Restriction Theory – A.I. & A.Mayeli, 2024)

Suppose that $f : \mathbb{Z}_N^d \to \mathbb{C}$ *is supported in* $E \subset \mathbb{Z}_N^d$, *and* $\hat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ *is supported in* $S \subset \mathbb{Z}_N^d$. *Suppose* S *satisfies the* (p, q) *restriction estimate with norm* $C_{p,q}$, $1 \leq p \leq q$, $p \leq 2$.

i) If $q \geq 2$, *then*

$$|E|^{\frac{2-p}{p}} \cdot |S| \geq \frac{N^d}{C_{p,q}^2}.$$

ii) If $1 < q < 2$, *then*

$$|E|^{\frac{(q'-p)q}{q'p}} \cdot |S| \geq \frac{N^d}{C_{p,q}^q}$$

From Restriction to Exact Recovery

Corollary

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ *with support* $supp(f) = E$*. Let* r *be another signal with support of the same size such that* $\widehat{r}(m) = \widehat{f}(m)$ *for* $m \notin S$*, and* 0 *otherwise. Suppose* $S \subset \mathbb{Z}_N^d$ *satisfies the* (p,q)*,* $p < 2$*, restriction estimate with uniform constant* $C_{p,q}$*. Then* f *can be reconstructed from* r *uniquely if*

$$|E|^{\frac{1}{p}} \cdot |S| < \frac{N^d}{2^{\frac{1}{p}} C_{p,q}},$$

or if

$$|E|^{\frac{2-p}{p}} \cdot |S| < \frac{N^d}{2^{\frac{2-p}{p}} C_{p,q}^2} \text{ when } q \geq 2,$$

and

$$|E|^{\frac{(q'-p)q}{q'p}} \cdot |S| < \frac{N^d}{2^{\frac{(q'-p)q}{q'p}} C_{p,q}^q} \text{ when } q \leq 2.$$

From additive energy to restriction

Theorem (A.I. & A. Mayeli, 2023)

Let $S \subset \mathbb{Z}_N^d$ *with the property that*

$$|S| = \Lambda_{size} N^{\frac{d}{2}},$$

and

$$|\{(x,y,x',y') \in U : x+y = x'+y'\}| \leq \Lambda_{energy} \cdot |U|^2$$

for every $U \subset S$.

Then S *satisfies* $(\frac{4}{3}, 2)$ *restriction with* $C_{p,q} = \Lambda_{size}^{-\frac{1}{2}} \cdot \Lambda_{energy}^{\frac{1}{4}}$, *i.e*

$$\left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^2 \right)^{\frac{1}{2}} \leq \Lambda_{size}^{-\frac{1}{2}} \cdot \Lambda_{energy}^{\frac{1}{4}} \cdot N^{-d} \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}} \right)^{\frac{3}{4}}.$$

Bourgain's Λ_q theorem - general formulation

- Jean Bourgain proved that if G is a locally compact abelian group, $\phi_1, \dots, \phi_n$ are orthogonal functions with $||\phi_j||_\infty \leq 1$, the for a generic set $S \subset \{1, 2, \dots, n\}$ of size $\approx n^{\frac{2}{q}}$, $q > 2$,

$$\left|\left|\sum_{i \in S} a_i \phi_i\right|\right|_{L^q(G)} \leq C(q) \cdot \left(\sum_{i \in S} |a_i|^2\right)^{\frac{1}{2}},$$

where $C(q)$ depends only on q.

Bourgain's Λ_q theorem - general formulation

- Jean Bourgain proved that if G is a locally compact abelian group, $\phi_1, \dots, \phi_n$ are orthogonal functions with $||\phi_j||_\infty \leq 1$, the for a generic set $S \subset \{1, 2, \dots, n\}$ of size $\approx n^{\frac{2}{q}}$, $q > 2$,

$$\left|\left|\sum_{i \in S} a_i \phi_i\right|\right|_{L^q(G)} \leq C(q) \cdot \left(\sum_{i \in S} |a_i|^2\right)^{\frac{1}{2}},$$

  where $C(q)$ depends only on q.

- As we shall see, this result has a beautiful built-in uncertainty principle.

Bourgain's Λ_q theorem

- It is a consequence of Bourgain's celebrated Λ_p theorem in locally compact abelian groups that if $f : \mathbb{Z}_N^d \to \mathbb{C}$ and $\widehat{f}$ **is supported in** S, then for a "generic" set of size $\approx N^{\frac{2d}{q}}$, $2 < q < \infty$,

$$\left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^q\right)^{\frac{1}{q}} \leq K_q(S) \left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2\right)^{\frac{1}{2}},$$

with $K_q(S)$ independent of N.

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with $K_q(S)$ independent of N.

- For such a set S it follows by duality that

$$\left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^2 \right)^{\frac{1}{2}} \leq C_{p,2} N^{-d} \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}}, \text{ with } p = q'.$$

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$$N^{-\frac{d}{q}} \cdot |E|^{\frac{1}{q}} \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^q \right)^{\frac{1}{q}}$$

$$\leq K_q(S) N^{-\frac{d}{2}} \cdot |E|^{\frac{1}{2}} \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^2 \right)^{\frac{1}{2}}.$$

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- Consequently, if we send a signal f supported on a set of size $o(N^d)$ via its Fourier transform, and the frequencies in a generic $S \subset \mathbb{Z}_N^d$ are missing, we can recover f exactly with very high probability.

Arithmetic matters

- In 2006, Terry Tao proved that if $f : \mathbb{Z}_p \to \mathbb{C}$, p prime, f is supported in E and $\widehat{f}$ is supported in S, then

$$|E| + |S| \geq p + 1.$$

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- The key element of the proof is a classical theorem due to Cebotarev which says that if $A, B \subset \mathbb{Z}_p$, $|A| = |B|$, then

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- Roy Meshulam used Tao's result and a beautiful iteration argument show that if $f : \mathbb{Z}_p^d \to \mathbb{C}$ is supported in E and $\widehat{f}$ is supported in S, then for $0 \leq j \leq d-1$,

$$p^j|E| + p^{d-j-1}|S| \geq p^d + p^{d-1}.$$

More arithmetic

Lemma

(A.I., A. Mayeli, and J. Pakianathan (2017)) [Magic Lemma] Suppose that $f : \mathbb{Z}_p^2 \to \mathbb{Q}$, p odd prime. Suppose that $\widehat{f}(m) = 0$ for some $m \neq (0,0)$. Then $\widehat{f}(rm) = 0$ for all $r \neq 0$. Moreover, if $f(x) = E(x)$, the indicator function of $E \subset \mathbb{Z}_p^2$, and $\widehat{E}(m) = 0$ for some $m \neq (0,0)$, then E is equidistributed on the p lines orthogonal to m.

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- Suppose that $\widehat{E}(m) = 0$, as above, with $m \neq (0,0)$ and let $r \neq 0$. We have

$$\widehat{E}(rm) = p^{-2} \sum_t \zeta^{\frac{t}{r}} n(t/r) = p^{-2} \sum_t \zeta^t n(t) = 0.$$

More arithmetic

Lemma

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- Suppose that $\widehat{E}(m) = 0$, as above, with $m \neq (0,0)$ and let $r \neq 0$. We have
$$\widehat{E}(rm) = p^{-2} \sum_t \zeta^{\frac{t}{r}} n(t/r) = p^{-2} \sum_t \zeta^t n(t) = 0.$$
- It follows that if $m \neq (0,0)$ is a zero of $\widehat{E}$, then so is every non-zero multiple of m.

Magic Lemma demystified

- Observe that

$$0 = \sum_t \zeta^t n(t) = n(0) + n(1)\zeta + n(2)\zeta^2 + \cdots + n(p-1)\zeta^{p-1}$$

says that ζ satisfies the polynomial of degree $p-1$ with coefficients given by $\{n(t)\}$.

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- The minimal polynomial of ζ is

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1}.$$

- We conclude that $n(t) = \mathit{constant}$, so E has the **same** number of points on lines $\perp m$. In particular, $|E|$ is a multiple of p.

FUP consequence of the Magic Lemma

- It is not difficult to see that if $f : \mathbb{Z}_p^2 \to \mathbb{Q}$ and $\widehat{f}$ vanishes on a random set S with $|S| = o(p^2)$, then with high probability, f is supported on all of $\mathbb{Z}_p^2$.

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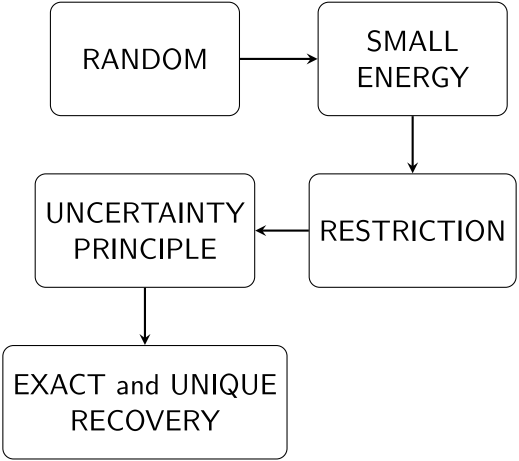
- The point is that it is highly unlikely that a randomly chosen set S of size $o(p^2)$ contains a full line through the origin with the origin removed.

Summary of connections



Proof of Energy → Restriction

- We have

$$\sum_{m \in S} |\widehat{f}(m)|^2 = \sum_m |\widehat{f}(m)|^2 S(m)$$

$$= \sum_m \widehat{f}(m) S(m) g(m),$$

where

$$g(m) = \overline{\widehat{f}(m)} S(m).$$

By definition of the Fourier transform, the right-hand side is equal to

$$N^{-d} \sum_m \sum_x \chi(-x \cdot m) f(x) S(m) g(m)$$

$$= \sum_x f(x) \widehat{gS}(x).$$

Proof of Additive Energy → Restriction (continued)

- By Holder's inequality, the quantity above is bounded by

$$\left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \cdot \left(\sum_{x \in \mathbb{Z}_N^d} |\widehat{gS}(x)|^4 \right)^{\frac{1}{4}}.$$

Proof of Additive Energy $\to$ Restriction (continued)

- By Holder's inequality, the quantity above is bounded by

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- Continuing, we have

$$\sum_{x\in\mathbb{Z}_N^d} |\widehat{gS}(x)|^4$$

$$= N^{-4d}\sum_x \sum_{m_1,m_2,m_3,m_4\in S} \chi(x\cdot(m_1+m_2-m_3-m_4))\prod_{i=1}^4 g(m_i)$$

$$= N^{-3d} \sum_{m_1+m_2=m_3+m_4;m_j\in S} g(m_1)g(m_2)g(m_3)g(m_4).$$

Proof of Energy $\to$ Restriction (continued)

- The modulus of this expression is bounded by

$$\Lambda_{energy} \cdot N^{-3d} \cdot \left(\sum_m |g(m)|^2 \right)^2,$$

where we have used Cauchy-Schwartz and the energy assumption.

Proof of Energy → Restriction (continued)

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- Going back, we see that the expression is bounded by

$$\left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \cdot \Lambda_{energy}^{\frac{1}{4}} \cdot N^{-\frac{3d}{4}} \cdot \left(\sum_m |g(m)|^2 \right)^{\frac{1}{2}}.$$

Proof of Energy $\to$ Restriction (continued)

- If we go back and unravel the definitions, we see that

$$\sum_m |g(m)|^2 \leq \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \cdot \Lambda_{energy}^{\frac{1}{4}} \cdot N^{-\frac{3d}{4}} \cdot \left(\sum_m |g(m)|^2 \right)^{\frac{1}{2}},$$

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- hence

$$\left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \cdot \frac{1}{|S|^{\frac{1}{2}}} \cdot \Lambda_{energy}^{\frac{1}{4}} \cdot N^{-\frac{3d}{4}}.$$

Proof of Energy → Restriction (finale)

- This expression equals

$$\Lambda_{energy}^{\frac{1}{4}} \cdot N^{-d} \cdot \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \cdot \frac{N^{\frac{d}{4}}}{|S|^{\frac{1}{2}}}$$

$$= \Lambda_{size}^{-\frac{1}{2}} \cdot \Lambda_{energy}^{\frac{1}{4}} \cdot N^{-d} \cdot \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}} \right)^{\frac{3}{4}},$$

as claimed.

Proof of Uncertainty Principle via Restriction I

- Suppose that f is supported in a set E, and $\widehat{f}$ is supported in a set S. Then by the Fourier Inversion Formula and the support condition,

$$f(x) = \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m)\widehat{f}(m) = \sum_{m \in S} \chi(x \cdot m)\widehat{f}(m).$$

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- By Holder's inequality,

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- By Holder's inequality,

$$|f(x)| \leq |S| \cdot \left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^q \right)^{\frac{1}{q}}.$$

- By the restriction bound assumption, this expression is bounded by

$$|S| \cdot C_{p,q} \cdot N^{-d} \cdot \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}},$$

Proof of Uncertainty Principle via Restriction I (continued)

- and by the support assumption, this quantity is equal to

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Proof of Uncertainty Principle via Restriction I (continued)

- and by the support assumption, this quantity is equal to

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- Putting everything together, we see that

$$|f(x)| \leq |S| \cdot C_{p,q} \cdot N^{-d} \cdot \left(\sum_{x \in E} |f(x)|^p \right)^{\frac{1}{p}}.$$

Proof of Uncertainty Principle via Restriction I (continued)

- and by the support assumption, this quantity is equal to

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- Putting everything together, we see that

$$|f(x)| \leq |S| \cdot C_{p,q} \cdot N^{-d} \cdot \left(\sum_{x \in E} |f(x)|^p \right)^{\frac{1}{p}}.$$

- Raising both sides to the power of p, summing over E, and dividing both sides of the resulting inequality by $\sum_{x \in E} |f(x)|^p$, we obtain

$$|S|^p \cdot |E| \cdot C_{p,q}^p \geq N^{dp}.$$

Proof of Uncertainty Principle via Restriction I (finale)

- or, equivalently,
$$|E|^{\frac{1}{p}} \cdot |S| \geq \frac{N^d}{C_{p,q}},$$
as desired.

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as desired.

- This completes the proof of the Uncertainty Principle via Restriction Theory.

Proof of Uncertainty Principle via Restriction II (definitions)

- Define

$$||f||_{L^p(E)} = \left(\sum_{x \in E} |f(x)|^p \right)^{\frac{1}{p}}, ||f||_{L^p(\mu_E)} = \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^p \right)^{\frac{1}{p}}.$$

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- Similarly define

$$||f||_{L^p(S)} = \left(\sum_{x \in S} |f(x)|^p \right)^{\frac{1}{p}}, ||f||_{L^p(\mu_S)} = \left(\frac{1}{|S|} \sum_{x \in S} |f(x)|^p \right)^{\frac{1}{p}}.$$

Proof of Uncertainty Principle via Restriction II: $q \geq 2$

- The restriction estimate takes the form

$$||\widehat{f}||_{L^q(\mu_S)} \leq C_{p,q} N^{-d} ||f||_{L^p(E)}.$$

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- Since $q > 2$,

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- Since $\widehat{f}$ is supported in S, and f is supported on E, Plancherel implies that

$$||\widehat{f}||_{L^2(\mu_S)} = |S|^{-\frac{1}{2}} \cdot N^{-\frac{d}{2}} ||f||_{L^2(E)}.$$

Proof of Uncertainty Principle via Restriction II: $q \geq 2$ (continued)

- Plugging this back into the restriction estimate, we see that

$$|S|^{-\frac{1}{2}} \cdot N^{-\frac{d}{2}} ||f||_{L^2(E)} \leq C_{p,q} N^{-d} ||f||_{L^p(E)}$$

$$\leq C_{p,q} N^{-d} |E|^{\frac{1}{p}-\frac{1}{2}} ||f||_{L^2(E)}.$$

Proof of Uncertainty Principle via Restriction II: $q \geq 2$ (continued)

- Plugging this back into the restriction estimate, we see that

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$$\leq C_{p,q} N^{-d} |E|^{\frac{1}{p}-\frac{1}{2}} ||f||_{L^2(E)}.$$

- Combining everything yields

$$|E|^{\frac{2-p}{p}} \cdot |S| \geq \frac{N^d}{C_{p,q}^2},$$

as claimed.

Proof of Uncertainty Principle via Restriction II: $q \leq 2$

- To handle the case $q < 2$, we shall need Hausdorff-Young. If $1 \leq p \leq 2$,

$$||\widehat{g}||_{L^{p'}(\mathbb{Z}_N^d)} \leq N^{-\frac{d}{p'}} ||g||_{L^p(\mathbb{Z}_N^d)}.$$

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- The Hausdorff-Young implies that the left hand side of the restriction inequality is bounded from **below** by (with $\widehat{f} = g$)

$$|S|^{-\frac{1}{q}} N^{\frac{d}{q}} ||\widehat{g}||_{L^{p'}(\mathbb{Z}_N^d)} = |S|^{-\frac{1}{q}} N^{-\frac{d}{q'}} ||f||_{L^{q'}(E)}.$$

Combining this with the restriction theorem bound, we get

$$|S|^{-\frac{1}{q}} N^{-\frac{d}{q'}} ||f||_{L^{q'}(E)} \leq C_{p,q} N^{-d} ||f||_{L^p(E)}$$

$$\leq C_{p,q} N^{-d} \cdot |E|^{\frac{1}{p} - \frac{1}{q'}} ||f||_{L^{q'}(E)}.$$

Proof of Uncertainty Principle via Restriction II: $q \leq 2$ (Finale)

- Cancelling the $L^{q'}$ norms, putting everything together and rearranging yields

$$|E|^{\frac{q(q'-p)}{pq'}} \cdot |S| \geq \frac{N^d}{C_{p,q}^q}.$$

Proof of Uncertainty Principle via Restriction II: $q \leq 2$ (Finale)

- Cancelling the $L^{q'}$ norms, putting everything together and rearranging yields

$$|E|^{\frac{q(q'-p)}{pq'}} \cdot |S| \geq \frac{N^d}{C_{p,q}^q}.$$

- An algebraic calculation shows that

$$\frac{q(q'-p)}{pq'} < \frac{1}{p},$$

we gain over the first restriction theory mechanism we described provided that $C_{p,q}$ is not too large.


Eh...
thanks
a lot!