A new approach to the Fourier Extension Problem for the Paraboloid

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joint work with

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(I) In $\mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$ define

$$\boxed{E_{d+1} f(x,t) = \int_{\mathbb{R}^d} f(\xi)\, e^{ix\cdot\xi}\, e^{it|\xi|^2}\, d\xi}$$

for f with $\mathrm{supp}(f) \subseteq B_1 = \{|x| \leq 1\}$.

Conjecture 1 (Stein)

$$\boxed{\mathbb{E}_{d+1} : L^{d} \to L^{a}}$$

Note that $\mathbb{E}_{d+1} f(x,t)$ solves the Schrödinger equation

$$\begin{cases} \partial_t u + i \Delta u = 0 \\ u|_{t=0} = \hat{f} \end{cases}.$$

- the case $\mathbb{R}^{1+1}$ is known Fefferman & Zygmund 70's.
- the rest of the cases $\mathbb{R}^{d+1}$ for $\boxed{d+1 \geq 3}$ are OPEN.

- Sample Recent Results:

① $\boxed{d+1=3}$ $L^{3+\frac{1}{14}}$ estimates (best is L^3)

Wang & Wu

② $\boxed{d+1=6}$ $L^{2+\frac{1}{2}}$ estimates (best is $L^{2+\frac{2}{5}}$)

Guth

③ $\boxed{d+1=19}$ $L^{2+\frac{1}{7}}$ estimates (best is $L^{2+\frac{1}{9}}$)

Hickman & Rogers

- Also note that $\langle \mathbb{E}_{d+1} f, g \rangle =$

$$= \int_{\mathbb{R}^d} f(\xi) \cdot \underline{\underline{\hat{g}(\xi, |\xi|^2)}} \, d\xi$$

(Restriction of $\hat{g}$ to the paraboloid)

So " Extension $\Longleftrightarrow$ Restriction "

Now, for $1 \le k \le d+1$ define $\mathbb{E}_{d+1}^{k\text{-lin}}$ by

$$\boxed{\begin{aligned} &\mathbb{E}_{d+1}^{k\text{-lin}} (f_1, \dots, f_k)(x,t) := \\ &\mathbb{E}_{d+1}(f_1)(x,t) \cdot \ldots \cdot \mathbb{E}_{d+1}(f_k)(x,t) \end{aligned}}$$

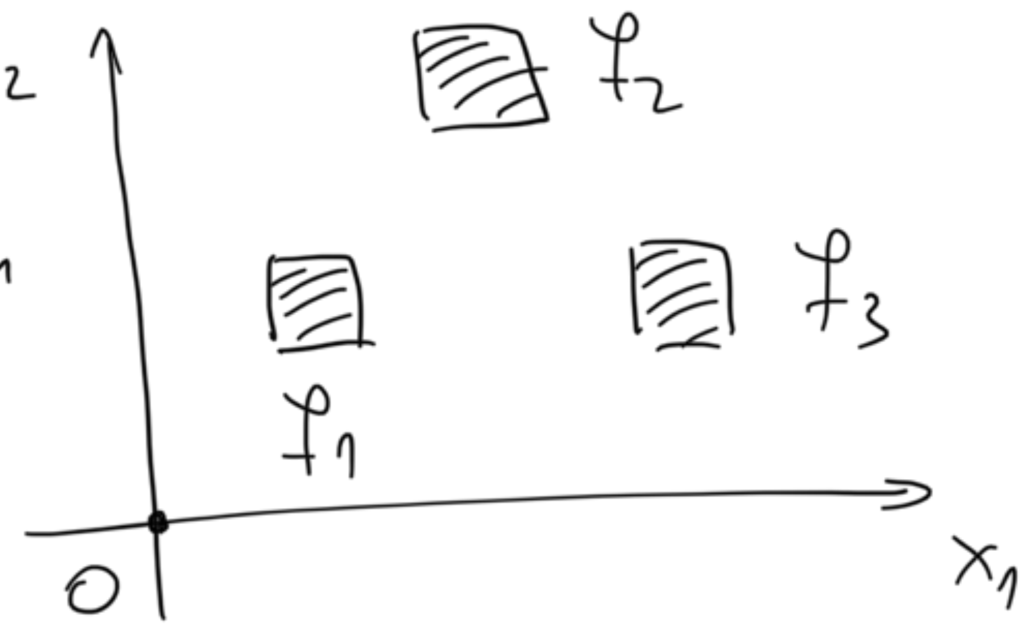
the product of k linear extensions.

Conjecture 2 (Multi-linear Restriction, L^2 input with transversality)

$$\boxed{\mathbb{E}^{T,k\text{-lin}}_{d+1} : L^2 \times \dots \times L^2 \longrightarrow L^{\frac{2(d+k+1)}{k(d+k-1)}+\varepsilon}}$$

T = transversality ; Example for $d=2$ and $k=3$:

- Kleinerman – Machedon
- Bennett



- $\boxed{k=1}$ Strichartz 70's
- $\boxed{k=2}$ Tao (also Wolff – cone case)
- $\boxed{k=d+1}$ Bennett – Carbery – Tao
- $\boxed{k=d}$ Bejenaru

Conjecture 3 (Multi-linear Restriction, $L^{\frac{2(d+1)}{kd}}$ target with or without transversality)

Both $\mathbb{E}^{k\text{-lin}}_{d+1}$ & $\mathbb{E}^{T,k\text{-lin}}_{d+1}$ should map

$$\boxed{L^{\frac{2(d+1)}{d}+\varepsilon} \times \dots \times L^{\frac{2(d+1)}{d}+\varepsilon} \longrightarrow L^{\frac{2(d+1)}{k\cdot d}+\varepsilon}}$$

II

- Oh – Recent partial results when $d=2$ and $k=2$

Conjecture 3 would follow from

Conjecture 4 (Hölder + Restriction, L^∞ input, no transversality)

$$\boxed{\mathbb{E}_{d+1}^{k-\text{lin}} : L^\infty \times \dots \times L^\infty \to L^{\frac{2(d+1)}{kd}+\varepsilon}}$$

- Bourgain proved that L^∞ is enough (Nikishin–Pisier factorization)

ASSUME from now on that one of the functions involved (say f_1) is a tensor (i.e. is of type $h(x_1,\dots,x_d)=h_1(x_1)\cdot\ldots\cdot h_d(x_d)$). THEN one can prove the following theorems:

Theorem 1 (Oliveira – M)

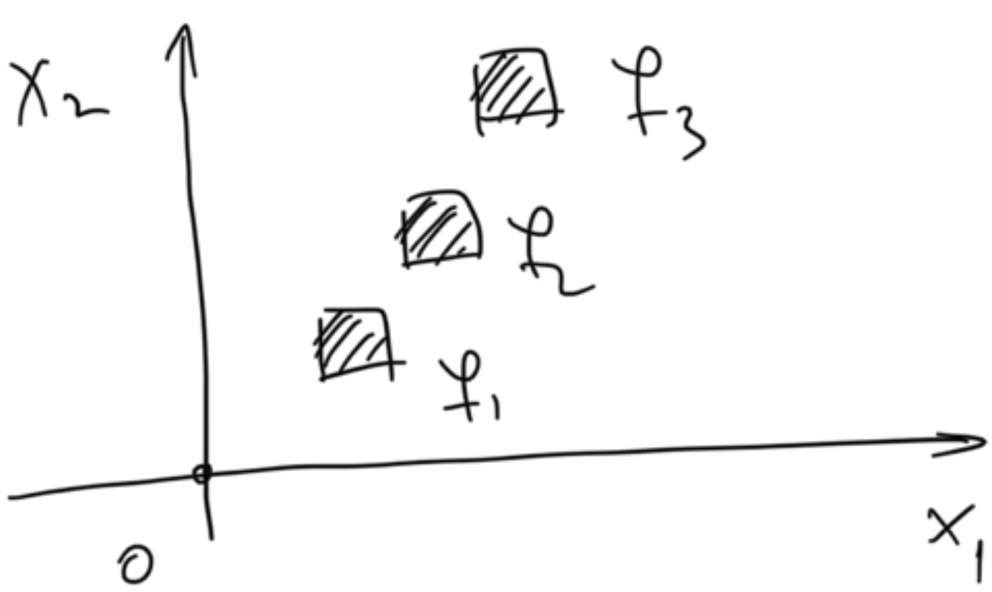
$$\boxed{\| \mathbb{E}_{d+1}(f_1)\|_{\frac{2(d+1)}{d}+\varepsilon} \lesssim \|f_1\|_{\frac{2(d+1)}{d}+\varepsilon}}$$

Theorem 2 (Oliveira – M)

$$\left\| E_{d+1}^{\text{weak-T},\, k\text{-lin}}(f_1, \dots, f_k) \right\|_{\frac{2(d+k+1)}{k(d+k-1)}+\varepsilon} \lesssim \prod_{j=1}^{k} \|f_j\|_2$$

- weak-T = weak transversality

Example for $k=3$ and $d=2$



weak-T

BUT

NOT T !

- Under weak-T thm 2 is SHARP.

Theorem 3 (Oliveira – M)

$$\left\| E_{d+1}^{\text{weak-T},\, k\text{-lin}}(f_1, \dots, f_k) \right\|_{\frac{2(d+1)}{kd}+\varepsilon} \lesssim \prod_{j=1}^{k} \|f_j\|_{p(k,d)}$$

where

$$p(k,d) = \begin{cases} \frac{4(d+1)}{d+k+1}, & 2 \le k < \frac{d}{2} \end{cases}$$

$$\left(\frac{4(d+1)}{2d-k+1}, \quad \frac{d}{2} \le k < d+1 \right)$$

- $p(k,d)$ is close to $\underline{\underline{4}}$ but smaller than it.

__Theorem 4__ (Oliveira – M)

$$\boxed{\left\| \mathbb{E}^{k\text{-}lin}_{d+1}(f_1, \dots, f_k) \right\|_{\frac{2(d+1)}{kd}+\varepsilon} \lesssim \prod_{j=1}^{k} \|f_j\|_4}$$

II Linearization & Wave packets decomposition

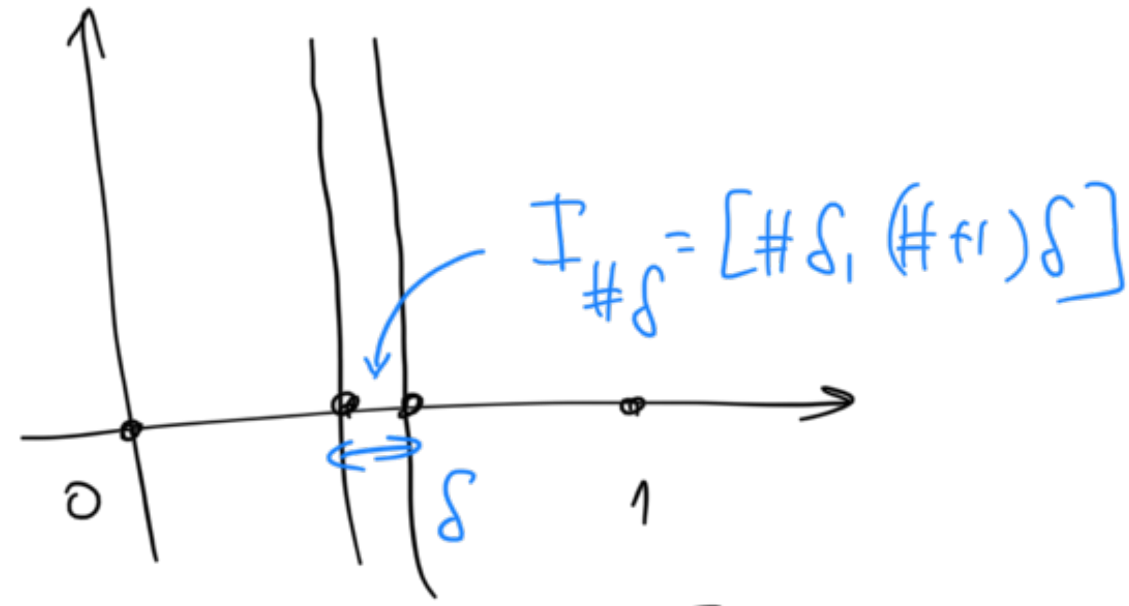
For simplicity let $\underline{\underline{d=1}}$. Recall

$$\boxed{\mathbb{E} f(x,t) = \int_{\mathbb{R}} f(\xi)\, e^{ix\cdot\xi}\, e^{it\xi^2}\, d\xi}$$

supp $f \subseteq [0,1]$. Let $R>0$ __large__ and fixed and assume $\boxed{|t| \le R}$

Let also $\boxed{\delta = \frac{1}{\sqrt{R}}}$. Decompose f

as follows:



$$f(\xi) = \sum_{\#=0}^{\sqrt{R}} f(\xi)\,\varphi_\delta(\xi - \#\delta) = \sum_{\#=0}^{\sqrt{R}} f_{\#\delta}(\xi)$$

On $I_{\#\delta}$ $\quad \boxed{e^{it\xi^2} \sim e^{2it\#\delta\xi}}$

So $\boxed{E(f_{\#\delta})(x,t)\,1_{[0,\frac{1}{\delta^2}]}(|t|) \sim \hat{f}(x + 2t\#\delta)}$

Split each $f_{\#\delta}(\xi)$ on $I_{\#\delta}$ as a Fourier series:

$$f_{\#\delta}(\xi) = \sum_{\ell\in\mathbb{Z}} \widehat{f_{\#\delta}}(\ell)\, e^{2\pi i \frac{\xi}{\delta}\ell} \cdot \varphi_\delta(\xi - \#\delta).$$

This gives

$$E(f_{\#\delta})(x,t) = \sum \hat{f}(\ell)\,\widehat{\varphi}\,(x - \ell\sqrt{R} + 2t\#$$

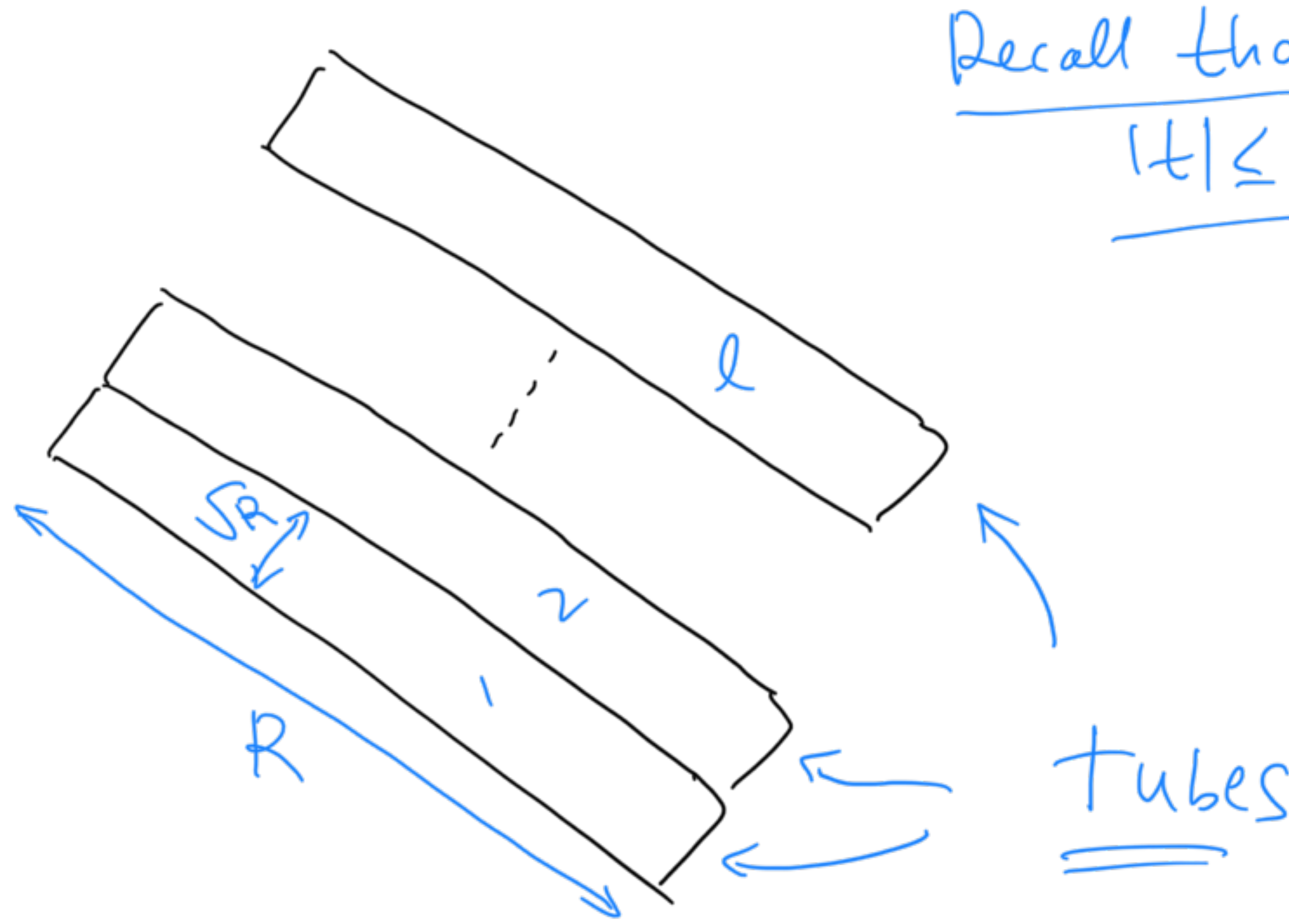
$$\#\theta / \quad \sum_{\ell\in\mathbb{Z}} f(t)\cdot \delta\left(x - \ell\sqrt{R} + \frac{2t\#}{\sqrt{R}}\right)$$

And $\underline{\underline{\delta}}$ is essentially supported on

$$-\sqrt{R} \le x - \ell\sqrt{R} + \frac{2t\#}{\sqrt{R}} \le \sqrt{R}$$

which is a strip of $\underline{\underline{\text{width}}} \sim \sqrt{R}$ along a vector $\perp \left(1, \frac{2\#}{\sqrt{R}}\right)$

Recall that $|t| \le R$.



If one does this for every $\# = \overline{0, \sqrt{R}}$ one obtains

$$\mathbb{E} f(x,t) \sim \sum_T c_T(f) \cdot \Phi_T =$$

$$\sqrt{R} \cdot \sum_T c_T(f) \cdot \tilde{\Phi}_T$$

L^2-normalized now in $\mathbb{R}^2$

for $(x,t) \in B_R \subseteq \mathbb{R}^2$.

- Extension Conjecture 1 is trivial for a single "strip". Say $\# = 0$.

then

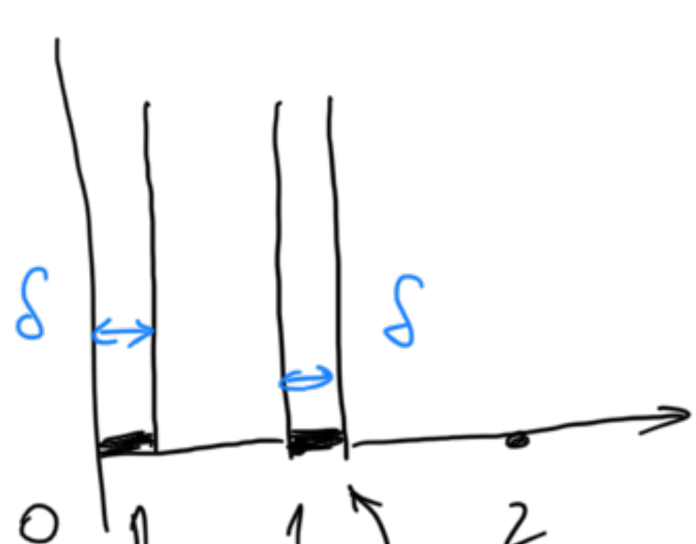
$$\| \widehat{f}_{0\delta}(x) 1_{[0,\frac{1}{\delta^2}]}(t) \|_{L^4_{x,t}} = \frac{1}{\delta^{1/2}} \| \widehat{f}_{0\delta} \|_4$$

H-Y $\qquad\qquad$ Hölder

$$\leq \frac{1}{\delta^{1/2}} \| f_{0\delta} \|_{4/3} \leq \frac{1}{\delta^{1/2}} \|f\|_4 \cdot \| 1_{[0,\delta]} \|_2$$

$$= \|f\|_4 \quad \blacksquare$$

- Same for the bilinear estimate



supp (f_1) supp (f_2)

$$\| \widehat{f}_1(x)\, \widehat{f}_2(x+2t)\, 1_{[0,\frac{1}{\delta^2}]}(t) \|_{L^2_{x,t}} \lesssim$$

$$\lesssim \|\widehat{f}_1\|_2 \cdot \|\widehat{f}_2\|_2 = \|f_1\|_2 \cdot \|f_2\|_2$$

- However, for $d \geq 2$ "single strip" localized extensions, behave better!

$$\| \widehat{f_{[0,\delta]\times[0,\delta]}}(x_1,x_2)\, 1_{[0,\frac{1}{\delta^2}]}(t) \|_{L^2_{x_1,x_2,t}} =$$

$$= \frac{1}{\delta} \| \widehat{f_{[0,\delta]\times[0,\delta]}} \|_{L^2} = \frac{1}{\delta} \| f_{[0,\delta]\times[0,\delta]} \|_2$$

$$\leq \frac{1}{\delta} \cdot \delta \|f\|_\infty = \|f\|_\infty$$

So $L^\infty \to L^2$ which, together with

$L^1 \to L^\infty$ imply the expected,

$L^3 \to L^3$ estimate.

III Discretization & Quadratically modulated wave packets

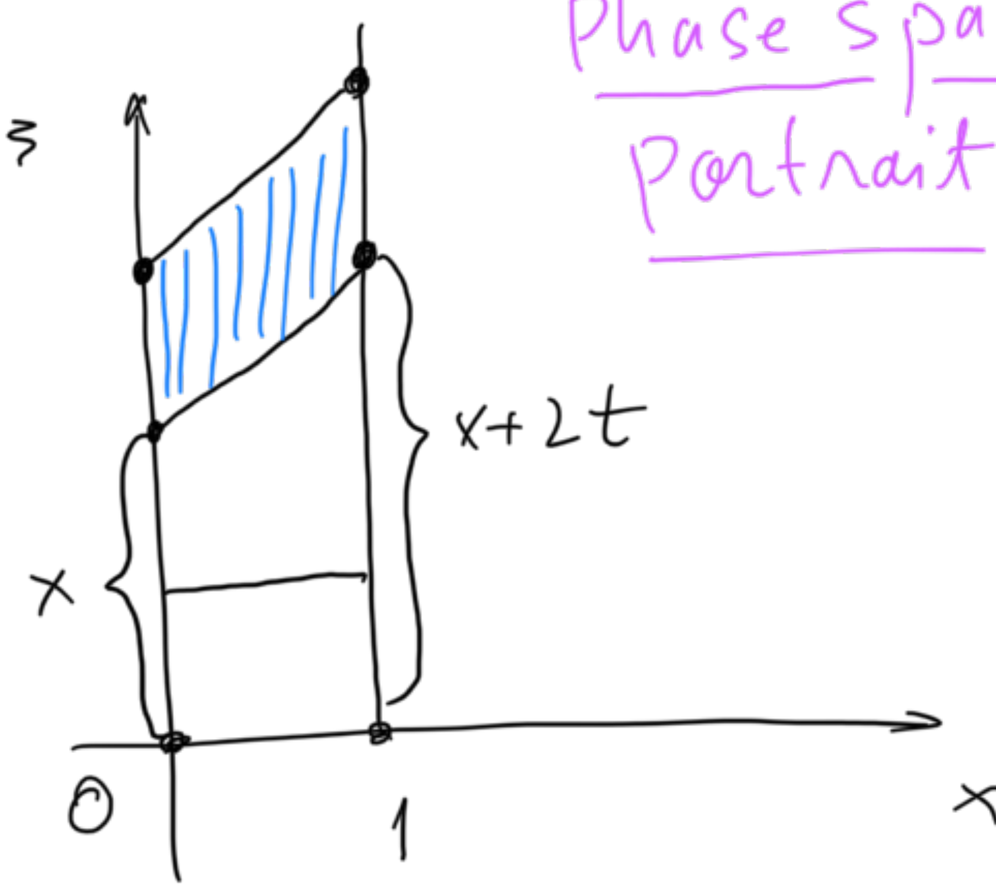
Again $d=1$ for simplicity. Recall

$$\boxed{Ef(x,t) = \int_{\mathbb{R}} f(\xi) e^{ix\xi} e^{it\xi^2} d\xi}$$

and $\mathrm{supp}(f) \subseteq [0,1]$.

Phase space portrait

$f(\xi) e^{ix\xi} e^{it\xi^2} \longrightarrow$



Slope

$$\boxed{\alpha \to x + 2\alpha t}$$

Discrete model:

$$\boxed{Ef = \sum_{n,m \in \mathbb{Z}} \langle f, \varphi_{n,m} \rangle 1_{I_n} \otimes 1_{I_m}}$$

where I ... $= [\ell, \ell_{...}]$ and when $I_\ell := [x, x+1]$

$$\boxed{\varphi_{M,m}(\xi) = \varphi(\xi) e^{iM\xi} e^{im\xi^2}}$$

- then, the $\mathbb{R}^{1+1}$ classical results become

(A) $$\boxed{\left\| \left(\langle f, \varphi_{M,m} \rangle \right)_{M,m} \right\|_{\ell^{4+\varepsilon}} \lesssim \| f \|_4}$$

(B) $$\boxed{\left\| \left(\langle f, \varphi_{M,m} \rangle \right)_{M,m} \right\|_{\ell^{6+\varepsilon}} \lesssim \| f \|_2}$$

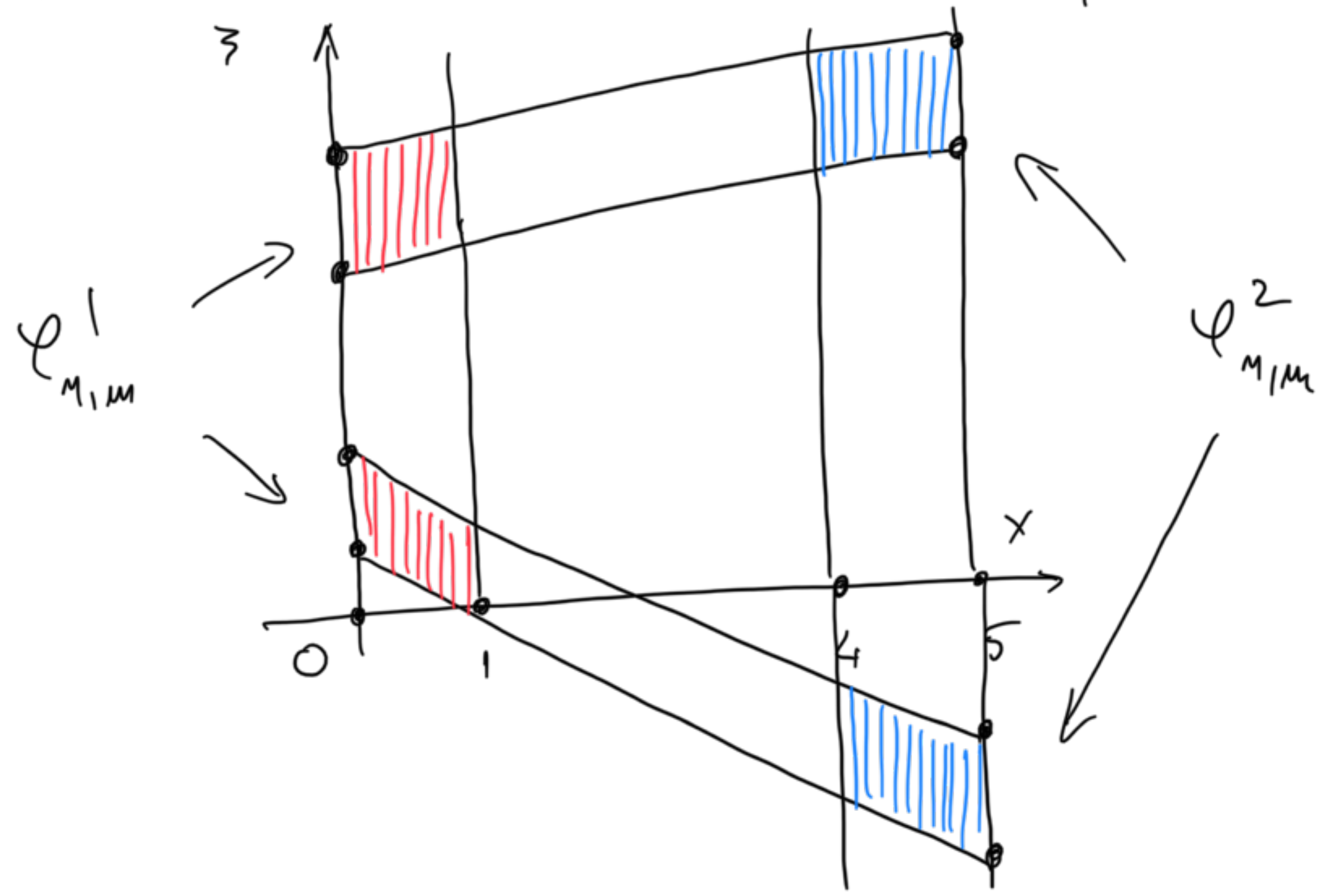
(C) $$\boxed{\left\| \left(\langle f, \varphi^1_{M,m} \rangle \langle g, \varphi^2_{M,m} \rangle \right)_{M,m} \right\|_{\ell^{2}} \lesssim \| f \|_2 \cdot \| g \|_2}$$

if $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$.

- Quick proof of (C):

$$\langle f, \varphi^1_{M,m} \rangle \langle g, \varphi^2_{M,m} \rangle = \langle f \otimes g, \varphi^1_{M,m} \otimes \varphi^2_{M,m} \rangle$$

and the family $\boxed{\left(\varphi^1_{M,m} \otimes \varphi^2_{M,m} \right)_{M,m}}$ is almost orthogonal, see picture:

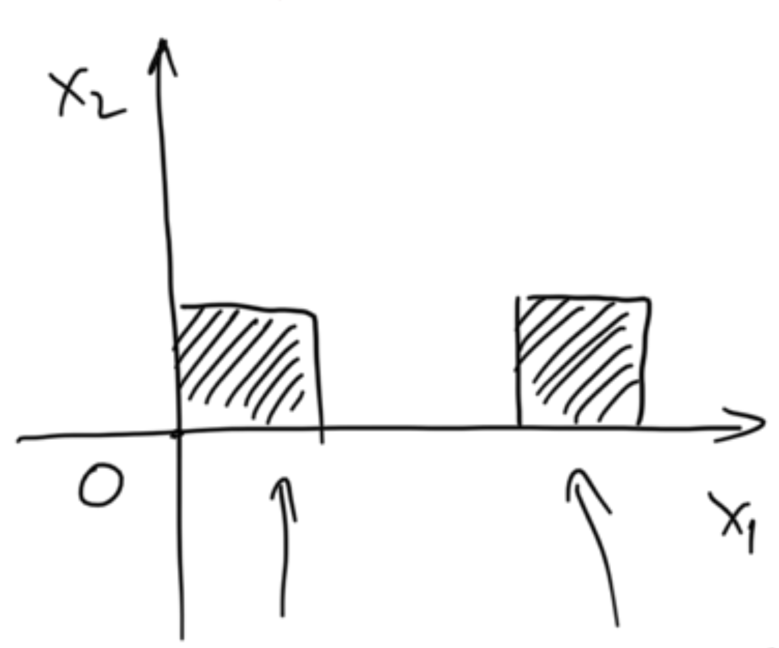


- Consider the $\mathbb{R}^{2+1}$ bilinear case.

$$(f_1, f_2) \to \sum_{n_1, n_2, m} \langle f_1, \varphi_{n_1,n_2,m} \rangle \langle f_2, \varphi_{n_1,n_2,m} \rangle 1_{I_{n_1}} \otimes 1_{I_{n_2}} \otimes 1_{I_m}$$

$$\|\langle f_1, \varphi_{n_1,m}\rangle\|_2 \quad \|\langle f_1, \varphi_{n_2,m}\rangle\|_2 \qquad \|\langle f_2, \varphi_{n_1,m}\rangle\|_2 \quad \|\langle f_2, \varphi_{n_2,m}\rangle\|_2$$

where $\boxed{\varphi_{n_1,n_2,m} = \varphi_{n_1,m} \otimes \varphi_{n_2,m}}$



WANT

$$L^2 \times L^2 \to L^{5/3+\varepsilon}$$

$supp(f_1)$ $supp(f_2)$

Let $\begin{cases} f_1(x_1,x_2) = 1_E(x_1,x_2) = \underline{1_{E_1}(x_1)\cdot 1_{E_2}(x_2)} \\ f_2(x_1,x_2) = 1_F(x_1,x_2) \\ h(x_1,x_2,t) = 1_H(x_1,x_2,t) \end{cases}$ tensor structure

And the goal is to estimate the 3-linear form $\underline{\Lambda(f_1,f_2,h)}$ naturally associated to the bilinear operator.

Define the level sets

$$A_1^{l_1} = \{(n_1,n_2,m) : |\langle f_1, \varphi_{n_1,n_2,m}\rangle| \sim 2^{-l_1}\}$$

$$A_2^{l_2} = \{(n_1,n_2,m) : |\langle f_2, \varphi_{n_1,n_2,m}\rangle| \sim 2^{-l_2}\}.$$

Similarly define

$$B_1^{r_1} = \{(n_1,m) : \|\langle f_1, \varphi_{n_1,m}\rangle\|_2 \sim 2^{-r_1}\}$$

p

$$f_1 \to \quad B_2^{r_2} = \{(n_2, m) : \| \langle f_1, \varphi_{n_2,m} \rangle \|_2 \sim 2^{-r_2}\}$$

$$f_2 \to \begin{cases} C_1^{s_1} = \{(n_1, m) : \| \langle f_2, \varphi_{n_1,m} \rangle \|_2 \sim 2^{-s_1}\} \\ C_2^{s_2} = \{(n_2, m) : \| \langle f_2, \varphi_{n_2,m} \rangle \|_2 \sim 2^{-s_2}\} \end{cases}$$

Finally define

$$D^k = \{(n_1, n_2, m) : | \langle h, 1_{I_{n_1}} \otimes 1_{I_{n_2}} \otimes 1_{I_m} \rangle | \sim 2^{-k}\}.$$

Hence

$$\boxed{|\Lambda(f_1, f_2, h)| \lesssim \sum_{\vec{l}, \vec{r}, \vec{s}, k} 2^{-l_1} 2^{-l_2} 2^{-k} \# (X^{\vec{l}, \vec{r}, \vec{s}, \vec{k}})}$$

Note: We further multi-linearized the 3-linear form with 4 "non-linear" quantities!

- Connections between scalar & vector valued expressions

$$l_1 \qquad 2^{-r_1} \quad 2^{-r_2}$$

$$2^{-s_1} = \frac{\alpha \cdot \alpha}{|E|^{1/2}}$$

$$2^{-l_2} \lesssim \frac{2^{-s_1}}{\left\| 1_{X^{l_2,s_1}} \right\|^{1/2}_{\ell^\infty_{n_1,m} \ell^1_{n_2}}} \quad \leftarrow \frac{4}{5}$$

and also
$$2^{-l_2} \lesssim \frac{2^{-s_2}}{\left\| 1_{X^{l_2,s_2}} \right\|_{\ell^\infty_{n_2,m} \ell^1_{n_1}}} \quad \leftarrow \frac{1}{5}$$

$$\left(\frac{1}{5} + \frac{4}{5} = 1 \right)$$

where
$$\begin{cases} X^{l_2,s_1} = A_2^{l_2} \cap C_1^{s_1} \\ X^{l_2,s_2} = A_2^{l_2} \cap C_2^{s_2} \end{cases}$$

- Estimates for $\#\left(X^{\vec{l},\vec{n},\vec{s},k} \right) := \#$

⊙₁ $\# \lesssim 2^k |H|$ ← trivial

⊙ $\| \quad \lesssim 2^{2n_1+2s_1} \| \quad \|$

$(\bullet_2)$ $\boxed{\# \lesssim \alpha \cdot \|1_{X^{l_2,s_1}}\|_{\ell^\infty_{u_1,m}\ell^1_{m_2}} \cdot |E| \cdot |F|}$

<u>Proof of $(\bullet_2)$</u>:

$$\# \le \sum_{u_1,m_2,m} 1_{X^{l_2,s_1}}(u_1,m_2,m) \cdot 1_{B_1^{r_1} \cap C_1^{s_1}}(u_1,m) \le$$

$$\le \|1_{X^{l_2,s_1}}\|_{\ell^\infty_{u_1,m}\ell^1_{m_2}} \cdot \|1_{B_1^{r_1} \cap C_1^{s_1}}\|_{\ell^1_{u_1,m}}$$

and

is $\lesssim \alpha^{2r_1+2s_1} \sum_{u_1,m} \|\langle f_1, \varphi_{u_1,m}\rangle\|_2 \cdot \|\langle f_2, \varphi_{u_1,m}\rangle\|_2$

$\lesssim \alpha^{2r_1+2s_1} |E| \cdot |F|$ by using

L^2-valued 2-linear transversality ∎

$(\bullet_3)$ $\boxed{\# \lesssim \|1_{X^{l_2,s_2}}\|_{\ell^\infty_{u_2,m}\ell^1_{m_1}} \cdot 2^{r_1} |E|^{1/2} \cdot 2^{5r_2+s_2} \cdot |E|^{5/2} \cdot |F|^{1/2}}$

Proof of $(\bullet_2)$:

Proof of ... .

$$\# \leq \sum_{u_1,u_2,m} 1_{B_2^{r_2} \cap C_2^{s_2}}(u_{2,m}) \cdot 1_{X^{l_2,s_2}}(u_1,u_{2,m}) \cdot 1_{B_1^{r_1}}(u_{1,m})$$

$$\leq \| 1_{X^{l_2,s_2}} \|_{\ell^\infty_{u_{2,m}} \ell^1_{u_1}} \cdot \| 1_{B_1^{r_1}} \|_{\ell^\infty_m \ell^1_{u_1}}^{1/2} \cdot \| 1_{B_2^{r_2} \cap C_2^{s_2}} \|_{\ell^1_{u_{2,m}}}$$

OK

then, for every fixed m :

Bessel

$$\| 1_{B_1^{r_1}} \|_{\ell^1_{u_1}} \lesssim 2^{2r_1} \sum_{u_1} \| \langle f_1, \varphi_{u_1,m} \rangle \|_2^2 \lesssim$$

$$\lesssim 2^{2r_1} \cdot |E| .$$

finally $\#(B_2^{r_2} \cap C_2^{s_2}) \lesssim 2^{5r_2 + s_2} \times$

$$\times \sum_{u_{2,m}} \| \langle f_1, \varphi_{u_{2,m}} \rangle \|_2^5 \cdot \| \langle f_2, \varphi_{u_{2,m}} \rangle \|_2 \lesssim$$

$$\lesssim 2^{5r_2 + s_2} \cdot \left(\sum_{u_{2,m}} \| \langle f_1, \varphi_{u_{2,m}} \rangle \|_2^6 \right)^{5/6} \times$$

$$\times \left(\sum_{u_{2,m}} \| \langle f_2, \varphi_{u_{2,m}} \rangle \|_2^6 \right)^{1/6} \lesssim$$

By L^2-valued Strichartz

$$2^{5n_2+s_2} \cdot |E|^{5/2} \cdot |F|^{1/2} \;\blacksquare.$$

- In the end, interpolate with weights $\frac{2}{5}^-$, $\frac{1}{5}^-$ and $\frac{2}{5}^+$ between ①, ② and ③ to obtain

$$\boxed{|\Lambda(f_1, f_2, h)| \lesssim |E|^{1/2} \cdot |F|^{1/2} \cdot |H|^{2/5^+}}$$

as desired... $\blacksquare$

This method also proves

$$\boxed{\mathbb{E}^{2\text{-}lin}_{2+1} : L^4 \times L^4 \to L^{3/2+\varepsilon}}$$

and

$$\boxed{\mathbb{E}^{3\text{-}lin} : L^4 \times L^4 \times L^4 \to L^{1+\varepsilon}}$$