Harmonic Analysis
and the Self-Improving Property of the Oscillation of a Function

Carlos Pérez

University of the Basque Country & BCAM

Harmonic Analytic Connections

Creswick, Victoria, 29 of May, 2024

Australia

Old, recent and very recent collaborators

- More recent results with:

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- More recent results with:

Iker Gardeazabal and Emiel Lorist

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Paul McManus

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improving the classical estimate of **Meyers-Ziemer theorem**

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$$I_\alpha(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n$$

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- equivalent to the **Isoperimetric inequality**

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- The point of the lecture is to avoid the use of fractional operators

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I will refer to it as the Brézis-Bourgain-Mironescu phenomenon.

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- **Method: Harmonic Analysis**

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- It is very simple to verify this condition

A first model theorem and a first geometric condition

Fix f and a_μ as above such that

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx \le a_\mu(Q) \qquad Q\in\mathcal{Q}$$

then

Thm (baby version)

$$\left\|f-f_Q\right\|_{L^{p^*,\infty}(Q,\frac{dx}{|Q|})} \le c_{n,p}\, a_\mu(Q)$$

- A consequence of the following geometric condition satisfied by the functional a_μ

$$\left(\sum_j a_\mu(Q_j)^{p^*}\frac{|Q_j|}{|Q|}\right)^{\frac{1}{p^*}} \le a_\mu(Q) \qquad (D_{p^*} \text{ condition})$$

- It is very simple to verify this condition
- The exponent p^* is optimal

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- method of proof is based on the **Moser iteration technique**

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and to consider other objects beyond the gradient

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- the sharpest $r = r_a$ is a sort of the “**Sobolev exponent**” of the functional a

A recent much sharper theorem

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• Notice that this yields the John-Nirenberg theorem as well with the sharp A_∞

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- Compare with the $D_r(w)$ condition:

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- The best possible constant c above is denoted by $\|a\|$

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- Letting $s \to \infty$ (which formally is $D_r(w)$) we recover the previous result

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- **A. Lerner, E. Lorist and S. Ombrosi** relax the A_∞ condition we imposed. They use sparse theory

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$$\left(\fint_Q |f(x) - f_Q|^{p_\delta^*}\, dx \right)^{\frac{1}{p_\delta^*}} \le c_n\, p_\delta^*\, c_n \ell(Q)^{\delta} \left(\fint_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, dx \right)^{1/p}$$

from which we get the global result,

$$\left(\int_{\mathbb{R}^n} |f(x)|^{p_\delta^*}\, dx \right)^{\frac{1}{p_\delta^*}} \le c_n\, p_\delta^* \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\delta p}}\, dy\, dx \right)^{1/p}$$

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- We can **self-improve** starting from this result

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• Motivated by a result by **Brezis-Van Schaftingen-Yung**

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This inequality is important in many applications.

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- From here we can recover the result of Trudinger.

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- The case $q = \frac{n-\delta}{n}$ corresponds to the Isoperimetric inequality already mentioned

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• Improves classical result of W. Ziemer and N. Meyers from the 60's. They just considered the case $q = 1$.

Isoperimetric inequalities and Extrapolation theory I

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Thank you