How to produce curved Kakeya sets that are small

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Contents

Hörmander type operators

- Joint with Shaoming Guo and Hong Wang (2022), we studied $L^p \to L^q$ mapping properties of Hörmander type operators.
- These include the key operators in Fourier restriction and Bochner-Riesz.

Hörmander type operators: setup

- We care about oscillatory integral operators mapping functions on $\mathbb{R}^{n-1}$ to functions on $\mathbb{R}^n$.
- For $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$, real $\phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ smooth in a neighborhood of supp$a$ and $\lambda > 1$, consider the operator

$$T^\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{2\pi \mathrm{i} \phi^\lambda(x;\xi)} a^\lambda(x;\xi) f(\xi) \mathrm{d}\xi$$

where $\phi^\lambda(x;\xi) = \lambda\phi(\frac{x}{\lambda};\xi)$ and $a^\lambda(x;\xi) = a(\frac{x}{\lambda};\xi)$.

Hörmander conditions

If we have

- (H1) The rank of $\nabla_x \nabla_\xi \phi$ is $n-1$ throughout supp$a$.
- (H2) For the *Gauss map* $G(x;\xi)$ with $G = \frac{G_0(x;\xi)}{|G_0(x;\xi)|}$ and

  $$G_0(x;\xi) = \wedge_{j=1}^{n-1} \partial_{\xi_j} \nabla_x \phi(x;\xi),$$

  we have

  $$\det(\nabla_\xi)^2 \langle \nabla_x \phi(x;\xi), G(x;\xi_0) \rangle|_{\xi=\xi_0} \neq 0$$

  ,

then T^λ is called a (family of) *Hörmander type operator*(s).

The positive definiteness condition

To make life easier, let us only care about Hörmander type operators that in addition satisfy:

- (H2+) $(\nabla_\xi)^2 \langle \nabla_x \phi(x;\xi), G(x;\xi_0)\rangle|_{\xi=\xi_0}$ is always positive definite.

(H2+) holds for the key operators of interest in Bochner-Riesz and Fourier restriction.

Central question

Question

For a family of Hörmander type operators T^λ satisfying (H2+), is it true that $\|T^\lambda\|_{L^p\to L^p} \lesssim_\varepsilon \lambda^\varepsilon$, $\forall p > \frac{2n}{n-1}$?

- Answer: Not necessarily (Bourgain (1991), Guth-Hickman-Iliopoulou (2017), see also Bourgain-Guth (2011) and Wisewell (2005)). Answer is known to be complicated.
- Motivations: Unifying Fourier restriction and Bochner-Riesz. Important check to various approaches for Bochner-Riesz.

Bourgain's condition

Our work was inspired by a 1991 paper of Bourgain.
Diffeomorphisms in x and in ξ (separately) can change ϕ to a *normal form* around any point (taken to 0) in supp$a$:

$$\phi(x;\xi) = x_1\xi_1 + \cdots + x_{n-1}\xi_{n-1} + x_n\langle A\xi, \xi\rangle + O(|x_n||\xi|^3 + |x|^2|\xi|^2).$$

We say ϕ satisfies *Bourgain's condition* at the point if in the above normal form, $\partial_{x_n}^2(\nabla_\xi)^2\phi|_{(0;0)}$ being a multiple of $\partial_{x_n}(\nabla_\xi)^2\phi|_{(0;0)}$.

- This is intrinsic.

Conjecture (Guo-Wang-Z. (2022))

For a family of Hörmander type operators T^λ satisfying (H2+), $\|T^\lambda\|_{L^p\to L^p} \lesssim_\varepsilon \lambda^\varepsilon$ holds for every $p > \frac{2n}{n-1}$ if and only if ϕ satisfies Bourgain's condition everywhere in supp$a$.

For the key operators in Bochner-Riesz and Fourier restriction, ϕ indeed satisfies Bourgain's condition!

Generic failure

Theorem (Guo-Wang-Z. (2022))

If Bourgain's condition fails at a point, then $\|T^\lambda\|_{L^p\to L^p} \lesssim_\varepsilon \lambda^\varepsilon$ *fails for* $p < \frac{2(2n^2+n-1)}{2n^2-n-2}$.

- This number is $> \frac{2n}{n-1}$.
- Generic failure in dimension 3 by Bourgain (1991).

Positive result

Theorem (Guo-Wang-Z. (2022))

If Bourgain's condition is satisfied everywhere in supp$a$*, then* $\|T^\lambda\|_{L^p\to L^p} \lesssim_\varepsilon \lambda^\varepsilon$ *holds for* $p > p_{n,\mathit{GWZ}}$.

- Asymptotically improves on both Bochner-Riesz and Fourier restriction in high dimensions

More motivation

- General Theory needed to study operators on Riemannian manifolds.
- For reduced Carleson-Sjölin operators for manifolds, Bourgain's condition $\Leftrightarrow$ constant sectional curvature. (Dai-Gong-Guo-Z., 2023).

Curved Kakeya sets

- Our results are related to the theory of curved Kakeya sets.
- $(x, t) \in \mathbb{R}^n$. $x \in \mathbb{R}^{n-1}$. $t \in \mathbb{R}$.
- Setup: For "frequency" $\xi \in [0, 1]^{n-1}$, we have a family of curves $(x, t)_{0 \leq t \leq 1}$ where $x = x(\xi, t, \omega)$ smooth.
- ω: "position parameter".
- Question: If we choose such a curve for each $\xi \in [0, 1]^{n-1}$, can the union have dimension $< n$?

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- Usual Kakeya: $x(\xi, t, \omega) = \omega + t\xi$.
- Oversimplification warning: In reality, the function x has some more constraints. e.g. one curve per direction per point.

Technical comments about oversimplification

- In all applications, $x = x(\xi, t, \omega)$ is determined by

$$\nabla_\xi \phi(x, t, \xi) = \omega.$$

- One curve per point per direction for nondegenerate ϕ, etc.

Large curved Kakeya sets

- Question: If we choose a curve for each $\xi \in [0,1]^{n-1}$, can the union have dimension $< n$?
- Example: for usual Kakeya $x(\xi, t, \omega) = \omega + t\xi$, it is conjectured the union has dimension n.

Small curved Kakeya sets

- Question: If we choose a curve for each $\xi \in [0,1]^{n-1}$, can the union have dimension $< n$?
- Example: for $x(\xi, t, \omega) = \omega + t\xi + t^2(0, \xi_1)$, $n = 3$, consider $\omega = (\xi_2, 0)$. We note that all $(\xi_2 + t\xi_1, t\xi_2 + t^2\xi_1, t)$ are on the surface $x_2 = x_1 x_3$.
- Hence in this case the curved Kakeya set can have dimension 2!

How to form a conjecture?

- For $\xi \in [0,1]^{n-1}$, we have a family of curves $(x,t)_{0\leq t\leq 1}$ where $x = x(\xi, t, \omega)$ smooth.
- Question: If we choose a curve for each $\xi \in [0,1]^{n-1}$, can the union have dimension $< n$?
- A reasonable guess (inspired by Katz-Rogers (2018)): The truth should not be too far from when $\omega = \omega(\xi)$ is a "nice" map (smooth, bounded degree algebraic, etc.).

Can the Kakeya set have dimension $< n$?

- For $\xi \in [0,1]^{n-1}$, we have a family of curves $(x,t)_{0\leq t\leq 1}$ where $x = x(\xi, t, \omega)$ smooth.
- Question: If we choose a curve for each $\xi \in [0,1]^{n-1}$, can the union have dimension $< n$?
- Pretending ω is nice, by calculus we can expect

$$\left|\bigcup_{\xi\in[0,1]^{n-1}}\{(x,t) : 0 \leq t \leq 1\}\right|$$
$$\sim \quad \int_{\xi\in[0,1]^{n-1}}\int_0^1 \left|\det\left(\nabla_\xi x + \nabla_\omega x \cdot \nabla_\xi \omega\right)\right| \mathrm{d}t\mathrm{d}\xi.$$

Can we make the integral small?

- By calculus we can expect

$$\left|\bigcup_{\xi\in[0,1]^{n-1}}\{(x,t):0\le t\le 1\}\right|$$
$$\sim \int_{\xi\in[0,1]^{n-1}}\int_0^1 \left|\det\left(\nabla_\xi x+\nabla_\omega x\cdot\nabla_\xi\omega\right)\right| \mathrm{d}t\mathrm{d}\xi.$$

- Key trick: Integrating in t first. The unknown $\nabla_\xi\omega$ becomes a constant for $0\le t\le 1$.
- If $\int_0^1 \left|\det\left(\nabla_\xi x+\nabla_\omega x\cdot M\right)\right| \mathrm{d}t \gtrsim 1$ for every matrix M, then good reasons to believe Kakeya holds. Can be verified under Bourgain's condition.
- Otherwise, no reason to expect Kakeya. Good chance to fail the analogue of Fourier Restriction Conjecture.

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- Otherwise, no reason to expect Kakeya. Good chance to fail the analogue of Fourier Restriction Conjecture.
- For technicality reasons, we often care about the bound $\int_0^{\delta^\varepsilon}\left|\det\left(\nabla_\xi x+\nabla_\omega x\cdot M\right)\right|\mathrm{d}t\gtrsim\delta^\varepsilon$. Allows us to assume everything is degree $O(1)$ polynomial.

Testing a good example (Kakeya)

- Is it true that $\int_0^1 |\det(\nabla_\xi x + \nabla_\omega x \cdot M)| \, \mathrm{d}t \gtrsim 1$ for $x(\xi, t, \omega) = \omega + t\xi$?
- The integrand is $|\det(tI + M)| = |P_M(t)|$ (monic, degree $n-1$). The average of this is $\gtrsim 1$ on $[0,1]$, independent of M.
- Key ingredient in Katz-Rogers' proof of the Polynomial Wolff Axiom.

Testing a bad example

- Is it true that $\int_0^1 \left|\det\left(\nabla_\xi x + \nabla_\omega x \cdot M\right)\right| \mathrm{d}t \gtrsim 1$ for $x(\xi, t, \omega) = \omega + t\xi + t^2(0, \xi_1)$?
- The integrand is $\left|\det\left(\begin{pmatrix} t & t^2 \\ 0 & t \end{pmatrix} + M\right)\right|$.
- This is identically 0 for $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$!

What to do when things look nice

- By Taylor, everything can be assumed to be polynomial of degree $O(1)$ and we honestly have

$$\begin{aligned} & \left|\bigcup_{\xi\in[0,1]^{n-1}}\{(x,t):0\leq t\leq 1\}\right| \\ \sim & \int_{\xi\in[0,1]^{n-1}}\int_0^1 \left|\det\left(\nabla_\xi x+\nabla_\omega x\cdot\nabla_\xi\omega\right)\right| \mathrm{d}t\mathrm{d}\xi \\ \gtrapprox & \quad 1. \end{aligned}$$

.

When things look nice...

- One can prove the analogues of Polynomial Wolff Axiom (Katz-Rogers, 2018) and nested Polynomial Wolff Axiom (Hickman-Rogers-Z. (2019), independently Zahl (2019)).
- Kakeya for $\omega \in C^{\alpha}, \alpha > 1 - \frac{1}{(n-1)^2}$ is known (Fu-Gan, 2023).

What to do when things look nasty

- The image of $\Psi : (\xi, t) \mapsto (x, t)$ has abnormally small measure.
- To control its δ-neighborhood volume, we need to understand the boundary of the image of the map Ψ.
- Contained in $\Psi(\text{Sing}\Psi) \bigcup \Psi(\partial[0,1]^n)$. Dimension is lower. Entropy bound by Yomdin-Comte (2004) that generalizes Wongkew (1993).
- Compare to Bourgain's work: for a fixed t he made the image of Ψ near a line.

When things look nasty...

- For Hörmander type operators, we know if Bourgain's condition fails at one point, we always can have a Kakeya compression that is significant enough to fail the analogue of Fourier restriction (Bourgain, 1991 for $n = 3$; Guo-Wang-Z., 2022).

Open problems

- For a particular $x(\xi, t, \omega)$, make the conjecture and prove it (Wisewell for some examples, 2005).
- When Bourgain's condition fails, improve our result to find an even larger Kakeya compression (so we know the analogue of Fourier restriction fails at an even higher p)?
- What do the set of all possible critical exponents of "Hörmander restriction" and "Hörmander Kakeya" look like? Countable? Finite? Very few limit points?
- Characterize the operator/the Kakeya setup when some particular exponent is attained?

Thank you!