IMPROVING THE RANGE OF p IN THE EXTENSION CONJECTURE FOR THE SPHERE, AND IN THE KAKEYA MAXIMAL CONJECTURE

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1. Introduction

The extension conjecture for the sphere $\mathbb{S}^{d-1} \in \mathbb{R}^d$ says that if E is the extension operator, defined for smooth functions h on $\mathbb{S}^{d-1}$ by

$$Eh(x) = \int_{\mathbb{S}^{d-1}} h(\xi) e^{2\pi i x\cdot\xi} d\sigma(\xi),$$

then E extends to a bounded linear operator from $L^p(\mathbb{S}^{d-1}, d\sigma)$ to $L^q(\mathbb{R}^d)$, whenever

$$q > \frac{2d}{d-1}, \quad p' \leq \frac{d-1}{d+1} q.$$

Here $d\sigma$ is the standard surface measure on $\mathbb{S}^{d-1}$. If the extension conjecture is true for a certain pair of exponents (p_0, q_0), then it is also true for all pairs of exponents (p, q_0) where $p > p_0$, since $L^p(\mathbb{S}^{d-1}, d\sigma)$ embeds continuously into $L^{p_0}(\mathbb{S}^{d-1}, d\sigma)$. Rather surprisingly, there is sometimes a way to reverse this implication: following Bourgain [1] (see the remark after his Proposition 6.47), we will use a factorization theorem of Pisier [5] (that has its origin in earlier works of Maurey and Nikishin), together with rotation invariance of the extension operator, to prove the following theorem.

Theorem 1. *Suppose $E\colon L^\infty(\mathbb{S}^{d-1}, d\sigma) \to L^q(\mathbb{R}^d)$ is bounded for some $q \geq 2$. Then E extends to a bounded linear operator from $L^{q,1}(\mathbb{S}^{d-1}, d\sigma)$ to $L^q(\mathbb{R}^d)$.*

Here $L^{q,1}(\mathbb{S}^{d-1}, d\sigma)$ is the Lorentz space, with quasi-norm

$$\begin{aligned}\|h\|_{L^{q,1}(\mathbb{S}^{d-1},d\sigma)} &= q\|t d\sigma(|h| > t)^{\frac{1}{q}}\|_{L^1(\mathbb{R}^+, \frac{dt}{t})} \\ &= q\int_0^\infty d\sigma\{\xi \in \mathbb{S}^{d-1} \colon |h(\xi)| > t\}^{\frac{1}{q}} dt.\end{aligned}$$

It then follows, by Marcinkiewicz interpolation with the trivial continuity of $E\colon L^\infty(\mathbb{S}^{d-1}, d\sigma) \to L^\infty(\mathbb{R}^d)$, that E maps $L^s(\mathbb{S}^{d-1}, d\sigma)$ boundedly to $L^s(\mathbb{R}^d)$ for all $s > q$.

A similar argument can also be used in the study of the Kakeya maximal conjecture. It says that for any

$$q \geq \frac{d}{d-1}, \quad p' \leq (d-1)q, \quad \beta > \frac{d}{q'} - 1,$$

there exists a constant $C_{p,q,\beta}$ such that for any $0 < \delta < 1$ and any family $\mathbb{T}$ of δ-separated $\delta^{d-1} \times 1$ tubes in $\mathbb{R}^d$, we have

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^d)} \leq C_{p,q,\beta} \delta^{-\beta} \left(\sum_{T \in \mathbb{T}} |T| \right)^{\frac{1}{p}}. \tag{1}$$

Here a family of δ-separated tubes is one where the directions of any different tubes from the family are at least δ from each other. When $p = \infty$, the term $\left(\sum_{T \in \mathbb{T}} |T|\right)^{1/p}$ above is interpreted to be 1. If (1) is true for a certain triple of exponents (p_0, q_0, β_0), then it is also true for all triples of exponents (p, q_0, β_0) where $p > p_0$, since we always have $\sum_{T \in \mathbb{T}} |T| \lesssim 1$, thanks to the fact that we have at most $\lesssim \delta^{-(d-1)}$ tubes in $\mathbb{T}$ should they be δ-separated. There is a partial converse to this implication:

Theorem 2. *Suppose (1) holds for certain exponents q and β with $p = \infty$. Then for any $\varepsilon > 0$, there exists a constant $C_{q,\beta,\varepsilon}$ such that*

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^d)} \leq C_{q,\beta,\varepsilon} \delta^{-\beta-\varepsilon} \left(\sum_{T \in \mathbb{T}} |T| \right)^{\frac{1}{q}}$$

for all $0 < \delta < 1$ and all family $\mathbb{T}$ of δ-separated $\delta^{d-1} \times 1$ tubes in $\mathbb{R}^d$.

One can also restate this in terms of the Kakeya maximal function; this is implicit in the proof given in Section 5.

We note that our proofs of Theorems 1 and 2 are not the shortest possible. See also lecture notes of Tao [6, Section 2], as well as the exposition in the book of Mattila [4, Propositions 19.9 and 22.7], for some more direct proofs. Also, sometimes it is possible to improve the range of exponents even further: for instance, Kim [3] improved the recent celebrated result of Guth [2], that says the extension operator for any compact smooth surface $S \subset \mathbb{R}^3$ with positive second fundamental form is bounded from $L^p(S)$ to $L^q(\mathbb{R}^3)$, whenever $q > 3.25$ and $p = \infty$. By refining the methods of Guth, Kim showed that the same remains true, whenever $q > 3.25$ and $p \leq q'/2$.

This note will be organized as follows. In Section 2, we will give an exposition of the Maurey-Nikishin-Pisier factorization theorem, including a full and direct proof of the part we will use. In Section 3, we give a corollary of the Maurey-Nikishin-Pisier factorization theorem for operators that commutes with rotations. We then apply this corollary to the extension problem for the sphere in Section 4, and to the Kakeya maximal conjecture in Section 5. Finally, in Section 6, we give some remarks about the implications of Theorems 1 and 2 on the Hausdorff dimension of a Kakeya set in $\mathbb{R}^d$.

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2. The Maurey-Nikishin-Pisier factorization theorem

The Maurey-Nikishin-Pisier factorization theorem concerns a necessarily and sufficient condition for a bounded linear operator T to be factorized through a weak L^p space after a change of measure. To appreciate this, let $(\Omega, d\mu)$ be any measure space (with a non-negative measure μ). Suppose $0 < r < p < \infty$, and $f \geq 0$ is a function on Ω with $\int_\Omega f d\mu \leq 1$. Then $(\Omega, f d\mu)$ is a finite measure space, and $L^{p,\infty}(\Omega, f d\mu)$ embeds into $L^r(\Omega, f d\mu)$. The map $M_{f^{1/r}}$, defined to be the multiplication map by $f^{1/r}$, is a continuous linear map from $L^r(\Omega, f d\mu)$ to $L^r(\Omega, d\mu)$. Thus $M_{f^{1/r}}$ is a continuous linear map from $L^{p,\infty}(\Omega, f d\mu)$ to $L^r(\Omega, d\mu)$.

If now X is a Banach (or quasi-Banach) space, and $\tilde{T} \colon X \to L^{p,\infty}(\Omega, f d\mu)$ is a bounded quasi-linear map (actually we do not need full quasi-linearity here; we just need $|T(\lambda x)| \leq |\lambda||Tx|$ a.e. for all $\lambda \in \mathbb{R}$, $x \in X$), then the map $T := M_{f^{1/r}} \circ \tilde{T}$ is bounded from X to $L^r(\Omega, d\mu)$, i.e.

$$\|Tx\|_{L^r(\Omega, d\mu)} \leq C \|x\|_X$$

for all $x \in X$. The Maurey-Nikishin-Pisier factorization theorem states when a bounded quasi-linear map $T \colon X \to L^r(\Omega, d\mu)$ can be factorized like this. More precisely, it states the following (c.f. Theorem 1.2 of [5]):

Theorem 3 (Maurey-Nikishin-Pisier)**.** *Suppose $(\Omega, d\mu)$ is a measure space, and $0 < r < p < \infty$. Let X be a Banach (or quasi-Banach) space, and $T \colon X \to L^r(\Omega, d\mu)$ be a bounded quasi-linear operator. The following are equivalent:*

(i) There is a constant C such that for any finite sequences $\{x_i\}$ in X, we have

$$\left\| \sup_i |T x_i| \right\|_{L^r(\Omega, d\mu)} \leq C \left(\sum_i \|x_i\|_X^p \right)^{1/p}$$

(ii) There exists a non-negative measurable function f on Ω with $\int_\Omega f d\mu = 1$, and a constant C', such that for any $x \in X$ and any measurable subset E of Ω, we have

$$\left(\int_E |Tx|^r d\mu \right)^{\frac{1}{r}} \leq C' \|x\|_X \left(\int_E f d\mu \right)^{\frac{1}{r} - \frac{1}{p}}.$$

(iii) There exists a non-negative function f on Ω with $\int_\Omega f d\mu = 1$, such that for any $x \in X$, we have $Tx = 0$ μ-almost everywhere on the zero set of f, and such that T admits a factorization

$$T = M_{f^{1/r}} \circ \tilde{T},$$

where $M_{f^{1/r}}$ is the multiplication operator by $f^{1/r}$, and $\tilde{T}$ is a bounded quasi-linear operator from X to $L^{p,\infty}(\Omega, f d\mu)$.

We will only use the implication (i) implies (ii), and we will give a direct and complete proof of this implication in what follows. Along the way, we will also observe that one may choose the constant C' in (ii) to be $2^{1/r}$ times the constant C in (i). This will be useful in our proof of Theorem 2.

The key for the proof that (i) implies (ii) is the Dunford-Pettis theorem, about *uniform integrability* of a family of functions.

Definition 1. *A family of integrable functions $\mathcal{F}$ on a measure space $(\Omega, d\mu)$ is said to be uniformly integrable, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever A is a measurable subset of Ω with $\mu(A) < \delta$, we have*

$$\int_A f d\mu < \varepsilon$$

for all $f \in \mathcal{F}$.

We first have the following sufficient condition for uniform integrability:

Lemma 1. *Suppose $(\Omega, d\mu)$ is a measure space, and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative measurable functions with $\int_\Omega f_n d\mu \leq 1$ for all $n \in \mathbb{N}$. Suppose for any sequence of mutually disjoint measurable subsets $\{A_n\}_{n\in\mathbb{N}}$ of Ω, we have*

$$\lim_{n\to\infty} \int_{A_n} f_n d\mu = 0.$$

Then $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable.

Proof. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is as given. If $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly integrable, then there exists $\varepsilon > 0$, such that for any $n \in \mathbb{N}$, there exists a measurable subset B_n of Ω, with $\mu(B_n) < 2^{-n}$, such that

$$\int_{B_n} f_n d\mu > \varepsilon.$$

We extract a subsequence B_{n_k} of B_n as follows: first let $n_1 = 1$ so that $B_{n_1} = B_1$. When n_k is chosen for some $k \geq 1$, we choose n_{k+1} large enough, so that

$$\int_{B_{n_k} \setminus \bigcup_{n \geq n_{k+1}} B_n} f_{n_k} d\mu > \frac{\varepsilon}{2}.$$

This is possible since $f_{n_k} \in L^1(\Omega, d\mu)$, and

$$\mu\left(\bigcup_{n\geq n_{k+1}} B_n\right) < \sum_{n\geq n_{k+1}} 2^{-n} = 2^{1-n_{k+1}} \to 0$$

as $n_{k+1} \to \infty$. We define a sequence of measurable subsets $\{A_n\}_{n\in\mathbb{N}}$ of Ω, by

$$A_n = \begin{cases} \emptyset & \text{if } n \neq n_k \text{ for any } k \in \mathbb{N} \\ B_{n_k} \setminus \bigcup_{n\geq n_{k+1}} B_n & \text{if } n = n_k \text{ for some } k \in \mathbb{N}. \end{cases}$$

Then $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of mutually disjoint measurable sets in Ω, with

$$\int_{A_n} f_n d\mu > \frac{\varepsilon}{2}$$

for infinitely many n's. But our assumption on $\{f_n\}_{n\in\mathbb{N}}$ says that $\int_{A_n} f_n d\mu$ should tend to zero as $n \to \infty$. This is a contradiction, so $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable. ☐

Next we need a consequence of uniform integrability from the Dunford-Pettis theorem:

Theorem 4 (Dunford-Pettis)**.** *Suppose* $(\Omega, d\mu)$ *is a probability measure space, and* $\mathcal{F}$ *is a bounded subset of* $L^1(\Omega, d\mu)$*. Then* $\mathcal{F}$ *is uniformly integrable, if and only if* $\mathcal{F}$ *is weakly compact in* $L^1(\Omega, d\mu)$*, i.e. for any sequence in* $\{f_n\}_{n\in\mathbb{N}}$ *in* $L^1(\Omega, d\mu)$*, there exists a subsequence* $\{f_{n_k}\}_{k\in\mathbb{N}}$ *and a function* $f \in L^1(\Omega, d\mu)$*, such that*

$$\lim_{k\to\infty} \int_\Omega f_{n_k}\, g\, d\mu = \int_\Omega f\, g\, d\mu$$

whenever $g \in L^\infty(\Omega, d\mu)$*.*

We will only need the forward implication, so let us focus only on that.

Proof of the forward implication. Suppose $\mathcal{F}$ is a bounded subset of $L^1(\Omega, d\mu)$ and is uniformly integrable. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$. Then by Banach-Alaoglu theorem, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$, and some $F \in L^\infty(\Omega, d\mu)^*$, such that

$$\lim_{k\to\infty} \int_\Omega f_{n_k}\, g\, d\mu = F(g)$$

for all $g \in L^\infty(\Omega, d\mu)$. We will define a new measure space $(\Omega, d\nu)$, whose measurable sets are precisely those of $(\Omega, d\mu)$, such that

$$\nu(A) := F(\chi_A)$$

for every μ-measurable subset A of Ω; here χ_A is the characteristic function of A. (We will just say "measurable" in lieu of "μ-measurable" since μ-measurability is the same as ν-measurability, and there is no danger of confusion here.) To verify that the above indeed defines ν as a measure, we need to check that ν is countably additive. By linearity of F, we see that ν is additive; if $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of mutually disjoint measurable subsets of Ω, and $A = \bigcup_{n\in\mathbb{N}} A_n$, then

$$\nu(A) = \sum_{n=1}^{N} \nu(A_n) + \nu\left(\bigcup_{n>N} A_n\right).$$

We claim that

$$\lim_{N\to\infty} \nu\left(\bigcup_{n>N} A_n\right) = 0. \tag{2}$$

To do so, note that since (Ω, μ) is a finite measure space, we have

$$\lim_{N\to\infty} \mu\left(\bigcup_{n>N} A_n\right) = 0.$$

Since $\{f_{n_k}\}_{k\in\mathbb{N}}$ is uniformly integrable, given any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever B is a measurable subset of Ω with $\mu(B) < \delta$, we have

$$\int_B f_{n_k} d\mu < \varepsilon$$

for all $k \in \mathbb{N}$. Now pick N large enough such that

$$\mu\left(\bigcup_{n>N} A_n\right) < \delta.$$

Then

$$\nu\left(\bigcup_{n>N} A_n\right) = \lim_{k\to\infty} \int_{\bigcup_{n>N} A_n} f_{n_k} d\mu \leq \varepsilon.$$

This shows that (2) holds, and one sees that ν is a measure on Ω.

Since simple functions are dense in $L^\infty(\Omega, d\mu)$, we see that

$$F(g) = \int_\Omega g \, d\nu$$

for every $g \in L^\infty(\Omega, d\mu)$. Also, ν is absolutely continuous with respect to μ: if A is a measurable subset of Ω with $\mu(A) = 0$, then

$$\nu(A) = \lim_{k\to\infty} \int_A f_{n_k} d\mu = 0.$$

It follows from the Radon-Nikodym theorem that there exists a function $f \in L^1(\Omega, d\mu)$ such that $d\nu = f d\mu$. In other words,

$$\lim_{k\to\infty} \int_\Omega f_{n_k} \, g \, d\mu = \int_\Omega f \, g \, d\mu$$

for all $g \in L^\infty(\Omega, d\mu)$, as desired. □

Proof of the implication (i) ⇒ (ii) in Theorem 3. For each $n \in \mathbb{N}$, let

$$C_n := \sup\left\{ \left\| \sup_{1\leq i\leq n} |Tx_i| \right\|_{L^r(\Omega, d\mu)} : x_1, \dots, x_n \in X \text{ with } \sum_{i=1}^n \|x_i\|_X^p = 1 \right\}.$$

Then (i) implies that the sequence $\{C_n\}_{n\in\mathbb{N}}$ is bounded above, and without loss of generality we assume that C_n increases to C and $C > 0$. Now for each $n \in \mathbb{N}$, let $x_1^{(n)}, \dots, x_n^{(n)} \in X$ be such that $\sum_{i=1}^n \|x_i^{(n)}\|_X^p = 1$, and

$$\left\| \sup_{1\leq i\leq n} |Tx_i^{(n)}| \right\|_{L^r(\Omega, d\mu)} \geq C_n - \frac{1}{n}.$$

We write

$$f_n := C^{-r} \sup_{1\leq i\leq n} |Tx_i^{(n)}|^r,$$

so that

$$C^{-r}\left(C_n - \frac{1}{n}\right)^r \leq \|f_n\|_{L^1(\Omega, d\mu)} \leq 1. \tag{3}$$

We claim that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable on $(\Omega, d\mu)$: indeed, for any non-negative sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, by assumption (i), we have

$$\left\| \sup_{1\leq n\leq N} \sup_{1\leq i\leq n} \lambda_n^{1/r} |Tx_i^{(n)}| \right\|_{L^r(\Omega, d\mu)} \leq C \left(\sum_{n=1}^N \sum_{i=1}^n \lambda_n^{p/r} \|x_i^{(n)}\|_X^p \right)^{1/p},$$

i.e.

$$\left\| \sup_{1\leq n\leq N} \lambda_n f_n \right\|_{L^1(\Omega, d\mu)} \leq \left(\sum_{n=1}^N \lambda_n^{p/r} \right)^{r/p}.$$

So if $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of mutually disjoint measurable subsets of Ω, then

$$\sum_{n=1}^{N} \lambda_n \int_{A_n} f_n d\mu \le \left(\sum_{n=1}^{N} \lambda_n^{p/r} \right)^{r/p}.$$

Since this is true for any non-negative sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, we see that the sequence $\{\int_{A_n} f_n d\mu\}_{n\in\mathbb{N}}$ is in ℓ^γ, where γ is the dual exponent to p/r. Note that $p/r \in (1,\infty)$, hence so is γ. In particular,

$$\lim_{n\to\infty} \int_{A_n} f_n d\mu = 0,$$

so Lemma 1 applies, and $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable on $(\Omega, d\mu)$. It follows from the Dunford-Pettis Theorem 4 that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$, and a function $f \in L^1(\Omega, d\mu)$, such that

$$\lim_{k\to\infty} \int_E f_{n_k}\, d\mu = \int_E f\, d\mu \tag{4}$$

whenever E is a measurable subset of Ω. Letting $E = \Omega$, and using (3), we see that

$$\int_\Omega f d\mu = 1.$$

Suppose now $x \in X$ and $\varepsilon > 0$. Then by (i), we have

$$\left\| \max\left\{ \varepsilon |Tx|, \sup_{1\le i\le n} |Tx_i^{(n)}| \right\} \right\|_{L^r(\Omega,d\mu)} \le C \left(\|\varepsilon x\|_X^p + \sum_{i=1}^{n} \|x_i^{(n)}\|_X^p \right)^{1/p} = C \left(\varepsilon^p \|x\|_X^p + 1 \right)^{1/p}.$$

Hence

$$\int_\Omega \max\{\varepsilon^r |Tx|^r, C^r f_n\} d\mu \le C^r \left(\varepsilon^p \|x\|_X^p + 1 \right)^{r/p}$$

If E is a measurable subset of Ω, then the left hand side above is at least

$$C^r \int_{\Omega\setminus E} f_n d\mu + \int_E \varepsilon^r |Tx|^r d\mu \ge \left(C_n - \frac{1}{n} \right)^r - C^r \int_E f_n d\mu + \varepsilon^r \int_E |Tx|^r d\mu,$$

so we get

$$\varepsilon^r \int_E |Tx|^r d\mu \le C^r \left(\varepsilon^p \|x\|_X^p + 1 \right)^{r/p} - \left(C_n - \frac{1}{n} \right)^r + C^r \int_E f_n d\mu.$$

Passing to limit along the subsequence $\{n_k\}_{k\in\mathbb{N}}$, then using (4), we see that

$$\begin{aligned} \varepsilon^r \int_E |Tx|^r d\mu &\le C^r \left(\varepsilon^p \|x\|_X^p + 1 \right)^{r/p} - C^r + C^r \int_E f d\mu \\ &\le C^r \varepsilon^p \|x\|_X^p + C^r \int_E f d\mu. \end{aligned}$$

(We used that $r/p < 1$ in the last inequality.) Setting $\varepsilon = \|x\|_X^{-1} \left(\int_E f d\mu \right)^{1/p}$, and taking $1/r$ power on both sides, we see that

$$\left(\int_E |Tx|^r d\mu \right)^{1/r} \le 2^{1/r} C \|x\|_X \left(\int_E f d\mu \right)^{\frac{1}{r} - \frac{1}{p}}.$$

This establishes (ii) with $C' = 2^{1/r} C$. $\square$

3. Maurey-Nikishin-Pisier factorization for rotationally invariant operators

We now specialize to the situation when $\Omega = \mathbb{S}^{d-1}$ and $d\mu = d\sigma$, the standard surface measure on $\mathbb{S}^{d-1}$. We will apply Theorem 3 to a bounded linear or sublinear operator $T\colon L^{q'}(\mathbb{R}^d) \to L^1(\mathbb{S}^{d-1}, d\sigma)$; in other words, we take $r = 1$ and $X = L^{q'}(\mathbb{R}^d)$ for some exponent q'. We will assume also that T commutes with rotations; i.e.

$$T(g \circ A) = (Tg) \circ A$$

for all $g \in L^{q'}(\mathbb{R}^d)$ and all A in the orthogonal group $O(d)$. We then have the following corollary.

Corollary 1. *Let $1 < p < \infty$, and let $T\colon L^{q'}(\mathbb{R}^d) \to L^1(\mathbb{S}^{d-1}, d\sigma)$ be a bounded linear or sublinear operator that commutes with all rotations on $\mathbb{R}^d$. Suppose furthermore that there exists a constant C such that for any finite sequences $\{g_i\}$ in $L^{q'}(\mathbb{R}^d)$, we have*

$$\left\| \sup_i |Tg_i| \right\|_{L^1(\mathbb{S}^{d-1}, d\sigma)} \leq C \left(\sum_i \|g_i\|^p_{L^{q'}(\mathbb{R}^d)} \right)^{\frac{1}{p}}. \tag{5}$$

Then T defines a bounded operator from $L^{q'}(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{S}^{d-1}, d\sigma)$, with a norm C' where $C' \leq 2C$.

Proof. By Theorem 3 (more precisely the implication (i) ⇒ (ii) there), there exists a non-negative measurable function f on $\mathbb{S}^{d-1}$, with $\int_{\mathbb{S}^{d-1}} f d\sigma = 1$, and a constant $C' \leq 2C$, such that for any $g \in L^{q'}(\mathbb{R}^d)$ and any measurable subset E of $\mathbb{S}^{d-1}$, we have

$$\int_E |Tg| d\sigma \leq C' \|g\|_{L^{q'}(\mathbb{R}^d)} \left(\int_E f d\sigma \right)^{1-\frac{1}{p}}. \tag{6}$$

Suppose now A is any rotation in $O(d)$. Then applying (6) to $g \circ A$ in place of g, and $A(E)$ in place of E, we obtain

$$\int_{A(E)} |Tg| \circ A \, d\sigma \leq C' \|g\|_{L^{q'}(\mathbb{R}^d)} \left(\int_{A(E)} f d\sigma \right)^{1-\frac{1}{p}}$$

for all $g \in L^{q'}(\mathbb{R}^d)$ and all measurable subsets E of $\mathbb{S}^{d-1}$, by invariance of T under A. Using the invariance of $d\sigma$ under rotations, we then get

$$\int_E |Tg| d\sigma \leq C' \|g\|_{L^{q'}(\mathbb{R}^d)} \left(\int_{A(E)} f d\sigma \right)^{1-\frac{1}{p}}$$

for all $g \in L^{q'}(\mathbb{R}^d)$ and all measurable subsets E of $\mathbb{S}^{d-1}$. We now average with respect to the Haar measure dA on $O(d)$: using Hölder's inequality to interchange the integral over A with the $1 - \frac{1}{p}$ power on the right hand side (note $0 < 1 - \frac{1}{p} < 1$), we obtain

$$\int_E |Tg| d\sigma \leq C' \|g\|_{L^{q'}(\mathbb{R}^d)} \left(\int_{A \in O(d)} \int_{A(E)} f d\sigma dA \right)^{1-\frac{1}{p}} \tag{7}$$

for all $g \in L^{q'}(\mathbb{R}^d)$ and all measurable subsets E of $\mathbb{S}^{d-1}$. The map

$$E \mapsto \int_{A \in O(d)} \int_{A(E)} f d\sigma dA$$

defines a rotationally invariant measure on measurable subsets E of $\mathbb{S}^{d-1}$, and this measure is 1 when $E = \mathbb{S}^{d-1}$ by our normalization $\int_{\mathbb{S}^{d-1}} f d\sigma = 1$. Hence we have

$$\int_{A \in O(d)} \int_{A(E)} f d\sigma dA = \sigma(E),$$

the standard surface measure of E. In other words, we have, from (7), that

$$\int_E |Tg| d\sigma \leq C' \|g\|_{L^{q'}(\mathbb{R}^d)} \sigma(E)^{1-\frac{1}{p}} \tag{8}$$

for all $g \in L^{q'}(\mathbb{R}^d)$ and all measurable subsets E of $\mathbb{S}^{d-1}$. Now given $g \in L^{q'}(\mathbb{R}^d)$, let

$$E = \{\xi \in \mathbb{S}^{d-1} \colon |Tg(\xi)| > \alpha\}.$$

Then the left hand side above is at least $\alpha\, \sigma(E)$. Thus we have

$$\alpha\, \sigma(E)^{\frac{1}{p}} \leq C' \|g\|_{L^{q'}(\mathbb{R}^d)}.$$

Since this is true for all $g \in L^{q'}(\mathbb{R}^d)$, we conclude that T is bounded from $L^{q'}(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{S}^{d-1}, d\sigma)$ with norm $\leq C'$. □

We remark that (8) is just (6) with $f \equiv 1$. The proof that (8) implies $T \colon L^{q'}(\mathbb{R}^d) \to L^{p,\infty}(\mathbb{S}^{d-1}, d\sigma)$ can be easily generalized to prove the implication (ii) ⇒ (iii) in Theorem 3.

4. Application to the Extension problem

In this section we prove Theorem 1. We use duality. The adjoint of the extension operator E for the sphere $\mathbb{S}^{d-1}$ is the restriction operator, given by $Rg = \widehat{g}|_{\mathbb{S}^{d-1}}$ on $\mathbb{R}^d$. If $E \colon L^\infty(\mathbb{S}^{d-1}, d\sigma) \to L^q(\mathbb{R}^d)$ is bounded, then $R \colon L^{q'}(\mathbb{R}^d) \to L^1(\mathbb{S}^{d-1}, d\sigma)$ is bounded. Clearly R commutes with all rotations on $\mathbb{R}^d$. Thus we are in position to apply Corollary 1 from the last section, for R in place of T. We will let $p = q'$, and then we need to check that

$$\left\| \sup_i |Rg_i| \right\|_{L^1(\mathbb{S}^{d-1}, d\sigma)} \leq C \left(\sum_i \|g_i\|_{L^{q'}(\mathbb{R}^d)}^{q'} \right)^{\frac{1}{q'}} \tag{9}$$

for any finite sequences $\{g_i\}$ in $L^{q'}(\mathbb{R}^d)$. To do so we use Khintchine's inequality: since the restriction operator is bounded from $L^{q'}(\mathbb{R}^d)$ to $L^1(\mathbb{S}^{d-1}, d\sigma$, we have, for any choice of signs $\{\varepsilon_i\}$'s, that

$$\int_{\mathbb{S}^{d-1}} \left| R\left(\sum_i \varepsilon_i g_i \right) \right| d\sigma \leq C \left\| \sum_i \varepsilon_i g_i \right\|_{L^{q'}(\mathbb{R}^d)}.$$

Taking expectations over all possible choices of signs, and using Hölder's inequality to interchange the expectation and the $1/q'$ power on the right hand side, we get

$$\int_{\mathbb{S}^{d-1}} \left(\sum_i |Rg_i|^2 \right)^{\frac{1}{2}} d\sigma \leq C \left(\int_{\mathbb{R}^d} \left(\sum_i |g_i|^2 \right)^{\frac{q'}{2}} dx \right)^{\frac{1}{q'}}.$$

The left hand side is bigger than the left hand side of (9), while the right hand side is less than the right hand side of (9) since $q' \leq 2$ and hence $\ell^{q'}$ embeds into ℓ^2 (this is where we use our assumption $q \geq 2$). Hence (9) follows, and Corollary 1 shows that R is bounded from $L^{q'}(\mathbb{R}^d)$ to $L^{q',\infty}(\mathbb{S}^{d-1}, d\sigma)$. This gives the desired boundedness of $E\colon L^{q,1}(\mathbb{S}^{d-1}, d\sigma) \to L^q(\mathbb{R}^d)$.

5. Application for the Kakeya maximal function

In this section we prove Theorem 2. Again we dualize. Following Bourgain [1], we define, for locally integrable functions g on $\mathbb{R}^d$ and $\delta > 0$, the Kakeya maximal operator

$$g_\delta^*(\xi) = \sup_T \frac{1}{|T|} \int_T |g(x)| dx, \quad \xi \in \mathbb{S}^{d-1}$$

where the supremum is over all $\delta^{d-1} \times 1$ cylinders whose direction is parallel to ξ.

Suppose now (1) holds for certain exponents q and β with $p = \infty$. Then by duality, for any $\varepsilon > 0$, there exists a constant $C_{q,\beta,\varepsilon}$ such that

$$\|g_\delta^*\|_{L^1(\mathbb{S}^{d-1}, d\sigma)} \leq C_{q,\beta,\varepsilon} \delta^{-\beta-\varepsilon} \|g\|_{L^{q'}(\mathbb{R}^d)} \tag{10}$$

for all $0 < \delta < 1$ and all $g \in L^{q'}(\mathbb{R}^d)$. Hence for any $0 < \delta < 1$,

$$Tg := g_\delta^*$$

defines a bounded sub-linear operator from $L^{q'}(\mathbb{R}^d)$ to $L^1(\mathbb{S}^{d-1}, d\sigma)$ with norm $\leq C_{q,\beta,\varepsilon} \delta^{-\beta-\varepsilon}$. Clearly T commutes with all rotations on $\mathbb{R}^d$. Thus we are in position to apply Corollary 1. We will let $p = q'$, and we will show that

$$\left\| \sup_i (g_i)_\delta^* \right\|_{L^1(\mathbb{S}^{d-1}, d\sigma)} \leq C_{q,\beta,\varepsilon} \delta^{-\beta-\varepsilon} \left(\sum_i \|g_i\|_{L^{q'}(\mathbb{R}^d)}^{q'} \right)^{\frac{1}{q'}} \tag{11}$$

for any finite sequences $\{g_i\}$ in $L^{q'}(\mathbb{R}^d)$. It then follows that

$$\|g_\delta^*\|_{L^{q',\infty}(\mathbb{S}^{d-1}, d\sigma)} \lesssim_q C_{q,\beta,\varepsilon} \delta^{-\beta-\varepsilon} \|g\|_{L^{q'}(\mathbb{R}^d)}$$

for all $g \in L^{q'}(\mathbb{R}^d)$, which by duality again implies that

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^q(\mathbb{R}^d)} \lesssim_q C_{q,\beta,\varepsilon} \delta^{-\beta-\varepsilon} \left(\sum_{T \in \mathbb{T}} |T| \right)^{\frac{1}{q}}$$

for all family $\mathbb{T}$ of δ-separated $\delta^{d-1} \times 1$ tubes in $\mathbb{R}^d$.

Hence it remains to establish (11). To do so, suppose $\{g_i\}$ is a finite sequence in $L^{q'}(\mathbb{R}^d)$, and $0 < \delta < 1$. For any $\xi \in \mathbb{S}^{d-1}$, we have

$$\sup_i [(g_i)_\delta^*(\xi)] \leq (\sup_i |g_i|)_\delta^*(\xi).$$

This is just another way of saying that

$$\sup_i \sup_{T // \xi} \frac{1}{|T|} \int_T |g_i| d\sigma \leq \sup_{T // \xi} \frac{1}{|T|} \int_T \sup_i |g_i| d\sigma,$$

which is clearly true. Hence by (10), applied to $g = \sup_i |g_i|$, we have

$$\left\| \sup_i (g_i)_\delta^* \right\|_{L^1(\mathbb{S}^{d-1}, d\sigma)} \leq C_{q,\beta,\varepsilon} \delta^{-\beta-\varepsilon} \left\| \sup_i |g_i| \right\|_{L^{q'}(\mathbb{R}^d)},$$

from which (11) follows readily since pointwisely $(\sup_i |g_i|)^{q'} \le \sum_i |g_i|^{q'}$.

6. Implications on the dimension of a Kakeya set

In this section, we make some remarks about the implications of a partial restriction estimate on the Hausdorff dimension of a Kakeya set in $\mathbb{R}^d$. A Kakeya set in $\mathbb{R}^d$ is a subset of $\mathbb{R}^d$ that contains a unit line segment in every possible direction. It is well-known that if equation (10) holds for some triple of exponents (p, q, β) with $p = q$, then the Hausdorff dimension of a Kakeya set in $\mathbb{R}^d$ is at least $d - \beta q'$.

Suppose the extension operator E for the sphere $\mathbb{S}^{d-1}$ is bounded from $L^p(\mathbb{S}^{d-1}, d\sigma)$ to $L^q(\mathbb{R}^d)$. Then by a standard argument involving Khintchine's inequality and the Knapp example, we have

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^{q/2}(\mathbb{R}^d)} \lesssim \delta^{-\beta} \left(\sum_{T \in \mathbb{T}} |T| \right)^{2/p} \tag{12}$$

with

$$\beta = 2 \left(\frac{d}{(q/2)'} - 1 \right)$$

for any $0 < \delta < 1$ and any family $\mathbb{T}$ of δ-separated $\delta^{d-1} \times 1$ tubes in $\mathbb{R}^d$. If further $p \le q$, then we may replace the exponent $2/p$ on the right hand side by $2/q$, and still maintain the inequality. Hence the Hausdorff dimension of a Kakeya set in $\mathbb{R}^d$ is at least

$$d - 2 \left(\frac{d}{(q/2)'} - 1 \right) \left(\frac{q}{2} \right)' = 2 \left(\frac{q}{2} \right)' - d.$$

The above argument does not apply if we do not have $p \le q$ in the inequality (12). Nevertheless, if we only know that E is bounded from $L^p(\mathbb{S}^{d-1}, d\sigma)$ to $L^q(\mathbb{R}^d)$ for some $p > q$ (with $q > \frac{2d}{d-1} > 2$; otherwise E cannot map boundedly into L^q), then we know E is bounded from $L^\infty(\mathbb{S}^{d-1}, d\sigma)$ to $L^q(\mathbb{R}^d)$, so Theorem 1 applies, and we deduce, as remarked immediately after Theorem 1, then E maps $L^s(\mathbb{S}^{d-1}, d\sigma) \to L^s(\mathbb{R}^d)$ for all $s > q$. From our discussion above, it then follows that the Hausdorff dimension of the Kakeya set in $\mathbb{R}^d$ is at least $2 \left(\frac{s}{2} \right)' - d$ for all $s > q$, i.e. at least $2 \left(\frac{q}{2} \right)' - d$.

Alternatively, if we only know that E is bounded from $L^\infty(\mathbb{S}^{d-1}, d\sigma)$ to $L^q(\mathbb{R}^d)$ for some $q > \frac{2d}{d-1}$, then from (12) we see that (11) holds with $p = \infty$, q replaced by $q/2$, and $\beta = 2 \left(\frac{d}{(q/2)'} - 1 \right)$. Thus by Theorem 2, for any $\varepsilon > 0$, we have

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^{q/2}(\mathbb{R}^d)} \lesssim_\varepsilon \delta^{-2\left(\frac{d}{(q/2)'} - 1 \right) - \varepsilon} \left(\sum_{T \in \mathbb{T}} |T| \right)^{2/q}$$

for any $0 < \delta < 1$ and any family $\mathbb{T}$ of δ-separated $\delta^{d-1} \times 1$ tubes in $\mathbb{R}^d$. This shows that the Hausdorff dimension of the Kakeya set in $\mathbb{R}^d$ is at least $d - \left[2 \left(\frac{d}{(q/2)'} - 1 \right) + \varepsilon \right] \left(\frac{q}{2} \right)'$ for all $\varepsilon > 0$, i.e. at least $2 \left(\frac{q}{2} \right)' - d$, yielding the same conclusion as before.

References

[1] J. Bourgain, *Besicovitch type maximal operators and applications to Fourier analysis*, Geom. Funct. Anal. **1** (1991), no. 2, 147–187.

[2] Larry Guth, *A restriction estimate using polynomial partitioning*, J. Amer. Math. Soc. **29** (2016), no. 2, 371–413.

[3] Jongchon Kim, *Some remarks on Fourier restriction estimates*, available at `https://arxiv.org/pdf/1702.01231.pdf`.

[4] Pertti Mattila, *Fourier Analysis and Hausdorff Dimension*, Cambridge studies in advanced mathematics, vol. 150, Cambridge University Press, Cambridge, 2015.

[5] Gilles Pisier, *Factorization of operators through $L_{p\infty}$ or L_{p1} and noncommutative generalizations*, Math. Ann. **276** (1986), no. 1, 105–136.

[6] Terence Tao, *Lecture notes for Math 254B*, available at `https://www.math.ucla.edu/~tao/254b.1.99s/notes6.dvi`.