On a class of pseudodifferential operators with mixed homogeneities

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Introduction

- Joint work with E. Stein (and an outgrowth of work of Nagel-Ricci-Stein-Wainger, to appear)
- Motivating question: what happens when one compose two operators of two different homogeneities?
- e.g. On $\mathbb{R}^N$ one can associate two different dilations: for $x = (x', x^N) \in \mathbb{R}^N$, $\lambda > 0$, one can define

$$\lambda \cdot x := (\lambda x', \lambda x^N) \quad \text{(isotropic)}$$

$$\lambda \odot x := (\lambda x', \lambda^2 x^N) \quad \text{(non-isotropic)}$$

- Associated to these are two norms, each homogeneous with respect to one of these dilations:

$$|x| = |x'| + |x^N|$$

$$\|x\| = |x'| + |x^N|^{1/2}$$

- There are also the dual norms, on the cotangent space of $\mathbb{R}^N$, given by

$$|\xi| = |\xi'| + |\xi_N|$$

$$\|\xi\| = |\xi'| + |\xi_N|^{1/2}$$

- One could look at multipliers $e(\xi)$, with

$$|\partial_{\xi'}^{\alpha} \partial_{\xi_N}^{\beta} e(\xi)| \lesssim |\xi|^{-|\alpha|-|\beta|},$$

  and their associated multiplier operators $\widehat{T_e f}(\xi) = e(\xi)\widehat{f}(\xi)$. (These are just standard isotropic singular integral operators.)
- One could also look at multipliers $h(\xi)$, with

$$|\partial_{\xi'}^{\alpha} \partial_{\xi_N}^{\beta} h(\xi)| \lesssim \|\xi\|^{-|\alpha|-2|\beta|},$$

  and their associated multiplier operators $\widehat{T_h f}(\xi) = h(\xi)\widehat{f}(\xi)$. (These are just non-isotropic singular integral operators, arising e.g. when one solves the heat equation.)

Some motivating Questions

- What happens when one compose T_e with T_h?
  What kind of operator do we get?
  What are the nature of the singularities of the multiplier, or the kernel, of $T_e T_h$?
- What mapping properties does $T_e T_h$ satisfy?
  When is it (say) weak-type (1,1)?
- The question of composition is quite easy, since we are dealing with convolution operators on an abelian group $\mathbb{R}^N$.
- The question of mapping properties was already studied in a paper of Phong and Stein in 1982.
- We are interested in these questions, because they serve as toy model problems for what one needs to do in more general settings.

- e.g. In several complex variables, in solving the $\overline{\partial}$-Neumann problem on a smooth strongly pseudoconvex domain in $\mathbb{C}^{n+1}$, one faces the problem of inverting the Calderon operator $\Box_+$.
- Roughly speaking, this amounts to composing an operator with isotropic homogeneity, with an operator with non-isotropic homogeneity:

$$\Box_+^{-1} \simeq \Box_- \Box_b^{-1}$$

  (at least when $n > 1$; c.f. Greiner-Stein 1977).
- Similar compositions arise in the study of the Hodge Laplacian on k-forms on the Heisenberg group $\mathbb{H}^n$ (Müller-Peloso-Ricci 2012)

The role of flag kernels

- The flag kernels are integral kernels that are singular along certain subspaces on $\mathbb{R}^N$ (e.g. Nagel-Ricci-Stein 2001, Nagel-Ricci-Stein-Wainger 2011, to appear).
- These are special cases of product kernels, which were studied by many authors (e.g. R. Fefferman-Stein 1982, Journe 1985, Nagel-Stein 2004).
- e.g. On the Heisenberg group $\mathbb{H}^n$, a flag kernel could be singular along the t-axis, and satisfies

$$|K(z,t)| \lesssim |z|^{-2n}(|z|^2+|t|)^{-1}$$

along with some corresponding differential inequalities and cancellation conditions.

- Simple examples of flag kernels include both singular integral kernels with isotropic homogeneities, and those with non-isotropic homogeneities.
- More sophisticated examples of flag kernels on $\mathbb{H}^n$ are given by the joint spectral multipliers $m(\mathcal{L}_0, iT)$, where m is a Marcinkiewicz multiplier, and $\mathcal{L}_0$ is the sub-Laplacian (Müller-Ricci-Stein 1995, 1996).
- The singular integral operators with flag kernels map L^p to L^p, for $1 < p < \infty$, and form an algebra under composition.
- Thus if we want to compose a singular integral with isotropic homogeneity, with one that has non-isotropic homogeneity, we could have composed them in the class of all flag kernels.
- But then we get as a result a flag kernel, which is singular along some subspaces (whereas our original kernels are both singular only at one point).

- It turns out one should consider the intersection of those flag kernels that are singular along the *t*-axis, with those that are singular along the *z*-axis, as in Nagel-Ricci-Stein-Wainger (to appear). This gives rise to operators with *mixed* homogeneities:

$$|K(z,t)| \lesssim (|z|+|t|)^{-2n}(|z|^2+|t|)^{-1}$$

along with differential inequalities and cancellation conditions.
- Our first result will be a pseudodifferential realization of the above operators of with mixed homogeneities.
- The goal is to write them as pseudodifferential operators:

$$T_a f(x) = \int_{\mathbb{H}^n} a(x,\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

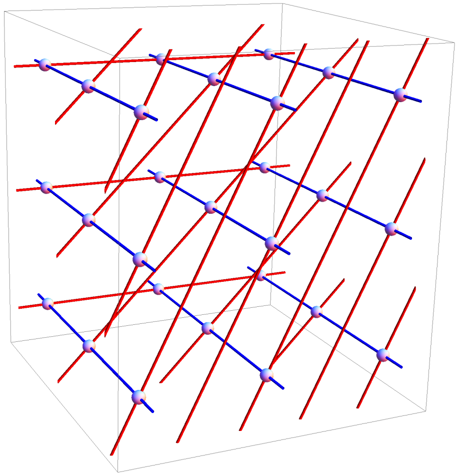
for some suitable symbols $a(x,\xi)$.

- Nagel-Stein (1979) and Beals-Greiner (1988) has realized those singular integrals with purely non-isotropic homogeneities as pseudodifferential operators.
- We will do so for singular integrals with mixed homogeneities; in doing so, we will also consider operators of all orders (not just order 0 ones, as singular integrals would be).
- Our results will actually hold in a more general setting, outside several complex variables; it will hold as long as a smooth distribution of tangent subspaces (of constant rank) is given on $\mathbb{R}^N$.
- We will also see some geometric invariance of our class of operators as we proceed.
- A very step-2 theory!

Our set-up

- Suppose on $\mathbb{R}^N$, we are given a (global) frame of tangent vectors, namely $X_1, \dots, X_N$, with $X_i = \sum_{j=1}^N A_i^j(x) \frac{\partial}{\partial x^j}$.
- We assume that all $A_i^j(x)$ are C^∞ functions, and that $\partial_x^J A_i^j(x) \in L^\infty(\mathbb{R}^N)$ for all multiindices J.
- We also assume that $|\det(A_i^j(x))|$ is uniformly bounded from below on $\mathbb{R}^N$.
- Let $\mathcal{D}$ be the distribution of tangent subspaces on $\mathbb{R}^N$ given by the span of $\{X_1, \dots, X_{N-1}\}$.
- Our constructions below seem to depend on the choice of the frame $X_1, \dots, X_N$, but ultimately the class of operators we introduce will only depend on $\mathcal{D}$.
- No curvature assumption on $\mathcal{D}$ is necessary!

An example: the contact distribution on $\mathbb{H}^1$



(Picture courtesy of Assaf Naor)

Geometry of the distribution

- We write $\theta^1, \ldots, \theta^N$ for the frame of cotangent vectors dual to $X_1, \ldots, X_N$.
- We will need a variable seminorm $\rho_x(\xi)$ on the cotangent bundle of $\mathbb{R}^N$, defined as follows.
- Given $\xi = \sum_{i=1}^N \xi_i dx^i$, and a point $x \in \mathbb{R}^N$, we write

$$\xi = \sum_{i=1}^{N} (M_x \xi)_i \theta^i.$$

Then

$$\rho_x(\xi) := \sum_{i=1}^{N-1} |(M_x \xi)_i|.$$

- We also write $|\xi| = \sum_{i=1}^N |\xi_i|$ for the Euclidean norm of ξ.

- The variable seminorm $\rho_x(\xi)$ induces a quasi-metric $d(x, y)$ on $\mathbb{R}^N$.
- If $x, y \in \mathbb{R}^N$ with $|x - y| < 1$, we write

$$d(x,y) := \sup\left\{\frac{1}{\rho_x(\xi) + |\xi|^{1/2}} : (x-y)\cdot\xi = 1\right\}.$$

- We also write $|x - y|$ for the Euclidean distance between x and y.

Our class of symbols with mixed homogeneities

- The symbols we consider will be assigned two different 'orders', namely m and n, which we think of as the 'isotropic' and 'non-isotropic' orders of the symbol respectively.
- In the case of the constant distribution, it is quite easy to define the class of symbols we are interested in: given $m, n \in \mathbb{R}$, if $a_0(x,\xi) \in C^\infty(T^*\mathbb{R}^N)$ is such that

$$|a_0(x,\xi)| \lesssim (1+|\xi|)^m (1+\|\xi\|)^n$$

$$|\partial_x^J \partial_{\xi'}^\alpha \partial_{\xi_N}^\beta a_0(x,\xi)| \lesssim_{J,\alpha,\beta} (1+|\xi|)^{m-\beta} (1+\|\xi\|)^{n-|\alpha|}$$

where

$$\|\xi\| = |\xi'| + |\xi_N|^{1/2} \quad \text{if } \xi = \sum_{i=1}^N \xi_i dx^i,$$

then we say $a_0 \in S^{m,n}(\mathcal{D}_0)$. (Note that in this situation, $1 + \|\xi\| \simeq 1 + \rho_x(\xi) + |\xi|^{1/2}$.)

- More generally, suppose $\mathcal{D}$ is a distribution as before (in particular, we fix the frame $X_1, \dots, X_N$, and its dual frame $\theta^1, \dots, \theta^N$). Given $x \in \mathbb{R}^N$ and

$$\xi = \sum_{i=1}^{N} \xi_i dx^i = \sum_{i=1}^{N} (M_x \xi)_i \theta^i,$$

write

$$M_x \xi = \sum_{i=1}^{N} (M_x \xi)_i dx^i.$$

Then we say $a \in S^{m,n}(\mathcal{D})$, if and only if

$$a(x, \xi) = a_0(x, M_x \xi)$$

for some $a_0 \in S^{m,n}(\mathcal{D}_0)$.

- e.g. Suppose we are on $\mathbb{R}^3$. Pick coordinates $x = (x^1, x^2, t)$, and dual coordinates $\xi = (\xi_1, \xi_2, \tau)$. Define

$$X_1 = \frac{\partial}{\partial x^1} + 2x^2 \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x^2} - 2x^1 \frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial t},$$

  and

$$\mathcal{D} := \text{span}\{X_1, X_2\}.$$

  Then identifying $\mathbb{R}^3$ with the Heisenberg group $\mathbb{H}^1$, $\mathcal{D}$ is sometimes called the contact distribution.

  We then have $a \in S^{m,n}(\mathcal{D})$, if and only if

$$a(x^1, x^2, t, \xi^1, \xi^2, \tau) = a_0(x^1, x^2, t, \xi_1 + 2x^2\tau, \xi_2 - 2x^1\tau, \tau)$$

  for some $a_0 \in S^{m,n}(\mathcal{D}_0)$.
- The condition $a \in S^{m,n}(\mathcal{D})$ can also be phrased in terms of suitable differential inequalities.

- e.g. Still consider the contact distribution $\mathcal{D}$ on $\mathbb{H}^1$.
  Recall coordinates $x = (x^1, x^2, t)$ on $\mathbb{H}^1$, and dual coordinates $\xi = (\xi_1, \xi_2, \tau)$. Let

$$D_\tau = \frac{\partial}{\partial \tau} - 2x^2 \frac{\partial}{\partial \xi_1} + 2x^1 \frac{\partial}{\partial \xi_2},$$

  and

$$D_1 = \frac{\partial}{\partial x^1} + 2\tau \frac{\partial}{\partial \xi_2}, \quad D_2 = \frac{\partial}{\partial x^2} - 2\tau \frac{\partial}{\partial \xi_1}, \quad D_3 = \frac{\partial}{\partial t}.$$

  Then $a \in S^{m,n}(\mathcal{D})$, if and only if $a(x,\xi) \in C^\infty(\mathbb{H}^1 \times \mathbb{R}^3)$ satisfies

$$|a(x,\xi)| \lesssim (1 + |\xi|)^m (1 + \rho_x(\xi) + |\xi|^{1/2})^n,$$

$$|\partial_\xi^\alpha D_\xi^\beta D_J a(x,\xi)| \lesssim (1 + |\xi|)^{m-\beta} (1 + \rho_x(\xi) + |\xi|^{1/2})^{n-|\alpha|},$$

  where D_J is composition of any number of the D_1, D_2, D_3.

- Back to the general set up. We call elements of $S^{m,n}(\mathcal{D})$ a symbol of order (m, n).
- Our class of symbols $S^{m,n}$ is quite big.
- $S^{m,0}$ contains every standard (isotropic) symbol of order m:

$$|a(x,\xi)| \lesssim (1+|\xi|)^m$$

$$|\partial_\xi^\alpha \partial_x^J a(x,\xi)| \lesssim_{\alpha,J} (1+|\xi|)^{m-|\alpha|}$$

- Also, $S^{0,n}$ contains the following class of symbols, which we think of as non-isotropic symbols of order n:

$$|a(x,\xi)| \lesssim (1+\rho_x(\xi)+|\xi|^{1/2})^n$$

$$|\partial_\xi^\alpha D_\xi^\beta D^J a(x,\xi)| \lesssim_{\alpha,\beta,J} (1+\rho_x(\xi)+|\xi|^{1/2})^{n-|\alpha|-2\beta}$$

- Many of the results below for $S^{m,n}$ has an (easier) counter-part for these non-isotropic symbols.

Our class of pseudodifferential operators with mixed homogeneities

- To each symbol $a \in S^{m,n}(\mathcal{D})$, we associate a pseudodifferential operator

  $$T_a f(x) = \int_{\mathbb{R}^N} a(x,\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi.$$

  We denote the set of all such operators $\Psi^{m,n}(\mathcal{D})$.
- We remark that $\Psi^{m,n}(\mathcal{D})$ depends only on the distribution $\mathcal{D}$, and not on the choice of the vector fields $X_1, \dots, X_N$ (nor on the choice of a coordinate system on $\mathbb{R}^N$).
  (Geometric invariance!)
- One sees that $\Psi^{m,n}(\mathcal{D})$ is the correct class of operators to study, via the following kernel estimates:

Theorem (Stein-Y. 2013)

If $T \in \Psi^{m,n}(\mathcal{D})$ with $m > -1$ and $n > -(N-1)$, then one can write

$$Tf(x) = \int_{\mathbb{R}^N} f(y)K(x,y)dy,$$

where the kernel $K(x,y)$ satisfies

$$|K(x,y)| \lesssim |x-y|^{-(N-1+n)} d(x,y)^{-2(1+m)},$$
$$|(X')^{\gamma}_{x,y}\partial^{\delta}_{x,y} K(x,y)| \lesssim |x-y|^{-(N-1+n+|\gamma|)} d(x,y)^{-2(1+m+|\delta|)}.$$

Here X' refers to any of the 'good' vector fields $X_1, \dots, X_{N-1}$ that are tangent to $\mathcal{D}$, and the subscripts x, y indicates that the derivatives can act on either the x or y variables.

- c.f. two-flag kernels of Nagel-Ricci-Stein-Wainger

Main theorems

Theorem (Stein-Y. 2013)

If $T_1 \in \Psi^{m,n}(\mathcal{D})$ *and* $T_2 \in \Psi^{m',n'}(\mathcal{D})$, *then*

$$T_1 \circ T_2 \in \Psi^{m+m',n+n'}(\mathcal{D}).$$

Furthermore, if T_1^* *is the adjoint of* T_1 *with respect to the standard* L^2 *inner product on* $\mathbb{R}^N$, *then we also have*

$$T_1^* \in \Psi^{m,n}(\mathcal{D}).$$

In particular, the class of operators $\Psi^{0,0}$ *form an algebra under composition, and is closed under taking adjoints.*

- Proof unified by introducing some suitable compound symbols.

Theorem (Stein-Y. 2013)

If $T \in \Psi^{0,0}(\mathcal{D})$, then T maps $L^p(\mathbb{R}^N)$ into itself for all $1 < p < \infty$.

- Operators in $\Psi^{0,0}$ are operators of type $(1/2, 1/2)$. As such they are bounded on L^2.
- But operators in $\Psi^{0,0}$ may not be of weak-type (1,1); this forbids one to run the Calderon-Zygmund paradigm in proving L^p boundedness $\rightarrow$ use Littlewood-Paley projections instead, and need to introduce some new strong maximal functions.
- Nonetheless, there are two very special ideals of operators inside $\Psi^{0,0}$, namely $\Psi^{\varepsilon,-2\varepsilon}$ and $\Psi^{-\varepsilon,\varepsilon}$ for $\varepsilon > 0$.
  (The fact that these are ideals of the algebra $\Psi^{0,0}$ follows from the earlier theorem). They satisfy:

Theorem (Stein-Y. 2013)

If $T \in \Psi^{\varepsilon,-2\varepsilon}$ or $\Psi^{-\varepsilon,\varepsilon}$ for some $\varepsilon > 0$, then

(a) *T is of weak-type (1,1), and*

(b) *T maps the Hölder space $\Lambda^\alpha(\mathbb{R}^N)$ into itself for all $\alpha > 0$.*

- An analogous theorem holds for some non-isotropic Hölder spaces $\Gamma^\alpha(\mathbb{R}^N)$.
- Proof of (a) by kernel estimates
- Proof of (b) by Littlewood-Paley characterization of the Hölder spaces Λ^α.

Theorem (Stein-Y. 2013)

Suppose $T \in \Psi^{m,n}$ *with*

$$m > -1, \quad n > -(N-1), \quad m+n \leq 0, \quad \text{and} \quad 2m+n \leq 0.$$

For $p \geq 1$*, define an exponent* p^* *by*

$$\frac{1}{p^*} := \frac{1}{p} - \gamma, \qquad \gamma := \min\left\{\frac{|m+n|}{N}, \frac{|2m+n|}{N+1}\right\}$$

if $1/p > \gamma$*. Then:*

(i) $T: L^p \to L^{p^*}$ *for* $1 < p \leq p^* < \infty$*; and*

(ii) *if* $\frac{m+n}{N} \neq \frac{2m+n}{N+1}$*, then* T *is weak-type* $(1, 1^*)$*.*

- These estimates for $S^{m,n}$ are better than those obtained by composing between the optimal results for $S^{0,n}$ and $S^{m,0}$.

- For example, take the example of the 1-dimensional Heisenberg group $\mathbb{H}^1$ (so $N = 3$).
- If T_1 is a standard (or isotropic) pseudodifferential operator of order $-1/2$ (so $T_1 \in \Psi^{-1/2,0}$), and T_2 is a non-isotropic pseudodifferential operator of order -1 (so $T_2 \in \Psi^{0,-1}$), then the best we could say about the operators individually are just

$$T_2 \colon L^{4/3} \to L^2, \quad T_1 \colon L^2 \to L^3.$$

But according to the previous theorem,

$$T_1 \circ T_2 \colon L^{4/3} \to L^4,$$

which is better.
- Proof of (i) by interpolation between $\Psi^{0,0}$ and $\Psi^{-1,-(N-1)}$.
- Proof of (ii) by kernel estimates.

Some applications

- One can localize the above theory of operators of class $\Psi^{m,n}$ to compact manifolds without boundary.
- Let $M = \partial\Omega$ be the boundary of a strongly pseudoconvex domain Ω in $\mathbb{C}^{n+1}$, $n \geq 1$.
- Then the Szegö projection S is an operator in $\Psi^{\varepsilon,-2\varepsilon}$ for all $\varepsilon > 0$; c.f. Phong-Stein (1977).
- Furthermore, the Dirichlet-to-$\overline{\partial}$-Neumann operator $\Box_+$, which one needs to invert in solving the $\overline{\partial}$-Neumann problem on Ω, has a parametrix in $\Psi^{1,-2}$; c.f. Greiner-Stein (1977), Chang-Nagel-Stein (1992).

Epilogue

- On $\mathbb{R}$, we know that if $Tf := f * |y|^{-1+\alpha}$, $0 < \alpha < 1$, then

$$\|Tf\|_{L^q} \leq C\|f\|_{L^p},$$

whenever

$$\frac{1}{q} = \frac{1}{p} - \alpha, \quad 1 < p < q < \infty.$$

There are many known proofs; below we sketch one that is perhaps less commonly known, and that is reminiscent of our proof of the smoothing properties for our class $\Psi^{m,n}$ above.

- The idea is to use complex interpolation: For $s \in \mathbb{C}$ with $0 \leq \mathrm{Re}\, s \leq 1$, let k_s be the tempered distribution

$$k_s(y) = (1+s)^{-1}\partial_y^2|y|^{1+s},$$

so that when $0 < \mathrm{Re}\, s \leq 1$, we have $k_s(y) = s|y|^{-1+s}$.

- Let $T_s f = f * k_s$ for $f \in \mathcal{S}$.
- Then when $\mathrm{Re}\, s = 1$, we have $T_s \colon L^1 \to L^\infty$, since it is then a convolution against a bounded function.
- Also when $\mathrm{Re}\, s = 0$, we have $T_s \colon L^r \to L^r$ for all $1 < r < \infty$, since k_s is then a Calderon-Zygmund kernel (one can check that $\widehat{k_s} \in L^\infty$ for such s, since k_s, being a derivative, satisfies a cancellation condition).
- Interpolation then shows that $T_\alpha \colon L^p \to L^q$, when

$$\frac{1}{q} = \frac{1}{p} - \alpha, \quad 1 < p < q < \infty.$$