Spectral projection theorems on compact manifolds

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This is a note on Christopher Sogge's paper, Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds, in *J. Funct. Anal.* 77, 123-138 (1988).

Let M be a smooth compact connected manifold without boundary of dimension $n \geq 2$. Let P be a second order elliptic self-adjoint operator on M whose coefficients are smooth and whose principal symbol is positive definite. It is known that $L^2(M)$ is the direct sum of the eigenspaces of P. Let

$$\lambda_1 < \lambda_2 < \dots$$

be the eigenvalues of P (each of which may repeat with high multiplicities), and for each positive integer k, let

$$\chi_k = \sum_{\sqrt{\lambda_j} \in [k-1,k)} \text{Proj}_{\lambda_j}$$

be the projection of $L^2(M)$ onto the sum of the eigenspaces whose eigenvalues λ_j satisfy $\sqrt{\lambda_j} \in [k-1, k)$. We shall be interested in the sharp bounds of χ_k on various L^p spaces. The main goal is the following theorem.

Theorem 1 (Spectral Projection). *For $f \in L^p(M)$,*

(a) If $1 \leq p \leq \frac{2n+2}{n+3}$, then

$$\|\chi_k f\|_{L^2} \leq C k^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} \|f\|_{L^p}.$$

(b) If $\frac{2n+2}{n+3} \leq p \leq 2$, then

$$\|\chi_k f\|_{L^2} \leq C k^{\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|f\|_{L^p}.$$

The interest of the theorem is in the growth of the bounds with k, since each χ_k is trivially bounded from L^p to L^2.

This theorem is closely related to the Euclidean restriction theorem that concerns the restriction of the Fourier transform of a function to a hypersurface, as we will see below. We shall also look at some examples, some corollaries, and finally get to its proof.

Notation. We will write

$$p_n = \frac{2n+2}{n+3},$$

the critical exponent in dimension n, and

$$\delta(p) = n\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}$$

for the sharp power, whenever $1 \leq p \leq p_n$.

1. Examples

Take $M = \mathbb{S}^n$, the standard sphere, and $P = -\Delta$ the Laplace-Beltrami operator on $\mathbb{S}^n$. Then it is known that $\lambda_j = j(j+n-1)$ with multiplicity $C^{n+j}_j - C^{n+j-2}_{j-2} \simeq j^{n-1}$, and χ_k is just the projection onto the eigenspace corresponding to λ_{k-1} for k large. The theorem in this case was known before Sogge's paper, and the theorem can be thought of a generalization of this special case. Recall that by Weyl, if (M, P) is as above, then the number of eigenvalues λ_j of P with $\sqrt{\lambda_j} \in [k-1, k)$ is $\simeq k^{n-1}$ as $k \to \infty$. So χ_k is the projection onto a portion of the frequency space that 'has the right size', and is a 'correct' generalization of $\mathrm{Proj}_{\lambda_{k-1}}$ on the sphere.

The flat torus is another easy example that we will compute shortly.

2. Relation with the Euclidean restriction theorem

The Euclidean restriction theorem says that if S is a compact hypersurface in $\mathbb{R}^n$ with nowhere vanishing Gaussian curvature, then the Fourier transform of any L^p function on $\mathbb{R}^n$ can be restricted meaningfully to the hypersurface S, whenever $1 \leq p \leq p_n$. This is remarkable since apriori the Fourier transform of such a function is only defined almost everywhere, and S has measure zero in $\mathbb{R}^n$. More precisely:

Theorem 2 (Euclidean restriction theorem)**.** *Let S be a compact hypersurface in $\mathbb{R}^n$ with nowhere vanishing Gaussian curvature. If*

$$1 \leq p \leq p_n, \quad q \leq \left(\frac{n-1}{n+1}\right) p'$$

and f is Schwarz, then

$$\|\hat{f}\|_{L^q(S)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$

The geometric assumption that S has nowhere vanishing Gaussian curvature is essential. The usual Cauchy sequence argument then allows one to define the restriction on S of the Fourier transform of a general L^p function on $\mathbb{R}^n$ for this range of p.

Historically the interest of this theorem was its relation with oscillatory integrals, which we are not going into. We shall just observe that the spectral projection theorem is a discrete analogue of the Euclidean restriction theorem, and that indeed the spectral theorem on the flat torus $[-\pi, \pi]^n$ implies the Euclidean restriction theorem for spheres upon rescaling.

2a. Discrete analogue of the Euclidean restriction theorem

According to the Euclidean restriction theorem

$$\int_S |\hat{f}(\xi)|^2 d\sigma(\xi) \leq C\|f\|^2_{L^p(\mathbb{R}^n)} \tag{1}$$

whenever $1 \leq p \leq p_n$ and f is Schwarz. Here $d\sigma$ is the surface measure on S. Define the continuous spectral projection operators ρ_k by

$$\rho_k f(x) = \int_{k-1 \leq |\xi| < k} \hat{f}(\xi) e^{ix\cdot\xi} d\xi$$

for f Schwarz. This is a spectral projection associated with the usual Laplacian on $\mathbb{R}^n$, and it is roughly the direct analog of χ_k that we defined above; note again the size of the annulus $\{k-1 \leq |\xi| < k\}$ is roughly k^{n-1}. Using (1), one can show

$$\|\rho_k f\|_{L^2(\mathbb{R}^n)} \leq Ck^{\delta(p)}\|f\|_{L^p(\mathbb{R}^n)} \tag{2}$$

for $1 \leq p \leq p_n$ and f Schwarz, which is a direct analogue of the spectral projection theorem.

Proof of (2).

$$\rho_k f(x) = \int_{1-\frac{1}{k} \leq |\xi| < 1} \hat{f}(k\xi) e^{ix\cdot\xi} k^n d\xi = \rho_{[1-\frac{1}{k},1]} g(kx),$$

where $g(y) := f(y/k)$ and $\rho_{[1-\frac{1}{k},1)}g(x) := \int_{1-\frac{1}{k}\leq|\xi|<1} \hat{g}(\xi)e^{2\pi i x\cdot\xi}d\xi$ is the obvious spectral projection onto the interval $[1-\frac{1}{k},1)$. Hence

$$\begin{aligned}
\|\rho_k f\|_{L^2(\mathbb{R}^n)} &= k^{-\frac{n}{2}} \left\|\rho_{[1-\frac{1}{k},1)}g\right\|_{L^2(\mathbb{R}^n)} \\
&= k^{-\frac{n}{2}} \left(\int_{1-\frac{1}{k}\leq|\xi|<1} |\hat{g}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&= k^{-\frac{n}{2}} \left(\int_{1-\frac{1}{k}}^{1} \int_{r\mathbb{S}^n} |\hat{g}(\xi)|^2 d\sigma(\xi)dr\right)^{\frac{1}{2}} \\
&\leq Ck^{-\frac{n}{2}} \left(\int_{1-\frac{1}{k}}^{1} \|g\|^2_{L^p(\mathbb{R}^n)}dr\right)^{\frac{1}{2}} \\
&= Ck^{-\frac{n}{2}-\frac{1}{2}}k^{\frac{n}{p}}\|f\|_{L^p(\mathbb{R}^n)} \\
&= Ck^{\delta(p)}\|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

□

2b. Spectral projection theorem implies Euclidean restriction theorem

Suppose we have the spectral projection theorem on the flat torus $([-\pi,\pi]^n,-\Delta)$:

$$\|\chi_k g\|_{L^2([-\pi,\pi]^n)} \leq Ck^{\delta(p)}\|g\|_{L^p([-\pi,\pi]^n)}, \quad 1\leq p\leq p_n.$$

Let $T >> 1$ and we shall rescale this to the torus $([-\pi T,\pi T]^n,\ -\Delta)$.

The eigenvalues of $-\Delta$ on $[-\pi,\pi]^n$ are precisely $\lambda_j = j^2$. So

$$\chi_k g(x) = \frac{1}{(2\pi)^n}\sum_{|m|=k-1}\hat{g}(m)e^{im\cdot x}$$

where $\hat{g}(m) = \int_{[-\pi,\pi]^n} g(x)e^{-im\cdot x}dx$. Now on the torus $[-\pi T,\pi T]^n$, the eigenvalues of $-\Delta$ are $\frac{j^2}{T^2}$, so if we write the spectral projection on $[-\pi T,\pi T]^n$ as $\tilde{\chi}_k$, then

$$\tilde{\chi}_k f(x) = \frac{1}{(2\pi T)^n}\sum_{(k-1)T\leq|m|<kT}\hat{f}\left(\frac{m}{T}\right)e^{i\frac{m}{T}\cdot x},$$

where $\hat{f}(m) = \int_{[-\pi T,\pi T]^n} f(x)e^{-im\cdot x}dx$.

Let now $f\in L^2([-\pi T,\pi T]^n)$. Rescale to get $g(x) := f(Tx)\in L^2([-\pi,\pi]^n)$. Then

$$\tilde{\chi}_k f(x) = \frac{1}{(2\pi T)^n}\sum_{(k-1)T\leq|m|<kT}\hat{f}\left(\frac{m}{T}\right)e^{i\frac{m}{T}\cdot x} = \frac{1}{(2\pi)^n}\sum_{(k-1)T\leq|m|<kT}\hat{g}(m)e^{im\cdot\frac{x}{T}}$$

so by orthogonality,

$$\begin{aligned}
\|\tilde{\chi}_k f\|_{L^2([-\pi T,\pi T]^n)} &= \frac{1}{(2\pi)^n}\left\|\sum_{(k-1)T\le|m|<kT}\hat{g}(m)e^{im\cdot x}\right\|_{L^2([-\pi,\pi]^n)} T^{\frac{n}{2}}\\
&\le \frac{1}{(2\pi)^n}\left(\sum_{(k-1)T\le l<kT}\|\chi_l g\|^2_{L^2([-\pi,\pi]^n)}\right)^{\frac{1}{2}} T^{\frac{n}{2}}\\
&\le C\left(T\left((kT)^{\delta(p)}\|g\|_{L^p([-\pi,\pi]^n)}\right)^2\right)^{\frac{1}{2}} T^{\frac{n}{2}}\\
&= Ck^{\delta(p)}\|f\|_{L^p([-\pi,\pi]^n)}
\end{aligned}$$

independent of T. Let now $f \in C_c^\infty(\mathbb{R}^n$. For T large so that the support of f is contained in $[-\pi T, \pi T]$, we apply the above inequality: since

$$\|\tilde{\chi}_k f\|_{L^2([-\pi T,\pi T]^n)} = \left(\frac{1}{(2\pi T)^n}\sum_{k-1\le\frac{|m|}{T}<k}\left|\hat{f}\left(\frac{m}{T}\right)\right|^2\right)^{\frac{1}{2}} \to \left(\int_{k-1\le|\xi|<k}|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

as $T\to\infty$, we get

$$\left(\int_{k-1\le|\xi|<k}|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \le Ck^{\delta(p)}\|f\|_{L^p(\mathbb{R}^n)}$$

from which we can recover the restriction theorem for spheres in $\mathbb{R}^n$ by running backwards the argument of Section 2a.

3. Equivalent versions and corollaries

There are several equivalent versions of the spectral projection theorem by duality and the T^*T lemma:

1. (Duality) T maps L^p to L^2 if and only if T^* maps L^2 to $L^{p'}$, with identical norms. Since χ_k is self-adjoint, we arrive at the following equivalent formulation of the spectral projection theorem: for $f \in L^p(M)$,
   (a) If $1 \le p \le p_n$, then
   $$\|\chi_k f\|_{L^{p'}} \le Ck^{\delta(p)}\|f\|_{L^2}.$$
   (b) If $p_n \le p \le 2$, then
   $$\|\chi_k f\|_{L^{p'}} \le Ck^{\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\|f\|_{L^2}.$$
   This is the version that we will prove below.
2. (T^*T lemma) T maps L^p to L^2 if and only if T^*T maps L^p to $L^{p'}$, with $\|T^*T\| = \|T\|^2$. Since $\chi_k^*\chi_k = \chi_k$, we arrive at the following equivalent formulation of the spectral projection theorem, although we shall not need it in sequel: for $f \in L^p(M)$,
   (a) If $1 \le p \le p_n$, then
   $$\|\chi_k f\|_{L^{p'}} \le Ck^{2\delta(p)}\|f\|_{L^p}.$$
   (b) If $p_n \le p \le 2$, then
   $$\|\chi_k f\|_{L^{p'}} \le Ck^{(n-1)\left(\frac{1}{p}-\frac{1}{2}\right)}\|f\|_{L^p}.$$

The spectral projection theorem allow us to bound the $L^{p'}$ norms of the eigenfunctions of P.

Corollary 3. *If $\{e_j\}$ is an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of P, then writing λ_j for the eigenvalue of e_j (so now the λ_j's may repeat), we have*

(a) If $1 \leq p \leq p_n$, then

$$\|e_j\|_{L^{p'}} \leq C(1+|\lambda_j|)^{\delta(p)}.$$

(b) If $p_n \leq p \leq 2$, then

$$\|e_j\|_{L^{p'}} \leq C(1+|\lambda_j|)^{\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}.$$

The spectral projection theorem also allows one to obtain results concerning Bochner-Riesz summability. Let (M,P) be as above. For convenience, assume that all eigenvalues of P are non-negative. Define the Bochner-Riesz summation operators by

$$S_R^\delta f(x) = \sum_j \left(1-\frac{\lambda_j}{R}\right)_+^\delta < f, e_j > e_j(x).$$

Then using the spectral projection theorem and additional arguments, one can show

Corollary 4. *If*

$$1 \leq p \leq p_n \quad \text{and} \quad \delta > \delta(p),$$

then

$$\|S_R^\delta f\|_{L^p(M)} \leq C\|f\|_{L^p(M)}$$

uniformly in R, and hence as $R \to \infty$,

$$S_R^\delta f \to f$$

in $L^p(M)$ for all $f \in L^p(M)$. The same is true if in the conditions p is replaced by p'.

See Sogge, On convergence of Riesz means on compact manifolds, *The Annals of Mathematics*, 2nd Ser., Vol. 126, No. 3 (Nov. 1987), 439-447.

4. Proof of the spectral projection theorem

4a. Strategy

To study χ_k, the main idea is to construct a parametrix for the operator $P-(k+i)^2$. We shall show that around each point there are smooth cut-off functions η_1, η_2 supported near that point such that

$$\eta_1(x)u(x) = T_1[\eta_2(P-(k+i)^2)u](x) + T_2(\eta_2 u)(x)$$

for some operators T_1 and T_2. Here T_1 is a parametrix for $P-(k+i)^2$, and T_2 is the error. We shall have control on both: T_1 will map L^2 to $L^{p'}$ with norm $Ck^{\delta(p)-1}$, while T_2 will map L^2 to $L^{p'}$ with norm $Ck^{\delta(p)}$, $p=p_n$. Apply this to $\chi_k u$; it follows that in every sufficiently small open set U we have

$$\|\chi_k u\|_{L^{p'}(U)} \leq Ck^{\delta(p)-1}\|(P-(k+i)^2)(\chi_k u)\|_{L^2(V)} + Ck^{\delta(p)}\|\chi_k u\|_{L^2(V)}$$

where V is a slightly bigger open neighborhood of U. By a covering argument (now M is compact), we obtain

$$\|\chi_k u\|_{L^{p'}(M)} \leq Ck^{\delta(p)-1}\|(P-(k+i)^2)(\chi_k u)\|_{L^2(M)} + Ck^{\delta(p)}\|\chi_k u\|_{L^2(M)}.$$

However, $\|\chi_k u\|_{L^2(M)} \leq \|u\|_{L^2(M)}$ and

$$\|(P-(k+i)^2)(\chi_k u)\|_{L^2(M)} \leq Ck\|u\|_{L^2(M)}$$

by orthogonality; indeed since now P is self-adjoint and has positive definite principal symbol, χ_k is basically the projection onto the eigenspaces with eigenvalues $(k-1)^2 \leq \lambda < k^2$. Hence if $u = \sum_j a_j e_j$, where $\{e_j\}$ is an orthonormal basis of eigenfunctions of P, then

$$\|(P-(k+i)^2)(\chi_k u)\|^2_{L^2(M)} = \sum_{(k-1)^2 \leq \lambda < k^2} |a_j|^2 |\lambda_j - (k+i)^2|^2 \leq Ck^2 \|u\|^2_{L^2(M)}.$$

As a result,

$$\|\chi_k u\|_{L^{p'}(M)} \leq Ck^{\delta(p)} \|u\|_{L^2(M)}$$

for $p = p_n$, as desired. The result for $p = 1$ can be proved similarly (and can be established using a different argument), while the case $p = 2$ is trivial. The rest then follows by complex interpolation.

So it remains to construct the parametrix T_1, the error T_2 and obtain their sharp bounds.

4b. Construction of parametrix

The motivation is that if P had constant coefficients in a coordinate chart, then a parametrix of $P - z$ can be constructed explicitly via the use of the Bessel potentials.

For any unreal complex number z, consider the Bessel potential on $\mathbb{R}^n$

$$F(x) = F(|x|) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{|\xi|^2 - z} e^{ix\cdot\xi} d\xi.$$

(We suppress the dependence of F on z in the notation for a moment.) We have on $\mathbb{R}^n$

$$(-\Delta - z)F(x) = \delta_0(x)$$

or as measures

$$(-\Delta - z)F(x)dx = \delta_0(x).$$

Here $-\Delta$ is the standard Laplacian with the standard flat metric. Now if we change coordinate on $\mathbb{R}^n$, we can bring the metric to any desired constant positive definite symmetric matrix g_{jk}. In the new (say y) coordinate, the Laplacian becomes $-\frac{\partial}{\partial y^j} g^{jk} \frac{\partial}{\partial y^k}$ and the function F becomes $F(|y|_g) = F((g_{jk}y^j y^k)^{\frac{1}{2}})$. The volume measure becomes $(\det g_{jk})^{\frac{1}{2}} dy$, so now

$$\left(-\frac{\partial}{\partial y^j} g^{jk} \frac{\partial}{\partial y^k} - z\right) F\left(|y|_g\right) (\det g_{jk})^{\frac{1}{2}} dy = \delta_0(y).$$

More generally, for any $x \in \mathbb{R}^n$,

$$\left(-\frac{\partial}{\partial y^j} g^{jk} \frac{\partial}{\partial y^k} - z\right) F\left(|x-y|_g\right) (\det g_{jk})^{\frac{1}{2}} dy = \delta_x(y). \tag{3}$$

So if P on M now had constant coefficients in a certain (say y) coordinate chart, then writing the principal symbol of P as $-\frac{\partial}{\partial y^j} g^{jk} \frac{\partial}{\partial y^k}$, we can construct a parametrix of P in that chart as follows: let η_1 be a cut-off function that is supported in that coordinate chart. Then for any u on M, we have

$$\int_M (P-z)(F(|x-y|_g)\eta_1(y)u(y)(\det g_{jk})^{\frac{1}{2}} dy = \eta_1(x)u(x)$$

on M, and thus integrating by parts

$$\eta_1(x)u(x) = \int_M \phi(y)F(|x-y|_g)(P-z)u(y)dy + \int_M \left(\phi_0(y)F(|x-y|_g) + \sum_{i=1}^n \phi_i(y) \frac{\partial}{\partial y^i} F(|x-y|_g)\right) u(y)dy$$

for some smooth functions ϕ and ϕ_i supported in the support of η_1. Now let η_2 be a cut-off function that is 1 on the support of η_1. Then the above equation can be written

$$\eta_1(x)u(x) = T_1(\eta_2(P-z)u)(x) + T_2(\eta_2 u)(x),$$

where

$$T_1 f(x) = \int_M \phi(y) F(|x-y|_g) f(y) dy$$

is the desired parametrix of $P - z$ and

$$T_2 f(x) = \int_M \left(\phi_0(y) F(|x-y|_g) + \sum_{i=1}^n \phi_i(y) \frac{\partial}{\partial y^i} F(|x-y|_g) \right) f(y) dy$$

is the desired error term (both depending on z).

It turns out that when P on M has variable coefficients, as long as we put a suitable metric on M and use the geodesic normal coordinate, the equation (3) remains approximately true, and the above argument works. More precisely, in a coordinate chart, let P be

$$-\frac{\partial}{\partial y^j} g^{jk}(y) \frac{\partial}{\partial y^k} + b^j(y) \frac{\partial}{\partial y^j} + c,$$

so that the principal symbol of P is $-\frac{\partial}{\partial y^j} g^{jk}(y) \frac{\partial}{\partial y^k}$ with g^{jk} self-adjoint positive definite, and put a Riemannian metric on that coordinate chart by setting it to be $g_{jk}(y) dy^j \otimes dy^k$. This is well-defined since the principal symbol of P is a well-defined (2,0)-tensor on M. Take a totally geodesic normal neighborhood U in this chart. Then for each point $x \in U$, we adopt the geodesic normal coordinate $\tilde{y}$ in U, so that $\tilde{y} = 0$ corresponds to the point x. (Note the definition of $\tilde{y}$ depends on the base point x.) For each point x, write the metric in the $\tilde{y}$ coordinates as $\tilde{g}_{jk}(\tilde{y}) d\tilde{y}^j \otimes d\tilde{y}^k$. Then

$$\tilde{g}_{jk}(\tilde{y}) \tilde{y}^k = \tilde{y}^j$$

for all points $\tilde{y}$ in U. This follows from the Gauss lemma, since

$$\tilde{g}_{jk}(\tilde{y}) \tilde{y}^k = g_{\tilde{y}} \left(\frac{\partial}{\partial \tilde{y}^j}, \tilde{y}^k \frac{\partial}{\partial \tilde{y}^k} \right)$$

and the second vector in the last expression is the tangent vector of a radial geodesic. It follows that if we write $|\tilde{y}|^2 = \sum_i (\tilde{y}^i)^2$, then for any function f defined in U,

$$\tilde{g}^{jk}(\tilde{y}) \frac{\partial}{\partial \tilde{y}^k} f(|\tilde{y}|) = \tilde{g}^{jk}(\tilde{y}) \frac{\tilde{y}^k}{|\tilde{y}|^2} f'(|\tilde{y}|) = \frac{\tilde{y}^j}{|\tilde{y}|^2} f'(|\tilde{y}|) = \frac{\partial}{\partial \tilde{y}^j} f(|\tilde{y}|).$$

Hence if η_1 is a smooth cut-off function supported in U, and F is the Euclidean Bessel potential defined as above, then

$$\begin{aligned} \left(-\frac{\partial}{\partial \tilde{y}^j} \tilde{g}^{jk}(\tilde{y}) \frac{\partial}{\partial \tilde{y}^k} - z \right) (\eta_1(\tilde{y}) F(|\tilde{y}|)) &= \eta_1(\tilde{y}) \left(-\frac{\partial^2}{\partial (\tilde{y}^j)^2} - z \right) F(|\tilde{y}|) + \phi_0(\tilde{y}) F(|\tilde{y}|) + \sum_{i=1}^n \phi_i(\tilde{y}) \frac{\partial}{\partial \tilde{y}^i} F(|\tilde{y}|) \\ &= \eta_1(\tilde{y}) \delta_0(\tilde{y}) + \phi_0(\tilde{y}) F(|\tilde{y}|) + \sum_{i=1}^n \phi_i(\tilde{y}) \frac{\partial}{\partial \tilde{y}^i} F(|\tilde{y}|) \end{aligned}$$

for some smooth cut-off functions ϕ_i, and modifying the ϕ_i's if necessary, we have

$$(P - z)(\eta_1(\tilde{y}) F(|\tilde{y}|)) = \eta_1(\tilde{y}) \delta_0(\tilde{y}) + \phi_0(\tilde{y}) F(|\tilde{y}|) + \sum_{i=1}^n \phi_i(\tilde{y}) \frac{\partial}{\partial \tilde{y}^i} F(|\tilde{y}|).$$

If for each pair of points y, $x \in U$ we now let $\tilde{y}$ be the normal coordinates of y in the normal coordinate chart centered at x and define

$$\begin{aligned} s(y,x) &= |\tilde{y}| \\ \eta_1(y,x) &= \eta_1(\tilde{y}) \\ \phi_i(y,x) &= \phi_i(\tilde{y}) \end{aligned}$$

then the above reads (upon going back to the y coordinate that we started with, modifying the ϕ_i's if necessary)

$$(P(y,D)-z)(\eta_1(y,x)F(s(y,x))) = \eta_1(x,x)\delta_x(y) + \phi_0(y,x)F(s(y,x)) + \sum_{i=1}^{n} \phi_i(y,x)\frac{\partial}{\partial y^i}F(s(y,x)).$$

Put in measure form,

$$\begin{aligned}&(P(y,D)-z)(\eta_1(y,x)F(s(y,x)))(\det g(y))^{\frac{1}{2}}dy \\ =&\eta_1(x,x)\delta_x(y) + \left(\phi_0(y,x)F(s(y,x)) + \sum_{i=1}^{n}\phi_i(y,x)\frac{\partial}{\partial y^i}F(s(y,x))\right)(\det g(y))^{\frac{1}{2}}dy.\end{aligned}$$

Hence for u defined in U, we have

$$\begin{aligned}\eta_1(x,x)u(x) =& \int_M \eta_1(y,x)(\det g(y))^{\frac{1}{2}}F(s(y,x))(P(y,D)-z)u(y)dy \\ &- \int_M (\det g(y))^{\frac{1}{2}}\left(\phi_0(y,x)F(s(y,x)) + \sum_{i=1}^{n}\phi_i(y,x)\frac{\partial}{\partial y^i}F(s(y,x))\right)u(y)dy.\end{aligned}$$

Modifying the ϕ_i's again,

$$\begin{aligned}\eta_1(x,x)u(x) =& \int_M \phi(y,x)F(s(y,x))[(P-z)u](y)dy \\ &+ \int_M \left(\phi_0(y,x)F(s(y,x)) + \sum_{i=1}^{n}\phi_i(y,x)\frac{\partial}{\partial y^i}F(s(y,x))\right)u(y)dy.\end{aligned}$$

Again taking $\eta_2(y)$ to be 1 on the support of $\eta_1(y)$, we have

$$\eta_1(x,x)u(x) = T_1(\eta_2(P-z)u)(x) + T_2(\eta_2 u)(x)$$

where

$$T_1 f(x) = \int_M \phi(y,x)F(s(y,x))f(y)dy$$

is the desired parametrix of $P-z$ and

$$T_2 f(x) = \int_M \left(\phi_0(y,x)F(s(y,x)) + \sum_{i=1}^{n}\phi_i(y,x)\frac{\partial}{\partial y^i}F(s(y,x))\right)f(y)dy$$

is the desired error term. The dependence of T_1 and T_2 on z is again suppressed in the notation.

4c. Boundedness of parametrix and error

To prove the desired boundedness of T_1 and T_2, we need some properties of the Bessel potentials F. We shall now emphasize the z dependence by writing F_k for the Bessel potential with $z=(k+i)^2$; in other words,

$$F_k(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{1}{|\xi|^2-(k+i)^2}e^{ix\cdot\xi}d\xi.$$

Lemma 5. *There is an absolute constant C such that for $|x|\le k^{-1}$,*

$$\begin{aligned}|F_k(x)| &\le C|x|^{-(n-2)} \quad \textit{if } n\ge 3\\ |F_k(x)| &\le C|\log|x|| \quad \textit{if } n=2\\ |\nabla F_k(x)| &\le C|x|^{-(n-1)} \quad \textit{if } n\ge 2\end{aligned}$$

Lemma 6. *For $|x| \geq k^{-1}$, $n \geq 2$,*

$$F_k(x) = k^{\frac{n-1}{2}-1}|x|^{-\frac{n-1}{2}}e^{-ik|x|}a_0(kx)$$
$$\frac{\partial}{\partial x^j}F_k(x) = k^{\frac{n-1}{2}}|x|^{-\frac{n-1}{2}}e^{-ik|x|}a_j(kx), \qquad j = 1, \ldots, n$$

where the a_j's are smooth radial functions that satisfy

$$\left|\frac{\partial^m}{\partial \rho^m}a_j(\rho)\right| \leq C_m|\rho|^{-m}.$$

These can be proved by integrating by parts and using methods of stationary phase.

We shall also need a lemma about oscillatory integral operators on $\mathbb{R}^n$, where the amplitude $a(x, y)$ is supported off the diagonal.

Lemma 7. *Let $a(x, y)$ be a compactly supported smooth function with support inside $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{1}{2} \leq |x - y| \leq 2\}$. Then there is a neighborhood of the function $s_0(x, y) = |x - y|$ in the C^∞ topology such that for any function $s(x, y)$ in that neighborhood we have*

$$\left\|\int_{\mathbb{R}^n} e^{i\lambda s(x,y)}a(x, y)f(y)dy\right\|_{L^{p'}(\mathbb{R}^n)} \leq C\lambda^{-\frac{n}{p'}}\|f\|_{L^{q'}(\mathbb{R}^n)},$$

where $1 \leq p \leq p_n$, $q = (\frac{n-1}{n+1})p'$, $\lambda > 0$, and C depends only on s and the bounds of finitely many derivatives of a.

We are now ready to prove the boundedness of T_1 and T_2. For simplicity, let $n \geq 3$. Write $p = p_n$, and we shall prove that T_1 maps L^2 to $L^{p'}$ with norm $Ck^{\delta(p)-1}$, while T_2 maps L^2 to $L^{p'}$ with norm $Ck^{\delta(p)}$.

Recall that the parametrix T_1 is

$$T_1 f(x) = \int_M \phi(y, x)F_k(s(y, x))f(y)dy.$$

Since $\phi(y, x)$ is supported in $U \times U$ where U is a small coordinate chart on M, one may identify U with a portion of $\mathbb{R}^n$; by taking a smaller coordinate chart if necessary, one may then assume $s(y, x) \simeq |y - x|$ where $|y - x|$ is the Euclidean distance between y and x. One may also think of this integral as one on $\mathbb{R}^n$, as we shall do from now on. Now we decompose T_1 dyadically based on the distance of (x, y) from the diagonal: Let $\varphi(t)$ be a smooth function with compact support in $[-2, 2]$ such that it is identically 1 in $[-1, 1]$. Let $\psi(t) = \varphi(t) - \varphi(2t)$ be supported in $[-\frac{1}{2}, 2]$. Then $\varphi(t) + \sum_{\nu=1}^{\infty}\psi(2^{-\nu}t) = 1$ on $\mathbb{R}$, and

$$T_1 f(x) = \int_{\mathbb{R}^n} \varphi(ks(y, x))\phi(y, x)F_k(s(y, x))f(y)dy + \sum_{\nu=1}^{C\log k}\int_{\mathbb{R}^n} \psi(2^{-\nu}ks(y, x))\phi(y, x)F_k(s(y, x))f(y)dy,$$

the second sum terminating after roughly the $\log k$-th term when one takes into account the fact that ϕ has compact support. We will use our information about F_k to estimate the $L^{p'}$ norm of the terms one by one:

- In the first term we use the Hausdorff-Young inequality: since we are to estimate the $L^{p'}$ of the first term in terms of the L^2 norm of f, we naturally look at the L^r norm of the kernel, where

$$1 + \frac{1}{p'} = \frac{1}{2} + \frac{1}{r}.$$

Then using the first estimate in Lemma 5, we have

$$\begin{aligned}\sup_x \|\varphi(ks(y,x))\phi(y,x)F_k(s(y,x))\|_{L^r(dy)} &\leq \sup_x \left(\int_{|x-y|\leq k^{-1}} |x-y|^{-(n-2)r} dy\right)^{\frac{1}{r}} \\ &= Ck^{n-2-\frac{n}{r}} \\ &= Ck^{n(\frac{1}{2}-\frac{1}{p'})-2} \\ &= Ck^{\delta(p)-1}k^{-\frac{1}{2}}\end{aligned}$$

and

$$\sup_y \|\varphi(ks(y,x))\phi(y,x)F_k(s(y,x))\|_{L^r(dx)} \leq Ck^{\delta(p)-1}k^{-\frac{1}{2}},$$

so

$$\left\|\int_{\mathbb{R}^n} \varphi(ks(y,x))\phi(y,x)F_k(s(y,x))f(y)dy\right\|_{L^{p'}} \leq Ck^{\delta(p)-1}k^{-\frac{1}{2}}\|f\|_{L^2}.$$

- For the ν-th term in the second sum, we use the asymptotic expansion of F in Lemma 6 and the oscillatory integral operator lemma after rescaling:

$$\begin{aligned}&\int_{\mathbb{R}^n} \psi(2^{-\nu}ks(y,x))\phi(y,x)F_k(s(y,x))f(y)dy \\ =&k^{\frac{n-1}{2}-1}\int_{\mathbb{R}^n} \psi(2^{-\nu}ks(y,x))\phi(y,x)s(y,x)^{-\frac{n-1}{2}}e^{-iks(y,x)}a_0(ks(y,x))f(y)dy \\ =&k^{\frac{n-1}{2}-1}S_{\nu,k}g(2^{-\nu}kx)\end{aligned}$$

where $g(y) = f(2^\nu k^{-1}y)$ and

$$\begin{aligned}S_{\nu,k}g(x) &= \int_{\mathbb{R}^n} \psi(s(y,x))\phi(2^\nu k^{-1}y, 2^\nu k^{-1}x)(2^\nu k^{-1}s(y,x))^{-\frac{n-1}{2}}e^{-i2^\nu s(y,x)}a_0(2^\nu s(y,x))g(y)(2^\nu k^{-1})^n dy \\ &= (2^\nu k^{-1})^{\frac{n+1}{2}}\int_{\mathbb{R}^n} a_{\nu,k}(y,x)g(y)e^{-i2^\nu s(y,x)}dy\end{aligned}$$

for some $a_{\nu,k}$ fulfilling the conditions of the oscillatory integral lemma. Note however that the derivatives of $a_{\nu,k}$'s are uniformly bounded independent of ν and k; this is because of the corresponding property of a_0, the uniform support property of $a_{\nu,k}$, and that $2^\nu k^{-1} \leq C$ in the range of ν that we can considering. Hence the $S_{\nu,k}$'s have $(L^2, L^{p'})$ norm approximately $C(2^\nu k^{-1})^{\frac{n+1}{2}}2^{-\nu\frac{n}{p'}}$, and

$$\begin{aligned}&\left\|\int_{\mathbb{R}^n} \psi(2^{-\nu}ks(y,x))\phi(y,x)F_k(s(y,x))f(y)dy\right\|_{L^{p'}} \\ \leq&Ck^{\frac{n-1}{2}-1}(2^\nu k^{-1})^{\frac{n}{p'}}\|Sg\|_{L^{p'}} \\ \leq&Ck^{\frac{n-1}{2}-1}(2^\nu k^{-1})^{\frac{n}{p'}}(2^\nu k^{-1})^{\frac{n+1}{2}}2^{-\nu\frac{n}{p'}}\|g\|_{L^2} \\ \leq&Ck^{\frac{n-1}{2}-1}(2^\nu k^{-1})^{\frac{n}{p'}}(2^\nu k^{-1})^{\frac{n+1}{2}}2^{-\nu\frac{n}{p'}}(2^\nu k^{-1})^{-\frac{n}{2}}\|f\|_{L^2} \\ =&Ck^{\delta(p)-1}(2^\nu k^{-1})^{\frac{1}{2}}\|f\|_{L^2}.\end{aligned}$$

Summing the geometric series, we get

$$\|T_1f\|_{L^{p'}} \leq Ck^{\delta(p)-1}\sum_{\nu=0}^{C\log k}(2^\nu k^{-1})^{\frac{1}{2}}\|f\|_{L^2} = Ck^{\delta(p)-1}\|f\|_{L^2}$$

as desired.

The first term of T_2 (involving no derivative of F_k) can be dealt with similarly as above. The second term of T_2, which involves the first derivatives of F_k, can be handled by applying the corresponding statements of Lemma 5 and 6. This concludes the proof of the sharp bounds of T_1 and T_2 and hence that of the spectral projection theorem.