A BRIEF INTRODUCTION TO SEVERAL COMPLEX VARIABLES

PO-LAM YUNG

Contents

This is the fourth talk in a sequence, but the material is largely independent of the previous talks. We will be mostly giving a brief first introduction to several complex variables, and at the very end I will tie everything together by mentioning some recent results in the subject that follow from the subelliptic real analysis we have seen in the previous talks.

1. The Cauchy-Riemann complex $\overline{\partial}$

Let Ω be a domain in $\mathbb{C}^{n+1}$ with smooth boundary, $n \geq 1$. We use the standard Euclidean coordinates on Ω: $z := (z^1, \dots, z^{n+1})$,

$$z^j := x^j + iy^j, \quad j = 1, \dots, n+1.$$

For $1 \leq j \leq n+1$, we introduce the following 1-forms with complex coefficients:

$$dz^j := dx^j + i dy^j, \quad d\overline{z}^j := dx^j - i dy^j.$$

A $(1,0)$ form is then a linear combination of the dz^j's at each point (where the coefficients may vary as the point varies). Similarly a $(0,1)$ form is a linear combination of the $d\overline{z}^j$'s at every point. We also introduce the following complex vector fields:

$$\frac{\partial}{\partial z^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \quad \frac{\partial}{\partial \overline{z}^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right).$$

(The coefficients were chosen so that the $\frac{\partial}{\partial z^j}$ and $\frac{\partial}{\partial \overline{z}^j}$'s are dual to the dz^j and $d\overline{z}^j$'s.) A holomorphic vector field is then a linear combination of the $\frac{\partial}{\partial z^j}$'s at each

Date: November 22, 2009.

point, and an anti-holomorphic vector field is a linear combination of the $\frac{\partial}{\partial \bar{z}^j}$'s at each point.[1] A (p,q) form is an alternating tensor product of p $(1,0)$ forms and q $(0,1)$ forms, and a $(0,0)$ form is just a function. From the analytic point of view, it is no loss in generality to consider only the case of $(0,q)$ forms, which we do from now on. At each point, a basis of $(0,q)$ forms is given by $\{d\bar{z}^I\}$, where

$$d\bar{z}^I := d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_q} \quad \text{if } I = (i_1, \dots, i_q),$$

and I runs over all strictly increasing multi-indices of length q (i.e. $i_1 < i_2 < \cdots < i_q$). We put a Hermitian metric on the space of $(0,q)$ forms by making the previous basis unitary at each point. We denote this pointwise Hermitian inner product by $\langle \cdot, \cdot \rangle$. This allows us to put a global Hermitian inner product on the space of all $(0,q)$ forms on Ω, by letting

$$(u,v) = \int_\Omega \langle u, v \rangle,$$

where the integration is with respect to the standard Euclidean measure on $\mathbb{C}^{n+1}$. We write $L^2_{(0,q)}(\Omega)$ for the space of all $(0,q)$ forms on Ω whose norm under the previous inner product is finite, and this is a Hilbert space.

We now define the *Cauchy-Riemann operator* $\overline{\partial}$ on Ω.

Definition 1. *The operator $\overline{\partial}$ is defined, in the distributional sense, by*

$$\overline{\partial} u := \sum_I \sum_{j=1}^{n+1} \frac{\partial u_I}{\partial \bar{z}^j} d\bar{z}^j \wedge d\bar{z}^I \qquad \textit{if } u = \sum_I u_I d\bar{z}^I.$$

Hereafter sums like $\sum_I$ will always mean sums over strictly increasing multi-indices. $\overline{\partial}$ sends $(0,q)$ forms to $(0,q+1)$ forms. A function u is said to be holomorphic if $\overline{\partial} u = 0$. Since we shall be working with the Hilbert space $L^2_{(0,q)}(\Omega)$ in a moment, from now on, however, unless otherwise specified, we shall take $\overline{\partial}$ to be the (unbounded) linear operator

$$\overline{\partial} \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$$

with domain

$$\mathrm{Dom}(\overline{\partial}) = \{u \in L^2_{(0,q)}(\Omega) \colon \text{the distributional } \overline{\partial} u \in L^2_{(0,q+1)}(\Omega)\}$$

so that the Hilbert space operator $\overline{\partial}$ agrees with the distributional $\overline{\partial}$ whenever the former is defined.

Another often useful reformulation of the definition of $\overline{\partial}$ is the following. If u is a function, we require $\overline{\partial} u$ to be the $(0,1)$ form satisfying

$$\overline{\partial} u(\overline{Z}) = \overline{Z}(u)$$

for all anti-holomorphic vector fields $\overline{Z}$; if u is a $(0,1)$ form, we require $\overline{\partial} u$ to be the $(0,2)$ form satisfying

$$(\overline{\partial} u)(\overline{Z}_1, \overline{Z}_2) = \overline{Z}_1(u(\overline{Z}_2)) - \overline{Z}_2(u(\overline{Z}_1)) - u([\overline{Z}_1, \overline{Z}_2])$$

[1] In more differential geometric terms, we are complexifying the tangent bundle of Ω and writing it as the direct sum of two subbundles, $T^{(1,0)}$ and $T^{(0,1)}$, where the fiber of $T^{(1,0)}$ at each point is the span of the $\frac{\partial}{\partial z^j}$'s, and the fiber of $T^{(0,1)}$ at each point is the span of the $\frac{\partial}{\partial \bar{z}^j}$'s. In other words, $T^{(1,0)}$ is the i-eigenspace of the complex structure J, and $T^{(0,1)}$ is the $-i$-eigenspace of J. The $(1,0)$ forms are then complexified 1-forms that annihilates $T^{(0,1)}$, and the $(0,1)$ forms are complexified 1-forms that annihilates $T^{(1,0)}$.

for all anti-holomorphic vector fields $\overline{Z}_1$, $\overline{Z}_2$. We also require that

$$\overline{\partial}(u \wedge v) = (\overline{\partial} u) \wedge v + (-1)^{q_1} u \wedge (\overline{\partial} v)$$

whenever u is a $(0, q_1)$ form and v is a $(0, q_2)$ form. This defines the same distributional $\overline{\partial}$ as above, and if $\overline{Z}_1, \dots, \overline{Z}_{n+1}$ is a local frame of anti-holomorphic vector fields, while $\overline{\omega}^1, \dots, \overline{\omega}^{n+1}$ is the dual frame of $(0,1)$ forms, then locally, in the sense of distribution, for $u = \sum_I u_I \overline{\omega}^I$, we have

$$\overline{\partial} u = \sum_I \sum_{j=1}^{n+1} \overline{Z}_j (u_I) \overline{\omega}^j \wedge \overline{\omega}^I + \text{ terms that are 0-th order in } u. \tag{1}$$

(The latter 0-th order terms arise when one takes $\overline{\partial} \overline{\omega}^I$, and are zero if $\overline{\partial} \overline{\omega}^j = 0$ for all j.) We can then define the Hilbert space operator $\overline{\partial}$ as above, and again from now on the symbol $\overline{\partial}$ shall refer to the Hilbert space operator.

One important property of the $\overline{\partial}$ operator is that it forms a *complex*: in other words, $\text{Range}(\overline{\partial}) \subseteq \text{Dom}(\overline{\partial})$, and

$$\overline{\partial} \circ \overline{\partial} = 0.$$

One fundamental question in several complex variables is to solve the following *inhomogeneous Cauchy-Riemann equation* for $u \in L^2_{(0,q)}(\Omega)$:

$$\overline{\partial} u = f. \tag{2}$$

In other words, we want to solve the above equation weakly for $u \in L^2_{(0,q)}(\Omega)$, assuming $f \in L^2_{(0,q+1)}(\Omega)$ is given. Since $\overline{\partial}$ forms a complex, this equation can only have a solution when the compatibility condition

$$\overline{\partial} f = 0$$

is satisfied, which we shall always assume from now on. Another way of viewing this is that this system of equations is over-determined, and some compatibility conditions must be imposed on the given data. The solution to the inhomogeneous Cauchy-Riemann equation, if exists, is not unique; if u is a solution, so is $u + v$ for any v in the kernel of $\overline{\partial}$. Thus if we have a solution to the inhomogeneous Cauchy-Riemann equation, we often orthogonally project to the orthogonal complement of the kernel of $\overline{\partial}$, and obtain the unique solution to the equation that is orthogonal to the kernel of $\overline{\partial}$. The latter solution is often called the *canonical solution*, and this is usually the one that we are interested in.

2. Geometry of the domain: Pseudoconvexity

It turns out that solutions to the inhomogeneous Cauchy-Riemann equation (2) may or may not exist in such a general formulation. To ensure the existence of solutions, one needs to impose a local geometric condition at each point on the boundary $\partial\Omega$ of Ω. This is usually formulated using the notion of *pseudoconvexity*, to which we now turn.

Again let Ω be a domain in $\mathbb{C}^{n+1}$ with smooth boundary. First we need the concept of a *holomorphic tangent vector* at each point on $\partial\Omega$.

Definition 2. *For each point $z \in \partial\Omega$, a holomorphic tangent vector at z is a holomorphic vector at z that is tangent to $\partial\Omega$. The space of all such is written $T^{(1,0)}_z(\partial\Omega)$.*

Hence $T_z^{(1,0)}(\partial\Omega)$ is a vector space of (complex) dimension n. Similarly we define $T_z^{(0,1)}(\partial\Omega)$, which is also n-dimensional, and we call each element in that an *anti-holomorphic tangent vector.* Note that the direct sum of these is not the full tangent space of $\partial\Omega$; one direction is always missing (by a simple count of dimensions). It is this failure of the holomorphic and anti-holomorphic tangent vectors to span the tangent space at the boundary that eventually contributes to the subelliptic nature of the subject, which we will see in a moment.

Now we shall define a Hermitian form on $T_z^{(1,0)}(\partial\Omega)$ at each $z \in \partial\Omega$. To do so, fix a smooth defining function ρ on Ω. This means that

$$\Omega = \{z \in \mathbb{C}^{n+1} \colon \rho(z) > 0\}$$

and that $|d\rho| \neq 0$ on the boundary of Ω.

Definition 3. *For each point $z \in \partial\Omega$, the* Levi form L_z *at z is a Hermitian form defined on $T_z^{(1,0)}(\partial\Omega)$, given by the restriction of the Hermitian matrix*

$$-\left(\frac{\partial^2\rho}{\partial z^j \partial \overline{z}^k}(z)\right)_{1\leq j,k\leq n+1}$$

to $T_z^{(1,0)}(\partial\Omega)$.

More explicitly, if $Z = \sum_{j=1}^{n+1} a^j \frac{\partial}{\partial z^j}$ and $W = \sum_{j=1}^{n+1} b^j \frac{\partial}{\partial z^j}$ are tangent to $\partial\Omega$ at z, then

$$L_z(Z,W) := -\sum_{j,k=1}^{n+1} a^j \overline{b^k} \frac{\partial^2\rho}{\partial z^j \partial \overline{z}^k}(z).$$

Alternatively, for holomorphic tangent vector fields Z and W to $\partial\Omega$,

$$L_z(Z,W) = -d(\overline{\partial}\rho)(Z,\overline{W})(z) = -(\overline{\partial}\rho)[Z,\overline{W}](z).$$

Now suppose locally T is a purely imaginary vector field on $\partial\Omega$ that is tangent to $\partial\Omega$ and that does not lie in the span of the holomorphic and anti-holomorphic tangent vectors. Then since $[Z,\overline{W}]$ is tangent to $\partial\Omega$, it is a linear combination of a holomorphic tangent vector, an anti-holomorphic vector and T. In the previous formula for $L_z(Z,W)$, however, it is only the last component that contributes, because $\overline{\partial}\rho$ annihilates any holomorphic and anti-holomorphic tangent vectors on $\partial\Omega$. Hence if we now choose a local frame of holomorphic tangent vectors $Z_1,\dots,Z_n$ to $\partial\Omega$ and let

$$[Z_j,\overline{Z}_k] = c_{jk}T \pmod{T^{1,0}(\partial\Omega)\oplus T^{0,1}(\partial\Omega)}, \quad 1\leq j,k\leq n, \tag{3}$$

then the Levi form on $\partial\Omega$ is given by

$$L_z(Z_j,Z_k) = -(\overline{\partial}\rho)(T)c_{jk}(z), \quad 1\leq j,k\leq n.$$

Now $-\overline{\partial}\rho(T)$ is real and nowhere vanishing, because T is purely imaginary and annihilates ρ. Replacing T by $-T$ if necessary, we may assume that $-\overline{\partial}\rho(T) > 0$, and thus the Levi form on on $\partial\Omega$ is essentially given by the $n\times n$ (Hermitian) matrix $(c_{jk}(z))$, which we call the *Levi matrix.*

The Levi form depends on the choice of the defining function ρ, and the Levi matrix depends on both the choice of ρ, $Z_1,\dots,Z_n$ and T, but the following concepts will be independent of all such:

Definition 4. *A domain Ω is said to be pseudoconvex at a point $z \in \partial\Omega$ if the Levi form at z (or the Levi matrix at z) is non-negative definite. It is simply said to be pseudoconvex if it is pseudoconvex at every point on the boundary.*

Definition 5. *A domain Ω is said to be strongly pseudoconvex at a point $z \in \partial\Omega$ if the Levi form at z (or the Levi matrix at z) is positive definite. It is simply said to be strongly pseudoconvex if it is strongly pseudoconvex at every point on the boundary.*

The names pseudoconvex and strongly pseudoconvex are chosen because every (smooth) convex domain is pseudoconvex, and every (smooth) strictly convex domain is strongly pseudoconvex.

There is actually a notion of pseudoconvexity for domains whose boundary is not smooth, but we shall not discuss that.

3. Solvability of the Cauchy-Riemann operator $\overline{\partial}$

The notion of pseudoconvexity is very important in several complex variables. This can be seen for instance from the following theorem, which underlies one of the starting point of the subject:

Theorem 1. *Let Ω be a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$. Then for all $q \geq 0$, for any $f \in L^2_{(0,q+1)}(\Omega)$ with $\overline{\partial} f = 0$, there exists $u \in L^2_{(0,q)}(\Omega)$ such that $\overline{\partial} u = f$.*

To prove the theorem, we shall need the theory of closed operators on Hilbert spaces. Recall that a densely defined linear operator $T \colon H_1 \to H_2$ between two Hilbert spaces is said to be *closed* if its graph is closed in $H_1 \times H_2$. It is easy to check that the operator $\overline{\partial} \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ we defined is a closed operator. At this point it is convenient to introduce the Hilbert space adjoint of $\overline{\partial}$, denoted by

$$\overline{\partial}^* \colon L^2_{(0,q+1)}(\Omega) \to L^2_{(0,q)}(\Omega).$$

In other words, if $f \in L^2_{(0,q+1)}(\Omega)$, then $f \in \mathrm{Dom}(\overline{\partial}^*)$ if and only if there is some $v \in L^2_{(0,q)}(\Omega)$ such that

$$(u, v) = (\overline{\partial} u, f) \quad \text{for all } u \in L^2_{(0,q)}(\Omega) \cap \mathrm{Dom}(\overline{\partial}),$$

and in this case we define

$$\overline{\partial}^* f = v.$$

Then $\overline{\partial}^*$ is also a densely defined, linear and closed operator. To get a sense of what this is, note that if $f = \sum_J f_J d\overline{z}^J \in L^2_{(0,q+1)}(\Omega) \cap \mathrm{Dom}(\overline{\partial}^*)$ is smooth up to the boundary, then an easy integration by parts argument shows that f has to satisfy the following *boundary condition*, namely that

$$\sum_{j=1}^{n+1} f_{jJ'} \frac{\partial \rho}{\partial z^j} = 0 \quad \text{on } \partial\Omega$$

for all strictly increasing multi-indices J' of length q, where ρ is a defining function of Ω with the additional property that $|\nabla\rho| = 1$ on the boundary (so that $\nabla\rho$ is the inward *unit* normal). Hereafter we shall write $f_{jJ'} = \varepsilon^{J, jJ'} f_J$ if $j \notin J'$, where J is the strictly increasing multi-index that is a permutation of $(j, J'_1, \dots, J'_q)$, and

$\varepsilon^{J,jJ'}$ is the sign of this permutation. We shall also let $f_{jJ'} = 0$ if $j \in J'$. In a sense that we shall make precise below, this is the condition that the normal components of f vanishes on the boundary of Ω. In fact for the above f one then also obtains

$$\overline{\partial}^* f = -\sum_{J'} \sum_{j=1}^{n+1} \frac{\partial f_{jJ'}}{\partial z^j} d\overline{z}^{J'} \quad \text{on } \Omega.$$

In terms of a local frame of $(0,1)$ forms $\overline{\omega}^1, \dots, \overline{\omega}^{n+1}$ and its dual frame of anti-holomorphic vector fields $\overline{Z}_1, \dots, \overline{Z}_{n+1}$, we can also write

$$\overline{\partial}^* f = -\sum_{J'} \sum_{j=1}^{n+1} Z_j(f_{jJ'})\overline{\omega}^{J'} + \text{ terms that are 0-th order in } f$$

if $f = \sum_I f_I \overline{\omega}^I$, by (1). Note that $\overline{\partial}^*$ forms a complex because $\overline{\partial}$ does.

Abstractly, it is easy to check that the orthogonal complement of (the closure of) the range of $\overline{\partial}$ in $L^2_{(0,q+1)}(\Omega)$ is the kernel of $\overline{\partial}^*$. Hence

$$L^2_{(0,q+1)}(\Omega) = \text{Kernel}(\overline{\partial}^*) \oplus \overline{\text{Range}(\overline{\partial})} \tag{4}$$

where $\oplus$ denotes an orthogonal direct sum.

Going back to the inhomogeneous Cauchy-Riemann equation $\overline{\partial} u = f$, suppose that $f \in L^2_{(0,q+1)}(\Omega)$ with $\overline{\partial} f = 0$. We want to find $u \in L^2_{(0,q)}(\Omega) \cap \text{Dom}(\overline{\partial})$ such that $\overline{\partial} u = f$. To find such an u amounts to showing that $f \in \text{Range}(\overline{\partial})$. But using (4) we can decompose

$$f = f_1 + f_2, \qquad f_1 \in \text{Kernel}(\overline{\partial}^*), \qquad f_2 \in \overline{\text{Range}(\overline{\partial})}.$$

Note that as a result $f_2 \in \text{Kernel}(\overline{\partial})$. Since we already have $f \in \text{Kernel}(\overline{\partial})$, we have $f_1 \in \text{Kernel}(\overline{\partial})$ as well. If we could show that

(i) $\text{Kernel}(\overline{\partial}) \cap \text{Kernel}(\overline{\partial}^*) = \{0\}$ on $L^2_{(0,q+1)}(\Omega)$, and
(ii) $\text{Range}(\overline{\partial})$ is closed in $L^2_{(0,q+1)}(\Omega)$,

then $f_1 = 0$, hence $f = f_2 \in \text{Range}(\overline{\partial})$ as desired. Hence we are reduced to showing (i) and (ii). This can be accomplished in one stroke if we could show the following *basic estimate* ($q \geq 0$):

$$\|f\|_{L^2} \leq C(\|\overline{\partial} f\|_{L^2} + \|\overline{\partial}^* f\|_{L^2}), \qquad f \in L^2_{(0,q+1)}(\Omega) \cap \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*). \tag{5}$$

In fact it is clear that (5) implies (i), and from (5) it follows that

$$\|f\|_{L^2} \leq C\|\overline{\partial}^* f\|_{L^2}$$

for all $f \in L^2_{(0,q+1)}(\Omega) \cap \text{Dom}(\overline{\partial}^*)$ orthogonal to the kernel of $\overline{\partial}^*$, so the range of $\overline{\partial}^*$ is closed in $L^2_{(0,q)}(\Omega)$, and (ii) follows. Hence it is tempted to prove the basic estimate on all bounded smooth pseudoconvex domains Ω.

It turns out that while the basic estimate does hold on all bounded smooth pseudoconvex domains Ω, it is not so easy to establish that directly. Here Hormander introduced a very clever trick, and he proved instead a *weighted* version of the basic estimate. From there one can still conclude the proof of Theorem 1, and in fact then one can conclude that the original basic estimate (5) holds as stated. We omit the details.

At this point we point out that the basic estimate is only a very weak estimate. In general, if $\overline{\partial} f$ and $\overline{\partial}^* f$ are in L^2, one is tempted to ask whether u would be gain some derivatives in L^2. It turns out that on strongly pseudoconvex domains one gains $1/2$ derivatives, as Kohn proved in the following theorem.

Theorem 2. *On every bounded smooth strongly pseudoconvex domain $\Omega \subseteq \mathbb{C}^{n+1}$, for any $q \geq 0$, we have*

$$\|f\|_{W^{\frac{1}{2},2}} \leq C(\|\overline{\partial} f\|_{L^2} + \|\overline{\partial}^* f\|_{L^2}), \quad f \in L^2_{(0,q+1)}(\Omega) \cap Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*).$$

Here $\|f\|_{W^{\frac{1}{2},2}}$ is the componentwise Sobolev norm of f. We omit the proof.

4. The Kohn Laplacian and the $\overline{\partial}$-Neumann problem

There is another system of equations in several complex variables that is closely related to the inhomogeneous Cauchy-Riemann equations (2). This is formulated using the *Kohn Laplacian*, which we denote by $\Box$. On $L^2_{(0,q)}(\Omega)$ ($q \geq 0$), this is defined as the Hilbert space operator

$$\Box := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}.$$

In other words, the domain of $\Box$ is given by

$$\text{Dom}(\Box) := \{U \in \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*) \colon \overline{\partial} U \in \text{Dom}(\overline{\partial}^*), \overline{\partial}^* U \in \text{Dom}(\overline{\partial})\},$$

and for $U \in \text{Dom}(\Box)$,

$$\Box U := \overline{\partial}\overline{\partial}^* U + \overline{\partial}^*\overline{\partial} U.$$

So $\Box$ sends $(0,q)$ forms to $(0,q)$ forms, and $\Box$ is a densely defined, linear, closed operator on $L^2_{(0,q)}(\Omega)$. In fact it is also (unbounded) self-adjoint on $L^2_{(0,q)}(\Omega)$.

The system of equations that we shall look at is just

$$\Box U = f; \tag{6}$$

in other words, given $f \in L^2_{(0,q)}(\Omega)$, we want to find $U \in L^2_{(0,q)}(\Omega) \cap \text{Dom}(\Box)$ such that $\Box U = f$. This is called the *$\overline{\partial}$-Neumann problem*, because if U were smooth, then the condition that $U \in \text{Dom}(\Box)$ is just the conditions that $U \in \text{Dom}(\overline{\partial}^*)$ and that $\overline{\partial} U \in \text{Dom}(\overline{\partial}^*)$, and the second one is a Neumann-type boundary condition on U: it says, in a sense that can be made precise, that the complex normal derivatives of the tangential components of U vanish on the boundary. It is this second boundary condition that is difficult to deal with, and that's why the word Neumann enters into the name of the equation.

To solve (6), observe that since $\Box$ is self-adjoint, we have the following orthogonal decomposition of $L^2_{(0,q)}(\Omega)$ for all $q \geq 0$:

$$L^2_{(0,q)}(\Omega) = \text{Kernel}(\Box) \oplus \overline{\text{Range}(\Box)}.$$

Hence for (6) to be solvable, we need f to be orthogonal to the kernel of $\Box$. But a moment's reflection reveals that

$$\text{Kernel}(\Box) = \text{Kernel}(\overline{\partial}) \cap \text{Kernel}(\overline{\partial}^*),$$

and if Ω is bounded smooth and pseudoconvex, this is equal to

$$\begin{cases} \{0\} & \text{if } q \geq 1 \\ \text{Kernel}(\overline{\partial}) & \text{if } q = 0 \end{cases}$$

by (i) of the previous section. In fact we have the following existence theorem for the $\overline{\partial}$-Neumann problem:

Theorem 3. *Let Ω be a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$.*
(a) If $q \geq 1$, then for any $f \in L^2_{(0,q)}(\Omega)$, there exists a unique $U \in L^2_{(0,q)}(\Omega) \cap Dom(\Box)$ such that $\Box U = f$.
(b) If $q = 0$, then for any $f \in L^2_{(0,q)}(\Omega)$ orthogonal to the kernel of $\overline{\partial}$, there exists some $U \in L^2_{(0,q)}(\Omega) \cap Dom(\Box)$ such that $\Box U = f$, and this U is unique modulo the kernel of $\overline{\partial}$.

To prove this theorem, all that one needs to prove now is that the range of $\Box$ is closed in $L^2_{(0,q)}(\Omega)$ for all $q \geq 0$, but this is a consequence of Theorem 1. There is even a formula for the solution operator[2] for $\Box$ in terms of the relative solution operators of $\overline{\partial}$, but we omit the details.

What is remarkable here is that conversely, if we can solve $\overline{\partial}$-Neumann problem, we can solve the inhomogeneous Cauchy-Riemann equations. Suppose $f \in L^2_{(0,q+1)}(\Omega)$ $(q \geq 0)$ satisfies $\overline{\partial} f = 0$. Let $U \in L^2_{(0,q+1)}(\Omega) \cap \mathrm{Dom}(\Box)$ solve
$$\Box U = f.$$
Then we claim that
$$u := \overline{\partial}^* U$$
is the canonical solution to
$$\overline{\partial} u = f.$$
In fact if we take $\overline{\partial}$ of both sides of the equation $\Box U = f$, we get
$$\overline{\partial}(\overline{\partial}^* \overline{\partial} U) = 0.$$
But then $\overline{\partial}^* \overline{\partial} U \in \mathrm{Kernel}(\overline{\partial}) \cap \mathrm{Kernel}(\overline{\partial}^*) = \{0\}$, and thus the equation $\Box U = f$ reduces to
$$\overline{\partial}\overline{\partial}^* U = f.$$
So if $u = \overline{\partial}^* U$, then $\overline{\partial} u = f$. Moreover, it is clear that $\overline{\partial}^* U$ is orthogonal to the kernel of $\overline{\partial}$. Thus $\overline{\partial}^* U$ is the canonical solution to $\overline{\partial} u = f$.

This is very nice because the system $\Box U = f$ is usually *easier* to deal with than the system $\overline{\partial} u = f$. On one hand this is because $\Box U = f$ is usually almost an uncoupled system: the number of unknown component functions in U is the same as the number of given component functions in f, and the system $\Box U = f$ is very often almost diagonal (in fact in some simple model case that we shall see in a moment, the system is entirely uncoupled to a number of *scalar* equations for the components of U). As such they are easier to manipulate. On the other hand, $\Box U = f$ is now a *boundary value problem*, because the condition that $U \in \mathrm{Dom}(\Box)$ is a boundary condition (at least for U that are smooth up to the boundary). There is a general paradigm for solving boundary value problems, as we shall discuss in the next section. For now, let us remember that when we studied the inhomogeneous Cauchy-Riemann equation, the canonical solution was the one that is orthogonal to the kernel of $\overline{\partial}$. That condition is very hard to use in general, because that is a global condition, and in particular is not preserved under localization. The boundary conditions of the $\overline{\partial}$-Neumann problem, on the other hand, are easier to deal with, because they are local in nature. Thus for instance, in the study of the regularity of the solutions to $\overline{\partial} u = f$, very often one first studies the regularity of the solutions to $\Box U = f$ and obtain the regularity of u from that of U.

[2]or relative solution operator if $q = 0$

5. Reduction to the boundary

There is a very general paradigm of solving boundary value problems by reducing it to a problem solely on the boundary. We shall describe one instance of this in the Euclidean setting.

Suppose we want to solve, on the upper half space $\mathbb{R}^{n+1}_+ = \{x^{n+1} > 0\}$, the following Neumann boundary value problem:

$$\begin{cases} \Delta u = f & \text{on } \mathbb{R}^{n+1}_+ \\ \frac{\partial u}{\partial x^{n+1}} = 0 & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases}$$

(Here u and f are just scalar-valued functions.) Then letting u_0 be the restriction of u to $\partial\mathbb{R}^{n+1}_+$, from $\Delta u = f$ we have

$$u = u_0 * P + Gf$$

where P is the Poisson kernel and G is the Green's operator on the upper half space. Hence to solve for u is to solve for u_0. But the only other requirement of u is that $\frac{\partial u}{\partial x^{n+1}} = 0$ on $\partial\mathbb{R}^{n+1}_+$. This is just the condition that

$$\frac{\partial(u_0 * P)}{\partial x^{n+1}}(x,0) = -\frac{\partial(Gf)}{\partial x^{n+1}}(x,0) \quad \text{for all } x \in \mathbb{R}^n.$$

The left-hand side of the equality is just

$$\begin{aligned} \frac{\partial}{\partial x^{n+1}}\bigg|_{x^{n+1}=0} \int_{\mathbb{R}^n} \hat{u_0}(\xi) e^{-2\pi x^{n+1}|\xi|} e^{2\pi i x\cdot\xi} d\xi &= \int_{\mathbb{R}^n} -2\pi|\xi|\hat{u_0}(\xi) e^{2\pi i x\cdot\xi} d\xi \\ &= -(-\Delta_{\mathbb{R}^n})^{\frac{1}{2}} u_0(x), \end{aligned}$$

where $\Delta_{\mathbb{R}^n}$ is the ordinary Laplacian on the boundary. Hence the condition on u_0 is that

$$-(-\Delta_{\mathbb{R}^n})^{\frac{1}{2}} u_0(x) = -\frac{\partial(Gf)}{\partial x^{n+1}}(x,0) \quad \text{for all } x \in \mathbb{R}^n,$$

or

$$-\Delta_{\mathbb{R}^n} u_0(x) = (-\Delta_{\mathbb{R}^n})^{\frac{1}{2}} \frac{\partial(Gf)}{\partial x^{n+1}}(x,0) \quad \text{for all } x \in \mathbb{R}^n,$$

and we have reduced the solution of the original boundary value problem to the solution of this elliptic problem on the boundary.

There is a similar story when we solve $\Box U = f$ for $U \in \text{Dom}(\Box)$, except that in this case the boundary conditions for U are not elliptic, meaning that the boundary operator that one needs to solve in this case is not elliptic. Nevertheless, when the domain Ω is say strongly pseudoconvex, one is then led to solve a subelliptic boundary operator[3], and one can obtain bounds for such, which in turn implies bounds for U.

6. The boundary tangential Cauchy-Riemann complex $\overline{\partial}_b$

It turns out that on the boundary of any smooth domain Ω in $\mathbb{C}^{n+1}$, there is another natural differential complex which is usually denoted as $\overline{\partial}_b$. To define this, let $z \in \partial\Omega$, and let $\Lambda^{(0,q)}_z(\mathbb{C}^{n+1})$ be the restriction of all $(0,q)$ forms in $\mathbb{C}^{n+1}$ to z. Consider the alternating algebra given by the direct sum of these, when q runs from 0 through $n+1$. Let I_z be the ideal of this algebra generated by all $(0,1)$

[3]This is basically the boundary Laplacian $\Box_b$ that we shall encounter in a moment.

forms that annihilates $T_z^{0,1}(\partial\Omega)$, and let $I_z^{(0,q)}$ be the vector space of $(0,q)$ forms in I_z. Then we define $\Lambda_z^{(0,q)}(\partial\Omega)$ to be the quotient

$$\Lambda_z^{(0,q)}(\partial\Omega) := \Lambda_z^{(0,q)}(\mathbb{C}^{n+1})/I_z^{(0,q)},$$

and a $(0,q)$ form on $\partial\Omega$ is by definition an association to each $z \in \partial\Omega$ an element of $\Lambda_z^{(0,q)}(\partial\Omega)$[4]. Note that on $\partial\Omega$, the highest non-trivial level of forms is given by $q = n$; in other words, there is no non-zero $(0, n+1)$ forms on $\partial\Omega$.

To define $\overline{\partial}_b$, let π be the natural projection map

$$\pi\colon \Lambda^{(0,q)}(\mathbb{C}^{n+1}) \to \Lambda^{(0,q)}(\partial\Omega).$$

Given any smooth $(0,q)$ form u_b on $\partial\Omega$, we pick a smooth $(0,q)$ form u on Ω such that

$$\pi(u) = u_b;$$

then we define $\overline{\partial}_b u_b$ by letting

$$\overline{\partial}_b u_b := \pi(\overline{\partial} u).$$

Obviously $\overline{\partial}_b$ sends a $(0,q)$ form on the boundary to a $(0,q+1)$ form on the boundary; it is easy to check that the right side is defined independent of the choice of u, and we extend the definition of $\overline{\partial}_b$ to distributions naturally. It is then clear that $\overline{\partial}_b$ also forms a complex:

$$\overline{\partial}_b \circ \overline{\partial}_b = 0.$$

Now we shall pick, at each point, a Hermitian metric on the space of all $(0,q)$ forms on $\partial\Omega$. Let ρ be a defining function of Ω for which $|d\rho| = 1$ near the boundary of Ω. Then let $\overline{\omega}^1, \dots, \overline{\omega}^n$ be $(0,1)$ forms on Ω such that $\overline{\omega}^1, \dots, \overline{\omega}^n, -2i\overline{\partial}\rho$ form a local unitary basis of $(0,1)$ forms near $\partial\Omega$. Take the restriction of $\overline{\omega}^1, \dots, \overline{\omega}^n$ to $\partial\Omega$, and project them onto $\partial\Omega$ using the map π. These then become $(0,1)$ forms on $\partial\Omega$, and we denote them by $\overline{\omega}_b^1, \dots, \overline{\omega}_b^n$. A basis of $(0,q)$ forms on $\partial\Omega$ is given by $\{\overline{\omega}_b^I\}$, where I runs over all strictly increasing multiindices of length q and

$$\overline{\omega}_b^I = \overline{\omega}_b^{i_1} \wedge \cdots \wedge \overline{\omega}_b^{i_q} \quad \text{if } I = (i_1, \dots, i_q).$$

(Again each $i_k \in \{1, \dots, n\}$.) The Hermitian metric on $(0,q)$ forms on $\partial\Omega$ is then defined by making $\{\overline{\omega}_b^I\}$ a unitary basis at each point of $\partial\Omega$, and we denote this by $\langle \cdot, \cdot \rangle$.

With this pointwise Hermitian metric, we define a global Hermitian inner product on $L^2_{(0,q)}(\partial\Omega)$, the space of all $(0,q)$ forms on $\partial\Omega$ with coefficients in L^2, by letting

$$(u_b, v_b) := \int_{\partial\Omega} \langle u_b, v_b \rangle d\sigma,$$

where $d\sigma$ is the induced surface measure on $\partial\Omega$. This makes $L^2_{(0,q)}(\partial\Omega)$ a Hilbert space, and allows us to talk about the Hilbert space operator $\overline{\partial}_b$. The Hilbert space adjoint of this is denoted by $\overline{\partial}_b^*$, and sends $(0,q+1)$ forms on the boundary to $(0,q)$ forms on the boundary; again it forms a complex because $\overline{\partial}_b$ does. We can again form the boundary Laplacian by letting

$$\Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b,$$

[4] In other words, $\Lambda^{(0,q)}(\partial\Omega)$ is a vector bundle over $\partial\Omega$, and a $(0,q)$ form on $\partial\Omega$ is a section of this vector bundle.

and this maps the space of $(0,q)$ forms on the boundary into itself. Note however that unlike the case of $\overline{\partial}^*$, now there is no boundary conditions for forms in the domain of $\overline{\partial}_b^*$ or $\Box_b$, because $\partial\Omega$ does not have a boundary. The subelliptic nature of the problem is now contained instead in the fact that $\Box_b$ is more or less like a sum of squares operator we have seen, where the vector fields do not span the full tangent space. It follows that the operator $\Box_b$ itself is not elliptic, but only at best subelliptic.

The formula that enters here are as follows. If $Z_1, \dots, Z_{n+1}$ is a local frame of holomorphic vectors on Ω in which the first n are tangent to $\partial\Omega$, and if $\overline{\omega}^1, \dots, \overline{\omega}^{n+1}$ is the dual frame of $(0,1)$ forms on Ω, then if $u_b = \sum_I u_I \overline{\omega}_b^I$ is a $(0,q)$ form on $\partial\Omega$, we have, by (1), that

$$\overline{\partial}_b u_b = \sum_I \sum_{j=1}^n \overline{Z}_j(u_I)\overline{\omega}_b^j \wedge \overline{\omega}_b^I + \text{ terms that are 0-th order in } u_b,$$

in the sense of distributions, and thus

$$\overline{\partial}_b^* u_b = -\sum_{J'} \sum_{j=1}^n Z_j(u_{jJ'})\overline{\omega}_b^{J'} + \text{ terms that are 0-th order in } u_b.$$

It follows that[5]

$$\Box_b u_b = -\sum_I \left(\sum_{\substack{1\le j\le n \\ j\in I}} \overline{Z}_j Z_j + \sum_{\substack{1\le j\le n \\ j\notin I}} Z_j \overline{Z}_j \right) u_I \overline{\omega}_b^I + \text{ terms that are first order in } u_b.$$

There are two reasons why one would like to study $\overline{\partial}_b$ and $\Box_b$. One is that these are natural boundary analogues of the operators $\overline{\partial}$ and $\Box$ that acts in the interior, In fact one can ask the analogues of the problems we asked before, namely to solve

$$\overline{\partial}_b u_b = f_b, \quad u_b \perp \text{Kernel of } \overline{\partial}_b$$

given that $\overline{\partial}_b f_b = 0$, and to solve

$$\Box_b U_b = f_b, \quad u_b \perp \text{Kernel of } \Box_b$$

given that f_b is orthogonal to the kernel of $\Box_b$. If we impose sufficient geometric condition on the boundary of Ω, the analysis of the latter is simpler because there are no boundary conditions for U_b to be in the domain of $\Box_b$; but as we have seen before, now the operator $\Box_b$ itself is not elliptic but only subelliptic, and we need a version of the subelliptic analysis we discussed in the previous talks for systems to analyze this. Another reason for analyzing $\overline{\partial}_b$ and $\Box_b$ is that this analysis actually *helps* us understand $\overline{\partial}$ and $\Box$. This is because $\Box_b$ is basically the operator that arise when we reduce the boundary value problem $\Box U = f$ to the boundary. It is the easiest to see this in the case of a special example, and this is the Heisenberg group that we have incidentally seen in the previous talks, which we discuss next.

[5] This latter expression for $\Box_b$, however, is usually too imprecise for any analysis.

7. A model case: the Heisenberg group

We shall now describe a particular example in several complex variables that is particularly easy to compute and serves as a good model for all strongly pseudoconvex domains. This is the upper half space $\mathcal{U}^{n+1} \subseteq \mathbb{C}^{n+1}$, defined by

$$\mathcal{U}^{n+1} := \{w = (w', w^{n+1}) \in \mathbb{C}^{n+1} \colon \text{Im } w^{n+1} > |w'|^2\}.$$

It plays the same role as the ordinary upper half-plane $\mathbb{R}^{n+1}_+ = \{x^{n+1} > 0\}$ in Euclidean harmonic analysis, which models any smooth domain in $\mathbb{R}^{n+1}$. In fact $\mathcal{U}^{n+1}$ is strongly pseudoconvex, and models any strongly pseudoconvex domains in $\mathbb{C}^{n+1}$. To see that $\mathcal{U}^{n+1}$ is strongly pseudoconvex, first choose a basis of $(0,1)$ forms on $\mathcal{U}^{n+1}$, namely

$$\overline{\omega}^j := d\overline{w}^j \quad (1 \leq j \leq n) \quad \text{and} \quad \overline{\omega}^{n+1} := -2i\overline{\partial}\rho = d\overline{w}^{n+1} + 2i \sum_{j=1}^n w^j d\overline{w}^j,$$

where $\rho := \text{Im } w^{n+1} - |w'|^2$ is a defining function for $\mathcal{U}^{n+1}$, and the corresponding dual basis of anti-holomorphic vector fields is then given by

$$\overline{Z}_j := \frac{\partial}{\partial \overline{w}^j} - 2iw^j \frac{\partial}{\partial \overline{w}^{n+1}} \quad (1 \leq j \leq n) \quad \text{and} \quad \overline{Z}_{n+1} := \frac{\partial}{\partial \overline{w}^{n+1}}.$$

Then for $1 \leq j, k \leq n$,

$$[Z_j, \overline{Z}_k] = -2i\delta_{jk} \left(\frac{\partial}{\partial \overline{w}^{n+1}} + \frac{\partial}{\partial w^{n+1}} \right),$$

so

$$\overline{\partial}\rho([Z_j, \overline{Z}_k]) = -\frac{1}{2i}\overline{\omega}^{n+1}([Z_j, \overline{Z}_k]) = \delta_{jk}$$

everywhere on $\mathcal{U}^{n+1}$, and in particular the Levi form is given by the $n \times n$ identity matrix at every point of $\partial\mathcal{U}^{n+1}$. The latter is certainly positive definite, and this proves that $\mathcal{U}^{n+1}$ is strongly pseudoconvex.

It is particularly easy to describe the $\overline{\partial}$ complex in terms of this basis, because $[\overline{Z}_j, \overline{Z}_k] = 0$ for all j, k. In particular, $\overline{\partial}\overline{\omega}^j = 0$ for all j, and thus by our alternative description of $\overline{\partial}$ given at the beginning, we have

$$\overline{\partial}u = \sum_I \sum_{j=1}^{n+1} \overline{Z}_j(u_I)\overline{\omega}^j \wedge \overline{\omega}^I \quad \text{if } u = \sum_I u_I \overline{\omega}^I,$$

at least in the sense of distributions.

To describe $\overline{\partial}^*$, we need to put a Hermitian metric at each point on the space of $(0,q)$ forms on $\mathcal{U}^{n+1}$. The metric we put now, however, is slightly different from the one that we have described so far: we are not going to put the Euclidean metric on $(0,q)$ forms. Rather, we are going to make $\{\overline{\omega}^I\}$ a unitary basis at every point of $\mathcal{U}^{n+1}$. Then we have an inner product on the space of all $(0,q)$ forms on $\mathcal{U}^{n+1}$, namely

$$(u,v) = \int_{\mathcal{U}^{n+1}} \sum_I u_I \overline{v_I} dw \quad \text{if } u = \sum_I u_I \overline{\omega}^I \text{ and } v = \sum_I v_I \overline{\omega}^I,$$

and $L^2_{(0,q)}(\mathcal{U}^{n+1})$ is the space of all $(0,q)$ forms whose norm under this inner product is finite. We can now define the Hilbert space operator $\overline{\partial}$, and its Hilbert space adjoint $\overline{\partial}^*$ as before.

Now the Hilbert space operator $\overline{\partial}^*$ has a very nice description in terms of this basis. If $f = \sum_I f_I \overline{\omega}^I$ on $\mathcal{U}^{n+1}$ is smooth up to the boundary and vanishes sufficiently fast at infinity, then $f \in \mathrm{Dom}\overline{\partial}^*$ if and only if

$$f_I = 0 \quad \text{on the boundary of } \mathcal{U}^{n+1} \text{ whenever } n+1 \in I.$$

In this case

$$\overline{\partial}^* f = -\sum_{J'} \sum_{j=1}^{n+1} Z_j(f_{jJ'}) d\overline{\omega}^{J'} \quad \text{on } \mathcal{U}^{n+1}.$$

Thus the Hilbert space operator $\Box$ can be described as follows: if U is smooth up to the boundary and vanishes at infinity, then a $(0,q)$ form $U \in \mathrm{Dom}(\Box)$ if and only if

$$\begin{cases} U_I = 0 & \text{on } \partial\mathcal{U}^{n+1} \text{ whenever } n+1 \in I, \\ \overline{Z}_{n+1} U_I = 0 & \text{on } \partial\mathcal{U}^{n+1} \text{ whenever } n+1 \notin I. \end{cases}$$

In this case, $\Box U$ is just

$$\Box U = \sum_{n+1 \notin I} \Box_q^{\tan}(U_I)\overline{\omega}_I + \sum_{n+1 \in I} \Box_q^{\mathrm{nor}}(U_I)\overline{\omega}_I \tag{7}$$

where $\Box_q^{\tan}$, $\Box_q^{\mathrm{nor}}$ are scalar differential operators acting on functions, defined by

$$\Box_q^{\tan} := \mathcal{L}_{n-2q} - \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \rho^2}\right)$$

and

$$\Box_q^{\mathrm{nor}} := \mathcal{L}_{n-2q+2} - \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \rho^2}\right).$$

Here the $\mathcal{L}_\alpha$'s are scalar tangential differential operators acting on functions, given by

$$\mathcal{L}_\alpha := -\frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z}_j + \overline{Z}_j Z_j\right) + i\alpha T$$

where

$$T = \frac{1}{2i}[\overline{Z}_j, Z_j] = 2\mathrm{Re}\left(\frac{\partial}{\partial \overline{w}^{n+1}}\right).$$

We can also write $\mathcal{L}_\alpha$ as

$$\mathcal{L}_\alpha = -\frac{1}{4}\sum_{j=1}^{n}\left(X_j^2 + Y_j^2\right) + i\alpha T \tag{8}$$

where X_j, Y_j are real vector fields such that $Z_j = \frac{1}{2}(X_j - iY_j)$. Thus to solve the $\overline{\partial}$-Neumann problem, we are reduced to solving the scalar equations

$$\Box_q^{\tan}\phi = \psi \quad \text{on } \mathcal{U}^{n+1}, \qquad \overline{Z}_{n+1}\phi = 0 \quad \text{on } \partial\mathcal{U}^{n+1} \tag{9}$$

and

$$\Box_q^{\mathrm{nor}}\phi = \psi \quad \text{on } \mathcal{U}^{n+1}, \qquad \phi = 0 \quad \text{on } \partial\mathcal{U}^{n+1}; \tag{10}$$

note how we have uncoupled the original system $\Box U = f$.

The boundary value problem (10) involving a Dirichlet boundary condition is actually elliptic, and as such it is easy to deal with. The problem (9) involving a complex Neumann condition, on the other hand, is not elliptic, because the boundary condition $\overline{Z}_{n+1}\phi = 0$ is not elliptic. In fact if we carry out the reduction to the boundary, the operator that interwines is precisely $\mathcal{L}_{n-2q}$, and from (8) we see that this is a variant of the sum of squares operators we have seen in the previous talks. This operator is not elliptic; it is only subelliptic. It is where the subelliptic analysis we have seen enters into the picture.

Note that it is here that we also see the role of the boundary Laplacian $\Box_b$ in this problem; in fact from the same derivation of the formula of $\Box$ in (7), forgetting about the contribution of Z_{n+1} and $\overline{Z}_{n+1}$, one sees that the boundary Laplacian $\Box_b$ acts on $(0,q)$ forms on the boundary by acting componentwise by $\mathcal{L}_{n-2q}$. In fact in our previous notation, $\overline{\partial}_b\overline{\omega}_b^j = 0$ for all j, and $\overline{\omega}_b^1, \dots, \overline{\omega}_b^n$ becomes a unitary basis of $(0,1)$ forms on $\partial\mathcal{U}^{n+1}$; thus at least in the sense of distributions,

$$\overline{\partial}_b u_b = \sum_I \sum_{j=1}^n \overline{Z}_j(u_I)\overline{\omega}_b^j \wedge \overline{\omega}_b^I \quad \text{if } u_b = \sum_I u_I\overline{\omega}_b^I,$$

$$\overline{\partial}_b^* f_b = -\sum_{J'} \sum_{j=1}^n Z_j(f_{jJ'})d\overline{\omega}_b^{J'} \quad \text{if } f_b = \sum_I f_I\overline{\omega}_b^I,$$

and if $U_b = \sum_I U_I\overline{\omega}_b^I$ is a $(0,q)$ form on $\partial\mathcal{U}^{n+1}$, then by the same computation as in the derivation of (7) (which we did not really carry out),

$$\Box_b U_b = \sum_I \mathcal{L}_{n-2q}(U_I)\overline{\omega}_b^I$$

The upshot is that in the aforementioned reduction of the boundary value problem (9) to the boundary, the operator that arises, namely $\mathcal{L}_{n-2q}$, is basically our boundary Laplacian $\Box_b$. Thus it is important to be able to solve $\Box_b$ and obtain estimates for that.

In the situation of the upper half-space $\mathcal{U}^{n+1}$ we are rather fortunate. This is because the boundary $\partial\mathcal{U}^{n+1}$ of $\mathcal{U}^{n+1}$ is also a Lie group: if $w = (w', w^{n+1}) \in \partial\mathcal{U}^{n+1}$, then it induces a biholomorphism of $\mathcal{U}^{n+1}$ into itself by *translation*, by sending

$$(\zeta', \zeta^{n+1}) \mapsto (\zeta' + w', \zeta^{n+1} + \text{Re } w^{n+1} + 2i\zeta' \cdot \overline{w'} + i|w'|^2).$$

There is a one-to-one correspondence between points on $\partial\mathcal{U}^{n+1}$ and biholomorphisms of $\mathcal{U}^{n+1}$ of this form, and the set of all biholomorphisms of $\mathcal{U}^{n+1}$ that arises as such form a group. This gives $\partial\mathcal{U}^{n+1}$ the structure of the Lie group, which is usually called the *Heisenberg group*[6]. In fact this is a homogeneous group, in the sense that it carries an automorphic dilation. With this structure of a homogeneous group on $\partial\mathcal{U}^{n+1}$, one can then solve $\mathcal{L}_{n-2q}$ (and hence $\Box_b$) on $\partial\mathcal{U}^{n+1}$ rather explicitly: this is because $\mathcal{L}_{n-2q}$ is a homogeneous left-invariant differential operator of degree 2, and we can solve this by convolving against a homogeneous distribution of degree $-(Q-2)$, where $Q(= 2n+2)$ is the homogeneous dimension of $\partial\mathcal{U}^{n+1}$. This is more or less what we have done in the second talk, and we omit the details.

8. Some regularity results for $\Box_b$ and $\Box$

We shall now describe some regularity results for $\Box_b$, and their consequences for the solutions to the $\overline{\partial}$-Neumann problem.

---

[6]The Euclidean analog of this is just that $\partial\mathbb{R}^{n+1}_+$ is diffeomorphic to $\mathbb{R}^n$, which is a Lie group in itself, and which acts on $\partial\mathbb{R}^{n+1}_+$ by translation in the obvious way.

Theorem 4. *Let Ω be a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Then for $q \geq 0$, there exists an operator K_b such that for all $f_b \in L^2_{(0,q)}(\partial\Omega)$,*

$$f_b = \Box_b K_b f_b + S_b f_b$$

where S_b is the orthogonal projection onto the kernel of $\Box_b$, and $K_b f_b$ is orthogonal to the kernel of $\Box_b$. Furthermore, if $Z_1, \ldots, Z_n$ is a local frame of holomorphic tangent vectors to $\partial\Omega$, we have

$$\|Q(Z, \overline{Z}) K_b f_b\|_{L^p} \lesssim \|f_b\|_{L^p}, \quad 1 < p < \infty,$$

where $Q(Z, \overline{Z})$ is any quadratic (non-commutative) polynomial in Z_1, ..., Z_n, $\overline{Z}_1$, ..., $\overline{Z}_n$.

By the previous paradigm for solving boundary value problems, one can then conclude the following:

Theorem 5. *Let Ω be a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Suppose $Z_1, \ldots, Z_n$ are linearly independent holomorphic vector fields on Ω that are tangent to $\partial\Omega$ on $\partial\Omega$, and Z_{n+1} is a holomorphic vector field on Ω that is linearly independent with $Z_1, \ldots, Z_n$. If $U \in L^2_{(0,q)}(\Omega) \cap Dom(\Box)$ for some $q \geq 1$, and $\Box U = f$ on Ω, then*

$$\|Q(Z, \overline{Z}) U\|_{L^p} + \|\overline{Z}_{n+1} U\|_{W^{1,p}} \lesssim \|f\|_{L^p}, \quad 1 < p < \infty,$$

where again $Q(Z, \overline{Z})$ is any quadratic (non-commutative) polynomial in Z_1, ..., Z_n, $\overline{Z}_1$, ..., $\overline{Z}_n$.

We omit the details. We shall, however, mention that there is the following remarkable generalization of Theorem 4 about the $\overline{\partial}_b$ complex on the boundary. To describe this, we need two definitions.

Definition 6. *Let Ω be a smooth domain in $\mathbb{C}^{n+1}$. It is said to be of finite commutator type m, if near each point on $\partial\Omega$, there is a local frame of holomorphic tangent vectors $Z_1, \ldots, Z_n$ such that the commutators of $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n$ of length $\leq m$ span the (complexified) tangent space to $\partial\Omega$ at that point.*

Note that there is just one direction in the tangent space of $\partial\Omega$ that is not spanned by $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n$. This is the direction T in our definition of the Levi matrix. If Ω is strongly pseudoconvex, then since the Levi matrix (c_{jk}) is positive definite at every point, in particular $c_{11} > 0$. Hence in this case T is in the span of commutators of $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n$ of length ≤ 2, and Ω is of finite commutator type 2.

Next we remember that the Levi form of a smooth domain is a Hermitian form at every point on the boundary. As such they can be diagonalized, and the eigenvalues are real. The eigenvalues are non-negative if the domain is pseudoconvex.

Definition 7. *Let Ω be a smooth pseudoconvex domain in $\mathbb{C}^{n+1}$. It is said to have comparable Levi eigenvalues if there is a constant $C > 0$ such that for any $z \in \partial\Omega$, and for any eigenvalues $\lambda_1(z)$, $\lambda_2(z)$ of the Levi form at z, we have $\lambda_1(z) \leq C\lambda_2(z)$.*

Again if Ω is a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^{n+1}$, then it has comparable Levi eigenvalues; also any pseudoconvex domain in $\mathbb{C}^2$ trivially have comparable Levi eigenvalues, because the Levi matrix in this case is just a 1×1 matrix.

The following theorem generalizes Theorem 4:

Theorem 6. *Let Ω be a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$. Suppose Ω is of finite commutator type, and Ω has comparable Levi eigenvalues. Then for $q \geq 0$, there exists an operator K_b such that for all $f_b \in L^2_{(0,q)}(\partial\Omega)$,*

$$f_b = \Box_b K_b f_b + S_b f_b$$

where S_b is the orthogonal projection onto the kernel of $\Box_b$, and $K_b f_b$ is orthogonal to the kernel of $\Box_b$. Furthermore, if $Z_1, \dots, Z_n$ is a local frame of holomorphic tangent vectors to $\partial\Omega$, we have

$$\|Q(Z, \overline{Z}) K_b f_b\|_{L^p} \lesssim \|f_b\|_{L^p}, \quad 1 < p < \infty,$$

where $Q(Z, \overline{Z})$ is any quadratic (non-commutative) polynomial in Z_1, $\dots$, Z_n, $\overline{Z}_1$, $\dots$, $\overline{Z}_n$.

This is a result of Christ [2] in the case $n = 2$, and a result of Koenig [6] when $n \geq 3$. From this one can also deduce a regularity theorem for the $\overline{\partial}$-Neumann problem on such domains, as was done in Koenig [7].

9. Sobolev inequalities for $(0, q)$ forms

We shall end by discussing some Sobolev inequalities for the $\overline{\partial}_b$ complex, following [9].

Theorem 7. *Let Ω be a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Then for all $(0, q)$ forms u_b on $\partial\Omega$ that are smooth up to the boundary, if $u_b \in$ Dom$(\overline{\partial}_b^*)$ and $q \neq 1$ nor $n - 1$, then*

$$\|u_b\|_{L^{\frac{Q}{Q-1}}} \lesssim \|\overline{\partial}_b u_b\|_{L^1} + \|\overline{\partial}_b^* u_b\|_{L^1}, \quad Q = 2n + 2.$$

More generally, we have:

Theorem 8. *Let Ω be a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$ of finite commutator type m, and that has comparable Levi eigenvalues. Then for all $(0, q)$ forms u_b on $\partial\Omega$ that are smooth up to the boundary, if $u_b \in$ Dom$(\overline{\partial}_b^*)$ and $q \neq 1$ nor $n - 1$, then*

$$\|u_b\|_{L^{\frac{Q}{Q-1}}} \lesssim \|\overline{\partial}_b u_b\|_{L^1} + \|\overline{\partial}_b^* u_b\|_{L^1}, \quad Q = 2n + m.$$

The proof of the latter is by means of duality, and the L^1 duality inequality we have seen last time. This is entirely analogous to the proof of the Euclidean theorem; one just need to observe that under our conditions for Ω, we can apply locally the L^1 duality inequality to the vector fields that are the real and imaginary parts of a basis of holomorphic tangent vector fields, and this is possible because they are of finite type m, and that they are linearly independent. The non-isotropic dimension would then be given by $Q = 2n + m$, because a commutator of length m is needed to span the missing direction. Finally, one can conclude the proof by duality using the estimates for the relative solution operators K_b in Theorem 6, because all the conditions of Ω in that theorem are satisfied here. The details are omitted.

This is a very rough outline of the big picture in several complex variables, and its relations to subelliptic analysis. Hopefully through these we have seen some interesting aspects of the subject, and below one can find some standard references in the subject should one wishes to pursue further.

References

[1] So-Chin Chen and Mei-Chi Shaw, *Partial differential equations in several complex variables*, AMS/IP Studies in Advanced Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2001.

[2] Michael Christ, *Regularity properties of the $\overline{\partial}_b$ equation on weakly pseudoconvex CR manifolds of dimension* 3, J. Amer. Math. Soc. **1** (1988), no. 3, 587–646.

[3] Gerald B. Folland and Joseph J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Princeton University Press, Princeton, N.J., 1972.

[4] Gerald B. Folland and Elias M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[5] Lars Hörmander, *An introduction to complex analysis in several variables*, North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990.

[6] Kenneth D. Koenig, *On maximal Sobolev and Hölder estimates for the tangential Cauchy-Riemann operator and boundary Laplacian*, Amer. J. Math. **124** (2002), no. 1, 129–197.

[7] ———, *A parametrix for the $\overline{\partial}$-Neumann problem on pseudoconvex domains of finite type*, J. Funct. Anal. **216** (2004), no. 2, 243–302.

[8] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993.

[9] Po-Lam Yung, *Sobolev inequalities for $(0,q)$ forms on CR manifolds of finite type*, to appear in Math. Res. Letts.