A SUBELLIPTIC L^1 DUALITY INEQUALITY

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In the last talk, we discussed some subelliptic real analysis associated to vector fields that satisfy Hormander's finite type condition. We established a Sobolev inequality for functions whose subelliptic gradient is in L^p, $1 \leq p < Q$, where Q is the (local) non-isotropic dimension associated to such a situation. Today we shall discuss a remedy of the failure of this embedding at the end-point when $p = Q$. This will have applications in several complex variables, as we will see next time.

1. Subelliptic Sobolev inequality

First we recall some results from last time. Let $X_1, \dots, X_n$ be some smooth real vector fields on $\mathbb{R}^N$. They are said to satisfy *Hormander's finite type condition* at the point 0 if they and their commutators of length $\leq r$ span the tangent space of $\mathbb{R}^N$ at 0 for some positive integer r. The smallest r for which this holds is called the *type* of $X_1, \dots, X_n$ at 0.

The trivial example is when there are N vector fields on $\mathbb{R}^N$, namely the coordinate vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}$. They are of finite type 1. In general n may be smaller than N, as in the following example:

Example 1.

$$\tilde{X} = \frac{\partial}{\partial x}, \quad \tilde{Y} = \frac{\partial}{\partial y} + x\frac{\partial}{\partial t} \quad \textit{on } \mathbb{R}^3.$$

These vector fields are of finite type 2 at every point.

Another example we considered is

Example 2.

$$X = \frac{\partial}{\partial x}, \quad Y = x\frac{\partial}{\partial t} \quad \textit{on } \mathbb{R}^2.$$

These vector fields are of finite type 2 at 0.

When $X_1, \dots, X_n$ satisfy Hormander's finite type condition, the sum of squares operator

$$L = \sum_{j=1}^{n} X_j^2$$

is hypoelliptic (in fact subelliptic), and we constructed a parametrix of L which allowed us to obtain sharp L^p estimates on the solutions to the equation

$$Lu = f.$$

From the size of the parametrix, we also deduced the following subelliptic Sobolev inequality for functions. The crucial notion here is that of a (local) non-isotropic dimension:

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Definition 1. *Suppose $X_1, \dots, X_n$ are smooth real vector fields that are of finite type r at 0. Take commutators of $X_1, \dots, X_n$ of length $\leq j$ and restrict them to 0; call the subspace of the tangent space at 0 that they span $V_j(0)$. Clearly $V_{j-1}(0) \subseteq V_j(0)$ for all j; we let*

$$n_1 = dim V_1(0) \quad and \quad n_j = dim V_j(0) - dim V_{j-1}(0) \quad for\ j \geq 2.$$

We then define the non-isotropic dimension Q at 0 to be

$$Q = \sum_{j=1}^{r} j n_j.$$

For example, in Example 1, $Q = 4$; in Example 2, $Q = 3$.

Theorem 1 (Subelliptic Sobolev inequality for functions)**.** *Let $X_1, \dots, X_n$ and Q be as in the previous definition. Then there exists a neighborhood Ω of 0 and $C > 0$ such that if $u \in C_c^\infty(\Omega)$ and $1 \leq p < Q$, then*

$$\|u\|_{L^{p^*}(\Omega)} \leq C \left(\|\nabla_b u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right), \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}$$

where $\nabla_b u = (X_1 u, \dots, X_n u)$. Moreover the inequality cannot hold for any bigger value of p^*.

This generalizes the classical Sobolev inequality on $\mathbb{R}^N$, and the fact that the inequality fails to hold for any bigger p^* shows that our notion of non-isotropic dimension Q is the correct one.

2. L^1-duality inequality

2.1. **Euclidean situation.** The previous subelliptic Sobolev inequality fails at the end-point $p = Q$. In the Euclidean space, this is just the well-known failure of the embedding $W^{1,N}$ into L^∞. In the first talk, we have already seen the following remedy of this failure:

Theorem 2 (L^1-duality inequality)**.** *Suppose $f = (f_1, \dots, f_N)$ is a divergence free vector field on $\mathbb{R}^N$, i.e.*

$$\sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} = 0$$

where each of the f_j are smooth and compactly supported. Then for any test function $\Phi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\left| \int_{\mathbb{R}^N} f_1 \Phi \right| \lesssim \|f\|_{L^1} \|\nabla \Phi\|_{L^N}.$$

This inequality appeared in the work of van Schaftingen [6], Lanzani-Stein [4] and Bourgain-Brezis [1], [2]. This is a very remarkable inequality. Using this estimate, we have already seen the proof of the following Sobolev inequality for differential forms on $\mathbb{R}^N$:

Corollary 1. *If u is a compactly supported smooth q form on $\mathbb{R}^N$, where $q \neq 1$ nor $N-1$, then*

$$\|u\|_{L^{\frac{N}{N-1}}} \lesssim \|du\|_{L^1} + \|d^* u\|_{L^1}.$$

One can also deduce, from the above theorem, an elliptic estimate in L^1 for the following system of equations:

Corollary 2. *Let U and F be smooth compactly supported vector fields on $\mathbb{R}^N$ such that $\Delta U = F$ (componentwise). If F is divergence free, then*

$$\|\nabla U\|_{L^{\frac{N}{N-1}}} \lesssim \|F\|_{L^1}.$$

This is remarkable because in general U is not in $W^{2,1}$; otherwise the inequality becomes trivial. The proof is just by duality: If Φ is a vector field on $\mathbb{R}^N$, then

$$(\Delta^{\frac{1}{2}} U, \Phi) = (\Delta U, \Delta^{-\frac{1}{2}} \Phi) = (F, \Delta^{-\frac{1}{2}} \Phi) \lesssim \|F\|_{L^1} \|\nabla \Delta^{-\frac{1}{2}} \Phi\|_{L^N} = \|F\|_{L^1} \|\Phi\|_{L^N}.$$

2.2. **Subelliptic case.** The goal today is to derive the following subelliptic version of L^1-duality inequality [7].

Theorem 3 (Subelliptic L^1-duality inequality)**.** *Let $X_1, \dots, X_n$ be smooth real vector fields in a neighborhood of 0 in $\mathbb{R}^N$. Suppose they are linearly independent at 0 and their commutators of length $\leq r$ span the tangent space at 0. Let Q be the non-isotropic dimension at 0. Then there exists a neighborhood U of 0 and $C > 0$ such that if*

$$X_1 f_1 + \cdots + X_n f_n = 0$$

on U with $f_1, \dots, f_n \in C_c^\infty(U)$, and if $\Phi \in C_c^\infty(U)$, then

$$\left| \int_U f_1(x) \Phi(x) dx \right| \leq C \|f\|_{L^1(U)} \left(\|\nabla_b \Phi\|_{L^Q(U)} + \|\Phi\|_{L^Q(U)} \right)$$

where $\nabla_b \Phi = (X_1 \Phi, \dots, X_n \Phi)$.

When the underlying space is a homogeneous group and when $X_1, \dots, X_n$ is a basis of left-invariant vector fields of degree 1 on the group, this theorem was proved by Chanillo and van Schaftingen [3]. This theorem, however, is more general because we only need the vector fields to satisfy Hormander's finite type condition. We shall make crucial use of their argument in the proof below.

For simplicity, we shall just look at the proof of the following model situation: Let

$$X = \frac{\partial}{\partial x}, \quad Y = x \frac{\partial}{\partial t} \quad \text{on } \mathbb{R}^2$$

as in Example 2. In this case the non-isotropic dimension Q at 0 is 3. Note that there is no structure of a Lie group on $\mathbb{R}^2$ that could make both X and Y left-invariant vector fields, because Y vanishes on only the t-axis. Hence the result of Chanillo-van Schaftingen does not apply. Nevertheless, we have:

Theorem 4. *If f_1, f_2 are smooth and compactly supported functions on $\mathbb{R}^2$ and*

$$X f_1 + Y f_2 = 0,$$

then for all $\Phi \in C_c^\infty(\mathbb{R}^2)$,

$$\left| \int_{\mathbb{R}^2} f_1 \Phi \right| \leq C \|f\|_{L^1(\mathbb{R}^2)} \|\nabla_b \Phi\|_{L^3(\mathbb{R}^2)}$$

where $\nabla_b \Phi = (X\Phi, Y\Phi)$.

Strictly speaking this is not a special case of Theorem 3, because X and Y are not linearly independent at 0; but this is where the main idea of the proof is the most transparent, and we shall only discuss the need for linear independence of the vector fields, as well as some other complications that arise in general, towards the end of the talk.

Before we describe the proof of this model theorem, it helps to remember how its Euclidean analog (i.e. Theorem 2) is proved.

Suppose $\sum_{j=1}^N \frac{\partial f_j}{\partial x_j} = 0$. To estimate $\int_{\mathbb{R}^N} f_1(x)\Phi(x)dx$, we first freeze the first coordinate x_1 and consider

$$\int_{\mathbb{R}^{N-1}} f_1(x_1,x')\Phi(x_1,x')dx'.$$

Since x_1 is now fixed, $\Phi(x_1,x')$ is just a function defined on the hyperplane over which we are integrating. For each positive parameter λ (which we shall choose in a minute depending on x_1), we shall decompose this function on the hyperplane as follows: we shall find two functions $\Phi_1^{x_1}$ and $\Phi_2^{x_1}$, defined on the hyperplane[1], such that

$$\Phi(x_1,x') = \Phi_1^{x_1} + \Phi_2^{x_1} \quad \text{on the hyperplane,}$$

i.e.

$$\Phi(x_1,x') = \Phi_1^{x_1}(x_1,x') + \Phi_2^{x_1}(x_1,x') \quad \text{for all } x' \in \mathbb{R}^{N-1},$$

and such that we control L^∞ norm of the first function and the gradient in L^∞ of the second; more precisely, we require

$$\begin{cases} \|\Phi_1^{x_1}\|_{L^\infty(dx')} & \leq C\lambda^{\frac{1}{N}}\|\nabla_{x'}\Phi\|_{L^N(dx')}(x_1) \\ \|\nabla_{x'}\Phi_2^{x_1}\|_{L^\infty(dx')} & \leq C\lambda^{\frac{1}{N}-1}\|\nabla_{x'}\Phi\|_{L^N(dx')}(x_1). \end{cases}$$

The power of λ here is dictated by homogeneity, and the decomposition can be achieved by decomposing Φ^{x_1} into the sum of its high frequency component and its low frequency component using Littlewood-Paley theory. We then need to estimate

$$\int_{\mathbb{R}^{N-1}} f_1(x_1,x')\Phi_1^{x_1}(x_1,x')dx' \quad \text{and} \quad \int_{\mathbb{R}^{N-1}} f_1(x_1,x')\Phi_2^{x_1}(x_1,x')dx'.$$

The first one is easy: we can just do

$$\|f_1\|_{L^1(dx')}(x_1)\|\Phi_1^{x_1}\|_{L^\infty(dx')}$$

and apply the estimate for $\Phi_1^{x_1}$; for the second integral, we apply fundamental theorem of calculus to f_1:

$$f_1(x_1,x') = \int_{-\infty}^{x_1} \frac{\partial f_1}{\partial x_1}(s,x')ds.$$

Plug this into the second integral, we are led to estimate

$$\int_{-\infty}^{x_1}\int_{\mathbb{R}^{N-1}} \frac{\partial f_1}{\partial x_1}(s,x')\Phi_2^{x_1}(x_1,x')dx'ds.$$

But using the divergence free condition on f, we can write $\frac{\partial f_1}{\partial x_1}$ as

$$\frac{\partial f_1}{\partial x_1} = -\sum_{j=2}^N \frac{\partial f_j}{\partial x_j}.$$

Since we are integrating over the whole space $\mathbb{R}^{N-1}$, we can then integrate by parts and let the derivatives fall on $\Phi_2^{x_1}$. All in all, this is bounded by

$$\|f\|_{L^1(dx)}\|\nabla_{x'}\Phi_2^{x_1}\|_{L^\infty(dx')}$$

[1]We are putting the superscript x_1 on the two functions in the decomposition to emphasize that the decomposition depends on x_1.

and we can again apply our defining estimate for $\nabla_{x'}\Phi_2^{x_1}$. This shows

$$\int_{\mathbb{R}^{N-1}} f_1(x_1,x')\Phi(x_1,x')dx'$$
$$\lesssim \left(\|f_1\|_{L^1(dx')}(x_1)\lambda^{\frac{1}{N}} + \|f\|_{L^1(dx)}\lambda^{\frac{1}{N}-1}\right)\|\nabla_{x'}\Phi\|_{L^N(dx')}(x_1)$$

for each fixed x_1. If we pick $\lambda = \lambda(x_1)$ such that this sum is minimized, and then integrate in x_1, applying Holder's inequality we can conclude the proof of Theorem 2.

To prove Theorem 4, we shall also freeze x and consider

$$\int_{\mathbb{R}} f_1(x,t)\Phi(x,t)dt.$$

We shall then need a decomposition lemma for Φ similar to the one above. But this time the argument cannot be as simple. The reason is that our vector fields have variable coefficients. In fact suppose for each x we have a decomposition of $\Phi(x,t) = \Phi_1^x(x,t) + \Phi_2^x(x,t)$ such that we control Φ_1^x in $L^\infty(dt)$ and $Y\Phi_2^x = x\frac{\partial\Phi_2^x}{\partial t}$ in $L^\infty(dt)$. Then if we apply fundamental theorem of calculus to f_1 to estimate

$$\int_{\mathbb{R}} f_1(x,t)\Phi_2^x(x,t)dt,$$

we get

$$f_1(x,t) = \int_{-\infty}^{x} (Xf_1)(s,t)ds$$

and thus using the divergence condition,

$$\int_{\mathbb{R}} f_1(x,t)\Phi_2^x(x,t)dt = \int_{-\infty}^{x}\int_{\mathbb{R}} -(Yf_2)(s,t)\Phi_2^x(x,t)dtds.$$

Now if we try to integrate by parts in Y, we are in trouble: because $(Yf_2)(s,t)$ is really $s\frac{\partial f_2}{\partial t}(s,t)$, and if we integrate by parts we get

$$\int_{-\infty}^{x}\int_{\mathbb{R}} f_2(s,t)s\frac{\partial\Phi_2^x}{\partial t}(x,t)dtds.$$

But we do not control $s\frac{\partial\Phi_2^x}{\partial t}(x,t)$! In fact what we control is only $(Y\Phi_2^x)(x,t)$, which is $x\frac{\partial\Phi_2^x}{\partial t}(x,t)$. Hence we need a different idea, and the correct decomposition lemma is the following:

Lemma 1 (Decomposition Lemma)**.** *Given any $\Phi \in C_c^\infty(\mathbb{R}^2)$, any $a \in \mathbb{R}$ and any $\lambda > 0$, there is a decomposition of the function Φ on the line $\{x = a\}$, given by*

$$\Phi(a,t) = \Phi_1^a(a,t) + \Phi_2^a(a,t) \quad \textit{for all } t,$$

and an extension of the second function Φ_2^a into the whole $\mathbb{R}^2$, such that

$$\begin{cases} \|\Phi_1^a\|_{L^\infty(dt)} & \leq C\lambda^{\frac{1}{3}}MI(a) \\ \|\nabla_b\Phi_2^a\|_{L^\infty(dxdt)} & \leq C\lambda^{\frac{1}{3}-1}MI(a) \end{cases},$$

where

$$I(x) = \|\nabla_b\Phi\|_{L^3(dt)}(x),$$

and M is the Hardy-Littlewood maximal function on $\mathbb{R}$.

Note that we are not only controlling $Y\Phi_2^a$ on the line $\{x = a\}$ (which would not be enough for the following argument), but also controlling the X and Y derivative of the extended Φ_2^a everywhere on $\mathbb{R}^2$. In the Euclidean situation this extension of the second function is also implicitly present. There we were just extending the function so that it is constant in the x_1 variable, thus making its derivative in x_1 direction identically zero. Such a simple extension is not sufficient in the subelliptic case; thus the proof of this lemma is quite a bit more involved than its Euclidean analogue, and it is why the maximal function enters.

Assuming this lemma for the moment, it is not hard to complete the proof of Theorem 4:

Proof of Theorem 4. Again we freeze $x = a$ and consider the integral

$$\int_{\mathbb{R}} f_1(a,t)\Phi(a,t)dt.$$

Let $\lambda > 0$ be a parameter, which we shall choose in a moment depending on a. On the line $\{x = a\}$, we have the following decomposition according to the Lemma:

$$\Phi(a,t) = \Phi_1^a(a,t) + \Phi_2^a(a,t).$$

Extend Φ_2^a to the whole $\mathbb{R}^2$ as in the Lemma. Then we need to estimate

$$\int_{\mathbb{R}} f_1(a,t)\Phi_1^a(a,t)dt + \int_{\mathbb{R}} f_1(a,t)\Phi_2^a(a,t)dt.$$

The first integral can be estimated by

$$\|f_1\|_{L^1(dt)}(a)\|\Phi_1^a\|_{L^\infty(dt)} \lesssim \|f_1\|_{L^1(dt)}(a)\lambda^{\frac{1}{3}}MI(a).$$

The second integral can be dealt with as follows: apply the fundamental theorem of calculus to the product $f_1\Phi_2^a$ (and not just f_1 itself!). Then

$$f_1(a,t)\Phi_2^a(a,t) = \int_{-\infty}^a (X(f_1\Phi_2^a))(s,t)ds.$$

Plug this into the second integral. Then that is equal to

$$\int_{-\infty}^a \int_{\mathbb{R}} (Xf_1)(s,t)\Phi_2^a(s,t) + f_1(s,t)(X\Phi_2^a)(s,t)dtds.$$

Now

$$Xf_1 = -Yf_2,$$

and this time we can integrate by parts in Y because both f_2 and Φ_2^a are evaluated at the same point (s,t). This gives

$$\int_{-\infty}^a \int_{\mathbb{R}} f_2(s,t)(Y\Phi_2^a)(s,t) + f_1(s,t)(X\Phi_2^a)(s,t)dtds.$$

Estimating using the Lemma, we can bound this by

$$\|f\|_{L^1(dxdt)}\lambda^{\frac{1}{3}-1}MI(a).$$

Together,

$$\int_{\mathbb{R}} f_1(a,t)\Phi(a,t)dt \lesssim \left(\|f_1\|_{L^1(dt)}(a)\lambda^{\frac{1}{3}} + \|f\|_{L^1(dxdt)}\lambda^{\frac{1}{3}-1}\right)MI(a)$$

for each fixed a. Picking $\lambda = \lambda(a)$ to optimize the sum, we see that

$$\left|\int_{\mathbb{R}} f_1(a,t)\Phi(a,t)dt\right| \le C\|f\|_{L^1(dxdt)}^{\frac{1}{3}}\|f_1\|_{L^1(dt)}^{\frac{2}{3}}(a)MI(a)$$

for all a. Integrating in a and applying Holder's inequality, we get the desired bound

$$\left|\int_{\mathbb{R}^2} f_1(a,t)\Phi(a,t)dadt\right| \leq C\|f\|_{L^1(dxdt)}\|MI\|_{L^3(dx)} \leq C\|f\|_{L^1(dxdt)}\|\nabla_b\Phi\|_{L^3(dxdt)}$$

because the maximal function M is bounded on $L^3(dx)$. □

To prove the decomposition lemma, we need again the idea of lifting to a homogeneous group [5]. Recall that $\mathbb{R}^3$ is a Lie group under the group law

$$(x,y,t)\cdot(u,v,w) = (x+u, y+v, t+w+xv).$$

We call this the Heisenberg group H. This is a homogeneous group in the sense that it carries an automorphic dilation:

$$\delta_\lambda(x,y,t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0.$$

The vector fields $\tilde{X} = \frac{\partial}{\partial x}$ and $\tilde{Y} = \frac{\partial}{\partial y} + x\frac{\partial}{\partial t}$ in Example 1 then form a basis of left-invariant vector fields of degree 1 on H. Now our vector fields $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$ can be lifted to $\tilde{X}$ and $\tilde{Y}$ on H via the map

$$\pi\colon H \to \mathbb{R}^2, \quad \pi(x,y,t) = (x,t),$$

in the sense that

$$d\pi(\tilde{X}) = X, \quad d\pi(\tilde{Y}) = Y.$$

The function Φ on $\mathbb{R}^2$ can also be lifted to a function $\tilde{\Phi}$ on H, by pulling back via the map π:

$$\tilde{\Phi} := \Phi \circ \pi.$$

Then clearly

$$\tilde{X}\tilde{\Phi} = \widetilde{X\Phi}, \quad \tilde{Y}\tilde{\Phi} = \widetilde{Y\Phi}.$$

The advantage of lifting, i.e. working with $\tilde{\Phi}$ on H rather than working with Φ on $\mathbb{R}^2$, is that we can take advantage of the convolution on H: for any two functions ϕ, η on H, we define their convolution by

$$(\phi * \eta)(\xi) = \int_H \phi(\xi\cdot\zeta)\eta(\zeta)d\zeta,$$

where $d\zeta = dxdydt$ is the Haar measure of H. (This is a slight change of notation from last time, but this is the convenient one for our present purpose.) The left-invariant vector fields on H works very well with this convolution: in fact

$$(\tilde{X}\phi) * \eta = -\phi * (\tilde{X}\eta), \quad (\tilde{Y}\phi) * \eta = -\phi * (\tilde{Y}\eta) \quad \text{on } H. \tag{1}$$

There is no way of defining any 'convolution' on $\mathbb{R}^2$ such that the analogue of this property is satisfied by our original vector fields X and Y on $\mathbb{R}^2$. This is the basic reason why we needed the lifting in the first place.

We need one more construction before we can prove the Decomposition Lemma. For $\lambda > 0$, let I_λ be the dilation of functions on H that preserves the L^1 norms:

$$(I_\lambda\eta)(x,y,t) = \lambda^{-4}\eta(\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t).$$

Then a simple calculation shows that

$$\frac{d}{d\lambda}I_\lambda\eta = \tilde{X}I_\lambda D_1\eta + \tilde{Y}I_\lambda D_2\eta \tag{2}$$

for some differential operators D_1 and D_2. (This is basically the subelliptic correspondence of the heuristic that any function ϕ that integrates to 0 on $\mathbb{R}^n$ can be written as a sum $\sum_j \frac{\partial}{\partial x_j}\phi_j$ for some ϕ_j, except that we are claiming that it suffices to use only the vector fields of degree 1, namely $\tilde{X}$ and $\tilde{Y}$ in our case.) In fact,

$$(I_\lambda \eta)(x,y,t) = \int_{\mathbb{R}^3} \hat{\eta}(\lambda\xi_1, \lambda\xi_2, \lambda^2\xi_3) e^{2\pi i(x\xi_1+y\xi_2+t\xi_3)} d\xi,$$

and if we differentiate with respect to λ, we get

$$\frac{d}{d\lambda} I_\lambda \eta = \lambda^{-1} I_\lambda \left(\frac{\partial}{\partial x}(x\eta) + \frac{\partial}{\partial y}(y\eta) + \frac{\partial}{\partial t}(2t\eta) \right).$$

Writing $\frac{\partial}{\partial y}$ as $\tilde{Y} - \frac{\partial}{\partial t}x$ and $\frac{\partial}{\partial t}$ as $[\tilde{X}, \tilde{Y}]$, this is equal to

$$\lambda^{-1} I_\lambda \left(\tilde{X}(x\eta) + \tilde{Y}(y\eta) + (\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X})((2t - xy)\eta) \right)$$

and if we now commute one of $\tilde{X}$ or $\tilde{Y}$ in each term outside I_λ, using that

$$\lambda^{-1} I_\lambda \tilde{X} = \tilde{X} I_\lambda, \quad \lambda^{-1} I_\lambda \tilde{Y} = \tilde{Y} I_\lambda,$$

we get that equal to $\tilde{X} I_\lambda D_1 \eta + \tilde{Y} I_\lambda D_2 \eta$ for some differential operators D_1 and D_2 as desired.

Proof of Lemma 1. By the invariance of the decomposition under the dilation

$$(x,t) \mapsto (\lambda x, \lambda^2 t),$$

without loss of generality we may assume that $\lambda = 1$. Given $\Phi \in C_c^\infty(\mathbb{R}^2)$, $a \in \mathbb{R}$, let η be any C_c^∞ bump function on H with $\int_H \eta = 1$. On the line $\{x = a\}$, we define

$$\Phi_2^a(a,t) = (\tilde{\Phi} * \eta)(a,y,t)$$

where $y \in \mathbb{R}$ is arbitrary (in fact the right side does not depend on y). We also define

$$\Phi_1^a(a,t) = \Phi(a,t) - \Phi_2^a(a,t).$$

Now extend Φ_2^a to $\mathbb{R}^2$ by setting

$$\Phi_2^a(a+s,t) = (\tilde{\Phi} * I_{\lambda_s}\eta)(a,y,t)$$

for any y, where

$$\lambda_s := \sqrt{1+s^2}.$$

We shall now derive the desired estimates on Φ_1^a and Φ_2^a.

First

$$(X\Phi_2^a)(a+s,t) = \frac{d}{ds}\Phi_2^a(a+s,t) = \frac{d}{ds}(\tilde{\Phi} * I_{\lambda_s}\eta)(a,y,t).$$

But by (2),

$$\frac{d}{ds}(\tilde{\Phi} * I_{\lambda_s}\eta) = \tilde{\Phi} * \left(\tilde{X} I_{\lambda_s}\eta_1 + \tilde{Y} I_{\lambda_s}\eta_2 \right) \frac{d\lambda_s}{ds}$$

for some bump functions η_1, η_2. The exact form of η_1 and η_2 are not important, and we shall simply abuse notation and write η for both of them. Integrating by parts using the crucial identities (1), we get

$$-\left(\tilde{X}\tilde{\Phi} + \tilde{Y}\tilde{\Phi} \right) * I_{\lambda_s}\eta \cdot \frac{d\lambda_s}{ds}.$$

Using $\tilde{X}\tilde{\Phi} = \widetilde{X\Phi}$, $\tilde{Y}\tilde{\Phi} = \widetilde{Y\Phi}$ and bounding $\frac{d\lambda_s}{ds} = \frac{s}{\sqrt{1+s^2}}$ by 1, we bound the above in absolute value by

$$(|\widetilde{X\Phi}| + |\widetilde{Y\Phi}|) * I_{\lambda_s}|\eta|.$$

Writing out the convolution, this is just

$$\int_H |\nabla_b \Phi|(a+u, t+w+av)\lambda_s^{-4}|\eta|(\lambda_s^{-1}u, \lambda_s^{-1}v, \lambda_s^{-2}w)dudvdw.$$

We estimate this in 3 steps: first apply Holder inequality in the integral in w, and bound this by

$$\int_{\mathbb{R}^2} I(a+u)\lambda_s^{-4+\frac{4}{3}} \left\|\eta\left(\lambda_s^{-1}u, \lambda_s^{-1}v, w\right)\right\|_{L^{3/2}(dw)} dudv;$$

remember

$$I(x) = \|\nabla_b \Phi\|_{L^3(dt)}(x).$$

Then the crucial step now is to observe that $I(a+u)$ does not depend on v, and thus in the integral, we can integrate v out (which is the variable we added in the lifting process), getting

$$\int_{\mathbb{R}} I(a+u)\lambda_s^{-4+\frac{4}{3}+1} \left\|\eta\left(\lambda_s^{-1}u, v, w\right)\right\|_{L^{3/2}(dw)L^1(dv)} du.$$

We can then bound this by the maximal function $\lambda_s^{-4+\frac{4}{3}+1+1}MI(a) = \lambda_s^{-\frac{2}{3}}MI(a)$, and since $\lambda_s \geq 1$, this proves the estimate for $X\Phi_2^a$.

To estimate $Y\Phi_2^a$, first observe that

$$(Y\Phi_2^a)(a+s,t) = (a+s)\frac{\partial}{\partial t}\left((\tilde{\Phi} * I_{\lambda_s}\eta)(a,y,t)\right),$$

and

$$a\frac{\partial}{\partial t}\left((\tilde{\Phi} * I_{\lambda_s}\eta)(a,y,t)\right) = \left(\left(a\frac{\partial}{\partial t}\tilde{\Phi}\right) * I_{\lambda_s}\eta\right)(a,y,t) = ((\tilde{Y}\tilde{\Phi}) * I_{\lambda_s}\eta)(a,y,t),$$

$$s\frac{\partial}{\partial t}\left((\tilde{\Phi} * I_{\lambda_s}\eta)(a,y,t)\right) = s\left(\left(\frac{\partial}{\partial t}\tilde{\Phi}\right) * I_{\lambda_s}\eta\right)(a,y,t) = s\left(([\tilde{X},\tilde{Y}]\tilde{\Phi}) * I_{\lambda_s}\eta\right)(a,y,t).$$

For the latter, we integrate by parts using our crucial identity (1), and get

$$\frac{s}{\lambda_s}\left(\tilde{X}\tilde{\Phi} * I_{\lambda_s}\eta_1 + \tilde{Y}\tilde{\Phi} * I_{\lambda_s}\eta_2\right)$$

for some bump functions η_1 and η_2; again we abuse notation and just write these as η. Bounding $\frac{s}{\lambda_s} = \frac{s}{\sqrt{1+s^2}}$ by 1, we get

$$|(Y\Phi_2^a)(a+s,t)| \leq (|\widetilde{X\Phi}| + |\widetilde{Y\Phi}|) * I_{\lambda_s}|\eta|(a,y,t).$$

We can then bound the latter as before by $MI(a)$, as desired.

Finally, to bound Φ_1^a on the line $\{x = a\}$, observe that

$$\Phi_1^a(a,t) = (\tilde{\Phi} - \tilde{\Phi} * \eta)(a,y,t) = -\int_0^1 \frac{d}{d\lambda}(\tilde{\Phi} * I_\lambda\eta)(a,y,t)d\lambda$$

because η is a bump function on H with integral 1, and $I_\lambda\eta$ converges weakly to the δ function at 0 as $\lambda \to 0$. We can carry out exactly what we did when we estimated $X\Phi_2^a$ to estimate $\frac{d}{d\lambda}(\tilde{\Phi} * I_\lambda)\eta(a,y,t)$; in fact

$$\left|\frac{d}{d\lambda}(\tilde{\Phi} * I_\lambda\eta)(a,y,t)\right| \leq C\lambda^{-\frac{2}{3}}MI(a).$$

Integrating in λ, we get the desired estimate for Φ_1^a.

$\square$

This completes the proof of our model theorem. We remark that the choice $\lambda_s = \sqrt{1+s^2}$ is just a convenient one; all we need for λ_s is that it is a smooth function of s, equals 1 at $s=0$, and grows like $|s|$ as $|s| \to \infty$.

We have chosen to estimate $Y\Phi_2^a$ in an ad hoc manner above, but there is a more conceptual way of doing it and that would carry through in general.

Several other difficulties need to be overcome if we were to prove the general Theorem 3:

(1) Suppose we are estimating $\int_{\mathbb{R}^N} f_1(x)\Phi(x)dx$, where $X_1 f_1 + \cdots + X_n f_n = 0$, and we want to freeze the x_1 variable. We would want to do so because we would want to change the X_1 derivative of f_1 into some X_j derivatives, $j \geq 2$, and integrate by parts. However, to do so, we need $X_2, \dots, X_n$ to be tangent to the hyperplanes where x_1 is constant. This is in general not possible (e.g. if the brackets of $X_2, \dots, X_n$ already span $\mathbb{R}^N$ at every point). Fortunately, when the $X_1, \dots, X_n$ are linearly independent, a perturbation argument works, and we can modify $X_2, \dots, X_n$ by a small multiple of X_1 to make them tangent to the hyperplanes where x_1 is constant. This goes back to an observation we made towards the end of the first talk, that proves if $X_1, \dots, X_N$ are linearly independent at 0 in $\mathbb{R}^N$ and $X_1 f_1 + \cdots + X_N f_N = 0$, then locally
$$\left| \int_{\mathbb{R}^N} f_1 \Phi \right| \lesssim \|f\|_{L^1} \left(\|\nabla \Phi\|_{L^N} + \|\Phi\|_{L^N} \right).$$
The above explains why we needed linear independence of the vector fields in Theorem 3; it is not known whether or not it is necessary.

Note that this problem does not arise in our simple model case, because $Y = x\frac{\partial}{\partial t}$ is tangent to the hyperplanes where x is constant, and we had no problem integrating by parts then.

(2) It is not always possible to lift vector fields satisfying Hormander's condition to left-invariant vector fields of degree 1 on a homogeneous group. What can be done, in general, is just to approximate, at every point, the lifted vector fields by left-invariant vector fields of a homogeneous group. The errors that arise in this approximation needs to be taken care of. But they are of lower degree in homogenity, and are not too difficult to control.

For our application next time, it is crucial to observe that we can allow, in the duality Theorem 3, that $X_1 f_1 + \cdots + X_n f_n$ be non-zero in L^1:

Theorem 5. *Let $X_1, \dots, X_n$ be as in Theorem 3. Let Q be the non-isotropic dimension at 0. Then there exists a neighborhood U of 0 and $C > 0$ such that if*
$$X_1 f_1 + \cdots + X_n f_n = g$$
on U with $f_1, \dots, f_n, g \in C_c^\infty(U)$, and if $\Phi \in C_c^\infty(U)$, then
$$\left| \int_U f_1(x)\Phi(x)dx \right| \leq C \left(\|f\|_{L^1} \|\Phi\|_{NL_1^Q} + \|g\|_{L^1} \|\Phi\|_{L^Q} \right)$$
where $\|\Phi\|_{NL_1^Q} := \|\nabla_b \Phi\|_{L^Q} + \|\Phi\|_{L^Q}$, and $\nabla_b \Phi = (X_1\Phi, \dots, X_n\Phi)$.

The proof of this theorem is the same as the proof of Theorem 3, except that in addition we need to control $\|\Phi_2^a\|_{L^\infty}$ in our decomposition lemma. We omit the details.

Next time we shall begin by discussing some backgrounds in several complex variables. We shall then apply this subelliptic duality inequality to prove some L^1 Sobolev inequalities for $(0, q)$ forms in that setting.

References

[1] Jean Bourgain and Haïm Brezis, *New estimates for elliptic equations and Hodge type systems*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 2, 277–315.

[2] ______, *New estimates for the Laplacian, the div-curl, and related Hodge systems*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 7, 539–543.

[3] Sagun Chanillo and Jean Van Schaftingen, *Subelliptic Bourgain-Brezis estimates on groups*, Math. Res. Lett. **16** (2009), no. 3, 487–501.

[4] Loredana Lanzani and Elias M. Stein, *A note on div curl inequalities*, Math. Res. Lett. **12** (2005), no. 1, 57–61.

[5] Linda P. Rothschild and Elias M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), no. 3-4, 247–320.

[6] Jean Van Schaftingen, *Estimates for L^1-vector fields*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 181–186.

[7] Po-Lam Yung, *Sobolev inequalities for $(0, q)$ forms on CR manifolds of finite type*, to appear in Math. Res. Letts.