THE SZEMEREDI-TROTTER THEOREM VIA POLYNOMIAL PARTITIONING

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Abstract. The following are notes taken (by Jianhui Li and Po-Lam Yung) from a mini-course given by Ruixiang Zhang, on a proof of the Szemeredi-Trotter theorem using polynomial partitioning. The technique of low-degree polynomial partitioning is also discussed. The note takers have also taken the opportunity to expand these notes slightly; in particular, they have also benefited from a blog post on the same topic by Terence Tao, and some exposition of incidence geometry by Larry Guth in his paper on restriction I.

A celebrated theorem in incidence geometry is the following theorem about incidences of points and lines in $\mathbb{R}^2$:

Theorem 1 (Szemeredi-Trotter)**.** *Let P be a finite set of points in $\mathbb{R}^2$, and L be a finite set of lines in $\mathbb{R}^2$. Let $I(P, L)$ be the set of incidences of P and L, i.e. $I(P, L) = \{(p, \ell) \in P \times L \colon p \in \ell\}$. Then*

$$|I(P, L)| \leq C(|P|^{2/3}|L|^{2/3} + |P| + |L|)$$

for some universal constant C.

To prove this, we begin by observing a trivial bound:

Lemma 2. *Let P and L be finite sets of points and lines in $\mathbb{R}^2$ respectively. Then*

$$|I(P, L)| \leq |P|^2 + |L| \tag{1}$$

and

$$|I(P, L)| \leq |L|^2 + |P|. \tag{2}$$

Proof. Let $L = L_1 \cup L_2$, where L_1 is the set of all lines in L that passes through at most one point in P, and L_2 is the set of all lines in L that passes through at least two points in P. Then

$$|I(P, L_1)| = \sum_{\ell \in L_1} |\ell \cap P| \leq |L_1| \leq |L|,$$

while

$$|I(P, L_2)| \leq |P|^2$$

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since one can define an injective map $I(P, L_2) \to P \times P$, so that an incidence $(p, \ell) \in I(P, L_2)$ is mapped into a pair $(p, p') \in P \times P$ with $p' \in \ell \backslash \{p\}$. (The injectivity of the map comes from the fact that two points in $\mathbb{R}^2$ determine a unique straight line through them.) Together, we see that

$$|I(P, L)| = |I(P, L_2)| + |I(P, L_1)| \leq |P|^2 + |L|.$$

Similarly, by writing $P = P_1 \cup P_2$, where P_1 is the set of all points in P that lie on at most one line in L, and P_2 is the set of all points in P that lie on at least two lines in L, we see that

$$|I(P, L)| = |I(P_1, L)| + |I(P_2, L)| \leq |L|^2 + |P|.$$

□

Next we note that a divide and conquer argument can yield a slightly better bound:

Proposition 3. *Let P and L be finite sets of points and lines in $\mathbb{R}^2$ respectively. Then*

$$|I(P, L)| \lesssim |P||L|^{1/2} + |L| \tag{3}$$

and

$$|I(P, L)| \lesssim |L||P|^{1/2} + |P|. \tag{4}$$

Proof. Indeed, let k be some positive integer to be determined, and partition $P = P_1 \cup \cdots \cup P_k$ so that each P_i has roughly $|P|/k$ points. Then

$$|I(P, L)| = \sum_{i=1}^{k} |I(P_i, L)|,$$

so applying the trivial bound (1) to each term on the right hand side, and using that $|P_i| \lesssim |P|/k$, we get

$$|I(P, L)| \lesssim k \left(\frac{|P|}{k} \right)^2 + k|L| = \frac{|P|^2}{k} + k|L|.$$

This is true for all positive integers k, so if $|P|^2 \geq |L|$, then we can take k to be roughly $|P|/|L|^{1/2}$. This gives

$$|I(P, L)| \lesssim |P||L|^{1/2} \qquad \text{if } |P|^2 \geq |L|;$$

but if $|P|^2 \leq |L|$, then (1) gives already that $|I(P, L)| \leq 2|L|$. Thus in either case (3) holds. Similarly one can prove (4) using (2), by partitioning the given collection of lines. □

The above divide-and-conquer is quite rough; we were basically partitioning the given points (or lines) randomly. One can substantially improve the efficiency of the divide-and-conquer process if we use *polynomial partitioning*, which we introduce below.

First we have to recall the Borsuk-Ulam theorem from algebraic topology:

Theorem 4 (Borsuk-Ulam). *Let $m \in \mathbb{N}$, and $F\colon \mathbb{S}^m \to \mathbb{R}^m$ be a continuous map. Then there exists $Q \in \mathbb{S}^m$ such that $F(Q) = F(-Q)$.*

We use this to prove the polynomial ham sandwich theorem:

Theorem 5 (Polynomial Ham Sandwich Theorem). *For any N open sets in $U_1, \dots, U_N$ of finite volume in $\mathbb{R}^d$, there exists a non-zero polynomial $Q \in \mathbb{R}[x_1, \dots, x_d]$, of degree $\lesssim N^{1/d}$, such that*

$$|U_j \cap Q_+| = |U_j \cap Q_-|$$

for all $1 \leq j \leq N$, where $Q_+ = \{x \in \mathbb{R}^d \colon Q(x) > 0\}$ and $Q_- = \{x \in \mathbb{R}^d \colon Q(x) < 0\}$.

Proof. Let D be the smallest positive integer for which $\binom{D+d}{d} > N$. Then $D \lesssim N^{1/d}$. The vector space $V := \{p \in \mathbb{R}[x_1, \dots, x_d] \colon \deg(p) \leq D\}$ has dimension $\binom{D+d}{d}$, and can hence be identified with $\mathbb{R}^{m+1}$ where $m := \binom{D+d}{d} - 1 \geq N$. A point $Q \in \mathbb{S}^m$ can then be identified with a polynomial $Q \in V$: this allows us to define a map $F\colon \mathbb{S}^m \to \mathbb{R}^m$, so that the j-th coordinate of $F(Q)$ is given by

$$|U_j \cap Q_+| - |U_j \cap Q_-|$$

for $j = 1, \dots, N$, and 0 for $j = N+1, \dots, m$. This map is continuous, and the Borsuk-Ulam theorem guarantees the existence of $Q \in \mathbb{S}^m$ such that $F(Q) = F(-Q)$. But this map is also odd by construction, so we conclude $F(Q) = 0$. We have thus a non-zero polynomial Q for which $|U_j \cap Q_+| = |U_j \cap Q_-|$ for all $1 \leq j \leq N$. □

We deduce the following corollary of the polynomial ham sandwich theorem:

Corollary 6. *Suppose $S_1, \dots, S_N$ are finite collection of points in $\mathbb{R}^d$, there exists a non-zero polynomial $Q \in \mathbb{R}[x_1, \dots, x_d]$, of degree $\leq A_d N^{1/d}$, such that for any $j = 1, \dots, N$, the sets $S_j \cap Q_+$ and $S_j \cap Q_-$ each contains at most $|S_j|/2$ points. Here A_d is a constant depending only on d.*

Proof. For each $\epsilon > 0$, let $U_1, \dots, U_N$ be an ϵ neighborhood of $S_1, \dots, S_N$. Let Q^ϵ be a non-zero polynomial of degree $D \lesssim N^{1/d}$, such that $|U_j \cap Q^\varepsilon_+| = |U_j \cap Q^\varepsilon_-|$ for all $1 \leq j \leq N$. We may assume that all the Q^ϵ are in $\mathbb{S}^m$, the unit sphere in the space of polynomials of degree $\leq D$. We may then find a sequence $\epsilon_k \to 0$, such that $Q^{\epsilon_k}(x)$ converges to a non-zero polynomial $Q(x)$ locally uniformly. If for some $j = 1, \dots, N$, the set $S_j \cap Q_+$ contains more than $|S_j|/2$ points, then the same would be true for Q^{ϵ_k} for all sufficiently large k, and this contradicts our choice of Q^ϵ. Similarly for $S_j \cap Q_-$. □

This in turn leads to the important cell decomposition theorem:

Theorem 7. *Given any N points in $\mathbb{R}^d$, and any $D > 1$, there exists a non-zero polynomial $Q \in \mathbb{R}[x_1, \dots, x_d]$, of degree $\leq C_d D$, such that $\mathbb{R}^d \setminus Z(Q)$ is the union of $< 2D^d$ open sets* *(called cells), and each cell contains $\leq N/D^d$ of the given points. Here C_d is a constant depending only on d.*

Proof. We claim there is a (large enough) constant B_d, such that given any N points in $\mathbb{R}^d$, and any $n \in \mathbb{N}$, there exists a non-zero polynomial $Q \in \mathbb{R}[x_1, \ldots, x_d]$, of degree $\leq B_d 2^{n/d}$, such that $\mathbb{R}^d \setminus Z(Q)$ can be written as the union of 2^n open sets, each of which contains $\leq N/2^n$ given points. If this is true, then given $D > 1$, we will pick $n \in \mathbb{N}$ such that $2^{n-1} < D^d \leq 2^n$ for some $n \in \mathbb{N}$, and apply this statement with this n. Then since $B_d 2^{n/d} = 2^{1/d} B_d 2^{(n-1)/d} < 2^{1/d} B_d D^d$, $2^n < 2D^d$ and $N/2^n \leq N/D^d$, we obtain our desired conclusion with $C_d = 2^{1/d} B_d$.

Let's establish the claim by induction on n. When $n = 1$ this follows directly from Corollary 6: we just use Corollary 6 to bisect 1 collection of points. If Q_n is the polynomial one obtains from the induction hypothesis for a certain $n \in \mathbb{N}$, and $S_1, \ldots, S_{2^n}$ are the given points in the 2^n open sets that make up $\mathbb{R}^d \setminus Z(Q_n)$, then to prove the claim for $n+1$, one just use Corollary 6 to bisect $S_1, \ldots, S_{2^n}$ by an additional bisecting polynomial, of degree $\leq A_d 2^{n/d}$, and multiply the bisecting polynomial to Q_n. This gives a polynomial Q_{n+1}, of degree at most

$$B_d 2^{n/d} + A_d 2^{n/d} = \frac{B_d + A_d}{2^{1/d}} 2^{(n+1)/d},$$

such that $\mathbb{R}^d \setminus Z(Q_{n+1})$ can be written as 2^{n+1} open sets, each of which contains $\leq N/2^{n+1}$ given points. One can ensure that $\frac{B_d + A_d}{2^{1/d}} \leq B_d$ by choosing B_d sufficiently large at the outset (since A_d is just a fixed dimensional constant coming from Corollary 6). This concludes our induction, and hence the proof of the theorem. □

We remark that the cells in the above theorem may not be connected. But a theorem of Oleinik-Petrovskii, Milnor, and Thom states the following:

Theorem 8 (Oleinik-Petrovsky, Milnor, Thom)**.** *Let $V \subset \mathbb{R}^d$ be an algebraic subset defined by equations of degrees $\leq D$. Then the number of connected components of V is at most $D(2D-1)^{d-1}$ (which in particular is $< 2^{d-1} D^d$).*

This implies that if $Z(Q)$ denotes the zero set of Q, then $\mathbb{R}^d \setminus Z(Q)$ has at most $\lesssim_d (\deg Q)^d$ connected components. As a result, by replacing $2D^d$ in the Theorem 7 by $C_d D^d$ for some dimensional constant C_d, we may assume that the cells there are all connected (this will not be necessary for our purposes below).

We would like to draw now a naive comparison between the cell decomposition Theorem 7 with the following simple fact from high school algebra

Fact. *If E is a finite subset of $\mathbb{R}$ that contains at most D points, then there exists a non-zero polynomial $P \in \mathbb{R}[x]$ of degree $\leq D$, that vanishes at every point of E.*

The cell decomposition Theorem 7 is a somewhat fancier theorem with a similar flavour: we are asking, in Theorem 7, not just the existence of a non-zero low-degree polynomial that does something to our given point set, but the existence of both some partitioning of our point set, and the existence of a non-zero low-degree polynomial that gives that particular partition. We note that the simple algebraic fact above (or its contrapositive) is also the key to Dvir's resolution of the finite field Kakeya conjecture.

We are now ready to prove the Szemeredi-Trotter theorem.

Proof of the Szemeredi-Trotter theorem. The strategy is to divide and conquer using the cell decomposition theorem. Let P and L be finite sets of points and lines in $\mathbb{R}^2$. We want to show the existence of some universal constant C, such that

$$|I(P,L)| \leq C(|P|^{2/3}|L|^{2/3} + |P| + |L|). \tag{5}$$

One can certainly find such C from Lemma 2 if $|P|^2 \leq |L|$ or $|L|^2 \leq (4C_2)^{3/2}|P|$, where C_2 is as in Theorem 7. We will show that by enlarging the constant C once if necessary, we can also have (5) when

$$|P|^2 \geq |L| \quad \text{and} \quad |L|^2 \geq (4C_2)^{3/2}|P|. \tag{6}$$

Assume now (6) holds. Let D be the least integer $> |P|^{2/3}|L|^{-1/3}$. Then since $|P|^2 \geq |L|$, we have $D > 1$, and hence $D \leq 2|P|^{2/3}|L|^{-1/3}$. We apply the cell decomposition Theorem 7 with this D to the point set P, to obtain a non-zero polynomial Q of degree $\leq C_2 D$, such that each cell that makes up $\mathbb{R}^2 \setminus Z(Q)$ contains $\leq |P|/D^2$ points from P. We partition the lines L into a disjoint union of L_a and L_c, where

$$L_a := \{\ell \in L \colon \ell \subset Z(Q)\}, \quad \text{and} \quad L_c := L \setminus L_a,$$

and partition the points P into a disjoint union

$$P = (P \cap Z(Q)) \cup (P \setminus Z(Q)).$$

This allows us to partition the incidences $I(P,L)$ into 3 contributions:

$$I(P,L) = I(P,L_a) \cup I(P \cap Z(Q), L_c) \cup I(P \setminus Z(Q), L_c).$$

We count them one by one.

Let's start with $I(P \setminus Z(Q), L_c)$. Let $\{O_i\}$ be a listing of all the cells that make up $\mathbb{R}^2 \setminus Z(Q)$. Then

$$|I(P \setminus Z(Q), L_c)| = \sum_i |I(P \cap O_i, L_c)| = \sum_i |I(P \cap O_i, L_{c,i})|,$$

where $L_{c,i} = \{\ell \in L_c \colon \ell \cap O_i \neq \emptyset\}$. By the trivial bound (1), we see that the latter is bounded by

$$\sum_i \left(|P \cap O_i|^2 + |L_{c,i}|\right). \tag{7}$$

Now $|P \cap O_i| \leq |P|/D^2 \leq |P|/(|P|^{2/3}|L|^{-1/3})^2 = |P|^{-1/3}|L|^{2/3}$, so

$$\sum_i |P \cap O_i|^2 \leq |P|^{-1/3}|L|^{2/3} \sum_i |P \cap O_i| = |P|^{2/3}|L|^{2/3};$$

also, each line in L_c can only intersect at most $\deg(Q) + 1$ of the cells $\{O_i\}$. Hence

$$\sum_i |L_{c,i}| \leq |L_c|(\deg(Q) + 1),$$

which is $\leq 2|L| \deg(Q) \leq 2C_2 D|L| \leq 4C_2|P|^{2/3}|L|^{2/3}$. Altogether this shows

$$|I(P \setminus Z(Q), L_c)| \leq (4C_2 + 1)|P|^{2/3}|L|^{2/3}. \tag{8}$$

(Incidentally, we note that the competition between the two terms in (7) is what dictates the choice of D in the application of the cell decomposition theorem.)

Next we count $I(P \cap Z(Q), L_c)$. Since each line $\ell \in L_c$ intersects $Z(Q)$ at at most $\deg(Q)$ points, we have

$$|I(P \cap Z(Q), L_c)| \leq |L_c| \deg(Q) \leq |L| \deg(Q),$$

which is $\leq C_2 D|L| \leq 2C_2|P|^{2/3}|L|^{2/3}$ by our bound on $\deg(Q)$ and on D. This gives

$$|I(P \cap Z(Q), L_c)| \leq 2C_2|P|^{2/3}|L|^{2/3}. \tag{9}$$

Finally we count $I(P, L_a)$. We proceed by induction on $|L|$: assume we have already the desired bound (5) for all families of points and lines that contain fewer lines than $|L|$. Note that Q cannot contain more than $\deg Q$ linear factors. Thus

$$|L_a| \leq \deg(Q) \leq C_2 D \leq 2C_2|P|^{2/3}|L|^{-1/3}.$$

By our assumption that $|L|^2 \geq (4C_2)^{3/2}|P|$, we have then

$$|L_a| \leq \frac{|L|}{2}.$$

So $|I(P, L_a)|$ can be estimated by the induction hypothesis, obtaining

$$|I(P, L_a)| \leq C(|P|^{2/3}|L_a|^{2/3} + |P| + |L_a|),$$

which gives

$$|I(P, L_a)| \leq C2^{-2/3}|P|^{2/3}|L|^{2/3} + C|P| + C|L|. \tag{10}$$

From (8), (9) and (10), we get

$$|I(P, L)| \leq (6C_2 + 1 + C2^{-2/3})|P|^{2/3}|L|^{2/3} + C(|P| + |L|).$$

If C were chosen large enough so that $6C_2 + 1 + C2^{-2/3} \leq C$, then we can close the induction, and conclude that (5) is true for our family P and L as well. This completes our proof. $\square$

We note that (8), (9) and (10) correspond to contributions from the cells, transversal contributions from the walls of the cells, and tangential contributions from the walls of the cells respectively. A similar trichotomy is present in the work of Guth on Fourier restriction in 3 dimensions.

The Szemeredi-Trotter theorem is a theorem about incidences of points and lines in $\mathbb{R}^2$. One can ask the same question in $\mathbb{R}^n$ where the dimension $d > 2$. The above proof of the Szemeredi-Trotter theorem uses polynomial partitioning with a polynomial of degree $\sim |P|^{2/3}|L|^{-1/3}$, which is medium sized if say $|P| \sim |L|$. If one tries to directly adapt the above proof to the higher dimensional case, one runs into difficulty dealing with the algebraic part of the incidences. (Varieties in higher dimensions are more complicated; in particular, it is no longer possible to bound the number of lines in the zero set of a polynomial by its degree any more. e.g. The (ruled) surface $z = xy$ in $\mathbb{R}^3$ contains infinitely many lines; this surface is also sometimes called the *regulus*.) It was an ingenious observation of Solomosi and Tao, that by carrying out the polynomial partitioning with a polynomial of lower degree, one can extend Szemeredi-Trotter theorem to higher dimensions, at the cost of a loss in power of ε. More precisely, they proved, among other things, the following theorem:

Theorem 9 (Cheap Szemeredi-Trotter theorem in $\mathbb{R}^d$). *Suppose $d \geq 2$. Let P be a finite set of points in $\mathbb{R}^d$, and L be finite number of lines in $\mathbb{R}^d$. Let $I(P, L)$ be the number of incidences of P and L. Then for any $\varepsilon > 0$, there exists a constant $C_{d,\varepsilon}$, depending only on d and ϵ, such that*

$$|I(P, L)| \leq C_{d,\varepsilon}|P|^{2/3+\epsilon}|L|^{2/3} + |P| + |L|. \tag{11}$$

Proof. Suppose $d \geq 2$, and $\varepsilon > 0$ are fixed. We proceed by induction on the number of points $|P|$. First, the inequality (1) also holds true in $\mathbb{R}^d$ with the same proof. So when $|P|$ is small, say $|P| \leq 1000$, we have the desired estimate (11) as long as $C_{d,\varepsilon}$ is larger than a fixed constant ($C_{d,\varepsilon} \geq 1000^{4/3} = 10000$ will do, since in this case $|P|^2 \leq 1000^{4/3}|P|^{2/3} \leq 1000^{4/3}|P|^{2/3}|L|^{2/3}$). We thus assume from now on that $C_{d,\varepsilon} \geq 10000$. Also, if $|P|^2 \leq |L|$, then $|P|^2 \leq |P|^{2/3}|L|^{2/3}$, so (11) follows from (1); similarly, if $|L|^2 \leq |P|$, then $|L|^2 \leq |P|^{2/3}L^{2/3}$, so (11) follows from (2). This shows that we may consider only the case

$$|P|^{1/2} \leq |L| \leq |P|^2.$$

Note that in this case

$$|P| \leq |P|^{2/3}|L|^{2/3} \quad \text{and} \quad |L| \leq |P|^{2/3}|L|^{2/3}. \tag{12}$$

Let $D > 1$ to be determined. Apply Theorem 7 with this D, we obtain a polynomial Q of degree $\leq C_d D$, such that each cell that makes up $\mathbb{R}^d \setminus Z(Q)$ contains $\leq |P|/D^d$ points in P. (D will depend only on d and ε; hence we call this low degree polynomial partitioning.) Similar to $\mathbb{R}^2$ case, we partition the lines L into a disjoint union of L_a and L_c, where

$$L_a := \{\ell \in L \colon \ell \subset Z(Q)\}, \quad \text{and} \quad L_c := L \setminus L_a,$$

and partition the points P into a disjoint union

$$P = (P \cap Z(Q)) \cup (P \setminus Z(Q)).$$

This allows us to partition the incidences $I(P, L)$ into 3 contributions:

$$I(P, L) = I(P, L_a) \cup I(P \cap Z(Q), L_c) \cup I(P \setminus Z(Q), L_c).$$

We count them one by one.

In the current set-up, the incidences in $I(P, L_a)$ are easier to count. Note that Q cannot have more that D linear factors. Thus

$$|L_a| \leq \deg(Q) \leq C_d D$$

So we can bound $|I(P, L_a)|$ trivially by

$$|I(P, L_a)| \leq C_d D|P|. \tag{13}$$

Next we count $I(P \cap Z(Q), L_c)$. Since each line $\ell \in L_c$ intersects $Z(Q)$ at at most $\deg(Q)$ points, we have

$$|I(P \cap Z(Q), L_c)| \leq |L_c| \deg(Q) \leq |L| \deg(Q) \leq C_d D|L|. \tag{14}$$

Finally we count $I(P \setminus Z(Q), L_c)$. We use our induction hypothesis: assume we have already the desired bound (11) for all families of points and lines that contain fewer points than $|P|$. Let $\{O_i\}$ be a listing of all the cells that make up $\mathbb{R}^d \setminus Z(Q)$. Let $L_{c,i} = \{\ell \in L_c \colon \ell \cap O_i \neq \emptyset\}$. We have

$$\begin{aligned}
|I(P \setminus Z(Q), L_c)| &= \sum_i |I(P \cap O_i, L_c)| \\
&= \sum_i |I(P \cap O_i, L_{c,i})| \\
&\leq \sum_i (C_{d,\varepsilon} |P \cap O_i|^{2/3+\epsilon} |L_{c,i}|^{2/3} + |P \cap O_i| + |L_{c,i}|). \qquad (15)
\end{aligned}$$

using the induction hypothesis (since $|P \cap O_i| \leq |P|/D^d$, which is $< |P|$ for all i). Now we consider the last two sums in (15): we have

$$\sum_i |P \cap O_i| \leq |P|,$$

and

$$\sum_i |L_{c,i}| \leq (\deg(Q) + 1)|L| \leq (C_d D + 1)|L| \leq 2C_d D|L|. \tag{16}$$

To consider the first sum in (15), note that by Theorem 7, the number of cells is $|\{O_i\}| \leq 2D^d$. Using this, and using again the estimate $|P \cap O_i| \leq |P|/D^d$, we get

$$\begin{aligned}\sum_i C_{d,\varepsilon}|P\cap O_i|^{2/3+\varepsilon}|L_{c,i}|^{2/3} &\leq \sum_i C_{d,\varepsilon}\left(\frac{|P|}{D^d}\right)^{2/3+\varepsilon}|L_{c,i}|^{2/3}\\ &\leq C_{d,\varepsilon}\left(\frac{|P|}{D^d}\right)^{2/3+\varepsilon}\left(\sum_i |L_{c,i}|\right)^{2/3}|\{O_i\}|^{1/3}\\ &\leq C_{d,\varepsilon}\left(\frac{|P|}{D^d}\right)^{2/3+\varepsilon}(2C_dD|L|)^{2/3}(2D^d)^{1/3}\\ &\leq 2C_{d,\varepsilon}C_d^{2/3}D^{-\varepsilon}|P|^{2/3+\varepsilon}|L|^{2/3}\end{aligned}$$

where we used (16) in the second-to-last line, and we used that $d \geq 2$ in the last line. Altogether, this shows that

$$|I(P\setminus Z(Q), L_c)| \leq 2C_{d,\varepsilon}C_d^{2/3}D^{-\varepsilon}|P|^{2/3+\varepsilon}|L|^{2/3}. \tag{17}$$

From (13), (14) and (17), we obtain

$$|I(P,L)| \leq 2C_{d,\varepsilon}C_d^{2/3}D^{-\varepsilon}|P|^{2/3+\varepsilon}|L|^{2/3} + (C_dD+1)|P| + 3C_dD|L|. \tag{18}$$

We now choose $D = D_{d,\varepsilon}$ so large, such that

$$2C_d^{2/3}D^{-\varepsilon} \leq \frac{1}{2}.$$

Then we choose $C_{d,\varepsilon}$ so large, so that

$$3C_dD \leq \frac{C_{d,\varepsilon}}{4}.$$

(18) together with our condition (12) then show that

$$|I(P,L)| \leq \frac{C_{d,\varepsilon}}{2}|P|^{2/3+\varepsilon}|L|^{2/3} + \frac{C_{d,\varepsilon}}{2}|P|^{2/3}|L|^{2/3} \leq C_{d,\varepsilon}|P|^{2/3+\varepsilon}|L|^{2/3},$$

which allows one to close the induction. □

We note that by losing this ε power of $|P|$ on the right hand side, we restrict ourselves to handling low-degree polynomials, for which the zero sets (and hence the treatment of $|I(P, L_a)|$) are simpler. This is particularly convenient in higher dimensions, where the algebraic geometry of zero sets gets complicated. This technique is called *low-degree polynomial partitioning*.