From cutting pancakes to Szemeredi-Trotter (and Ham Sandwiches too)

Po-Lam Yung

March 14, 2017

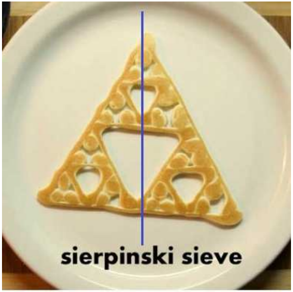
- How would you cut the following pancake into two equal halves using a knife (a straight line)?

  (Throughout the talk, 'two equal halves' mean 'two parts that have equal areas'.)

"Sierpinski Sieve," a pancake by Nathan Shields (www.10minutemath.com)

Photo credit: Nathan Shields (from AMS Mathematical Imagery)

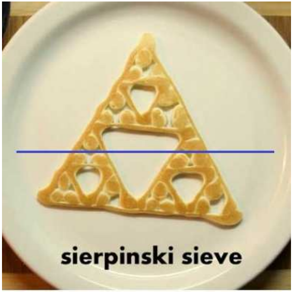
- How would you cut the following pancake into two equal halves using a knife (a straight line)?
- Maybe like this:



"Sierpinski Sieve," a pancake by Nathan Shields (www.10minutemath.com)

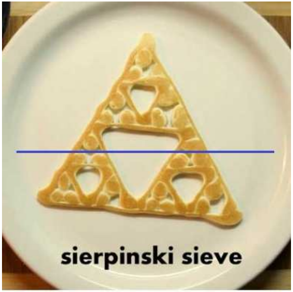
Any other ways?

- What if I insist that you cut it horizontally?



"Sierpinski Sieve," a pancake by Nathan Shields (www.10minutemath.com)

- What if I insist that you cut it horizontally?
- Yes, we use the intermediate value theorem!



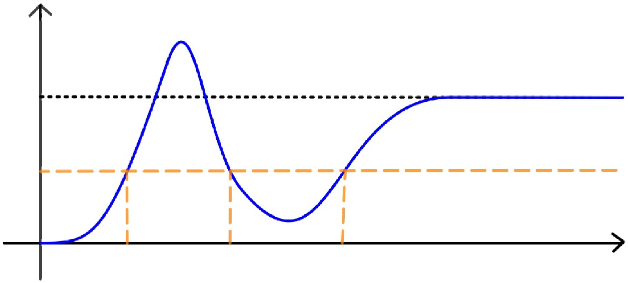
"Sierpinski Sieve," a pancake by Nathan Shields (www.10minutemath.com)

- The intermediate value theorem guarantees the following:

  Theorem. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and if there exists $a, b \in \mathbb{R}$ such that

$$\begin{cases} f(x) = 0 & \text{for all } x \leq a \\ f(x) = 1 & \text{for all } x \geq b, \end{cases}$$

  then there exists $c \in (a, b)$ such that $f(c) = 1/2$.



- To use this theorem to cut the pancake horizontally, imagine a horizontal straight line moving from bottom up.
- Let $f(x)$ be the percentage of the pancake below the straight line when the straight line is at 'height' x.

"Sierpinski Sieve," a pancake by Nathan Shields (www.10minutemath.com)

- To use this theorem to cut the pancake horizontally, imagine a horizontal straight line moving from bottom up.
- Let $f(x)$ be the percentage of the pancake below the straight line when the straight line is at 'height' x.
- Then f is continuous, and there exists $a, b \in \mathbb{R}$ such that

$$\begin{cases} f(x) = 0 & \text{for all } x \leq a \\ f(x) = 1 & \text{for all } x \geq b. \end{cases}$$

- So the intermediate value theorem guarantees the existence of a 'height' c where the straight line divides the pancake into two equal halves.

- Instead of holding the knife horizontally, we may also insist that we hold the knife at a fixed angle. One can cut the pancake into two equal halves no matter what the angle is.

"Sierpinski Sieve," a pancake by Nathan Shields (www.10minutemath.com)

- Instead of holding the knife horizontally, we may also insist that we hold the knife at a fixed angle. One can cut the pancake into two equal halves no matter what the angle is.
- For our pancake, at each fixed angle, there is actually a unique way of cutting the pancake into two equal halves. This follows from the following variant of the earlier result:

Theorem. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, and that there exists $a, b \in \mathbb{R}$ such that

$$\begin{cases} f(x) = 0 & \text{for all } x \leq a \\ f(x) = 1 & \text{for all } x \geq b. \end{cases}$$

If f is *strictly increasing* on $[a, b]$, then there exists a *unique* $c \in (a, b)$ such that $f(c) = 1/2$.

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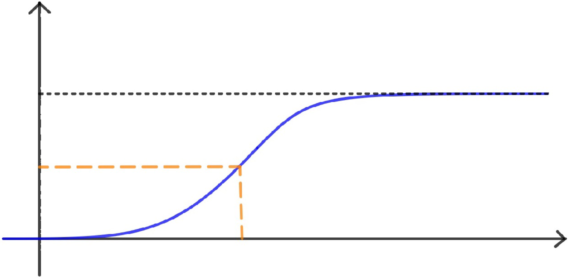
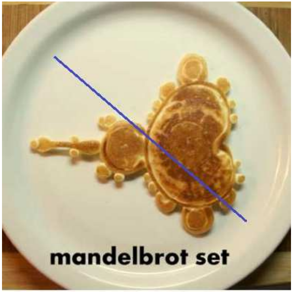


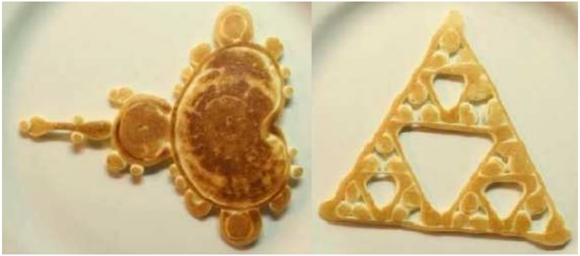
We may now cut any bounded open connect pancakes:

Theorem. For any bounded open connected set Ω in $\mathbb{R}^2$, and any angle $\alpha \in [0, \pi]$, there exists a unique straight line that makes an angle α with the horizontal axis, and that cuts Ω into two subsets of equal areas.



"Mandelbrot Set," a pancake by Nathan Shields (www.10minutemath.com)

- What if we have two pancakes (possibly of different shapes)?
- Can you *simultaneously* cut each of them into two equal halves using a knife (a straight line)?



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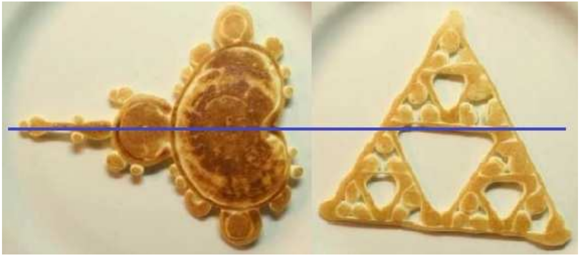

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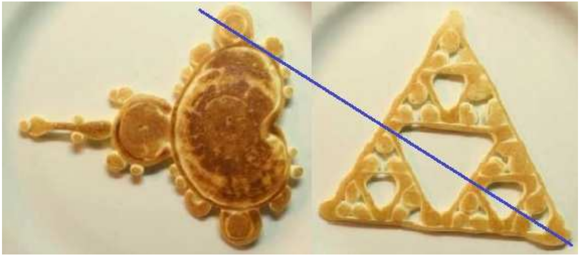
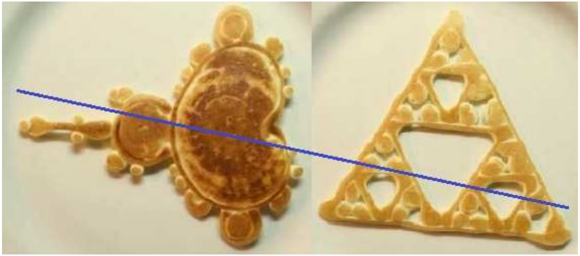


- Turns out that it is always possible to bisect two pancakes simultaneously using a straight line.

  Theorem. For any bounded open connected sets Ω_1 and Ω_2 in $\mathbb{R}^2$, there exists a straight line that cuts both Ω_1 and Ω_2 into two subsets of equal areas.



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Proof. For each angle $\alpha \in [0, \pi]$, find the unique straight line L_α that makes an angle α with the horizontal axis, and that bisects the first set Ω_1.

Consistently choose the positive and negative sides of the line L_α, and let $g(\alpha)$ be the percentage of Ω_2 on the positive side of L_α, minus the percentage of Ω_2 on the negative side of L_α.

Then g is a continuous function on $[0, \pi]$, and $g(0) = -g(\pi)$, so the intermediate value theorem again guarantees the existence of some $\alpha_0 \in [0, \pi]$, for which $g(\alpha_0) = 0$.

Then L_{α_0} bisects both Ω_1 and Ω_2!

- Theorem. For any bounded open connected sets Ω_1 and Ω_2 in $\mathbb{R}^2$, there exists a straight line that cuts both Ω_1 and Ω_2 into two subsets of equal areas.
- One can also drop the assumption that the sets are connected.
- But it certainly does not work for three pancakes in $\mathbb{R}^2$!

- There is a generalization to higher dimensions, using algebraic topology: it's called the ham sandwich theorem.

  Theorem. For any bounded open sets $\Omega_1, \Omega_2, \dots, \Omega_d$ in $\mathbb{R}^d$, there exists a (flat) hyperplane that cuts all of them into two subsets of equal areas.

  (And there is no guarantee that one can simultaneously bisect $d + 1$ sets in $\mathbb{R}^d$ using only a hyperplane!)

- Note that a ham sandwich typically consists of a ham and two slices of bread in $\mathbb{R}^3$. The theorem guarantees that one can always simultaneously bisect the ham and the two slices of bread by a flat knife in one cut!
- The proof of the theorem involves algebraic topology, which we will omit. But we can give some heuristics why this result is plausible, by counting dimensions.

- We know the system

$$\begin{cases} 3x + 4y = 6 \\ 7x - 2y = 8 \end{cases}$$

has a unique solution, but the system

$$\begin{cases} 3x + 4y = 6 \\ 7x - 2y = 8 \\ x + y = 9 \end{cases}$$

has no solution.

- This is because the latter system has too many equations: it is 3 equations in 2 unknowns, and

$$3 > 2.$$

- Generally speaking, we expect m equations in n unknowns to be solvable, only when $m \leq n$.
- Back to the ham sandwich theorem: we asserted that one can bisect any d bounded open sets in $\mathbb{R}^d$ by a hyperplane.
- This is plausible, because a hyperplane in $\mathbb{R}^d$ is of the form

$$a_1 x_1 + \cdots + a_d x_d = b,$$

and hence the set of all hyperplanes in $\mathbb{R}^d$ is d-dimensional.
- In other words, to determine a hyperplane in $\mathbb{R}^d$ is to determine d unknowns.
- To determine d unknowns, we can put at most d conditions on the unknowns. Requiring the hyperplane to bisect d different sets is exactly d conditions, so maybe this is doable. (It is indeed doable using the Borsuk-Ulam theorem in algebraic topology.)

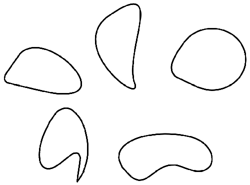
- It also seems plausible now that one cannot bisect $d+1$ sets in $\mathbb{R}^d$ using a single hyperplane: that would require solving for d unknowns under $d+1$ constraints!
- What if we really want to simultaneously bisect N sets in $\mathbb{R}^d$, where $N > d$?
- Now that we want to put N constraints on the unknowns, we had better have at least N variables. We can do so if we are not only looking at hyperplanes, but algebraic hypersurfaces of higher degree!

- For example in $\mathbb{R}^2$: a quadratic hypersurface in $\mathbb{R}^2$ is just a quadratic curve, of the form

$$ax^2 + bxy + cy^2 + dx + ey = f.$$

So the space of quadratic hypersurfaces in $\mathbb{R}^2$ is 5-dimensional.

- It turns out that given any 5 bounded open sets in $\mathbb{R}^2$, there exists a quadratic hypersurface that cuts all of them into two subsets of equal areas.



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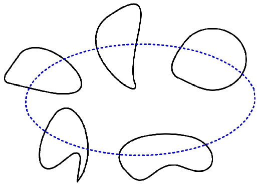

- More generally, we have the following polynomial ham sandwich theorem:

  Theorem. Given any N bounded open sets in $\mathbb{R}^d$, there exists a polynomial $Q(x)$ of degree $\lesssim N^{1/d}$ on $\mathbb{R}^d$, such that the zero set $Z(Q)$ of Q cuts all N bounded open sets into two subsets of equal areas.

- From this one can deduce a corollary:

  Corollary. Given any N collection of points $S_1, \ldots, S_N$ in $\mathbb{R}^d$, there exists a real polynomial $Q(x)$ of degree $\lesssim N^{1/d}$ on $\mathbb{R}^d$, not identically zero, such that for all $1 \le j \le N$, the sets

$$\{x \in \mathbb{R}^d \colon Q(x) > 0\} \quad \text{and} \quad \{x \in \mathbb{R}^d \colon Q(x) < 0\}$$

  each contains at most $|S_j|/2$ points from S_j.

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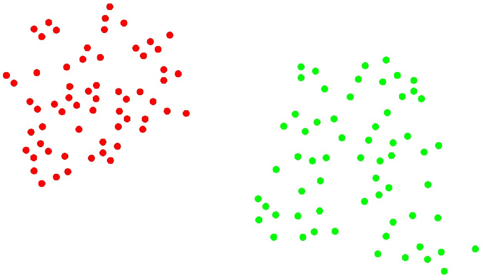
- Note that some of the points in S_j may lie on $Z(Q)$.



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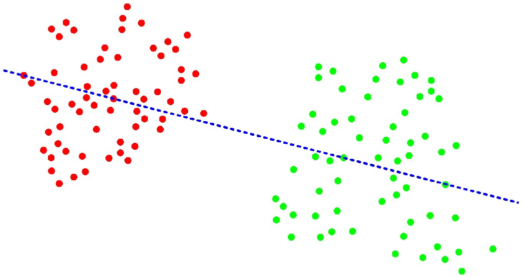

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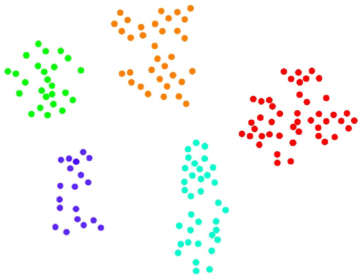

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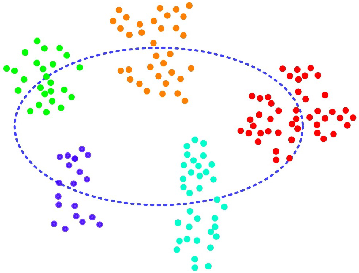

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each contains at most $|S_j|/2$ points from S_j.
- If we just have one collection of points in $\mathbb{R}^d$, but we want to divide it up evenly into 2^n subcollections, then we use the above theorem repeatedly:

  Polynomial partitioning Theorem. Given any N points in $\mathbb{R}^d$, and any positive integer n, there exists a real polynomial $Q(x)$ of degree $\lesssim 2^{n/d}$, not identically zero, such that $\mathbb{R}^d \setminus Z(Q)$ can be written as the union of 2^n open sets, each of which contains at most $N/2^n$ of the given points.

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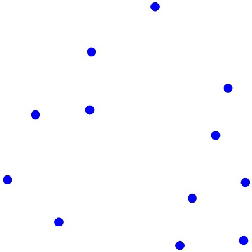

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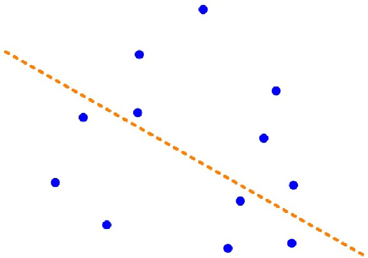

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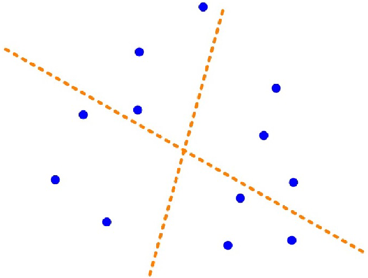

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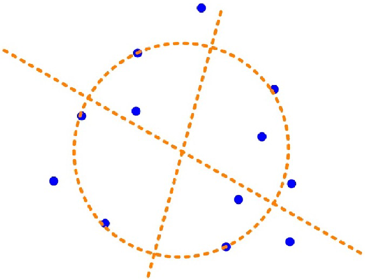
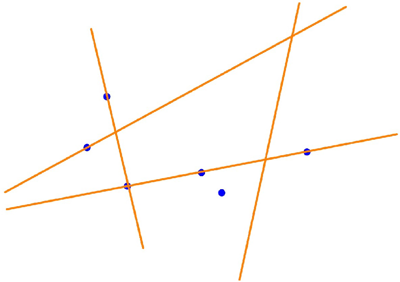


- We will use this to give a heuristic answer to the following question in combinatorial geometry:

  Given a finite set of points P and a finite set of lines L in $\mathbb{R}^2$, what is the maximum number of incidences between P and L?

- An incidence is a pair (p, ℓ) from $P \times L$ such that $p \in \ell$. We denote the set of all incidences between P and L by $I(P, L)$.



Trivial bound 1:

$$|I(P, L)| \leq |P|^2 + |L|.$$

Proof. Given P and L, let L_1 be those lines in L that pass through at most one point from P, and L_2 be the rest of lines in L.

Then $|I(P, L)| = |I(P, L_1)| + |I(P, L_2)|$.

But $|I(P, L_1)| \leq |L_1| \leq |L|$ by definition of L_1.

Also we claim $|I(P, L_2)| \leq |P|^2$: indeed for each incidence $(p, \ell) \in I(P, L_2)$, there exists some $p' \in P$, not equal to p, such that p' also lies in ℓ.

One can thus construct a map $(p, \ell) \in I(P, L_2) \mapsto (p, p') \in P \times P$, which is injective (since if (p, ℓ) is mapped to (p, p'), then ℓ must be the unique straight line passing through p and p'.

Thus $|I(P, L_2)| \leq |P|^2$. Together $|I(P, L)| \leq |P|^2 + |L|$.

Trivial bound 2:

$$|I(P, L)| \leq |L|^2 + |P|.$$

Proof. Given P and L, let P_1 be those points in P that lie on at most one line from L, and P_2 be the rest of points in P.

Then $|I(P, L)| = |I(P_1, L)| + |I(P_2, L)|$.

But using a similar argument as before, $|I(P_1, L)| \leq |P|$, and $|I(P_2, L)| \leq |L|^2$. (The key is that two intersecting lines intersect at at most one point.)

Together $|I(P, L)| \leq |L|^2 + |P|$.

- Trivial bounds:

$$|I(P,L)| \leq |P|^2 + |L| \quad \text{and} \quad |I(P,L)| \leq |L|^2 + |P|.$$

- So when $|P| \simeq |L| \simeq N$ for some large number N, then

$$|I(P,L)| \lesssim N^2.$$

- But this is far from best possible: it turns out we have the following celebrated result:

  Szemeredi-Trotter Theorem.

$$|I(P,L)| \lesssim |P|^{2/3}|L|^{2/3} + |P| + |L|.$$

- Szemeredi-Trotter Theorem.

$$|I(P,L)| \lesssim |P|^{2/3}|L|^{2/3} + |P| + |L|.$$

- So when $|P| \simeq |L| \simeq N$ for some large number N, then

$$|I(P,L)| \lesssim N^{4/3}.$$

- This is best possible, since if
  - P is the grid of all integer points in $[1, N] \times [1, 2N^2]$, and
  - L is the set of all lines of slopes $1, 2, \ldots, N$ passing through $(1, j)$ for $1 \leq j \leq N^2$,

  then there are $2N^3$ points, N^3 lines, and each of the N^3 lines in L passes through N points from P, so

$$|I(P,L)| \simeq N^4 \simeq |P|^{2/3}|L|^{2/3}.$$

- Instead of giving a full proof of the Szemeredi-Trotter theorem, we give some heuristics for its validity.
- In particular, we focus on why we could possibly have the exponent $2/3$.
- Let n be a positive integer to be determined later.
- Let P be a set of points P in $\mathbb{R}^2$.
- We apply our earlier polynomial partitioning theorem, to obtain a real polynomial $Q(x)$ of degree $\lesssim 2^{n/2}$ on $\mathbb{R}^2$, not identically zero, such that $\mathbb{R}^2 \setminus Z(Q)$ can be written as the union of 2^n open sets $\{O_i\}$, each of which contains at most $|P|/2^n$ points from P.
- Here's a slight lie: let's assume, for simplicity, that none of the lines in L are contained entirely in $Z(Q)$.

- Then a point in P is either in $Z(Q)$ or one of the open sets O_i's, so

$$|I(P,L)| = |I(P \cap Z(Q), L)| + \sum_i |I(P \cap O_i, L)|.$$

- Each line in L intersects $Z(Q)$ at at most $\deg(Q) \lesssim 2^{n/2}$ points, so
$$|I(P \cap Z(Q), L)| \lesssim 2^{n/2}|L|.$$
- For each i, let L_i be the set of all lines in L that passes through the open set O_i. Then $|I(P \cap O_i, L)| = |I(P \cap O_i, L_i)|$.
- The trivial bound gives

$$|I(P \cap O_i, L_i)| \leq |P \cap O_i|^2 + |L_i|,$$

so
$$|I(P,L)| \lesssim 2^{n/2}|L| + \sum_i (|P \cap O_i|^2 + |L_i|).$$

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- Now $|P \cap O_i| \leq |P|/2^n$ by definition of O_i, so

$$\sum_i |P \cap O_i|^2 \leq \frac{|P|}{2^n} \sum_i |P \cap O_i| \leq \frac{|P|^2}{2^n}.$$

- Also a line in L can pass through at most $\deg(Q) + 1 \lesssim 2^{n/2}$ open sets O_i's, so a double counting argument gives

$$\sum_i |L_i| = \sum_i \sum_{\substack{\ell \in L \\ \ell \cap O_i \neq \emptyset}} 1 = \sum_{\ell \in L} \sum_{i:\ \ell \cap O_i \neq \emptyset} 1 \lesssim \sum_{\ell \in L} 2^{n/2} = 2^{n/2}|L|.$$

- Altogether

$$|I(P,L)| \lesssim 2^{n/2}|L| + \frac{|P|^2}{2^n}.$$

- So we have
$$|I(P,L)| \lesssim 2^{n/2}|L| + \frac{|P|^2}{2^n}$$
for any positive integer n.
- If we could choose n so that
$$2^n \simeq \frac{|P|^{4/3}}{|L|^{2/3}},$$
we would have
$$|I(P,L)| \lesssim |P|^{2/3}|L|^{2/3}.$$
Thus the power $2/3$ in the Szemeredi-Trotter theorem sounds reasonable.