The Kohn Laplacian on blow-ups of pseudohermitian CR manifolds of dimension 3

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A toy problem

- $(\mathbb{S}^3, \hat{\theta})$: standard round sphere $\{|\zeta| = 1\}$ in $\mathbb{C}^2$,

$$\hat{\theta} := i(\bar{\partial} - \partial)|\zeta|^2$$

  compact strongly pseudoconvex pseudohermitian CR manifold.
- $(\mathbb{H}^1, \theta)$: Heisenberg group $\simeq \mathbb{C} \times \mathbb{R}$,

$$\theta := dt + i(zd\overline{z} - \overline{z}dz).$$

  non-compact.
- The two structures are 'conformally equivalent'
- Write $\hat{\Box}_b$ for the Kohn Laplacian on functions on $\mathbb{S}^3$, and $\Box_b$ for the Kohn Laplacian on $\mathbb{H}^1$. We know very well how to solve $\hat{\Box}_b$ since $\mathbb{S}^3$ is compact.
- Question: Is there a way to solve $\Box_b$ on $\mathbb{H}^1$, using the conformal equivalence of $\mathbb{H}^1$ with $\mathbb{S}^3$?

Set-up

- $\hat{M}$: a compact strongly pseudoconvex CR manifold of dimension 3; e.g. $\hat{M} = \mathbb{S}^3 \subset \mathbb{C}^2$.
- $\hat{\theta}$: a real contact 1-form on $\hat{M}$ such that

$$\text{kernel}(\hat{\theta}) = T^{1,0} \oplus T^{0,1}.$$

- $\hat{\theta}$ defines the Levi metric on $\hat{M}$:

$$\langle Z, W\rangle_{\hat{\theta}} := 2id\hat{\theta}(Z, \bar{W})$$

  for all $Z, W \in T^{1,0}$;
- Hence one defines the Carnot-Caratheodory distance $\hat{\rho}(\cdot,\cdot)$, the Webster scalar curvature $\hat{R}$, etc. Also the dual metric on the space of $(0,1)$ forms.
- $(\hat{M}, \hat{\theta})$ is called a pseudohermitian CR manifold.

- Take $\hat{\theta} \wedge d\hat{\theta}$ as the standard volume form on $\hat{M}$.
- Define L^p spaces of functions:

$$\|f\|^p_{L^p(\hat{M})} = \int_{\hat{M}} |f|^p \hat{\theta} \wedge d\hat{\theta}$$

and L^p spaces of $(0,1)$ forms:

$$\|\alpha\|^p_{L^p_{(0,1)}(\hat{M})} = \int_{\hat{M}} |\alpha|^p_{\hat{\theta}} \, \hat{\theta} \wedge d\hat{\theta}.$$

- Define a closed linear operator $\hat{\overline{\partial}}_b\colon L^2(\hat{M}) \to L^2_{(0,1)}(\hat{M})$:

  We say $u \in \mathrm{Dom}(\hat{\overline{\partial}}_b)$, if and only if there exists $u_n \in C^\infty(\hat{M})$ such that $u_n \to u$ in $L^2(\hat{M})$, and $\hat{\overline{\partial}}_b u_n$ converges to some α in $L^2_{(0,1)}(\hat{M})$. In that case we define $\hat{\overline{\partial}}_b u = \alpha$.
- We assume that

$$\hat{\overline{\partial}}_b\colon L^2(\hat{M}) \to L^2_{(0,1)}(\hat{M}) \text{ has closed range.}$$

- Analysis on $(\hat{M}, \hat{\theta})$ is then well-understood; for example, one can solve

$$\hat{\Box}_b u = (I - \hat{S})f,$$

  where $\hat{\Box}_b = \hat{\overline{\partial}}_b^{\,*}\hat{\overline{\partial}}_b$, and $\hat{S}$ is Szego projection on $(\hat{M}, \hat{\theta})$.
- We now turn to a blow-up of $\hat{M}$.

The blow-up

- Fix $p \in \hat{M}$, let $M := \hat{M} \setminus \{p\}$.
- Let $\hat{\rho}(\cdot) = \hat{\rho}(\cdot, p)$, and $\rho(\cdot) = \dfrac{1}{\hat{\rho}(\cdot)}$.
- Let G be a strictly positive smooth function on M such that

$$G(\cdot) \simeq |\hat{\rho}(\cdot)|^{-2}.$$

- We assume the existence of a CR function h on M such that $G \simeq |h|$ on M.
- Let $\theta = G^2 \hat{\theta}$. Then (M, θ) is a non-compact strongly pseudoconvex pseudohermitian CR manifold, with its own Levi metric $\langle \cdot, \cdot \rangle_\theta$ and volume form $\theta \wedge d\theta$.
- Motivated by considerations related to a positive mass theorem in 3-dim CR geometry (Cheng-Malchiodi-Yang), we want to understand analysis on (M, θ).

- e.g. $\hat{M} = \mathbb{S}^3 \subset \mathbb{C}^2$, $\hat{\theta} = i(\overline{\partial} - \partial)|\zeta|^2$, $p = (0, -1)$, $G =$ Green's function of conformal Laplacian on $\hat{M}$ with pole p, then $G = |h|$ with

$$h(\zeta_1, \zeta_2) = \frac{1}{1 + \zeta_2}.$$

Then (M, θ) is isometric to the Heisenberg group $(\mathbb{H}^1, \theta_0)$, where $\theta_0 = dt + i(zd\overline{z} - \overline{z}dz)$; in fact the map

$$\zeta \in \mathbb{S}^3 \setminus \{p\} \mapsto (z, t) \in \mathbb{H}^1$$

$$z = \frac{\zeta_1}{1 + \zeta_2}, \qquad t = -\mathrm{Re}\frac{1 - \zeta_2}{1 + \zeta_2}$$

is an isometry between (M, θ) and $(\mathbb{H}^1, \theta_0)$.

- Identifying M with $\mathbb{H}^1$, we have $\rho(z, t) \simeq (|z|^4 + |t|^2)^{1/4}$.
- We want to introduce and solve $\Box_b$ on (M, θ).

- Extend $\overline{\partial}_b$ so that it becomes a closed linear operator

$$\overline{\partial}_b \colon L^2(M) \to L^{4/3}_{(0,1)}(M);$$

in other words, $u \in \mathrm{Dom}(\overline{\partial}_b)$, if and only if there exists $u_n \in C_c^\infty(M)$ such that $u_n \to u$ in $L^2(M)$, and $\overline{\partial}_b u_n$ converges to some α in $L^{4/3}_{(0,1)}(M)$. In that case we define $\overline{\partial}_b u = \alpha$.
- The kernel of this operator is then a closed subspace of $L^2(M)$. Let

$$S \colon L^2(M) \to L^2(M)$$

be orthogonal projection onto this subspace.

- Similarly, extend the formal adjoint of $\overline{\partial}_b$ with respect to the metric θ so that it becomes a closed linear operator

$$\overline{\partial}_b^*\colon L^2_{(0,1)}(M) \to L^{4/3}(M),$$

and define orthogonal projection

$$S_1\colon L^2_{(0,1)}(M) \to L^2_{(0,1)}(M)$$

onto the kernel of this extended $\overline{\partial}_b^*$.
- Define, for $u \in C^\infty(M)$, that

$$\Box_b u := \overline{\partial}_b^* \overline{\partial}_b u.$$

Theorem

Assume in addition that $G = |h|$ *for some CR function* h *on* M*. If* f *is a smooth function on* M *that satisfies*

$$|f(x)| \lesssim \rho(x)^{-3} \quad \textit{and} \quad Sf = 0,$$

then there exists a smooth function u *on* M *such that*

$$\Box_b u = f \quad \textit{and} \quad |u(x)| \lesssim \rho(x)^{-1}.$$

- Remark: In joint work with Hsiao, we hope to prove a version of this theorem where this extra condition $G = |h|$ is removed (i.e. where one only assumes $G \simeq |h|$.)

Two approaches

- Direct one: Reduce the solution of $\Box_b$ to the solution of $\hat{\Box}_b$;
- More robust approach: solve $\Box_b u = f$ by first solving

$$\overline{\partial}_b^* v = f,$$

then solving

$$\overline{\partial}_b u = v.$$

The solution of the latter two are in turn reduced to the solutions of $\hat{\overline{\partial}}_b^{\,*}$ and $\hat{\overline{\partial}}_b$; only the solution of the second equation needs $G = |h|$.

  - If one could extend S_1 so that it becomes a bounded operator on $L^p_{(0,1)}(M)$ for some $p \in (1,2)$, show that

$$|S_1 v(x)|_\theta \lesssim \rho(x)^{-2} \quad \text{whenever} \quad |v(x)|_\theta \lesssim \rho(x)^{-2},$$

and show that S_1 is pseudolocal, then one can get rid of the extra assumption $G = |h|$ using the more robust approach.

Outline of the talk

- Some L^2 theory on $(\hat{M}, \hat{\theta})$
- Some L^p theory on $(\hat{M}, \hat{\theta})$
- Some L^p theory on (M, θ)
- Advantages of $G = |h|$
- Conclusion of proof of theorem
  - Will assume only $G \simeq |h|$ until we need $G = |h|$, and we will state carefully when we need $G = |h|$.

L^2 theory on $(\hat{M}, \hat{\theta})$

- Consider the closed linear operators

$$\hat{\overline{\partial}}_b \colon L^2(\hat{M}) \to L^2_{(0,1)}(\hat{M}), \quad \hat{\overline{\partial}}_b^{\,*} \colon L^2_{(0,1)}(\hat{M}) \to L^2(\hat{M})$$

- There exists bounded linear operators

$$\hat{K}_0 \colon L^2(\hat{M}) \to L^2_{(0,1)}(\hat{M}), \quad \hat{K}_1 \colon L^2_{(0,1)}(\hat{M}) \to L^2(\hat{M})$$

  such that

$$\hat{\overline{\partial}}_b \hat{K}_1 = Id - \hat{S}_1, \quad \text{and} \quad \hat{\overline{\partial}}_b^{\,*} \hat{K}_0 = Id - \hat{S}$$

  on $L^2_{(0,1)}(\hat{M})$ and $L^2(\hat{M})$ respectively, where

$$\hat{S} \colon L^2(\hat{M}) \to L^2_{(0,1)}(\hat{M}), \quad \hat{S}_1 \colon L^2_{(0,1)}(\hat{M}) \to L^2_{(0,1)}(\hat{M})$$

  are the Szego projections on functions and (0,1) forms respectively.

L^p theory on $(\hat{M}, \hat{\theta})$

- For $1 < p < \infty$, $\hat{S}$ extends boundedly to $L^p(\hat{M})$, and $\hat{S}_1$ extends continuously to $L^p_{(0,1)}(\hat{M})$.
- For $1 < p < 4$, let p^* be the Sobolev exponent

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{4}.$$

Then $\hat{K}_0$ extends continuously to an operator

$$\hat{K}_0 \colon L^p(\hat{M}) \to L^{p^*}_{(0,1)}(\hat{M}),$$

and

$$\hat{K}_1 \colon L^p_{(0,1)}(\hat{M}) \to L^{p^*}(\hat{M}).$$

- Now consider closed linear extensions

$$\overline{\hat{\partial}}_b : L^{p^*}(\hat{M}) \to L^p_{(0,1)}(\hat{M})$$

$$\overline{\hat{\partial}}_b^{\,*} : L^{p^*}_{(0,1)}(\hat{M}) \to L^p(\hat{M})$$

Then the identities

$$\overline{\hat{\partial}}_b \hat{K}_1 = Id - \hat{S}_1,$$

$$\overline{\hat{\partial}}_b^{\,*} \hat{K}_0 = Id - \hat{S}$$

continue to hold on $L^p_{(0,1)}(\hat{M})$ and $L^p(\hat{M})$ respectively, $1 < p < 4$.

- It follows that if $\hat{N} := \hat{K}_1 \hat{K}_0$, then

$$\hat{\Box}_b \hat{N} = Id - \hat{S} \quad \text{on } L^p(\hat{M}), \quad 1 < p < 4,$$

where the kernel of $\hat{N}$ satisfies

$$|\hat{N}(x,y)| \lesssim \hat{\rho}(x,y)^{-2}.$$

The bounds on $\hat{N}$ allows one to solve $\hat{\Box}_b$ with estimates.

- In other words, to solve $\hat{\Box}_b u = f$, one would need to make sure first that $f \in L^p$ for some $1 < p < 4$, and that

$$\hat{S} f = 0.$$

- Remark: If $f \in L^2(\hat{M})$, then the last condition means that f is orthogonal to CR functions. But if $f \in L^p(\hat{M})$ for some $p < 2$, then such an interpretation is not available, and one must prove $\hat{S} f = 0$ by other means.

Lemma

If $F \in L^{q^{**}}(\hat{M})$ *for some* $q \in (1, 4/3)$*, and*

$$\hat{S}F = 0,$$

then $\bar{h}F \in L^p(\hat{M})$ *for all* $p \in (1, q)$*, and*

$$\hat{S}(\bar{h}F) = 0.$$

Proof.

$$\begin{aligned}
\hat{S}F = 0 &\Rightarrow F = \hat{\bar{\partial}}_b^{\,*} v \quad \text{for some } v \in L^{q^{***}} \\
&\Rightarrow \bar{h}F = \hat{\bar{\partial}}_b^{\,*}(\bar{h}v) \\
&\Rightarrow \hat{S}(\bar{h}F) = \hat{S}\hat{\bar{\partial}}_b^{\,*}(\bar{h}v) = 0.
\end{aligned}$$

□

Similarly,

Lemma

If $\alpha \in L^{q^{**}}_{(0,1)}(\hat{M})$ *for some* $q \in (1, 4/3)$, *and*

$$\hat{S}_1 \alpha = 0,$$

then $h\alpha \in L^{p}_{(0,1)}(\hat{M})$ *for all* $p \in (1, q)$, *and*

$$\hat{S}_1 (h\alpha) = 0.$$

- We remark that we have already considered two different $\hat{\overline{\partial}}_b$'s on $L^2(\hat{M})$, namely

$$\hat{\overline{\partial}}_b \colon L^2(\hat{M}) \to L^2_{(0,1)}(\hat{M}) \quad \text{and} \quad \hat{\overline{\partial}}_b \colon L^2(\hat{M}) \to L^{4/3}_{(0,1)}(\hat{M}).$$

  Their kernels are the same closed subspace of $L^2(\hat{M})$, so there is no ambiguity in defining the Szego projection $\hat{S}$ on $L^2(\hat{M})$. Similarly for $\hat{S}_1$ on $L^2_{(0,1)}(\hat{M})$.

L^p theory on (M, θ)

- Now we turn to the blown-up manifold, namely (M, θ). Recall $M = \hat{M} \setminus \{p\}$, $\theta = G^2 \hat{\theta}$,

$$G(x) \simeq \hat{\rho}(x, p)^{-2} \simeq |h(x)|$$

for some CR function h on M, and later we will assume $G = |h|$.

- We have closed linear operators

$$\overline{\partial}_b \colon L^2(M) \to L^{4/3}_{(0,1)}(M), \quad \text{and} \quad \overline{\partial}_b^* \colon L^2_{(0,1)}(M) \to L^{4/3}(M).$$

We relate them to the corresponding operators on $\hat{M}$: formally we have $\overline{\partial}_b = \hat{\overline{\partial}}_b$, and $\overline{\partial}_b^* = G^{-4} \hat{\overline{\partial}}_b^{\,*} (G^2 \cdot)$.

Proposition

The following are equivalent:

(a) u *is in the domain of* $\overline{\partial}_b : L^2(M) \to L^{4/3}_{(0,1)}(M)$, *and* $\overline{\partial}_b u = \alpha$;

(b) $h^2 u$ *is in the domain of* $\hat{\overline{\partial}}_b : L^2(\hat{M}) \to L^{4/3}_{(0,1)}(\hat{M})$, *and* $\hat{\overline{\partial}}_b(h^2 u) = h^2 \alpha$.

Proposition

The following are equivalent:

(a) v *is in the domain of* $\overline{\partial}_b^* : L^2_{(0,1)}(M) \to L^{4/3}(M)$, *and* $\overline{\partial}_b^* v = f$;

(b) $\overline{h}^{-1} G^2 v$ *is in the domain of* $\hat{\overline{\partial}}_b^{\,*} : L^2_{(0,1)}(\hat{M}) \to L^{4/3}(\hat{M})$, *and* $\hat{\overline{\partial}}_b^{\,*}(\overline{h}^{-1} G^2 v) = \overline{h}^{-1} G^4 f$.

Corollary

The following are equivalent:

(a) u *is in the kernel of* $\overline{\partial}_b \colon L^2(M) \to L^{4/3}_{(0,1)}(M)$

(b) $h^2 u$ *is in the kernel of* $\hat{\overline{\partial}}_b \colon L^2(\hat{M}) \to L^{4/3}_{(0,1)}(\hat{M})$.

Corollary

The following are equivalent:

(a) v *is in the kernel of* $\overline{\partial}_b^* \colon L^2_{(0,1)}(M) \to L^{4/3}(M)$;

(b) $\overline{h}^{-1} G^2 v$ *is in the kernel of* $\hat{\overline{\partial}}_b^{\,*} \colon L^2_{(0,1)}(\hat{M}) \to L^{4/3}(\hat{M})$.

Corollary

The following are equivalent:

(a) $\overline{\partial}_b \colon L^2(M) \to L^{4/3}_{(0,1)}(M)$ *has closed range;*

(b) $\hat{\overline{\partial}}_b \colon L^2(\hat{M}) \to L^{4/3}_{(0,1)}(\hat{M})$ *has closed range.*

Corollary

The following are equivalent:

(a) $\overline{\partial}_b^* \colon L^2_{(0,1)}(M) \to L^{4/3}(M)$ *has closed range;*

(b) $\hat{\overline{\partial}}_b^{\ *} \colon L^2_{(0,1)}(\hat{M}) \to L^{4/3}(\hat{M})$ *has closed range.*

Advantages of $G = |h|$, Part I

Proposition

If $G = |h|$, then formally we have

$$\Box_b u = \bar{h}^{-1} h^{-2} \hat{\Box}_b (hu)$$

Proof.

In fact, $\bar{\partial}_b^* v = \bar{h} G^{-4} \hat{\bar{\partial}}_b^{\,*} (\bar{h}^{-1} G^2 v) = \bar{h}^{-1} h^{-2} \hat{\bar{\partial}}_b^{\,*} (hv)$, so

$$\begin{aligned}\Box_b u = \bar{\partial}_b^* \bar{\partial}_b u &= \bar{h}^{-1} h^{-2} \hat{\bar{\partial}}_b^{\,*} (h \bar{\partial}_b u) \\ &= \bar{h}^{-1} h^{-2} \hat{\bar{\partial}}_b^{\,*} \hat{\bar{\partial}}_b (hu) = \bar{h}^{-1} h^{-2} \hat{\Box}_b (hu).\end{aligned}$$

□

Conclusion of proof

- Remember we wanted to solve $\Box_b u = f$, when $|f(x)| \lesssim \rho(x)^{-3}$ and $Sf = 0$. We saw it suffices to solve
$$\hat{\Box}_b(hu) = \bar{h}h^2 f.$$
- Now $|\bar{h}h^2 f|(x) \lesssim \hat{\rho}(x,p)^{-3}$, so in particular is not in $L^2(\hat{M})$. But using previous lemma about $\hat{S}$, we have (using $Sf = 0$ and modulo some details)
$$\hat{S}(\bar{h}h^2 f) = 0.$$
- Hence there exists $\hat{u}$ such that $\hat{\Box}_b \hat{u} = \bar{h}h^2 f$; in fact from the size of $\bar{h}h^2 f$, we can choose $\hat{u}$ such that $|\hat{u}(x)| \lesssim \hat{\rho}(x,p)^{-1}$. Let now $u = h^{-1}\hat{u}$. Then
$$\Box_b u = f, \quad \text{and} \quad |u(x)| \lesssim |h(x)|^{-1}\hat{\rho}(x,p)^{-1} = \rho(x)^{-1}.$$

Advantage of $G = |h|$, Part II

Lemma

When

$$G = |h|,$$

we have

$$Sf = h^{-2}\hat{S}(h^2 f) \quad \text{for all } f \in L^2(M),$$

and

$$S_1 v = h^{-1}\hat{S}_1(hv) \quad \text{for all } v \in L^2_{(0,1)}(M).$$

- c.f. also Hirachi (1993)

The more robust approach, without $G = |h|$

- If one can show, without $G = |h|$, that
  - S_1 extends so that it becomes a bounded operator
    $$S_1 \colon L^p_{(0,1)}(M) \to L^p_{(0,1)}(M) \quad \text{for some } p \in (1,2);$$
  - S_1 is pseudolocal; and
  - $$|S_1 v(x)|_\theta \lesssim \rho(x)^{-2} \quad \text{whenever} \quad |v(x)|_\theta \lesssim \rho(x)^{-2},$$

  then one can get rid of the extra assumption $G = |h|$ in the theorem by first solving
  $$\overline{\partial}_b^* v = f,$$
  then solving
  $$\overline{\partial}_b u = (Id - S_1)v.$$