Introduction to Fourier decoupling

Po-Lam Yung

Australian National University

June 12, 2023

Motivations

- Fourier decoupling is a useful tool for many purposes.
- Helps count solutions to Diophantine systems such as

$$\begin{cases} x_1 + \cdots + x_s &= x_{s+1} + \cdots + x_{2s} \\ x_1^2 + \cdots + x_s^2 &= x_{s+1}^2 + \cdots + x_{2s}^2 \\ & \vdots \\ x_1^k + \cdots + x_s^k &= x_{s+1}^k + \cdots + x_{2s}^k \end{cases}$$

  with all variables $x_i \in \{1, \ldots, N\}$.
- More generally, estimates moments of exponential sums:

$$\int_{[0,1]^k} \Big| \sum_{n=1}^{N} a_n e^{2\pi i \gamma(n) \cdot x} \Big|^p dx, \quad \gamma(n) := (n, n^2, \ldots, n^k).$$

- Also estimates L^p norm of solutions to the periodic Schrödinger equation on the torus $\mathbb{R}^d / \mathbb{Z}^d$.

Motivations (continued)

- Spacetime estimates for solutions to the wave equation in $\mathbb{R}^d$:

$$\partial_t^2 u = \Delta_x u, \quad u(x,0) = f(x), \quad \partial_t u(x,0) = 0.$$

  What is the minimal regularity s so that

$$\Big(\int_{\mathbb{R}^d} \int_1^2 |u(x,t)|^p dx dt \Big)^{1/p} \lesssim \|f\|_{W^{s,p}}?$$

  (Original motivation of Wolff who initiated decoupling.)

- Other connections to geometric measure theory, e.g. the Falconer distance conjecture: If

$$\Delta(E) := \{|x-y| \colon x, y \in E\} \subset [0, \infty)$$

  for any set $E \subset \mathbb{R}^d$, what is the minimal value of s so that $\Delta(E)$ has positive Lebesgue measure for any $E \subset \mathbb{R}^d$ with Hausdorff dimension s?

Connections to other areas

- Fourier decoupling can be seen as an outgrowth of the study of the Fourier restriction conjecture.
- The restriction conjecture says if S is the paraboloid in $\mathbb{R}^n$, given by $\{(\xi, |\xi|^2) \colon \xi \in [0,1]^{n-1}\}$, then the restriction map

$$f \mapsto \widehat{f}|_S$$

  initially defined for Schwartz f on $\mathbb{R}^n$, extends to a bounded linear map from $L^p(\mathbb{R}^n)$ to $L^1(S)$ for $1 \leq p < \frac{2n}{n+1}$.
- Conjecture holds for $n = 2$, remains open in dimensions $n \geq 3$.
- Fourier decoupling uses tools from Fourier restriction theory, and can in turn be used to study Fourier restriction.
- But additional ideas / tools are seemingly necessary to resolve the restriction conjecture in full.
- e.g. The restriction conjecture implies the Kakeya conjecture, about incidence of thin tubes in $\mathbb{R}^n$. Decoupling alone does not seem to capture that.
- Decoupling also benefited from advances in number theory.

What is decoupling?

- Recall: $L^2(\mathbb{R}^n)$ is a Hilbert space.
- Given N orthogonal functions $f_1, \ldots, f_N$ on $\mathbb{R}^n$, we have

$$\Big\| \sum_{n=1}^{N} f_n \Big\|_{L^2} = \Big(\sum_{n=1}^{N} \|f_n\|_{L^2}^2 \Big)^{1/2}.$$

- In general we can't replace L^2 with other L^p where $p \neq 2$.
- Nevertheless, sometimes we can beat the trivial bound

$$\Big\| \sum_{n=1}^{N} f_n \Big\|_{L^p} \leq N^{\frac{1}{2}} \Big(\sum_{n=1}^{N} \|f_n\|_{L^p}^2 \Big)^{1/2}$$

  obtained via Minkowski inequality + Hölder. Note the f_n's are no longer coupled together on the right hand side above.
- Underlying mechanism: $f_1, \ldots, f_N$ will be (sums of) wave packets with different orientations.
- Decoupling captures the interference between such waves.

Superposition of waves

- Define Fourier transform on $\mathbb{R}^n$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}.$$

- Fourier inversion (for Schwartz f) says

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,$$

  i.e. $f =$ superpositions of waves $e^{2\pi i x \cdot \xi}$, with $\xi \in \operatorname{supp} \widehat{f}$.
- Think of $e^{2\pi i x \cdot \xi} = \cos(2\pi x \cdot \xi) + i \sin(2\pi x \cdot \xi)$ as waves travelling in direction ξ (draw their crests and troughs).

Grouping neighbouring frequencies together

- To formulate decoupling, start with $f \in \mathcal{S}(\mathbb{R}^n)$ so that $\widehat{f}$ is supported in a small neighborhood of a compact manifold S.
- Example S:
  1. unit paraboloid $\{(\xi, |\xi|^2) \colon \xi \in [0,1]^{n-1}\}$
  2. unit light cone $\{(\xi, |\xi|) \colon 1 \leq |\xi| \leq 2\}$
  3. unit moment curve $\{(\xi, \xi^2, \ldots, \xi^n) \colon \xi \in [0,1]\}$.
- We will cover supp $\widehat{f}$ with finitely overlapping rectangular boxes $\{\theta\}$ and let[1]

  $$\widehat{f_\theta} := \widehat{f} 1_\theta$$

  so that f_θ is a superposition of waves of similar frequencies (all contained in θ).
- Can we set it up so that $\|f\|_{L^p}$ is controlled by $\left(\sum_\theta \|f_\theta\|_{L^p}^2 \right)^{\frac{1}{2}}$?

[1]Fine print: Usually we take a partition of unity $\{\eta_\theta\}$ subordinate to the cover $\{\theta\}$ and let $\widehat{f_\theta} := \widehat{f}\eta_\theta$ instead, so that f_θ is Schwartz and $f = \sum_\theta f_\theta$.

Some heuristics

- Let θ be a rectangular box in $\mathbb{R}^n$.
- Let's gain some intuition about f_θ if supp $\widehat{f_\theta} \subset \theta$.
- First, in dimension $n = 1$, one can compute the inverse Fourier transform

$$\mathcal{F}^{-1} 1_{[0,1]}(x) = \int_0^1 e^{2\pi i x \xi} d\xi = \frac{e^{2\pi i x} - 1}{2\pi i x} = e^{i\pi x} \frac{\sin \pi x}{2\pi x}.$$

  We would like to think of this inverse Fourier transform as $1_{[0,1]}(x)$, even though it is not exactly true.
- Accepting this heuristic, the inverse Fourier transform of $1_{[0,1]^n}(\xi)$ is $1_{[0,1]^n}(x)$.
- Similarly, for every rectangular box $\theta \subset \mathbb{R}^n$ containing a point ω_θ, we think of the inverse Fourier transform of $1_\theta(\xi)$ as

$$|\theta^*|^{-1} e^{2\pi i \omega_\theta \cdot x} 1_{\theta^*}(x)$$

  where θ^* is the dual box to θ, which passes through 0, has the same orientation of θ and dimensions reciprocal to those of θ.

The uncertainty principle

- If $\widehat{f_\theta}$ is supported in a rectangular box θ containing 0, then $\widehat{f_\theta} = \widehat{f_\theta} 1_\theta$, so with our heuristic,

$$f_\theta(x) = f_\theta * |\theta^*|^{-1} 1_{\theta^*}(x).$$

- This suggests us to tile $\mathbb{R}^n$ by translates of θ^* and think of $f_\theta(x)$ as a constant (namely, its average) on each translate.
- If θ does not contain 0, then by modulating f_θ we can show instead that $|f_\theta|$ is morally constant on translates of θ^*.
- f_θ restricted to each translate of θ^* is called a wave packet; so f_θ is a sum of wave packets, all with the same orientation.
- In decoupling we usually have a family of boxes $\{\theta\}$ and a Schwartz family $\{f_\theta\}_\theta$ with supp $\widehat{f_\theta} \subset \theta$. Decoupling captures the interference patterns arising from summing f_θ's where the θ's in the sum have different orientations.

Decoupling for the paraboloid

- Let $d \geq 1$, $S =$ unit paraboloid in $\mathbb{R}^{d+1}$, and $0 < \delta \ll 1$.
- Cover δ neighborhood of S by rectangular boxes $\{\theta\}$ of dimensions $\delta^{1/2} \times \ldots \delta^{1/2} \times \delta$, that are 'tangent to S'.
- Suppose for each θ in this collection, f_θ is a Schwartz function on $\mathbb{R}^{d+1}$ with supp $\widehat{f_\theta} \subset \theta$. Let $f = \sum_\theta f_\theta$.
- What is the best constant D such that

$$\|f\|_{L^p(\mathbb{R}^{d+1})} \leq D\Big(\sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R}^{d+1})}\Big)^{1/2}?$$

- D depends on d, p and δ. Think d fixed, write $D = D_p(\delta)$.
- Since $\#\theta = \delta^{-d/2}$, trivial bound is

$$D_p(\delta) \leq (\delta^{-d/2})^{1/2},$$

and this is sharp at $p = \infty$.

A sharp example

- Consider the example $f_\theta := |\theta|^{-1}\mathcal{F}^{-1}1_\theta$. Clearly supp $\widehat{f_\theta} \subset \theta$.
- $|f(x)| = |\sum_\theta f_\theta(x)|$ is $\gtrsim \delta^{-d/2}$ for $|x| \lesssim 1$, so $\|f\|_{L^p} \gtrsim \delta^{-d/2}$.
- On the other hand,

$$\Big(\sum_\theta \|f_\theta\|_{L^p}^2\Big)^{1/2} \simeq (\delta^{-d/2})^{1/2}(\delta^{-(d+2)/2})^{1/p}.$$

- Hence

$$D_p(\delta) \gtrsim \frac{\delta^{-d/2}}{(\delta^{-d/2})^{1/2}(\delta^{-(d+2)/2})^{1/p}} = \delta^{-\frac{1}{2}(\frac{d}{2}-\frac{d+2}{p})}.$$

- In other words, if $\frac{d}{2} - \frac{d+2}{p} \geq 0$, i.e. if $p \geq \frac{2(d+2)}{d}$, then the best one can hope for is

$$\|f\|_{L^p(\mathbb{R}^{d+1})} \lesssim \delta^{-\frac{1}{2}(\frac{d}{2}-\frac{d+2}{p})}\Big(\sum_\theta \|f_\theta\|_{L^p(\mathbb{R}^{d+1})}^2\Big)^{1/2}.$$

  The loss in power of δ^{-1} cannot be removed unless

$$p \leq \frac{2(d+2)}{d}.$$

Theorem (Bourgain-Demeter 2014)

Suppose $f_\theta \in \mathcal{S}(\mathbb{R}^{d+1})$ *with supp* $\widehat{f_\theta} \subset \theta$ *for all* θ*, and let* $f = \sum_\theta f_\theta$*. Then for* $2 \le p \le \frac{2(d+2)}{d}$ *and any* $\varepsilon > 0$*,*

$$\|f\|_{L^p(\mathbb{R}^{d+1})} \lesssim_\varepsilon \delta^{-\varepsilon} \Big(\sum_\theta \|f_\theta\|_{L^p(\mathbb{R}^{d+1})}^2 \Big)^{1/2}.$$

Also for $p \ge \frac{2(d+2)}{d}$ *and any* $\varepsilon > 0$*,*

$$\|f\|_{L^p(\mathbb{R}^{d+1})} \lesssim_\varepsilon \delta^{-\frac{1}{2}(\frac{d}{2} - \frac{d+2}{p}) - \varepsilon} \Big(\sum_\theta \|f_\theta\|_{L^p(\mathbb{R}^{d+1})}^2 \Big)^{1/2}.$$

- In other words, one gets the best possible estimates, apart from the $\delta^{-\varepsilon}$ loss.
- The key is to prove the theorem for $p = \frac{2(d+2)}{d}$ (deferred). The rest follows from interpolation.

Connection to Strichartz estimates

- $p = \frac{2(d+2)}{d}$ is the Tomas-Stein / Strichartz exponent.
- Strichartz inequality says if u solves the Schrödinger equation $i\partial_t u = \Delta_x u$ on $\mathbb{R}^{d+1}$ and $u(x,0) = g(x)$ then for this p

$$\|u(x,t)\|_{L^p(\mathbb{R}^{d+1})} \lesssim \|g(x)\|_{L^2(\mathbb{R}^d)}.$$

- Note that the space-time Fourier transform of $u(x,t)$ is supported on a paraboloid in $\mathbb{R}^{d+1}$: in fact

$$u(x,t) = \int_{\mathbb{R}^d} \widehat{g}(\xi) e^{2\pi i(x\cdot\xi + 2\pi t|\xi|^2)} d\xi.$$

- Curvature of this paraboloid makes the Schrödinger equation dispersive, which makes Strichartz inequality possible.
- Not a coincidence that the Strichartz exponent shows up in decoupling: the latter implies some forms of Strichartz.