Introduction to Fourier decoupling

Po-Lam Yung

Australian National University

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From last time

- Main heuristics:

$$\mathcal{F}^{-1}1_{[-1/2,1/2]} = \frac{\sin \pi x}{\pi x}$$

  is ~ 1 on $[-1/2, 1/2]$ and ~ 0 away from it, so we pretend

$$\mathcal{F}^{-1}1_{[-1/2,1/2]} = 1_{[-1/2,1]/2}.$$

- If θ is a rectangular box and $\omega_\theta \in \theta$, then

$$\mathcal{F}^{-1}\Big(|\theta|^{-1}1_\theta\Big) = e^{2\pi i \omega_\theta \cdot x} 1_{\theta^*}(x)$$

  where θ^* is a dual rectangular box through 0, with dimensions reciprocal to those of θ. (Draw it!)
- If θ is a rectangular box and supp $\widehat{f_\theta} \subset \theta$, then $|f_\theta|$ is constant on each translate of θ^*. (Draw the tiling.)
- Decoupling captures cancellations inside $\sum_\theta f_\theta$, when we have many boxes θ in many different orientations.

Decoupling for the parabola

- Tile a δ-neighborhood of the unit parabola in $\mathbb{R}^2$ by $N := \delta^{-\frac{1}{2}}$ many rectangles $\{\theta\}$ of dimensions $\delta^{1/2} \times \delta$.

Theorem (Bourgain-Demeter 2014)

Suppose $f_\theta \in \mathcal{S}(\mathbb{R}^2)$ *with supp* $\widehat{f_\theta} \subset \theta$ *for all* θ. *Then for* $p \geq 6$,

$$\| \sum_\theta f_\theta \|_{L^p(\mathbb{R}^2)} \lessapprox N^{\frac{1}{2} - \frac{3}{p}} \Big(\sum_\theta \| f_\theta \|^2_{L^p(\mathbb{R}^2)} \Big)^{1/2}.$$

- Here $\lessapprox$ means $\lesssim_\varepsilon N^\varepsilon$.
- Estimate beats the trivial bound $N^{\frac{1}{2}}$, and the semi-trivial bound $N^{\frac{1}{2} - \frac{1}{p}}$ obtained by interpolating between L^2 and L^∞.
- Estimate sharp up to N^ε loss by considering $f_\theta = |\theta|^{-1} 1_\theta$, i.e.

$$\sum_\theta f_\theta(x) = \sum_\theta e^{2\pi i \omega_\theta \cdot x} 1_{\theta^*}(x).$$

(Draw it - it looks like a bush.)

Connection to Strichartz

- $p = 6$ is the Tomas-Stein / Strichartz exponent in $\mathbb{R}^2$.
- Strichartz inequality says if u solves the Schrödinger equation $i\partial_t u = \partial_x^2 u$ on $\mathbb{R}^{1+1}$ and $u(x, 0) = g(x)$ then

$$\|u(x,t)\|_{L^6(\mathbb{R}^2)} \lesssim \|g(x)\|_{L^2(\mathbb{R})}.$$

- Base line: $u(x,t) = e^{it\partial_x^2} g(x)$ is in $L^2(dx)$ for every time t.
- Strichartz says for most time t, $u(x,t)$ is in $L^6(dx)$ as well - solution spreads out.
- Curvature of this paraboloid makes the Schrödinger equation dispersive, which makes Strichartz inequality possible.
- $p = 6$ is the correct exponent for Strichartz on $\mathbb{R}^{1+1}$.
- Not a coincidence that the Strichartz exponent shows up in decoupling: decoupling implies some forms of Strichartz.

- Decoupling for the paraboloid implies discrete Strichartz.
- If $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $g \in L^2(\mathbb{T})$, then the solution $u(x,t)$ to the Schrödinger equation $i\partial_t u = \partial_x^2 u$ with initial data g obeys

$$\|u(x,t)\|_{L^6([0,1]^2)} \lessapprox \|g(x)\|_{L^2([0,1])}$$

  whenever supp $\widehat{g} \subset [-N, N]$.
- Discrete Strichartz is harder to prove than the original one, because waves exhibit less dispersion on the compact manifold $\mathbb{T}$. In fact, examples show the N^ε factor cannot be removed.
- Can be reformulated as an exponential sum estimate, since if

$$g(x) = \sum_{n=-N}^{N} b_n e(nx) \qquad \Longrightarrow \qquad u(x,t) = \sum_{n=-N}^{N} b_n e(nx+n^2 t).$$

  (Here $e(t) := e^{2\pi i t}$.)
- In other words, discrete Strichartz just says

$$\Big\| \sum_{n=-N}^{N} b_n e(nx + n^2 t) \Big\|_{L^6([0,1]^2)} \lessapprox \Big(\sum_{n=-N}^{N} |b_n|^2 \Big)^{1/2}$$

  for all finite sequences $\{b_n\} \subset \mathbb{C}$.

Why does decoupling implies discrete Strichartz?

- The point of decoupling is to replace L^6 norm of a sum, by ℓ^2 norm of the L^6 norm of the pieces.
- By rescaling the frequencies (n, n^2) back to $(\frac{n}{N}, \frac{n^2}{N^2})$, one can actually apply decoupling for the parabola, and deduce

$$
\begin{aligned}
&\Big\| \sum_{n=-N}^{N} b_n e(nx + n^2 t) \Big\|_{L^6([0,1]^2)} \\
\lessapprox &\Big(\sum_{n=-N}^{N} \Big\| b_n e(nx + n^2 t) \Big\|_{L^6([0,1]^2)}^2 \Big)^{1/2} \\
= &\Big(\sum_{n=-N}^{N} |b_n|^2 \Big)^{1/2}.
\end{aligned}
$$

The first inequality has to be justfied via a change of variables $(x, t) \mapsto (\frac{x}{N}, \frac{t}{N^2})$, and using periodicity.

Why is decoupling easier than discrete Strichartz?

- The formulation of decoupling allows easy access to a useful tool called *induction on scales*.
- Fix $p = 6$. Our goal is to bound $D(\delta) := D_p(\delta)$, which is the best constant for which

$$\|f\|_{L^p} \leq D(\delta)\Big(\sum_{\theta} \|f_\theta\|_{L^p}^2\Big)^{1/2},$$

  whenever $f = \sum_\theta f_\theta$, supp $\widehat{f_\theta} \subset \theta$, and θ cover a δ neighborhood of the parabola in $\mathbb{R}^2$.
- $D(1)$ is trivial: when $\delta = 1$ there are only $O(1)$ many θ's.
- Let's say by induction we already understand $D(\delta_1)$ for some $1 \geq \delta_1, \delta_2 \gg \delta$ with $\delta = \delta_1\delta_2$.
- Then we can cover the δ_1 neighborhood of the parabola by boxes $\{\tau\}$ of dimension $\delta_1^{1/2} \times \delta_1$, and let $f_\tau := \sum_{\theta\subset\tau} f_\theta$. (Draw it.)

- Information about $D(\delta_1)$ tells us

$$\|f\|_{L^p} \le D(\delta_1)\Big(\sum_{\tau} \|f_\tau\|_{L^p}^2\Big)^{1/2}.$$

- With some work, information about $D(\delta_2)$ will tell us

$$\|f_\tau\|_{L^p} \le D(\delta_2)\Big(\sum_{\theta\subset\tau} \|f_\theta\|_{L^p}^2\Big)^{1/2} \quad \text{for all } \tau.$$

  (Technically, we use the affine symmetry of the parabola here.)
- Together we get

$$\|f\|_{L^p} \le D(\delta_1)D(\delta_2)\Big(\sum_{\theta} \|f_\theta\|_{L^p}^2\Big)^{1/2},$$

  i.e. $D(\delta) \le D(\delta_1)D(\delta_2)$.

- $D_p(\delta) \leq D_p(\delta_1) D_p(\delta_2)$ if $\delta = \delta_1 \delta_2$.
- This is not quite a proof for the desired bound for $D_p(\delta)$, since we have no base case (given $\varepsilon > 0$, one needs some δ_0 so that $D_p(\delta_0) \leq \delta_0^{-\varepsilon}$ first).
- But it explains why decoupling might be 'easy'; in fact, this observation is what motivated the formulation of decoupling.
- For contrast, such an induction proof does not work if:
  a) we try to prove discrete Strichartz directly; or
  b) we are interested in bounding $\|f\|_{L^p}$ by $\left\|(\sum_{\theta} |f_\theta|^2)^{1/2}\right\|_{L^p}$.
- More about the actual proof of decoupling next time.

Why is decoupling a good proof of discrete Strichartz?

- Discrete Strichartz for $\mathbb{T}$ was known to Bourgain a long time ago using a trick from number theory.
- But the above proof via decoupling has 2 advantages.
- First it generalizes to give discrete Strichartz for all higher dimensional torus $\mathbb{T}^d$ (even those with irrational periods).
- Second for $\mathbb{T}$, the N^ε loss from decoupling can actually be improved to $(\log N)^c$, and it would yield automatically an improved discrete Strichartz where the loss is only $(\log N)^c$ (Guth, Maldague and Wang; Guo, Li and myself).

Decoupling for the circle

- Similar decoupling holds with the unit paraboloids replaced by the unit spheres.
- 2-d: Tile a δ neighborhood of the unit circle by $N := \delta^{-1/2}$ many rectangles $\{\theta\}$ of dimensions $\delta^{1/2} \times \delta$.

Theorem (Bourgain-Demeter 2014 + Pramanik-Seeger 2007)

Suppose $f_\theta \in \mathcal{S}(\mathbb{R}^2)$ with supp $\widehat{f_\theta} \subset \theta$ for all θ. Then for $p \geq 6$,

$$\| \sum_\theta f_\theta \|_{L^p(\mathbb{R}^2)} \lessapprox N^{\frac{1}{2}-\frac{3}{p}} \Big(\sum_\theta \|f_\theta\|_{L^p(\mathbb{R}^2)}^2 \Big)^{1/2}.$$

- Difficulty: Circle not affine invariant (unlike parabola)
- 'Proof': Circle can be approximated by parabola locally.
- Theorem can be rescaled: for $R \gg 1$, tile a 1 neighborhood of a circle of radius R by $N := R^{1/2}$ many rectangles $\{\theta\}$ of sizes $R^{1/2} \times 1$. Then the above theorem continues to hold.
- The rescaled theorem is equivalent to the original one. It is sharp thanks to the bush example again. (Draw it.)

A case with no non-trivial decoupling

- Tile a dyadic annulus of radius R on $\mathbb{R}^n$ by $N := R^{\frac{n-1}{2}}$ many sectors $\{\theta\}$ of dimensions $R^{1/2} \times \cdots \times R^{1/2} \times R$. (Draw it.)

Theorem

Suppose $F_\theta \in \mathcal{S}(\mathbb{R}^n)$ with supp $\widehat{F_\theta} \subset \theta$ for all θ. Then for $p \geq 2$,

$$\|\sum_\theta F_\theta\|_{L^p(\mathbb{R}^n)} \leq N^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_\theta \|F_\theta\|_{L^p(\mathbb{R}^n)}^2 \Big)^{1/2}.$$

- 'Proof': Interpolation between L^2 (orthogonality) and L^∞ (Minkowski inequality).
- Theorem optimal by a bush example! Set $F_\theta = |\theta|^{-1} 1_\theta$. Then

$$\|\sum_\theta F_\theta\|_{L^p(\mathbb{R}^n)} \gtrsim N(R^{-n})^{\frac{1}{p}}, \quad \Big(\sum_\theta \|F_\theta\|_{L^p(\mathbb{R}^n)}^2 \Big)^{\frac{1}{2}} \sim N^{\frac{1}{2}} R^{-\frac{n+1}{2p}},$$

and their ratio is $\gtrsim N^{\frac{1}{2}} R^{-\frac{n-1}{2p}} = N^{\frac{1}{2}-\frac{1}{p}}$.

Consequences for the wave equation

- Question: How much can the solution of the wave equation $\partial_t^2 u = \Delta_x u$ concentrates in space given its initial data?
- One way is measure $\|u(x,1)\|_{L^p(\mathbb{R}^n)}$. This gets large as $p \to \infty$ if solution concentrates in space.
- Our last theorem help us estimate $\|u(x,1)\|_{L^p(\mathbb{R}^n)}$.
- For simplicity, let $u(x,t) = e^{it\sqrt{-\Delta}} f(x)$.

Proposition

If supp $\widehat{f} \subset \{|\xi| \simeq R\}$, then for $p \geq 2$,

$$\|e^{i\sqrt{-\Delta}} f(x)\|_{L^p(\mathbb{R}^n)} \leq R^{(n-1)(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^p(\mathbb{R}^n)}.$$

- The proof relies on the trivial decoupling above, and on the heuristic that if θ is a sector as before and supp $\widehat{f_\theta} \subset \theta$, then $e^{i\sqrt{-\Delta}} f_\theta$ is morally a translate of f_θ in a direction given by θ.

- More precisely, decompose $f = \sum_\theta f_\theta$ according to the previous theorem. Let $F_\theta := e^{i\sqrt{-\Delta}} f_\theta$ so that $\operatorname{supp} \widehat{F_\theta} \subset \theta$.
- Since $F_\theta = e^{i\sqrt{-\Delta}} f_\theta$ is morally just a translate of f_θ, we have $\|F_\theta\|_{L^p} = \|f_\theta\|_{L^p}$ for all θ.
- For $p \geq 2$, 'decoupling' + above fact shows

$$\begin{aligned}\|e^{i\sqrt{-\Delta}} f\|_{L^p} = \|\sum_\theta F_\theta\|_{L^p} &\leq N^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_\theta \|F_\theta\|_{L^p}^2\Big)^{1/2} \\ &\leq N^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_\theta \|f_\theta\|_{L^p}^2\Big)^{1/2}.\end{aligned}$$

- Apply Holder to see

$$\Big(\sum_\theta \|f_\theta\|_{L^p}^2\Big)^{1/2} \leq N^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_\theta \|f_\theta\|_{L^p}^p\Big)^{1/p}.$$

- The $\ell^p L^p$ norm on the right is $\leq \|f\|_{L^p}$ by interpolation between L^2 and L^∞. Remembering $N = R^{\frac{n-1}{2}}$, this completes the proof.

- One can prove similarly the following small extension:

Proposition

If supp $\widehat{f} \subset \{|\xi| \simeq R\}$, *then for* $p \geq q \geq 2$,

$$\|e^{i\sqrt{-\Delta}} f(x)\|_{L^p(\mathbb{R}^n)} \leq R^{\frac{n-1}{2} - \frac{n}{p} + \frac{1}{q}} \|f\|_{L^q(\mathbb{R}^n)}.$$

- The case $p = q$ is just our earlier proposition.
- The proof goes as follows: as before

$$\|e^{i\sqrt{-\Delta}} f\|_{L^p} \leq N^{\frac{1}{2} - \frac{1}{p}} \Big(\sum_\theta \|f_\theta\|_{L^p}^2 \Big)^{1/2}.$$

But the uncertainty principle asserts that $|f_\theta|$ is constant on boxes with measure $R^{-\frac{n+1}{2}}$. Thus using $\ell^q \subset \ell^p$ when $p \geq q$, we see that $\|f_\theta\|_{L^p} \leq R^{\frac{n+1}{2}(\frac{1}{q} - \frac{1}{p})} \|f_\theta\|_{L^q}$. Finally use

$$\Big(\sum_\theta \|f_\theta\|_{L^q}^2 \Big)^{1/2} \leq N^{\frac{1}{2} - \frac{1}{q}} \Big(\sum_\theta \|f_\theta\|_{L^q}^q \Big)^{1/q} \leq N^{\frac{1}{2} - \frac{1}{q}} \|f\|_{L^q}.$$

- The proposition is sharp if the $F_\theta = e^{i\sqrt{-\Delta}} f_\theta$ is from the bush example.
- That means $f = \sum_\theta f_\theta$ is basically the characteristic function of the annulus $\{1 \leq |\xi| \leq 1 + \frac{1}{R}\}$. (Spacetime picture/movie)
- The proposition can be rephrased by saying that

  $$\|e^{i\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{W^{s,q}(\mathbb{R}^n)}$$

  if $p \geq q \geq 2$, $s = \frac{n-1}{2} - \frac{n}{p} + \frac{1}{q}$, and supp $\widehat{f}$ is contained in a dyadic annulus.
- The last assumption turns out to be unnecessary (Miyachi, Peral).
- It was also known that $e^{i\sqrt{-\Delta}}$ can be replaced by any Fourier integral operators of order 0 (Seeger, Sogge, Stein).
- These results that involves all frequencies are best captured in some function spaces that are introduced by Smith, & Hassell, Portal and Rozendaal (+ later joint work with myself).