Introduction to Fourier decoupling

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For those interested, I highly recommend:

- Larry Guth's ICM 2022 video + survey
  Decoupling estimates in Fourier analysis arXiv:2207.00652

Fixed time estimates for the wave equation

- Consider the initial value problem for the wave equation

$$\partial_t^2 U = \Delta_x U, \quad U(x,0) = f(x), \quad \partial_t U(x,0) = 0.$$

- Its solution is $U(x,t) = \frac{1}{2}(e^{it\sqrt{-\Delta}}f(x) + e^{-it\sqrt{-\Delta}}f(x))$, where

$$e^{\pm it\sqrt{-\Delta}}f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i(\pm t|\xi| + x\cdot\xi)} d\xi.$$

- Without loss of generality we consider $u(x,t) = e^{it\sqrt{-\Delta}}f(x)$.
- By Plancherel,

$$\|u(x,t)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \quad \forall t,$$

  but it was known that $\|u(x,t)\|_{L^p(\mathbb{R}^n)}$ is in general not bounded by $\|f\|_{L^p(\mathbb{R}^n)}$ if $p \neq 2$ and $t \neq 0$.
- Take $t = 1$. Question: What is the smallest $s = s(p,n) > 0$, so that

$$\|u(x,1)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^n)}?$$

- Peral and Miyachi showed independently that

$$\|u(x,1)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^n)} \quad \text{if} \quad s=(n-1)|\frac{1}{2}-\frac{1}{p}|.$$

- Let's say $p \geq 2$. Consider a special case where $\widehat{f}$ is supported in a dyadic annulus $A_R := \{R \leq |\xi| \leq 2R\}$, $R \gg 1$.
- Then the above result is equivalent to saying that

$$\|e^{i\sqrt{-\Delta}} f(x)\|_{L^p(\mathbb{R}^n)} \lesssim R^{(n-1)(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^p(\mathbb{R}^n)}.$$

- This actually follows from a semi-trivial decoupling theorem for the sectorial decomposition of A: Decompose A_R into a disjoint union of $N := R^{\frac{n-1}{2}}$ sectors $\{\theta\}$ of size $(R^{\frac{1}{2}})^{n-1} \times R$.

Theorem

Suppose $F_\theta \in \mathcal{S}(\mathbb{R}^n)$ with supp $\widehat{F_\theta} \subset \theta$ for all θ. Then for $p \geq 2$,

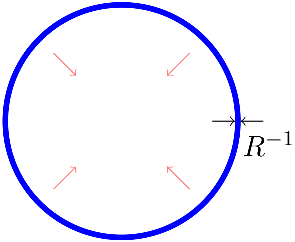
$$\|\sum_\theta F_\theta\|_{L^p(\mathbb{R}^n)} \lesssim N^{\frac{1}{2}-\frac{1}{p}}\Big(\sum_\theta \|F_\theta\|^2_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{2}} \lesssim N^{2(\frac{1}{2}-\frac{1}{p})}\Big(\sum_\theta \|F_\theta\|^p_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{p}}.$$

Waves focusing at a point

$$\|e^{i\sqrt{-\Delta}}f(x)\|_{L^p(\mathbb{R}^n)} \lesssim R^{(n-1)(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^p(\mathbb{R}^n)}, \quad p \geq 2.$$

- Suppose supp $\widehat{f} \subset A_R$. Why is the above inequality sharp?
- The example comes from waves focusing at a point:

$$|f(x)| = 1_{1 \leq |x| \leq 1+\frac{1}{R}}, \quad |e^{i\sqrt{-\Delta}}f(x)| \simeq R^{\frac{n-1}{2}} 1_{|x| \leq \frac{1}{R}}$$



- The height $R^{\frac{n-1}{2}}$ at time 1 can be explained by conservation of energy $\|u(x,1)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$.

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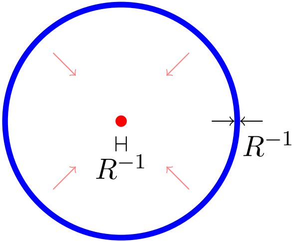

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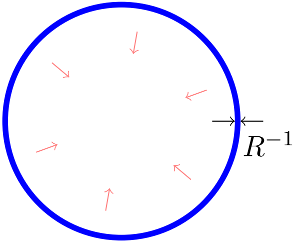
- The height at the center can also be understood using wave packets.



$|f| \simeq 1$

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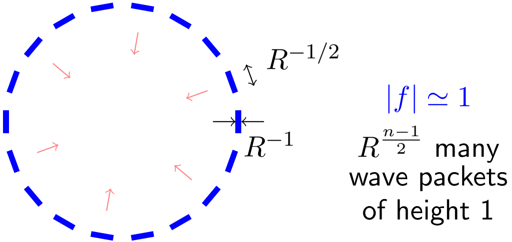

- The wave packets have Fourier supports in sectors of size $(R^{\frac{1}{2}})^{n-1} \times R$ inside the annulus A_R. Each wave packet moves in a direction dictated by its Fourier support as t varies.

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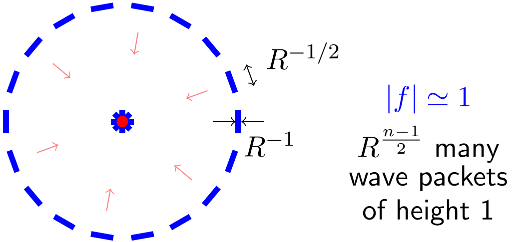

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Spacetime estimates for the wave equation

- Earlier we fix $t = 1$ and quantify concentration in space of the solution u, by estimating $\|u(x,t)\|_{L^p(\mathbb{R}^n)}$.
- The worst case is the example where waves focus to a single point at time $t = 1$.
- That example does not stay focused for very long.
- They stay focused for time $1 \leq t \leq 1 + \frac{1}{R}$, and then disperse again afterwards.
- One may ask what if we quantify spacetime concentration of the solution u, by measuring $\|u(x,t)\|_{L^p(\mathbb{R}^n \times [1,2])}$.

- The focusing example had supp $\widehat{f} \subset \{|\xi| \simeq R\}$ and

$$\|u(x,t)\|_{L^p(\mathbb{R}^n\times[1,2])} \simeq R^{(n-1)(\frac{1}{2}-\frac{1}{p})} R^{-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)}.$$

  Thus one might wonder whether it always hold that

$$\|u(x,t)\|_{L^p(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{W^{\sigma,p}(\mathbb{R}^n)}, \quad \sigma = (n-1)(\frac{1}{2}-\frac{1}{p})-\frac{1}{p}.$$

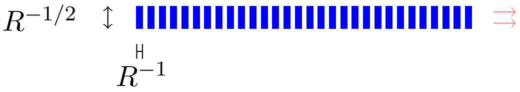
  In particular, maybe

$$\|u(x,t)\|_{L^p(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \text{if } p = \frac{2n}{n-1}?$$

- This turns out to be false when $n \geq 2$ if we omit ε losses. We have worse examples coming from Kakeya sets in $\mathbb{R}^n$.

Kakeya sets and wave trains

- Let $R \gg 1$, $n \geq 2$. A wave train in $\mathbb{R}^n$ consisting of waves of frequency $\sim R$ looks like:



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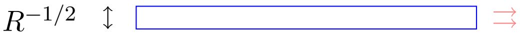
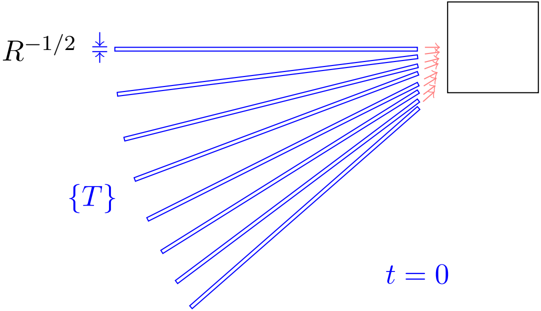


- It concentrates around a cylinder of size $(R^{-\frac{1}{2}})^{n-1} \times 1$.

Wave trains moving in different directions

- Consider many wave trains, initially concentrated on disjoint cylinders T of diameter $R^{-1/2}$ and height 2 at time $t = 0$, which travel in $R^{-1/2}$ separated directions into a unit cube.
- For any time $1 \leq t \leq 2$, each wave train occupies a fixed cylinder $\tilde{T}$ of height ~ 1 in the unit cube.



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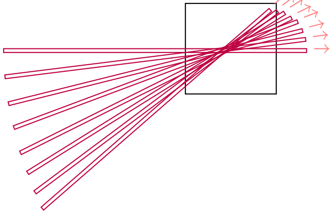

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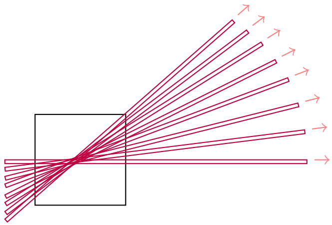

$t = 2$

Wave trains moving in different directions

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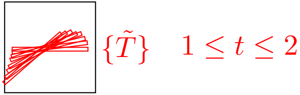

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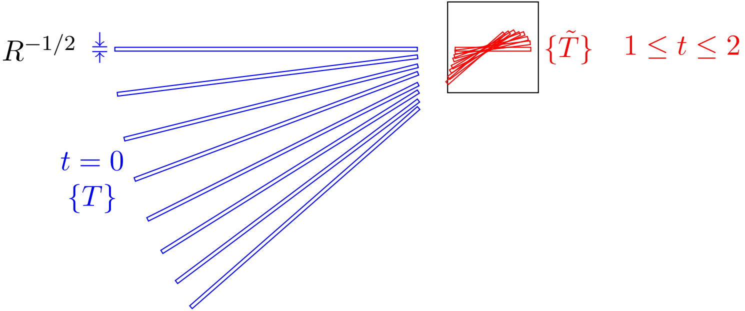

Connections to incidence geometry

- It can be arranged so that the $\tilde{T}$'s overlap quite a lot more than the initial T's (which are disjoint): we can have

$$\text{Volume}\Big(\bigcup T\Big) \gtrsim \frac{\log R^{1/2}}{\log\log R^{1/2}} \cdot \text{Volume}\Big(\bigcup \tilde{T}\Big)$$

Then for $p \geq 2$, conservation of energy gives

$$\|u(x,t)\|_{L^p(\mathbb{R}^n)} \gtrsim \Big(\frac{\log R^{1/2}}{\log\log R^{1/2}}\Big)^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for every } 1 \leq t \leq 2.$$

As a result,

$$\|u(x,t)\|_{L^p(\mathbb{R}^n \times [1,2])} \gtrsim \Big(\frac{\log R^{1/2}}{\log\log R^{1/2}}\Big)^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)}.$$

The local smoothing conjecture

- At $p = \frac{2n}{n-1}$, the tube example has

$$\|u(x,t)\|_{L^p(\mathbb{R}^n \times [1,2])} \gtrsim \left(\frac{\log R^{1/2}}{\log \log R^{1/2}}\right)^{\frac{1}{2n}} \|f\|_{L^p(\mathbb{R}^n)}.$$

- This example is worse than the example of wave focusing at a point, where we had no loss in $\log R$.
- It motivates the local smoothing conjecture, which says

$$\|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^n \times [1,2])} \lesssim_\varepsilon \|f\|_{W^{\sigma+\varepsilon,p}(\mathbb{R}^n)}$$

  whenever $\sigma = (n-1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{p}$, $p \geq \frac{2n}{n-1}$ and $\varepsilon > 0$. (This ε loss is necessary in light of the above tube example, at least when $p = \frac{2n}{n-1}$.)
- Conjecture true in dimensions $n = 1$ (classical), $n = 2$ (Guth, Wang, Zhang 2019), open in all higher dimensions.
- It is a difficult conjecture: it implies e.g. the Kakeya conjecture in $\mathbb{R}^n$.

The Kakeya conjecture

- Issue: If we can produce another pattern of thin cylinders in $\mathbb{R}^n$ that point in separated directions but overlap even more than the previous example, we might produce a counterexample to the local smoothing conjecture!
- In other words, in order to prove the local smoothing conjecture, we will have to rule out the possibility of having lots of thin cylinders in $\mathbb{R}^n$ that point in separated directions but overlap significantly more than the previous example.
- This is the content of the Kakeya conjecture. One form of it states: Any collection of cylinders in $\mathbb{R}^n$ with diameter $R^{-1/2}$ and height 1 that point in $R^{-1/2}$ separated directions cannot overlap too much: their union has measure $\gtrsim_\varepsilon R^{-\varepsilon}$ $\forall \varepsilon > 0$!
- The Kakeya conjecture is open in dimensions $n \geq 3$ and is and considered very difficult (despite much recent progress, by Katz, Zahl, Hickman, Rogers, Zhang, building upon earlier work of Bourgain, Wolff, Katz, Guth, Tao...).

Decoupling for the cone

- Let $n \geq 2$, $R \gg 1$, $S = \{(\xi, |\xi|) \colon R \leq |\xi| \leq 2R\}$ be a truncated cone in $\mathbb{R}^{n+1}$. Note S has one flat direction.
- Cover 1 neighborhood of S by rectangular boxes $\{\theta\}$ of dimensions $1 \times R^{1/2} \times \dots R^{1/2} \times R$, that are 'tangent to S'.
- Pramanik and Seeger (2007) developed a machinery for reducing the following theorem to decoupling for paraboloid:

Theorem (Bourgain-Demeter 2014, Pramanik-Seeger 2007)

Suppose $u_\theta \in \mathcal{S}(\mathbb{R}^{n+1})$ with supp $\widehat{u_\theta} \subset \theta$ for all θ. Then for $p \geq \frac{2(n+1)}{n-1}$ and any $\varepsilon > 0$,

$$\Big\| \sum_\theta u_\theta \Big\|_{L^p(\mathbb{R}^{n+1})} \lesssim_\varepsilon R^{\frac{1}{2}(\frac{n-1}{2} - \frac{n}{p}) + \varepsilon} \Big(\sum_\theta \|u_\theta\|^2_{L^p(\mathbb{R}^{n+1})} \Big)^{1/2}.$$

Application to local smoothing

- Using decoupling for the cone, one can make some progress on the local smoothing conjecture.
- It shows that in the partial range $p \geq \frac{2(n+1)}{n-1}$ (the Tomas-Stein exponent), one has the desired bound

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^n\times[1,2])} \lesssim_\varepsilon \|f\|_{W^{\sigma+\varepsilon,p}(\mathbb{R}^n)}, \quad \sigma = (n-1)(\frac{1}{2}-\frac{1}{p})-\frac{1}{p}.$$

  (Conjecture was for all $p \geq \frac{2n}{n-1}$, the restriction exponent.)
- In fact, one decomposes $f = \sum_\theta f_\theta$ using a sum over sectors, and applies the decoupling for the light cone with

$$u_\theta(x,t) = 1_{[1,2]}(t)e^{it\sqrt{-\Delta}}f_\theta(x),$$

  which will satisfy $\operatorname{supp}\widehat{u_\theta} \subset \theta$.

Decoupling for the moment curve

- To estimate the number of solutions to the Vinogradov system

$$\begin{cases} x_1 + \cdots + x_s &= x_{s+1} + \cdots + x_{2s} \\ x_1^2 + \cdots + x_s^2 &= x_{s+1}^2 + \cdots + x_{2s}^2 \\ &\vdots \\ x_1^k + \cdots + x_s^k &= x_{s+1}^k + \cdots + x_{2s}^k \end{cases}$$

  with all variables $x_i \in \{1, \dots, N\}$, it suffices to estimate

$$\Big\| \sum_{n=1}^{N} e(\gamma(n) \cdot x) \Big\|_{L^p([0,1]^k)}^p, \quad p = 2s$$

  (exponential sum estimates again!). Here

$$\gamma(t) := (t, t^2, \dots, t^k)$$

  is the degree k moment curve.
- Bourgain, Demeter and Guth achieved this by proving an suitable decoupling theorem for the degree k moment curve.

- Wooley actually proved the bound for the exponential sum for $k = 3$ case first, using number theory.
- He subsequently extended his number theory methods to all degrees $k \geq 4$.
- Using his insights, we have been able to give a simpler proof of decoupling for the degree k moment curve for all k (joint work with Shaoming Guo, Zane Kun Li, Pavel Zorin-Kranich).
- Below we describe a bit that proof, in the (much easier) case $k = 2$ (due to Zane Kun Li).

Decoupling for the parabola in $\mathbb{R}^2$

- Consider the unit parabola $P := \{(t, t^2) \colon t \in [0, 1]\}$.
- Cover a δ neighborhood of the parabola by $\delta^{-1/2}$ many rectangles of size $\delta^{1/2} \times \delta$ 'tangent' to the parabola.
- Suppose supp $\widehat{f_\theta} \subset \theta$ for every θ.
- Let $D(\delta)$ be the best constant so that

$$\Big\| \sum_{\theta} f_\theta \Big\|_{L^6(\mathbb{R}^2)} \leq D(\delta) \Big(\sum_{\theta} \|f_\theta\|^2_{L^6(\mathbb{R}^2)} \Big)^{1/2}.$$

- We want to show $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ for all $\varepsilon > 0$.

Idea 1: Bootstrap

$$\Big\| \sum_\theta f_\theta \Big\|_{L^6(\mathbb{R}^2)} \le D(\delta) \Big(\sum_\theta \|f_\theta\|^2_{L^6(\mathbb{R}^2)} \Big)^{1/2}.$$

- Trivial bound: $D(\delta) \le \delta^{-1/4}$ by Cauchy-Schwarz.
- So we may suppose $D(\delta) \lesssim \delta^{-\eta}$ for some $\eta > 0$, and show that $D(\delta) \lesssim \delta^{-\eta/2}$.
  (White lie: Actually can only show $D(\delta) \lesssim \delta^{-\eta - e^{-1/\eta}}$.)

Idea 2: Affine invariance

- Recall $P := \{(t, t^2) \colon t \in [0,1]\}$ is the unit parabola.
- Let $P_I := \{(t, t^2) \colon t \in I\}$ be a parabolic arc over an interval I.
- Any parabolic arc P_I can be mapped bijectively onto the unit parabola P by an affine transformation in $\mathbb{R}^2$.
  (Draw it: first consider $I = [0, b]$.)
- A δ neighborhood of the arc P_I is mapped bijectively onto a $|I|^{-2}\delta$ neighborhood of the P.
- If $\{\theta'\}$ are $\delta^{1/2} \times \delta$ rectangles covering P_I, and supp $\widehat{f_{\theta'}} \subset \theta'$, then

$$\Big\| \sum_{\theta'} f_{\theta'} \Big\|_{L^6(\mathbb{R}^2)} \leq D(|I|^{-2}\delta) \Big(\sum_{\theta'} \|f_{\theta'}\|_{L^6(\mathbb{R}^2)}^2 \Big)^{1/2}.$$

Idea 3: Many scales

- Back to the situation where we have all rectangles $\{\theta\}$ covering a δ neighborhood of the unit parabola P.
- Write $f = \sum_\theta f_\theta$. Let $J \gg 1$ to be determined.
- Introduce many scales $1 > \delta_1 > \delta_2 > \cdots > \delta_J = \delta$, so that

$$\delta_j := \delta^{2^{j-J}}.$$

- For any $j = 1, \ldots, J$, cover a δ_j neighborhood of P by rectangles $\{\tau_j\}$ of size $\delta_j^{1/2} \times \delta_j$.
- For each τ_j, let $f_{\tau_j} := \sum_{\theta \subset \tau_j} f_\theta$ so that $f = \sum_{\tau_j} f_{\tau_j}$ $\forall j$.
- By induction on scales, we may assume we have a good bound for $D(\delta')$ for all $1 \geq \delta' > \delta$.

Idea 4: Bilinearize

- Recall many scales $1 > \delta_1 > \delta_2 > \cdots > \delta_J = \delta$, so that

$$\delta_j := \delta^{2^{j-J}}.$$

- In order to estimate

$$\|f\|_{L^6} = \Big(\int_{\mathbb{R}^2} |f|^6\Big)^{1/6} = \Big(\int_{R^2} |\sum_{\tau_1} f_{\tau_1}|^4 |\sum_{\tau_2} f_{\tau_2}|^2\Big)^{1/6},$$

we might want to be able to estimate

$$\Big(\int_{\mathbb{R}^2} |f_{\tau_1}|^4 |f_{\tau_2}|^2\Big)^{1/6} \lesssim \delta^{-\eta/2} \Big(\sum_{\theta} \|f_\theta\|^2_{L^6(\mathbb{R}^2)}\Big)^{1/2}$$

where τ_1 and τ_2 are $\geq \delta_1^{1/2}$ apart. Turns out this is enough.

Idea 5: Hölder's inequality

- We estimate using Cauchy-Schwarz:

$$\int_{\mathbb{R}^2} |f_{\tau_1}|^4 |f_{\tau_2}|^2 \leq \Big(\int_{\mathbb{R}^2} |f_{\tau_1}|^2 |f_{\tau_2}|^4 \Big)^{1/2} \Big(\int_{\mathbb{R}^2} |f_{\tau_1}|^6 \Big)^{1/2}.$$

- The second factor is bounded using affine invariance by

$$D(\delta_1^{-1}\delta)^3 \Big(\sum_{\theta} \|f_\theta\|^2_{L^6(\mathbb{R}^2)} \Big)^{3/2},$$

and $D(\delta_1^{-1}\delta)$ is something we assume we understand because we can induct on scale (note $\delta_1^{-1}\delta > \delta$). It remains to bound the first factor.

Idea 6: L^2 orthogonality / 1 dimensional decoupling

- In the first factor

$$\Big(\int_{\mathbb{R}^2} |f_{\tau_1}|^2 |f_{\tau_2}|^4\Big)^{1/2},$$

the Fourier support of f_{τ_2} is smaller. By affine invariance again, we may assume $\tau_2 = [0, \delta_2^{1/2}] \times [0, \delta_2]$. We decompose

$$f_{\tau_1} = \sum_{\tau_3 \subset \tau_1} f_{\tau_3}.$$

- Since τ_1 is transverse to τ_2, the relevant τ_3 looks like a rectangle with vertical side length $\delta_3^{1/2} = \delta_2$ (and horizontal side length δ_3). Hence $\{f_{\tau_3} f_{\tau_2}^2\}_{\tau_3 \subset \tau_1}$ form an orthogonal family. We get

$$\Big(\int_{\mathbb{R}^2} |f_{\tau_1}|^2 |f_{\tau_2}|^4\Big)^{1/2} \le \Big(\sum_{\tau_3 \subset \tau_1} \int_{\mathbb{R}^2} |f_{\tau_3}|^2 |f_{\tau_2}|^4\Big)^{1/2}.$$

- Note the coarsest scale went down from scale δ_1 to a finer scaler δ_2.

Idea 7: Iteration

- Repeat Steps 5 and 6 many times to go from scale δ_1 to δ_2 to $\delta_3 \dots$ until we decouple down to scale $\delta_J = \delta$.
- A lot of book keeping! Eventually get

$$\Big(\int_{\mathbb{R}^2} |f_{\tau_1}|^4 |f_{\tau_2}|^2 \Big)^{1/6} \lesssim \delta^{-\frac{1}{2^J}} \prod_{j=1}^{J} D(\delta_j^{-1}\delta)^{\frac{1}{2^j}} \Big(\sum_{\theta} \|f_\theta\|^2_{L^6(\mathbb{R}^2)} \Big)^{1/2}$$

- Apply bootstrap hypothesis $D(\delta_j^{-1}\delta) \lesssim (\delta_j^{-1}\delta)^{-\eta}$ with $\delta_j = \delta^{2^{j-J}}$; compute the product in terms of δ, η and J. This beats the bootstrap assumption if J be large enough.

Other possible proofs

- The original proof of decoupling for the parabola by Bourgain and Demeter used incidence geometry.
- They needed to study how transverse tubes in $\mathbb{R}^2$ intersect each other.
- Yet another proof by Guth, Maldague and Wang tried to classify points in $\mathbb{R}^2$ according to the finest scale at which constructive interference happens (if at all). It was inspired by ideas in combinatorics and gives the best known bound.

Summary

- Decoupling is some form of orthogonality in L^p, $p \geq 2$.
- Underlying mechanism: superposition of waves packets with varying orientations.
- More precisely: Decoupling happens when we sum functions which have disjoint Fourier supports along curved manifolds.
- They can be used to study a range of problems, from bounding exponential sums to studying the wave equation.
- Many other applications that we did not have time to describe.

- A large part of cleverness is in formulating decoupling, in a way that can be proved by induction.
- Decoupling has various limitations too; e.g. it is tied to the Tomas-Stein exponent, and many harder problems are tied instead to the restriction exponent.
- Decoupling also seems to fail to capture interesting Kakeya phenomena.
- This is both good and bad: good that it explains why it is relatively easy to prove, bad because this is not the most powerful tool one can have.