Techniques for proving decoupling inequalities

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Decoupling inequalities

- Given a Schwartz function f on $\mathbb{R}^n$ and an exponent p, we seek a decomposition of f into a sum: $f = \sum_{j=1}^{N} f_j$, so that

$$\|f\|_{L^p} \leq D_p(N) \Big(\sum_j \|f_j\|_{L^p}^2 \Big)^{1/2}.$$

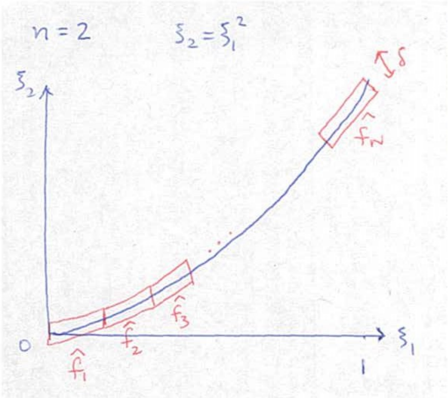
- If $p = 2$ and $f = \sum_j f_j$ is an orthogonal decomposition, then $D_p(N) = 1$.
- Minkowski inequality for L^p + Cauchy-Schwarz shows that the inequality always holds with $D_p(N)$ replaced by $N^{\frac{1}{2}}$ if $p \geq 2$:

$$\|f\|_{L^p} \leq \sum_j \|f_j\|_{L^p} \leq N^{\frac{1}{2}} \Big(\sum_j \|f_j\|_{L^p}^2 \Big)^{1/2}.$$

  (This is usually sharp at $p = \infty$)
- Often better bounds are possible for intermediate p's $\rightarrow$ powerful tools in PDEs (e.g. local smoothing, discrete restriction), analytic number theory, geometric measure theory.

Examples of non-trivial decoupling

- Start with f whose Fourier transform is supported in a δ neighbourhood of a curved, compact submanifold S in $\widehat{\mathbb{R}^n}$.
- We cut the neighbourhood into N boxes 'tangent' to S, and let f_j be the frequency localization of f to the j-th box.
- Paraboloids in $\widehat{\mathbb{R}^n}$ (Bourgain-Demeter):



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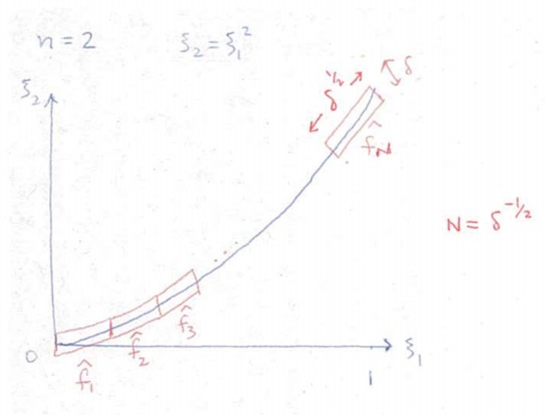
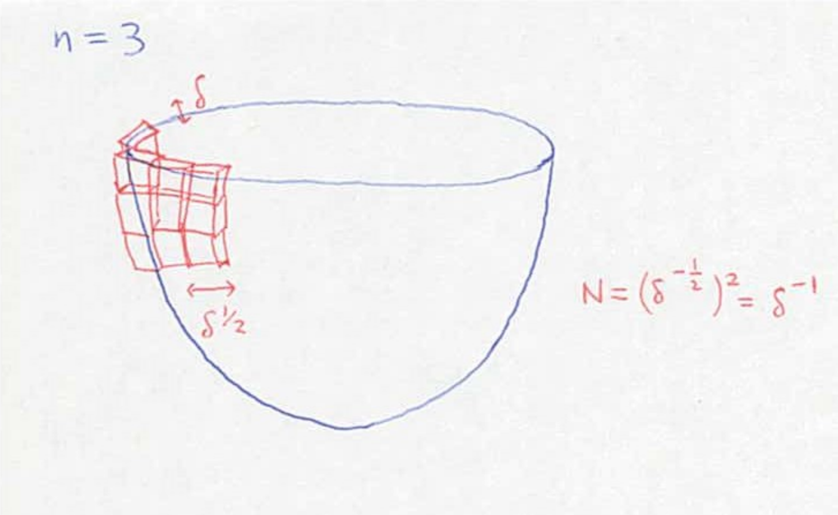
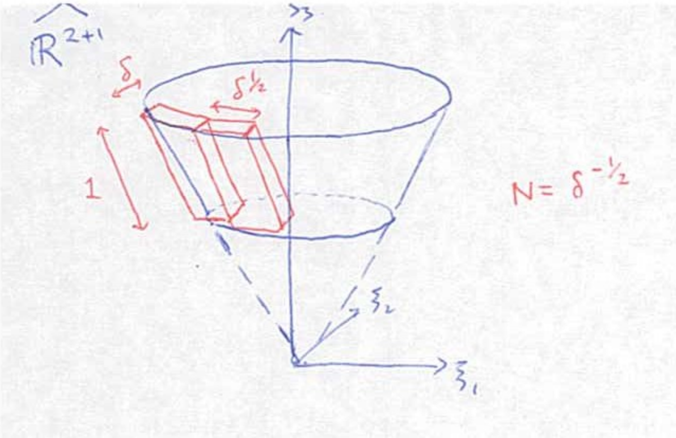
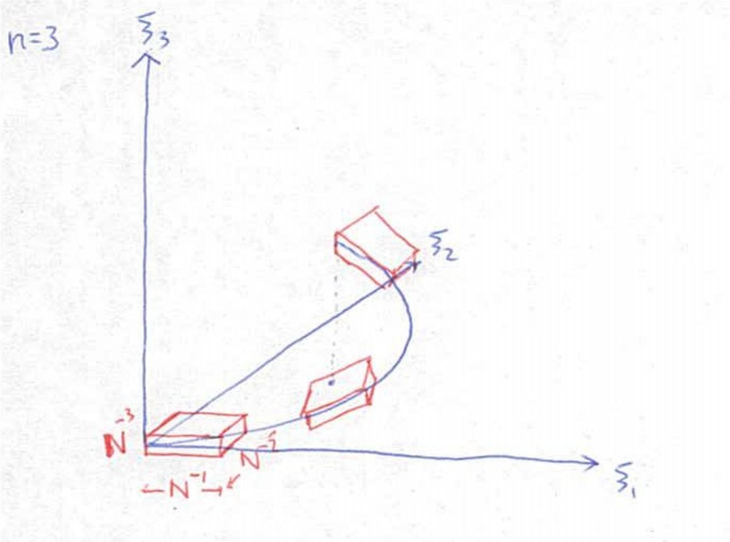



n = 3
δ
δ^{1/2}
N = (δ^{-1/2})^2 = δ^{-1}

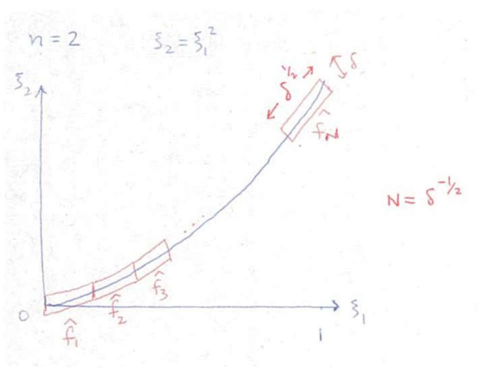
- Cones in $\widehat{\mathbb{R}^{n+1}}$ (Wolff, Wolff-Łaba, Pramanik-Seeger, Bourgain-Demeter):



- Moment curve $\{(t, t^2, \dots, t^n) \colon t \in [0,1]\}$ in $\widehat{\mathbb{R}^n}$ (Bourgain-Demeter-Guth, Wooley, Guo-Li-Y.-Zorin-Kranich):



The parabola case



- For every $\varepsilon > 0$, Bourgain-Demeter proved that

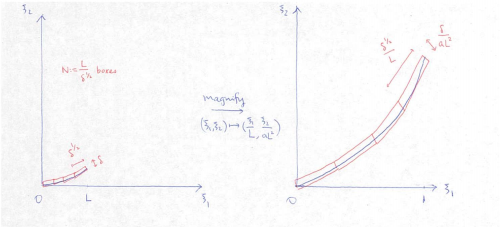
$$\|f\|_{L^p} \lesssim_\varepsilon N^\varepsilon \Big(\sum_{j=1}^{N} \|f_j\|_{L^p}^2 \Big)^{1/2} \quad \text{if } 2 \le p \le 6.$$

- Let's take this for granted, and see how we can use it to prove new decoupling inequalities.

Parabolic rescaling: length of parabola doesn't matter

- Let $1 \leq a \leq 2$. If $\widehat{f}$ is supported in a δ neighbourhood of a short parabolic arc $\{(t, at^2) \colon 0 \leq t \leq L\}$, then the number of $\delta^{1/2} \times \delta$ boxes covering this δ neighbourhood is $\simeq L/\delta^{1/2}$.
- These boxes rescale under $(\xi_1, \xi_2) \mapsto (\frac{\xi_1}{L}, \frac{\xi_2}{aL^2})$ to boxes covering a $\frac{\delta}{aL^2}$ neighbourhood of the unit parabola, so decoupling for the unit parabola applies to $f(\frac{x_1}{L}, \frac{x_2}{aL^2})$.
- This allows one to decouple f into $N := \frac{L}{\delta^{1/2}}$ many pieces, and obtain the corresponding decoupling inequality:

$$\|f\|_{L^p} \lesssim_\varepsilon N^\varepsilon \Big(\sum_{j=1}^{N} \|f_j\|_{L^p}^2 \Big)^{1/2} \quad \text{if } 2 \leq p \leq 6.$$



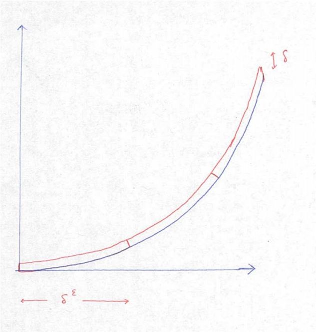
Pramanik-Seeger iteration: Decoupling for C^3 curves

- Let $\widehat{f}$ be supported in a δ neighbourhood of a C^3 curve $(t, \gamma(t))$ in $\widehat{\mathbb{R}^2}$ with $1 \leq |\gamma''(t)| \leq 2$ for all t (e.g.: a circle).
- Cut this neighbourhood into $N = \delta^{-1/2}$ boxes of size $\delta^{1/2} \times \delta$.
- A nice argument of Pramanik and Seeger allows one to decouple $f = \sum_{j=1}^{N} f_j$ by frequency localizing f to these boxes, and obtain

$$\|f\|_{L^p} \lesssim_{\varepsilon} N^{\varepsilon} \Big(\sum_{j=1}^{N} \|f_j\|_{L^p}^2 \Big)^{1/2} \quad \text{if } 2 \leq p \leq 6.$$

- A similar argument allows one to obtain decoupling for the sphere in $\widehat{\mathbb{R}^n}$, or the cone in $\widehat{\mathbb{R}^{n+1}}$.

- Proof: First trivially decouple f into $\delta^{-\varepsilon}$ many big pieces.



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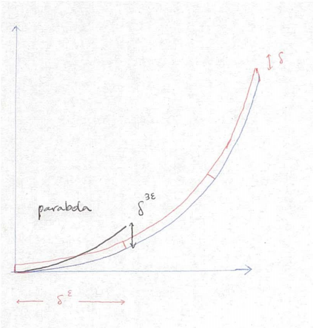

- By Taylor expanding γ, we see that each big piece of f is supported in a $\delta^{3\varepsilon}$ neighbourhood of a short parabolic arc.

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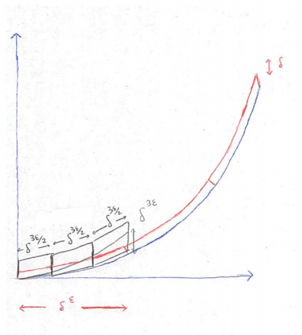

- By Taylor expanding γ, we see that each big piece of f is supported in a $\delta^{3\varepsilon}$ neighbourhood of a short parabolic arc.
- Bourgain-Demeter allows one to decouple big piece of f into $\delta^{\varepsilon}/(\delta^{3\varepsilon})^{1/2} = \delta^{-\varepsilon/2}$ many smaller pieces.
- The Fourier supports of the smaller pieces are even better approximated by short parabolic arcs.
- Repeat until one reaches boxes of size $\delta^{1/2} \times \delta$.

Fubini's theorem: Lifting to higher dimensions

- From the above, we can decouple functions on $\mathbb{R}^2$ whose Fourier support lies in a δ neighbourhood of the curve γ_0:

$$t \mapsto (t^2, t^3), \quad t \in [1/2, 1].$$

- We can lift it trivially to higher dimensions as follows.
- If f is a function on $\mathbb{R}^3$, then for fixed x_1, the Fourier support of $f(x_1, \cdot, \cdot)$ is contained in the projection of the Fourier support of f onto the ξ_2, ξ_3 plane:

$$\int_{\mathbb{R}^2} f(x_1, x_2, x_3) e^{-2\pi i(x_2\xi_2 + x_3\xi_3)} dx_2 dx_3 = \int_{\mathbb{R}} \widehat{f}(\xi_1, \xi_2, \xi_3) e^{2\pi i x_1 \xi_1} d\xi_1.$$

- If such projection is in a δ neighbourhood of γ_0, then we can decouple $f(x_1, \cdot, \cdot)$ into $\delta^{-1/2}$ many pieces for each x_1.
- Integrating with respect to x_1 and using Fubini's theorem, we can now decouple f in $\mathbb{R}^3$.
- This corollary of parabola decoupling is what one needs to prove decoupling for the moment curve in $\mathbb{R}^3$.

Proof of decoupling for the moment curve in $\mathbb{R}^3$

- Let $\gamma(t) = (t, t^2, t^3)$ for $t \in [0, 1]$ be the moment curve in $\mathbb{R}^3$.
- Partition $[0, 1]$ into N intervals $\{I\}$.
- For each I, and let θ_I be a box of size $\frac{1}{N} \times \frac{1}{N^2} \times \frac{1}{N^3}$ tangent to $\gamma(I)$, and f be a function with Fourier support in $\bigcup_I \theta_I$.
- Below we sketch a proof how one decouples f into $\sum_I f_I$, where f_I is the frequency localization of f to θ_I. Goal:

$$\|f\|_{L^{12}} \lesssim_\varepsilon N^\varepsilon \quad \text{if} \left(\sum_I \|f_I\|_{L^{12}}^2 \right)^{1/2} = 1.$$

- Notations: I is always one of these intervals of length $1/N$. For interval $J \subset [0, 1]$ with length $\geq 1/N$, write

$$f_J := \sum_{I \subset J} f_I,$$

where sum is over the subintervals I contained in J.
Also write $|J|$ for the length of an interval J.
$D(N)$ is the best constant in our main inequality.

Bilinearization

- Fix $\varepsilon > 0$. Since $f = \sum_{|J|=N^{-\varepsilon}} f_J$, if we expand $\|f\|_{L^{12}} = (\int |f|^{12})^{\frac{1}{12}}$, we need to estimate terms like

$$\left(\int |f_{J_1}|^6 |f_{J_2}|^6 \right)^{\frac{1}{12}} = \left\| |f_{J_1}|^{\frac{1}{2}} |f_{J_2}|^{\frac{1}{2}} \right\|_{L^{12}} \qquad (*)$$

  where $|J_1| = |J_2| = N^{-\varepsilon}$ are at distance $\geq N^{-\varepsilon}$.
- These are the main terms. Fix such J_1, J_2 from now on.
- For $0 < a, b \leq 1$ introduce

$$Q_{a,b} := \left\| \left\| \left\| |f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} \right\|_{L^{12}} \right\|_{\ell^4_{|K_1|=N^{-a}}} \right\|_{\ell^4_{|K_2|=N^{-b}}}$$

  where in the ℓ^4 norms, we sum over partitions $\{K_i\}$ of J_i.
- $Q_{\varepsilon,\varepsilon}$ is precisely the main term (*) we want to estimate.
- $Q_{1,1}$ is bounded by $\left(\sum_I \|f_I\|^2_{L^{12}} \right)^{1/2}$ since Hölder gives

$$\left\| |f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} \right\|_{L^{12}} \leq \|f_{K_1}\|^{1/2}_{L^{12}} \|f_{K_2}\|^{1/2}_{L^{12}}.$$

$$Q_{a,b} := \left\| \left\| \left\| |f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} \right\|_{L^{12}} \right\|_{\ell^4_{|K_1|=N^{-a}}} \right\|_{\ell^4_{|K_2|=N^{-b}}}$$

- We want to bound $Q_{\varepsilon,\varepsilon}$ by $Q_{1,1}$.
- To do so, for $0 < b \leq 1$ we bound $Q_{b,b}$ by $Q_{3b,3b}$, and then iterate:
  we bound $Q_{\varepsilon,\varepsilon}$ by $Q_{3\varepsilon,3\varepsilon}$, which in turn is bounded by $Q_{9\varepsilon,9\varepsilon}$, $Q_{27\varepsilon,27\varepsilon}$ etc until we reach $Q_{1,1}$.
- It will also be useful to consider, for $0 < a, b \leq 1$, a quantity

$$P_{a,b} := \left\| \left\| \left\| |f_{K_1}|^{\frac{1}{6}} |f_{K_2}|^{\frac{5}{6}} \right\|_{L^{12}} \right\|_{\ell^{12}_{|K_1|=N^{-a}}} \right\|_{\ell^{\frac{12}{5}}_{|K_2|=N^{-b}}} + \text{sym}$$

where sym is a similar term with roles of K_1 and K_2 reversed.

Bounding $Q_{b,b}$ by $Q_{3b,3b}$

- Three tools:
  1. Hölder's inequality
  2. L^2 orthogonality
  3. Decoupling for parabola in $\mathbb{R}^2$
- Four steps:
  1. $Q_{b,b} \leq P_{b,b}$ via Hölder
  2. $P_{b,b} \lesssim P_{3b,b}$ via L^2 orthogonality
  3. $P_{3b,b} \leq Q_{b,3b}^{1/3} D(N^{1-b})^{2/3}$ via Hölder
  4. $Q_{b,3b} \lesssim_{\varepsilon'} N^{\varepsilon'} Q_{3b,3b}$ via parabola decoupling
- Altogether, one gets

$$Q_{b,b} \lesssim_{\varepsilon'} N^{\varepsilon'} Q_{3b,3b}^{1/3} D(N^{1-b})^{2/3}$$

  which one can then bootstrap to show $D(N) \lesssim_{\varepsilon} N^{\varepsilon}$.

Hölder's inequality

- If $\theta \in [0,1]$ then for any p we have

$$\| |F|^\theta |G|^{1-\theta} \|_{L^p} \le \|F\|_{L^p}^\theta \|G\|_{L^p}^{1-\theta}. \tag{H1}$$

- Also works for ℓ^p norm of sequences:

$$\left\| |a_K|^\theta |b_K|^{1-\theta} \right\|_{\ell_K^p} \le \|a_K\|_{\ell_K^p}^\theta \|b_K\|_{\ell_K^p}^{1-\theta}.$$

- More generally, if $\theta \in [0,1]$ and $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$, then

$$\left\| |a_K|^\theta |b_K|^{1-\theta} \right\|_{\ell_K^p} \le \|a_K\|_{\ell_K^q}^\theta \|b_K\|_{\ell_K^r}^{1-\theta}. \tag{H2}$$

(The earlier inequality is the special case $q = r = p$.)

Step 1: $Q_{b,b} \le P_{b,b}$ via Hölder

$$Q_{b,b} = \left\| \left\| \left\| |f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} \right\|_{L^{12}} \right\|_{\ell^4_{|K_1|=N^{-b}}} \right\|_{\ell^4_{|K_2|=N^{-b}}}$$

$$P_{b,b} = \left\| \left\| \left\| |f_{K_1}|^{\frac{1}{6}} |f_{K_2}|^{\frac{5}{6}} \right\|_{L^{12}} \right\|_{\ell^{12}_{|K_1|=N^{-b}}} \right\|_{\ell^{\frac{12}{5}}_{|K_2|=N^{-b}}} + \text{sym}$$

- To bound $Q_{b,b}$ by $P_{b,b}$, first write

$$|f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} = (|f_{K_1}|^{\frac{1}{6}} |f_{K_2}|^{\frac{5}{6}})^{\frac{1}{2}} (|f_{K_2}|^{\frac{1}{6}} |f_{K_1}|^{\frac{5}{6}})^{\frac{1}{2}}$$

  and apply (H1) with $\theta = 1/2$, $p = 12$:

$$\left\| |f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} \right\|_{L^{12}} \le \left\| |f_{K_1}|^{\frac{1}{6}} |f_{K_2}|^{\frac{5}{6}} \right\|_{L^{12}}^{\frac{1}{2}} \left\| |f_{K_2}|^{\frac{1}{6}} |f_{K_1}|^{\frac{5}{6}} \right\|_{L^{12}}^{\frac{1}{2}}.$$

- Now take $\ell^4_{K_1}$ and then $\ell^4_{K_2}$ norm of both sides, and apply (H2) with $\theta = \frac{1}{2}$ and $\frac{1}{4} = \frac{\theta}{12} + \frac{1-\theta}{\frac{12}{5}}$ or $\frac{1}{4} = \frac{\theta}{\frac{12}{5}} + \frac{1-\theta}{12}$. Together with Minkowski for moving the $\ell^{12}_{K_2}$ norm inside the $\ell^{12/5}_{K_1}$ norm in the second term, we get $Q_{b,b} \le P_{b,b}$.

Step 2: $P_{b,b} \lesssim P_{3b,b}$ via L^2 orthogonality

$$P_{a,b} = \left\| \left\| \left\| |f_{K_1}|^{\frac{1}{6}} |f_{K_2}|^{\frac{5}{6}} \right\|_{L^{12}} \right\|_{\ell^{12}_{|K_1|=N^{-a}}} \right\|_{\ell^{\frac{12}{5}}_{|K_2|=N^{-b}}} + \text{sym}$$

- The key is that for each $K_2 \subset J_2$ with $|K_2| = N^{-b}$, we have
$$\left\| \left\| |f_{K_1}|^{\frac{1}{6}} |f_{K_2}|^{\frac{5}{6}} \right\|_{L^{12}} \right\|_{\ell^{12}_{|K_1|=N^{-b}}} \leq \left\| \left\| |f_{K_1}|^{\frac{1}{6}} |f_{K_2}|^{\frac{5}{6}} \right\|_{L^{12}} \right\|_{\ell^{12}_{|K_1|=N^{-3b}}}.$$
- Upon taking power 6 of both sides, this is the same as
$$\left\| \left\| |f_{K_1}| |f_{K_2}|^5 \right\|_{L^2} \right\|_{\ell^2_{|K_1|=N^{-b}}} \leq \left\| \left\| |f_{K_1}| |f_{K_2}|^5 \right\|_{L^2} \right\|_{\ell^2_{|K_1|=N^{-3b}}}.$$
- This follows from the fact that for each $K_1 \subset J_1$,
$$\left\| |f_{K_1}| |f_{K_2}|^5 \right\|_{L^2} \leq \left\| \left\| |f_{K_1'}| |f_{K_2}|^5 \right\|_{L^2} \right\|_{\ell^2_{K_1' \subset K_1, |K_1'|=N^{-3b}}},$$
which in turn is a consequence of L^2 orthogonality (and local constancy for $|f_{K_2}|^5$ at the correct scale).

Step 3: $P_{3b,b} \le Q_{b,3b}^{1/3} D(N^{1-b})^{2/3}$ via Hölder

$$P_{3b,b} = \left\| \left\| \left\| |f_{K_1}|^{\frac{1}{6}} |f_{K_2}|^{\frac{5}{6}} \right\|_{L^{12}} \right\|_{\ell^{12}_{|K_1|=N^{-3b}}} \right\|_{\ell^{\frac{12}{5}}_{|K_2|=N^{-b}}} + \text{sym}$$

$$Q_{3b,3b} = \left\| \left\| \left\| |f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} \right\|_{L^{12}} \right\|_{\ell^{4}_{|K_1|=N^{-3b}}} \right\|_{\ell^{4}_{|K_2|=N^{-3b}}}$$

- To bound $P_{3b,b}$ by $Q_{3b,3b}$, first write

  $$|f_{K_1}|^{\frac{1}{6}} |f_{K_2}|^{\frac{5}{6}} = (|f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}})^{\frac{1}{3}} (|f_{K_2}|)^{\frac{2}{3}}$$

  and apply (H1) with $\theta = 1/3$, $p = 12$:

  $$\left\| |f_{K_1}|^{\frac{1}{6}} |f_{K_2}|^{\frac{5}{6}} \right\|_{L^{12}} \le \left\| |f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} \right\|_{L^{12}}^{\frac{1}{3}} \|f_{K_2}\|_{L^{12}}^{\frac{2}{3}}.$$

- Now take $\ell^{12}_{K_1}$ and then $\ell^{12/5}_{K_2}$ norm of both sides, and apply (H2) with $\theta = 1/3$ and $\frac{1}{\frac{12}{5}} = \frac{\theta}{4} + \frac{1-\theta}{2}$ in doing the latter.

  By parabolic rescaling, we get $P_{3b,b} \le Q_{b,3b}^{1/3} D(N^{1-b})^{2/3}$.

Step 4: $Q_{b,3b} \lesssim_{\varepsilon'} N^{\varepsilon'} Q_{3b,3b}$ via parabola decoupling

$$Q_{a,3b} = \left\| \left\| \left\| |f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} \right\|_{L^{12}} \right\|_{\ell^4_{|K_1|=N^{-a}}} \right\|_{\ell^4_{|K_2|=N^{-3b}}}$$

- The key is that for each $K_2 \subset J_2$ with $|K_2| = N^{-3b}$, we have
$$\left\| \left\| |f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} \right\|_{L^{12}} \right\|_{\ell^4_{|K_1|=N^{-b}}} \lesssim_{\varepsilon'} N^{\varepsilon'} \left\| \left\| |f_{K_1}|^{\frac{1}{2}} |f_{K_2}|^{\frac{1}{2}} \right\|_{L^{12}} \right\|_{\ell^4_{|K_1|=N^{-3b}}}$$
- Upon taking power 2 of both sides, this is the same as
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which in turn is a consequence of the corollary of parabola decoupling we mentioned earlier (and local constancy for $|f_{K_2}|$ at the correct scale).

Remarks

- The above proof is from joint work with Shaoming Guo, Zane Kun Li and Pavel Zorin-Kranich.
- It was inspired by Wooley's work on efficient congruencing.
- The original proof of Bourgain, Demeter and Guth uses a multilinear reduction (as opposed to bilinear) and uses incidence estimates (multilinear Kakeya).
- We actually have $D(N) \lesssim \exp(C\frac{\log N}{\log\log N})$ (Schippa 2023).
- This is in turn based on an improved decoupling constant for the parabola at $p = 6$, due to Guth, Maldague and Wang. They proved a slightly easier estimate using square functions:

$$\|f\|_{L^6} \lesssim (\log N)^c \|f\|_{L^2}^{1/3} \Big(\sum_{j=1}^{N} \|f_j\|_{L^\infty}^2 \Big)^{1/3}$$

and then upgraded it to decoupling.

Questions

- We know the decoupling constant for the parabola in $\widehat{\mathbb{R}}^2$ (for $p = 6$) is just a power of $\log N$. What about the circle?
- Can one give a direct proof of decoupling for the circle in $\widehat{\mathbb{R}}^2$ without the Pramanik-Seeger iteration?
- Analogous question for the cone in $\widehat{\mathbb{R}^{2+1}}$?
- Small cap decoupling for the moment curve in $\widehat{\mathbb{R}}^n$, $n \geq 4$?
- What other tools do we have when decoupling alone does not give the optimal estimate?