Improved discrete restriction for the parabola

Po-Lam Yung

Australian National University
The Chinese University of Hong Kong

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Discrete restriction

- For every positive integer M, let $K(M)$ be the best constant so that

$$\left\| \sum_{n=1}^{M} b_n e(nx + n^2 t) \right\|_{L^6([0,1]^2)} \leq K(M) \left(\sum_{n=1}^{M} |b_n|^2 \right)^{1/2}$$

for every $b_1, \dots, b_M \in \mathbb{C}$, where $e(t) := \exp(2\pi i t)$.

- Study of $K(M)$ is motivated by the study of the periodic Schrödinger equation on $\mathbb{T}$ (dating back to Bourgain 1993): The solution to the initial value problem

$$\begin{cases} 2\pi i \partial_t u = \partial_x^2 u \\ u(x,0) = \sum_{n=1}^{M} b_n e(nx) \end{cases}$$

is precisely $\sum_{n=1}^{M} b_n e(nx + n^2 t)$.

Bounds on $K(M)$

- Bourgain showed that

$$(\log M)^{1/6} \lesssim K(M) \leq \exp(O(\frac{\log M}{\log\log M})).$$

- Building upon Bourgain and Demeter's work on Fourier decoupling for the parabola, and Wooley's work on efficient congruencing (as in the exposition of Pierce), Li gave a new proof of this upper bound of $K(M)$.
- Very recently Guth, Maldague and Wang improved the best constant for the Fourier decoupling inequality for the parabola, sharpening the upper bound of $K(M)$ to

$$K(M) \lesssim (\log M)^c$$

  for some finite, but unspecified constant c.
- In joint work with Shaoming Guo and Zane Kun Li, we improve this bound further, to

$$K(M) \lesssim_\varepsilon (\log M)^{2+\varepsilon}$$

  for every $\varepsilon > 0$.

Overview of our proof

- Our proof of the upper bound for $K(M)$ follows closely that of Guth, Maldague and Wang.
- In particular, we use a decomposition of square functions into high and low frequencies, which we will explain in due course.
- The main new difference is that we work p-adically.
- Indeed, we observed that if we are only interested in moments of exponential sums, then since $p = 6$ is an even integer, we do not need the full power of Fourier decoupling on the parabola in $\mathbb{R}^2$.
- Rather, we study Fourier decoupling for the parabola in $\mathbb{Q}_p^2$.
- In place of p, we write q for any odd prime, and work on $\mathbb{Q}_q^2$.
- We will describe some Fourier analysis on $\mathbb{Q}_q^2$, discuss why it is advantageous to work on $\mathbb{Q}_q^2$ over $\mathbb{R}^2$, and explain how the proof of Guth, Maldague and Wang works.

The q-adic field $\mathbb{Q}_q$

- From now on, q is a fixed odd prime.
- $\mathbb{Q}_q$ is the set of all formal power series

$$\sum_{j=k}^{\infty} a_j q^j, \quad \text{where } k \in \mathbb{Z} \text{ and } a_j \in \{0, \ldots, q-1\} \text{ for all } j.$$

- It is a field of characteristic zero if we add and multiply with carries.
- The q-adic absolute value on $\mathbb{Q}_q$ is given by $|0| = 0$ and $|\sum_{j=k}^{\infty} a_j q^j| = q^{-k}$ if $a_k \neq 0$. It satisfies an ultrametric inequality:

$$|x + y| \leq \max\{|x|, |y|\} \quad \text{for all } x, y \in \mathbb{Q}_q$$

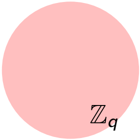
  with equality if $|x| \neq |y|$.
- The ring of q-adic integers is then $\mathbb{Z}_q := \{x \in \mathbb{Q}_q : |x| \leq 1\}$.
- We write dx for the Haar measure on $\mathbb{Q}_q$ with $\int_{\mathbb{Z}_q} dx = 1$.

Geometry of $\mathbb{Q}_q$

- If $a \in \mathbb{Q}_q$, then the q-adic interval of 'length' $q^{-\ell}$ around a is the set

  $$B(a, q^{-\ell}) := \{x \in \mathbb{Q}_q \colon |x - a| \leq q^{-\ell}\} = \{x \in \mathbb{Q}_q \colon x \equiv a \pmod{q^\ell}\}.$$

  ($x \equiv a \pmod{q^\ell}$ means $q^{-\ell}(x - a) \in \mathbb{Z}_q$.)
- Any two q-adic intervals of the same length are either disjoint or equal. Indeed, if $|b - a| \leq q^{-\ell}$, then $B(a, q^{-\ell}) = B(b, q^{-\ell})$, and if $|b - a| > q^{-\ell}$, then $B(a, q^{-\ell}) \cap B(b, q^{-\ell}) = \emptyset$.
- Each interval of length $q^{-\ell}$ is the disjoint union of q equidistant sub-intervals of lengths $q^{-\ell-1}$.
- For example, $\mathbb{Z}_q = B(1, q^{-1}) \sqcup B(2, q^{-1}) \sqcup \cdots \sqcup B(q, q^{-1})$, a disjoint union of q equidistant intervals of length q^{-1}.



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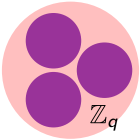

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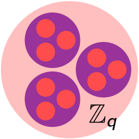

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Fourier analysis on $\mathbb{Q}_q$

- A Schwartz function on $\mathbb{Q}_q$ is a finite linear combination of characteristic functions of intervals (over $\mathbb{C}$).
- Let $\chi\colon \mathbb{Q}_q \to \mathbb{C}^\times$ be the additive character on $\mathbb{Q}_q$ given by

$$\chi\left(\sum_{j=k}^{\infty} a_j q^j\right) = e\left(\sum_{j=k}^{-1} a_j q^j\right).$$

- The Fourier transform on $\mathbb{Q}_q$ is defined for all Schwartz functions by

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_q} f(x)\chi(-x\xi)dx, \quad \xi \in \mathbb{Q}_q.$$

  (Henceforth we work only with Schwartz functions.)
- The Fourier inversion formula then says

$$f(x) = \int_{\mathbb{Q}_q} \widehat{f}(\xi)\chi(x\xi)d\xi.$$

- The convolution on $\mathbb{Q}_q$ is defined by $f * g(x) = \int_{\mathbb{Q}_q} f(x-y)g(y)dy$. It interacts well with the Fourier transform: $\widehat{f * g} = \widehat{f}\widehat{g}$.

Properties of the Fourier transform on $\mathbb{Q}_q$

- We have $\widehat{1_{\mathbb{Z}_q}} = 1_{\mathbb{Z}_q}$. Indeed,

$$\widehat{1_{\mathbb{Z}_q}}(\xi) = \int_{\mathbb{Z}_q} \chi(-x\xi)dx = \begin{cases} 1 & \text{if } |\xi| \leq 1 \\ 0 & \text{if } |\xi| > 1 \end{cases}$$

  because $x \mapsto \chi(-x\xi)$ defines a character on the compact group $\mathbb{Z}_q$, which is non-trivial if and only if $|\xi| > 1$.

- Also, if $M_a f(x) := \chi(-ax)f(x)$, then

$$\widehat{M_a f}(\xi) = \widehat{f}(\xi + a),$$

  and if $D_{q^\ell} f(x) = f(q^\ell x)$, then

$$\widehat{D_{q^\ell} f}(\xi) = q^{-\ell} D_{q^{-\ell}} \widehat{f}(\xi).$$

The uncertainty principle for the Fourier transform on $\mathbb{Q}_q$

- As a result, we can prove rigorously the uncertainty principle for the Fourier transform on $\mathbb{Q}_q$:
- If $\ell \in \mathbb{Z}$ and $\widehat{f}$ is compactly supported on a q-adic interval of length $q^{-\ell}$, then $|f|$ is a constant on every q-adic interval of length q^{ℓ}.
- Proof: First suppose $\widehat{f}$ is compactly supported on $\mathbb{Z}_q$. Then $\widehat{f} = \widehat{f} 1_{\mathbb{Z}_q} = \widehat{f} \widehat{1_{\mathbb{Z}_q}}$, so

$$f(x) = f * 1_{\mathbb{Z}_q}(x) = \int_{|x-y| \leq 1} f(y) dy$$

which is constant on q-adic intervals of length 1.

If now $\widehat{f}$ is compactly supported on $B(a, q^{-\ell})$ for some $a \in \mathbb{Q}_q^2$ and $\ell \in \mathbb{Z}$, then $\widehat{f}(a + q^{-\ell}\xi) = q^{\ell} \widehat{D_{q^{\ell}} M_a f}(\xi)$ is supported on $\mathbb{Z}_q$. So its inverse Fourier transform $q^{\ell} \chi(-aq^{\ell}x) f(q^{\ell}x)$ is constant on intervals of length 1. Hence $|f(q^{\ell}x)|$ is constant on intervals of length 1, which means $|f(x)|$ is constant on intervals of length q^{ℓ}.

Fourier analysis on $\mathbb{Q}_q^2$

- $\mathbb{Q}_q^2$ is a (2-dimensional) vector space over $\mathbb{Q}_q$.
- Define norm $|x-a| = \max\{|x_1-a_1|, |x_2-a_2|\}$ if $x, a \in \mathbb{Q}_q^2$.
- Balls are squares, written $B(a,r) = \{x \in \mathbb{Q}_q^2 \colon |x-a| \leq r\}$.
- $\mathbb{Q}_q^2$ is equipped with a Haar measure $dx = dx_1 dx_2$.
- Schwartz functions on $\mathbb{Q}_q^2$ are just finite linear combinations of characteristic functions of rectangles (products of q-adic intervals).
- The Fourier transform on $\mathbb{Q}_q^2$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_q^2} f(x)\chi(-x\cdot\xi)dx$$

  where $x\cdot\xi = x_1\xi_1 + x_2\xi_2$ if $x = (x_1,x_2), \xi = (\xi_1,\xi_2) \in \mathbb{Q}_q^2$.
- Fourier inversion reads

$$f(x) = \int_{\mathbb{Q}_q^2} \widehat{f}(\xi)\chi(x\cdot\xi)d\xi,$$

  and one can formulate a rigorous version of the uncertainty principle.
- One advantage of working over $\mathbb{Q}_q^2$ as opposed to $\mathbb{R}^2$ is that the uncertainty principle over $\mathbb{Q}_q^2$ is so clean. We thereby avoid a lot of technical difficulties Guth, Maldague and Wang encountered on $\mathbb{R}^2$.

Geometry in $\mathbb{Q}_q^2$

- Another slight advantage of working in $\mathbb{Q}_q^2$ is that the geometry trivializes completely when we zoom in to the right scale.
- Let $R \in q^{-2\mathbb{N}}$. We will be dealing with Schwartz functions on $\mathbb{Q}_q^2$ whose Fourier transform is supported on a R^{-1} neighborhood of the unit parabola:

$$\{(\xi_1, \xi_2) \in \mathbb{Q}_q^2 \colon \xi_1 \in \mathbb{Z}_q, |\xi_2 - \xi_1^2| \leq R^{-1}\}.$$

- If we restrict ξ_1 to a q-adic interval θ of length $R^{-1/2}$, the set

$$\{(\xi_1, \xi_2) \in \mathbb{Q}_q^2 \colon \xi_1 \in \theta, |\xi_2 - \xi_1^2| \leq R^{-1}\}$$

actually becomes a parallelogram: it is equal to

$$P_\theta := \{(\xi_1, \xi_2) \in \mathbb{Q}_q^2 \colon |\xi_1 - a| \leq R^{-1/2}, |\xi_2 - 2a\xi_1 + a^2| \leq R^{-1}\}$$

for any $a \in \theta$. A dual paralleogram to P_θ is then

$$P_\theta^* := \{(x, t) \in \mathbb{Q}_q^2 \colon |x + 2at| \leq R^{1/2}, |t| \leq R\}.$$

- The uncertainty principle shows that if $\widehat{f}$ is supported in P_θ, then $|f|$ is constant on translates of P_θ^*.

Main theorem

- Theorem. (Guo-Li-Y.) Let $R \in q^{-2\mathbb{N}}$. Let $\{\theta\}$ be a partition of $\mathbb{Z}_q$ into a disjoint union of q-adic intervals of length $R^{-1/2}$, and for each θ, let f_θ be a Schwartz function on $\mathbb{Q}_q^2$ whose Fourier transform is supported in the parallelogram P_θ. Write

  $$f := \sum_{|\theta|=R^{-1/2}} f_\theta.$$

  Then for every $\varepsilon > 0$, there exists C_ε such that

  $$\int_{\mathbb{Q}_q^2} |f|^6 \leq C_\varepsilon (\log R)^{12+\varepsilon} \left(\sum_{|\theta|=R^{-1/2}} \|f_\theta\|^2_{L^\infty(\mathbb{Q}_q^2)} \right)^2 \int_{\mathbb{Q}_q^2} |f|^2.$$

- Guth, Maldague and Wang proved a similar theorem on $\mathbb{R}^2$, with $(\log R)^c$ in place of $(\log R)^{12}$.
- Since the P_θ's are disjoint (they have disjoint projections onto the ξ_1 axis), we have

  $$\int_{\mathbb{Q}_q^2} |f|^2 = \sum_{|\theta|=R^{-1/2}} \int_{\mathbb{Q}_q^2} |f_\theta|^2.$$

Relevance for discrete restriction

- Suppose now $M \in q^{\mathbb{N}}$ and $b_1, \ldots, b_M \in \mathbb{C}$.
- Write $R = M^2$. Partition $\mathbb{Z}_q$ into disjoint union of M q-adic intervals $\{\theta\}$, each of length $R^{-1/2} = M^{-1}$. Then each θ contains exactly one number from $\{1, \ldots, M\}$. Let

$$f_\theta(x,t) = b_n \chi(nx + n^2 t) 1_{|(x,t)| \le R^{10}}$$

  where n is the unique element in $\{1, \ldots, M\} \cap \theta$. Then $\widehat{f_\theta}$ is supported in $B((n, n^2), R^{-10}) \subset P_\theta$.
- For $f = \sum_\theta f_\theta$, we have

$$\int_{\mathbb{Q}_q^2} |f|^6 = R^{10} \int_{[0,1]^2} \Big| \sum_{n=1}^{M} b_n e(nx + n^2 t) \Big|^6 dxdt$$

  (indeed this works with 6 replaced by any even integer) and

$$\int_{\mathbb{Q}_q^2} |f|^2 = R^{10} \sum_{n=1}^{M} |b_n|^2, \qquad \sum_{|\theta| = R^{-1/2}} \|f_\theta\|^2_{L^\infty(\mathbb{Q}_q^2)} = \sum_{n=1}^{M} |b_n|^2.$$

  Thus asserted bound for $K(M)$ follows from our theorem on $\mathbb{Q}_q^2$.

Iterating through many scales

- Bourgain and Demeter proved the first Fourier decoupling theorem for the parabola by working at many (physical) scales: they used

$$R > \frac{R}{K} > \frac{R}{K^2} > \frac{R}{K^3} > \cdots > 1$$

  where K is a large but fixed constant (independent of R).
- Guth, Maldague and Wang proved their theorem using *fewer* scales:

$$R > \frac{R}{(\log R)^6} > \frac{R}{(\log R)^{12}} > \frac{R}{(\log R)^{18}} > \cdots > 1$$

- We proved our theorem using roughly the same number of scales:

$$R > \frac{R}{(\log R)^{\varepsilon}} > \frac{R}{(\log R)^{2\varepsilon}} > \frac{R}{(\log R)^{3\varepsilon}} > \cdots > 1$$

  where $\varepsilon > 0$ is a small parameter.
- Our proof is, for the most part, parallel to that of Guth, Maldague and Wang.

An L^4 square function estimate

- The idea of Guth, Maldague and Wang is that to estimate $\int |f|^6$ one should exploit efficiently what we know at L^4.
- For instance, a classical estimate of Fefferman and Cordoba says that

$$\int_{\mathbb{Q}_q^2} |f|^4 \leq 2 \int_{\mathbb{Q}_q^2} \Big(\sum_{|\theta|=R^{-1/2}} |f_\theta|^2 \Big)^2.$$

- We want to show

$$\int_{\mathbb{Q}_q^2} |f|^6 \lessapprox \Big(\sum_{|\theta|=R^{-1/2}} \|f_\theta\|^2_{L^\infty(\mathbb{Q}_q^2)} \Big)^2 \int_{\mathbb{Q}_q^2} |f|^2$$

  where $\lessapprox$ means $\lesssim (\log R)^c$.
- By pigeonholing, we may assume that $\|f_\theta\|_{L^\infty(\mathbb{Q}_q^2)}$ are all comparable to each other (as long as they are not zero).

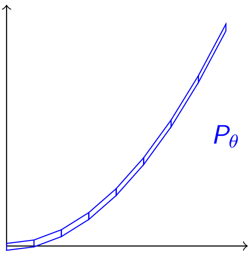
- Now
$$\int_{\mathbb{Q}_q^2} |f|^6 \leq \|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \int_{\mathbb{Q}_q^2} |f|^4 \leq \|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \int_{\mathbb{Q}_q^2} \Big(\sum_{|\theta|=R^{-1/2}} |f_\theta|^2 \Big)^2.$$

- If (miracle!) $\int_{\mathbb{Q}_q^2} \left(\sum_{|\theta|=R^{-1/2}} |f_\theta|^2 \right)^2 \lessapprox \int_{\mathbb{Q}_q^2} \sum_{|\theta|=R^{-1/2}} |f_\theta|^4$, then
$$\int_{\mathbb{Q}_q^2} |f|^6 \lessapprox \|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \sum_{|\theta|=R^{-1/2}} \int_{\mathbb{Q}_q^2} |f_\theta|^4$$
$$\leq \|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \sup_{\theta'} \|f_{\theta'}\|^2_{L^\infty(\mathbb{Q}_q^2)} \sum_{|\theta|=R^{-1/2}} \int_{\mathbb{Q}_q^2} |f_\theta|^2$$
$$= \|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \sup_{\theta'} \|f_{\theta'}\|^2_{L^\infty(\mathbb{Q}_q^2)} \int_{\mathbb{Q}_q^2} |f|^2.$$

- But $\sup_{\theta'} \|f_{\theta'}\|_{L^\infty(\mathbb{Q}_q^2)} \lesssim \|f_\theta\|_{L^\infty(\mathbb{Q}_q^2)}$ as long as $\|f_\theta\|_{L^\infty(\mathbb{Q}_q^2)} \neq 0$. Thus
$$\|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \sup_{\theta} \|f_\theta\|^2_{L^\infty(\mathbb{Q}_q^2)} \lesssim \Big(\sum_{|\theta|=R^{-1/2}} \|f_\theta\|_{L^\infty(\mathbb{Q}_q^2)} \Big)^2 \sup_{\theta'} \|f_{\theta'}\|^2_{L^\infty(\mathbb{Q}_q^2)}$$
$$\lesssim \Big(\sum_{|\theta|=R^{-1/2}} \|f_\theta\|^2_{L^\infty(\mathbb{Q}_q^2)} \Big)^2.$$

Does miracle happen?

- The miracle $\int_{\mathbb{Q}_q^2} \left(\sum_{|\theta|=R^{-1/2}} |f_\theta|^2 \right)^2 \lessapprox \int_{\mathbb{Q}_q^2} \sum_{|\theta|=R^{-1/2}} |f_\theta|^4$ happens when the $|f_\theta|^2$'s are (almost) orthogonal to each other.
- Let's look at the Fourier support of $|f_\theta|^2$ as θ varies.
- $\widehat{|f_\theta|^2} = \widehat{f_\theta} * \widehat{\overline{f_\theta}}$ is supported on $P_\theta + (-P_\theta) := P_\theta^o$ where P_θ^o is a translate of P_θ that contains the origin.
- The P_θ^o's are all contained inside a ball of radius $R^{-1/2}$ centered at the origin. They overlap a lot near the origin as θ varies, but overlap less and less as one moves away from the origin.
- So miracle happens away from the origin, but not so much near it.



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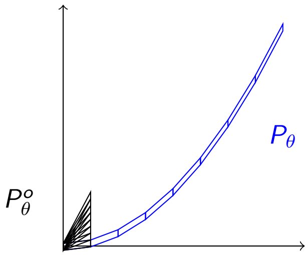

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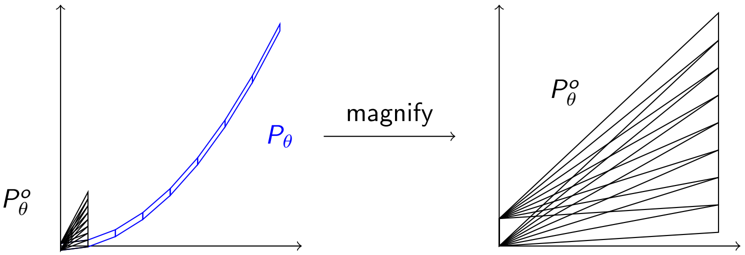

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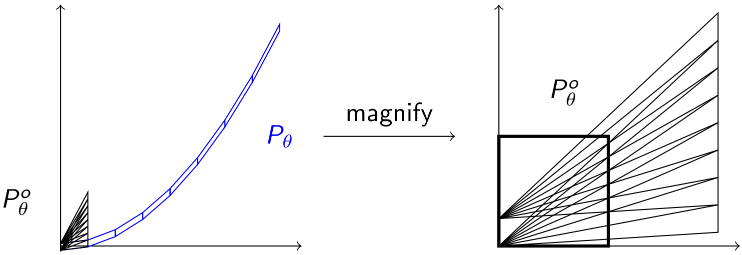


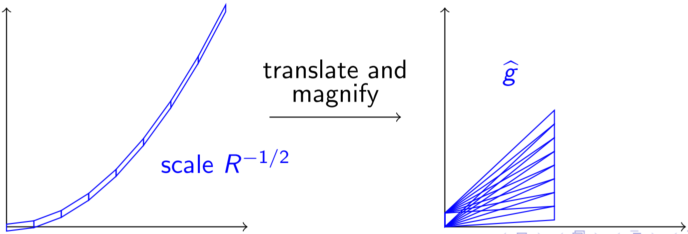
Introducing intermediate scales help

- We were trying to pass from $|f| = |\sum_{|\theta|=R^{-1/2}} f_\theta|$ to the square function $g := \sum_{|\theta|=R^{-1/2}} |f_\theta|^2$.
- It helps to introduce an intermediate scale R_* satisfying $R^{1/2} \le R_* \le R$, and the corresponding square function

$$g_* := \sum_{|\tau|=R_*^{-1/2}} |f_\tau|^2, \qquad f_\tau = \sum_{\substack{\theta\subset\tau \\ |\theta|=R^{-1/2}}} f_\theta.$$

- It is certainly easier to pass from $|f|$ to g_* than from $|f|$ to g.
- But g can be recovered as the low frequency part of g_*: Let's write

$$g_* = g_*^{low} + g_*^{high}, \quad \widehat{g_*^{low}} := \widehat{g_*} 1_{|\xi|\le R^{-1/2}}, \quad \widehat{g_*^{high}} := \widehat{g_*} 1_{|\xi|> R^{-1/2}}.$$



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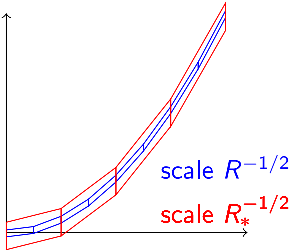

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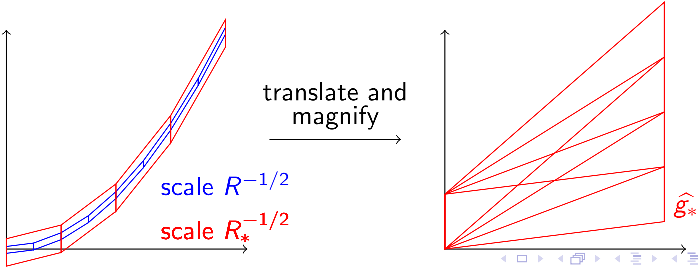

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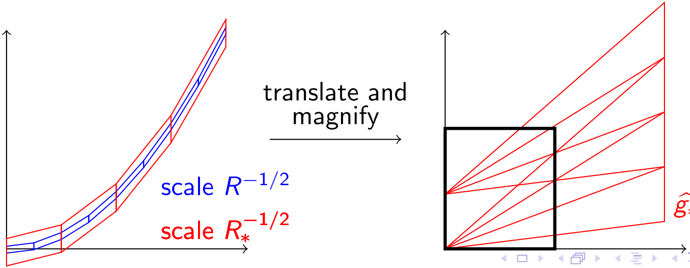

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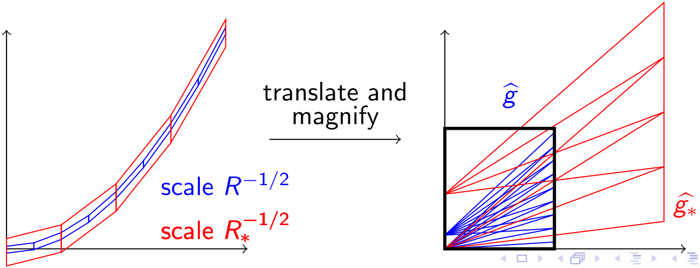

Advantages of the high-low decomposition

- Then g_*^{low} is precisely g. This is because

$$\widehat{g_*^{low}} = \sum_{|\tau|=R_*^{-1/2}} \widehat{|f_\tau|^2} 1_{|\xi|\le R^{-1/2}} = \sum_{|\tau|=R_*^{-1/2}} \sum_{\substack{\theta,\theta'\subset\tau \\ |\theta|=|\theta'|=R^{-1/2}}} \widehat{f_\theta \overline{f_{\theta'}}} 1_{|\xi|\le R^{-1/2}}$$

and $\widehat{f_\theta \overline{f_{\theta'}}} 1_{|\xi|\le R^{-1/2}} \begin{cases} = \widehat{f_\theta} * \widehat{\overline{f_{\theta'}}} 1_{|\xi|\le R^{-1/2}} = 0 & \text{if } \theta \neq \theta' \\ = \widehat{|f_\theta|^2} 1_{|\xi|\le R^{-1/2}} = \widehat{|f_\theta|^2} & \text{if } \theta = \theta'. \end{cases}$.

- On the other hand, $\widehat{g_*^{high}} = \sum_{|\tau|=R_*^{-1/2}} \widehat{|f_\tau|^2} 1_{|\xi|>R^{-1/2}}$ is a sum of functions whose support overlaps at most $(R/R_*)^{1/2}$ times, so

$$\int_{\mathbb{Q}_q^2} |g_*^{high}|^2 \le (R/R_*)^{1/2} \sum_{|\tau|=R_*^{-1/2}} \int_{\mathbb{Q}_q^2} |f_\tau|^4$$

which by Hölder's inequality is

$$\le (R/R_*)^2 \sum_{|\theta|=R^{-1/2}} \int_{\mathbb{Q}_q^2} |f_\theta|^4.$$

Putting these together

- Let $H := \{x \in \mathbb{Q}_q^2 \colon g_*(x) \leq A|g_*^{high}(x)|\}$ where A is a constant to be determined. We write $\int_{\mathbb{Q}_q^2} |f|^6 = \int_H |f|^6 + \int_{H^c} |f|^6$.
- For the first term, if we had (problem 1!) $\int_H |f|^4 \leq \int_H g_*^2$, then

$$\int_H |f|^6 \leq \|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \int_H |f|^4 \leq_? \|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \int_H g_*^2 \leq A^2 \|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \int_{\mathbb{Q}_q^2} |g_*^{high}|^2$$

which is bounded by

$$\leq A^2 (R/R_*)^2 \|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \sum_{|\theta| = R^{-1/2}} \int_{\mathbb{Q}_q^2} |f_\theta|^4$$

$$\leq A^2 (R/R_*)^2 \|f\|^2_{L^\infty(\mathbb{Q}_q^2)} \sup_{\theta'} \|f_{\theta'}\|^2_{L^\infty(\mathbb{Q}_q^2)} \sum_{|\theta| = R^{-1/2}} \int_{\mathbb{Q}_q^2} |f_\theta|^2$$

$$\leq A^2 (R/R_*)^2 \Big(\sum_{|\theta| = R^{-1/2}} \|f_\theta\|^2_{L^\infty(\mathbb{Q}_q^2)} \Big)^2 \int_{\mathbb{Q}_q^2} |f|^2.$$

This is good if A and R/R_* are (small) powers of $\log R$.

- For the second term, note if $x \notin H$, then $|g_*^{high}(x)| \leq A^{-1} g_*(x)$, so

$$g_*(x) = g_*^{low}(x) + g_*^{high}(x) \leq g_*^{low}(x) + A^{-1} g_*(x)$$

which implies

$$g_*(x) \leq (1 - A^{-1})^{-1} g_*^{low}(x) = (1 - A^{-1})^{-1} \sum_{|\theta| = R^{-1/2}} |f_\theta|^2.$$

Hence

$$\begin{aligned} \int_{H^c} |f|^6 &\leq R_*^{3/2} \int_{H^c} g_*^3 \qquad \text{(Hölder)} \\ &\leq R_*^{3/2} (1 - A^{-1})^{-3} \int_{\mathbb{Q}_q^2} \Big(\sum_{|\theta| = R^{-1/2}} |f_\theta|^2 \Big)^3 \\ &\leq R_*^{3/2} (1 - A^{-1})^{-3} \Big(\sum_{|\theta| = R^{-1/2}} \|f_\theta\|^2_{L^\infty(\mathbb{Q}_q^2)} \Big)^2 \int_{\mathbb{Q}_q^2} |f|^2. \end{aligned}$$

- Altogether, assuming problem 1 is solved, we have

$$\int_{\mathbb{Q}_q^2} |f|^6 \leq (A^2 (R/R_*)^2 + R_*^{3/2} (1 - A^{-1})^{-3}) \Big(\sum_{|\theta| = R^{-1/2}} \|f_\theta\|^2_{L^\infty(\mathbb{Q}_q^2)} \Big)^2 \int_{\mathbb{Q}_q^2} |f|^2.$$

Problem 2: $R_*^{3/2}$ is much bigger than $\log R$!

The ways out: many scales and bilinearize (plus pruning)

- To solve problem 2, we need to introduce $N \sim \log R$ scales, and apply the Hölder's inequality only once on the set of all x that is not in the high set at any scale. The loss incurred when Hölder is applied once is insignificant, so we are good there.
- But since we needed N many scales, instead of $(1 - A^{-1})^{-3}$ we accumulate a factor of $(1 - A^{-1})^{-3N}$. Thus when we defined the high sets, we want A to be large, so that $(1 - A^{-1})^{-3N}$ is bounded; this suggests that we choose $A \sim N \sim \log R$.
- To solve problem 1, we really can't say $\int_H |f|^4 \leq \int_H g_*^2$ via Fefferman and Cordoba. But we can bilinearize and use bilinear restriction instead: this gives us access to smaller physical scales at which we have a nice partition of H.
- In fact we bilinearize a little more efficiently than Guth, Maldague and Wang, using a Whitney decomposition as in previous work with Zorin-Kranich.
- The introduction of many scales also requires one to bound $\|f\|_{L^\infty(\mathbb{Q}_q^2)}^2 \sup_{|\tau|=R_k^{-1/2}} \|f_\tau\|_{L^\infty(\mathbb{Q}_q^2)}^2$ for many intermediate R_k's. This requires a careful wave packet pruning process, which we omit.

Thank you for your attention!

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