Sobolev inequalities for $(0, q)$ forms on CR manifolds of finite type

Po-Lam Yung

Princeton University

October 2, 2009

Introduction

- Goal: to study Sobolev inequalities for differential forms
- 3 parts of the talk:
  1. Known result: the exterior derivative d in $\mathbb{R}^N$ (elliptic complex)
  2. Corresponding result for $\overline{\partial}_b$ complex (subelliptic)
  3. A key element in the proof: a decomposition lemma
- Shall focus almost entirely on the L^1 theory only

Introduction

- Goal: to study Sobolev inequalities for differential forms
- 3 parts of the talk:
  1. Known result: the exterior derivative d in $\mathbb{R}^N$ (elliptic complex)
  2. Corresponding result for $\overline{\partial}_b$ complex (subelliptic)
  3. A key element in the proof: a decomposition lemma
- Shall focus almost entirely on the L^1 theory only

Introduction

- Goal: to study Sobolev inequalities for differential forms
- 3 parts of the talk:
  1. Known result: the exterior derivative d in $\mathbb{R}^N$ (elliptic complex)
  2. Corresponding result for $\overline{\partial}_b$ complex (subelliptic)
  3. A key element in the proof: a decomposition lemma
- Shall focus almost entirely on the L^1 theory only

Introduction

- Goal: to study Sobolev inequalities for differential forms
- 3 parts of the talk:
  1. Known result: the exterior derivative d in $\mathbb{R}^N$ (elliptic complex)
  2. Corresponding result for $\overline{\partial}_b$ complex (subelliptic)
  3. A key element in the proof: a decomposition lemma
- Shall focus almost entirely on the L^1 theory only

Introduction

- Goal: to study Sobolev inequalities for differential forms
- 3 parts of the talk:
    1. Known result: the exterior derivative d in $\mathbb{R}^N$ (elliptic complex)
    2. Corresponding result for $\overline{\partial}_b$ complex (subelliptic)
    3. A key element in the proof: a decomposition lemma
- Shall focus almost entirely on the L^1 theory only

The elliptic complex

- Work of Bourgain-Brezis, Lanzani-Stein and van Schaftingen
- Set-up: Introduce componentwise L^p norm on the space of q forms on $\mathbb{R}^N$
- d: Hodge de-Rham exterior derivative
  $d : q$ forms $\rightarrow (q+1)$ forms
- d^*: adjoint of d under the Euclidean inner product
  $d^* : q$ forms $\rightarrow (q-1)$ forms
- Question: Suppose u is a q form on $\mathbb{R}^N$ and du, $d^*u \in L^1$. What can we say about u?
- If $q = 0$, du is just the gradient of u, so

$$du \in L^1 \Rightarrow u \in L^{\frac{N}{N-1}}.$$

The elliptic complex

- Work of Bourgain-Brezis, Lanzani-Stein and van Schaftingen
- Set-up: Introduce componentwise L^p norm on the space of q forms on $\mathbb{R}^N$
- d: Hodge de-Rham exterior derivative
  $d : q$ forms $\to (q+1)$ forms
- d^*: adjoint of d under the Euclidean inner product
  $d^* : q$ forms $\to (q-1)$ forms
- Question: Suppose u is a q form on $\mathbb{R}^N$ and du, $d^*u \in L^1$. What can we say about u?
- If $q = 0$, du is just the gradient of u, so

$$du \in L^1 \Rightarrow u \in L^{\frac{N}{N-1}}.$$

The elliptic complex

- Work of Bourgain-Brezis, Lanzani-Stein and van Schaftingen
- Set-up: Introduce componentwise L^p norm on the space of q forms on $\mathbb{R}^N$
- d: Hodge de-Rham exterior derivative
  $d : q$ forms $\to (q+1)$ forms
- d^*: adjoint of d under the Euclidean inner product
  $d^* : q$ forms $\to (q-1)$ forms
- Question: Suppose u is a q form on $\mathbb{R}^N$ and du, $d^*u \in L^1$. What can we say about u?
- If $q = 0$, du is just the gradient of u, so

$$du \in L^1 \Rightarrow u \in L^{\frac{N}{N-1}}.$$

The elliptic complex

- Work of Bourgain-Brezis, Lanzani-Stein and van Schaftingen
- Set-up: Introduce componentwise L^p norm on the space of q forms on $\mathbb{R}^N$
- d: Hodge de-Rham exterior derivative
  $d : q$ forms $\rightarrow (q+1)$ forms
- d^*: adjoint of d under the Euclidean inner product
  $d^* : q$ forms $\rightarrow (q-1)$ forms
- Question: Suppose u is a q form on $\mathbb{R}^N$ and du, $d^*u \in L^1$. What can we say about u?
- If $q = 0$, du is just the gradient of u, so

$$du \in L^1 \Rightarrow u \in L^{\frac{N}{N-1}}.$$

The elliptic complex

- Work of Bourgain-Brezis, Lanzani-Stein and van Schaftingen
- Set-up: Introduce componentwise L^p norm on the space of q forms on $\mathbb{R}^N$
- d: Hodge de-Rham exterior derivative
  $d : q \text{ forms} \to (q+1) \text{ forms}$
- d^*: adjoint of d under the Euclidean inner product
  $d^* : q \text{ forms} \to (q-1) \text{ forms}$
- Question: Suppose u is a q form on $\mathbb{R}^N$ and du, $d^*u \in L^1$. What can we say about u?
- If $q = 0$, du is just the gradient of u, so

$$du \in L^1 \Rightarrow u \in L^{\frac{N}{N-1}}.$$

The elliptic complex

- Work of Bourgain-Brezis, Lanzani-Stein and van Schaftingen
- Set-up: Introduce componentwise L^p norm on the space of q forms on $\mathbb{R}^N$
- d: Hodge de-Rham exterior derivative
  $d : q$ forms $\to (q+1)$ forms
- d^*: adjoint of d under the Euclidean inner product
  $d^* : q$ forms $\to (q-1)$ forms
- Question: Suppose u is a q form on $\mathbb{R}^N$ and du, $d^*u \in L^1$. What can we say about u?
- If $q = 0$, du is just the gradient of u, so

$$du \in L^1 \Rightarrow u \in L^{\frac{N}{N-1}}.$$

The elliptic complex

- Work of Bourgain-Brezis, Lanzani-Stein and van Schaftingen
- Set-up: Introduce componentwise L^p norm on the space of q forms on $\mathbb{R}^N$
- d: Hodge de-Rham exterior derivative
  $d : q \text{ forms} \to (q+1) \text{ forms}$
- d^*: adjoint of d under the Euclidean inner product
  $d^* : q \text{ forms} \to (q-1) \text{ forms}$
- Question: Suppose u is a q form on $\mathbb{R}^N$ and du, $d^*u \in L^1$. What can we say about u?
- If $q = 0$, du is just the gradient of u, so

$$du \in L^1 \Rightarrow u \in L^{\frac{N}{N-1}}.$$

The elliptic complex

- Work of Bourgain-Brezis, Lanzani-Stein and van Schaftingen
- Set-up: Introduce componentwise L^p norm on the space of q forms on $\mathbb{R}^N$
- d: Hodge de-Rham exterior derivative
  $d : q$ forms $\rightarrow (q+1)$ forms
- d^*: adjoint of d under the Euclidean inner product
  $d^* : q$ forms $\rightarrow (q-1)$ forms
- Question: Suppose u is a q form on $\mathbb{R}^N$ and du, $d^*u \in L^1$. What can we say about u?
- If $q = 0$, du is just the gradient of u, so

$$du \in L^1 \Rightarrow u \in L^{\frac{N}{N-1}}.$$

More generally

Theorem (Sobolev inequality for Hodge d)

If u is a compactly supported smooth q form on $\mathbb{R}^N$, and if $q \neq 1$ nor $N-1$, then

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|du\|_{L^1} + \|d^*u\|_{L^1}\right).$$

- Result not true if $q = 1$ or $N-1$ ('the forbidden degrees', dual to each other)
- Essence of the theorem is contained in the following L^1-duality inequality:

More generally

Theorem (Sobolev inequality for Hodge d)

If u is a compactly supported smooth q form on $\mathbb{R}^N$, and if $q \neq 1$ nor $N-1$, then

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|du\|_{L^1} + \|d^*u\|_{L^1}\right).$$

- Result not true if $q = 1$ or $N-1$ ('the forbidden degrees', dual to each other)
- Essence of the theorem is contained in the following L^1-duality inequality:

More generally

Theorem (Sobolev inequality for Hodge d)

If u is a compactly supported smooth q form on $\mathbb{R}^N$, and if $q \neq 1$ nor $N-1$, then

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|du\|_{L^1} + \|d^* u\|_{L^1}\right).$$

- Result not true if $q = 1$ or $N-1$ ('the forbidden degrees', dual to each other)
- Essence of the theorem is contained in the following L^1-duality inequality:

More generally

Theorem (Sobolev inequality for Hodge d)

If u is a compactly supported smooth q form on $\mathbb{R}^N$, and if $q \neq 1$ nor $N-1$, then

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|du\|_{L^1} + \|d^* u\|_{L^1}\right).$$

- Result not true if $q = 1$ or $N-1$ ('the forbidden degrees', dual to each other)
- Essence of the theorem is contained in the following L^1-duality inequality:

Theorem (L^1-duality inequality)

If $f = (f_1, \dots, f_N)$ is a divergence free vector field on $\mathbb{R}^N$, i.e. if

$$\sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} = 0$$

with $f_j \in C_c^\infty$, then for any $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^N} f_1 \Phi \right| \leq C \|f\|_{L^1} \|\nabla \Phi\|_{L^N}.$$

- Remedy of failure of embedding of $W^{1,N}$ into L^∞ on $\mathbb{R}^N$.
- Relevant to previous Sobolev inequality for q forms because every component of du and d^*u is a component of a divergence free vector field, to which we can apply this duality inequality.

Theorem (L^1-duality inequality)

If $f = (f_1, \dots, f_N)$ *is a divergence free vector field on* $\mathbb{R}^N$*, i.e. if*

$$\sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} = 0$$

with $f_j \in C_c^\infty$*, then for any* $\Phi \in C_c^\infty$*,*

$$\left| \int_{\mathbb{R}^N} f_1 \Phi \right| \leq C \|f\|_{L^1} \|\nabla \Phi\|_{L^N}.$$

- Remedy of failure of embedding of $W^{1,N}$ into L^∞ on $\mathbb{R}^N$.
- Relevant to previous Sobolev inequality for q forms because every component of du and d^*u is a component of a divergence free vector field, to which we can apply this duality inequality.

Theorem (L^1-duality inequality)

If $f = (f_1, \dots, f_N)$ *is a divergence free vector field on* $\mathbb{R}^N$*, i.e. if*

$$\sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} = 0$$

with $f_j \in C_c^\infty$*, then for any* $\Phi \in C_c^\infty$*,*

$$\left| \int_{\mathbb{R}^N} f_1 \Phi \right| \leq C \|f\|_{L^1} \|\nabla \Phi\|_{L^N}.$$

- Remedy of failure of embedding of $W^{1,N}$ into L^∞ on $\mathbb{R}^N$.
- Relevant to previous Sobolev inequality for q forms because every component of du and d^*u is a component of a divergence free vector field, to which we can apply this duality inequality.

Theorem (L^1-duality inequality)

If $f = (f_1, \dots, f_N)$ is a divergence free vector field on $\mathbb{R}^N$, i.e. if

$$\sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} = 0$$

with $f_j \in C_c^\infty$, then for any $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^N} f_1 \Phi \right| \leq C \|f\|_{L^1} \|\nabla \Phi\|_{L^N}.$$

- Remedy of failure of embedding of $W^{1,N}$ into L^∞ on $\mathbb{R}^N$.
- Relevant to previous Sobolev inequality for q forms because every component of du and d^*u is a component of a divergence free vector field, to which we can apply this duality inequality.

Theorem (L^1-duality inequality)

If $f = (f_1, \ldots, f_N)$ *is a divergence free vector field on* $\mathbb{R}^N$*, i.e. if*

$$\sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} = 0$$

with $f_j \in C_c^\infty$*, then for any* $\Phi \in C_c^\infty$*,*

$$\left| \int_{\mathbb{R}^N} f_1 \Phi \right| \leq C \|f\|_{L^1} \|\nabla \Phi\|_{L^N}.$$

- Remedy of failure of embedding of $W^{1,N}$ into L^∞ on $\mathbb{R}^N$.
- Relevant to previous Sobolev inequality for q forms because every component of du and d^*u is a component of a divergence free vector field, to which we can apply this duality inequality.

- Example: $q = 0$, u is a function, $du = \sum \frac{\partial u}{\partial x_j} dx_j$.
  Each component of du is a component of a divergence free vector field: e.g. $\frac{\partial u}{\partial x_2}$ satisfies

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial u}{\partial x_1}\right) = 0.$$

  This is because $d \circ d = 0$.
- Similar pheonomenon for $d^* u$, since $d^* \circ d^* = 0$.
- Works as long as du is not top form and $d^* u$ is not a function, which is why we needed $q \neq 1$ nor $N - 1$.

- Example: $q = 0$, u is a function, $du = \sum \frac{\partial u}{\partial x_j} dx_j$.
  Each component of du is a component of a divergence free vector field: e.g. $\frac{\partial u}{\partial x_2}$ satisfies

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial u}{\partial x_1}\right) = 0.$$

  This is because $d \circ d = 0$.
- Similar pheonomenon for $d^* u$, since $d^* \circ d^* = 0$.
- Works as long as du is not top form and $d^* u$ is not a function, which is why we needed $q \neq 1$ nor $N - 1$.

- Example: $q = 0$, u is a function, $du = \sum \frac{\partial u}{\partial x_j} dx_j$.
  Each component of du is a component of a divergence free vector field: e.g. $\frac{\partial u}{\partial x_2}$ satisfies
  $$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial u}{\partial x_1}\right) = 0.$$
  This is because $d \circ d = 0$.
- Similar pheonomenon for $d^* u$, since $d^* \circ d^* = 0$.
- Works as long as du is not top form and $d^* u$ is not a function, which is why we needed $q \neq 1$ nor $N - 1$.

- Example: $q = 0$, u is a function, $du = \sum \frac{\partial u}{\partial x_j} dx_j$.
  Each component of du is a component of a divergence free vector field: e.g. $\frac{\partial u}{\partial x_2}$ satisfies

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial u}{\partial x_1}\right) = 0.$$

  This is because $d \circ d = 0$.
- Similar pheonomenon for $d^* u$, since $d^* \circ d^* = 0$.
- Works as long as du is not top form and $d^* u$ is not a function, which is why we needed $q \neq 1$ nor $N - 1$.

- Example: $q = 0$, u is a function, $du = \sum \frac{\partial u}{\partial x_j} dx_j$.
  Each component of du is a component of a divergence free vector field: e.g. $\frac{\partial u}{\partial x_2}$ satisfies

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial u}{\partial x_1}\right) = 0.$$

  This is because $d \circ d = 0$.
- Similar pheonomenon for $d^* u$, since $d^* \circ d^* = 0$.
- Works as long as du is not top form and $d^* u$ is not a function, which is why we needed $q \neq 1$ nor $N - 1$.

The subelliptic complex

- M: boundary of a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$
- Question: Suppose u is $(0, q)$ form on M, and $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^1$. What can you say about u?
- Problem is subelliptic in nature: $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^p$, $1 < p < \infty$ does NOT imply $u \in W^{1,p}$
- Will associate to M a non-isotropic dimension $Q > \dim_{\mathbb{R}}(M)$ and obtain a corresponding Sobolev inequality
- Recall that in Sobolev inequalities, the bigger the dimension, the less one gains in exponent
- But this is in the nature of subelliptic analysis, and we cannot hope to gain as much as in the elliptic setting

The subelliptic complex

- M: boundary of a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$
- Question: Suppose u is $(0, q)$ form on M, and $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^1$. What can you say about u?
- Problem is subelliptic in nature: $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^p$, $1 < p < \infty$ does NOT imply $u \in W^{1,p}$
- Will associate to M a non-isotropic dimension $Q > \dim_{\mathbb{R}}(M)$ and obtain a corresponding Sobolev inequality
- Recall that in Sobolev inequalities, the bigger the dimension, the less one gains in exponent
- But this is in the nature of subelliptic analysis, and we cannot hope to gain as much as in the elliptic setting

The subelliptic complex

- M: boundary of a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$
- Question: Suppose u is $(0,q)$ form on M, and $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^1$. What can you say about u?
- Problem is subelliptic in nature: $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^p$, $1 < p < \infty$ does NOT imply $u \in W^{1,p}$
- Will associate to M a non-isotropic dimension $Q > \dim_{\mathbb{R}}(M)$ and obtain a corresponding Sobolev inequality
- Recall that in Sobolev inequalities, the bigger the dimension, the less one gains in exponent
- But this is in the nature of subelliptic analysis, and we cannot hope to gain as much as in the elliptic setting

The subelliptic complex

- M: boundary of a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$
- Question: Suppose u is $(0,q)$ form on M, and $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^1$. What can you say about u?
- Problem is subelliptic in nature: $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^p$, $1 < p < \infty$ does NOT imply $u \in W^{1,p}$
- Will associate to M a non-isotropic dimension $Q > \dim_{\mathbb{R}}(M)$ and obtain a corresponding Sobolev inequality
- Recall that in Sobolev inequalities, the bigger the dimension, the less one gains in exponent
- But this is in the nature of subelliptic analysis, and we cannot hope to gain as much as in the elliptic setting

The subelliptic complex

- M: boundary of a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$
- Question: Suppose u is $(0, q)$ form on M, and $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^1$. What can you say about u?
- Problem is subelliptic in nature: $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^p$, $1 < p < \infty$ does NOT imply $u \in W^{1,p}$
- Will associate to M a non-isotropic dimension $Q > \dim_{\mathbb{R}}(M)$ and obtain a corresponding Sobolev inequality
- Recall that in Sobolev inequalities, the bigger the dimension, the less one gains in exponent
- But this is in the nature of subelliptic analysis, and we cannot hope to gain as much as in the elliptic setting

The subelliptic complex

- M: boundary of a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$
- Question: Suppose u is $(0,q)$ form on M, and $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^1$. What can you say about u?
- Problem is subelliptic in nature: $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^p$, $1 < p < \infty$ does NOT imply $u \in W^{1,p}$
- Will associate to M a non-isotropic dimension $Q > \dim_{\mathbb{R}}(M)$ and obtain a corresponding Sobolev inequality
- Recall that in Sobolev inequalities, the bigger the dimension, the less one gains in exponent
- But this is in the nature of subelliptic analysis, and we cannot hope to gain as much as in the elliptic setting

The subelliptic complex

- M: boundary of a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$
- Question: Suppose u is $(0, q)$ form on M, and $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^1$. What can you say about u?
- Problem is subelliptic in nature: $\overline{\partial}_b u, \overline{\partial}_b^* u \in L^p$, $1 < p < \infty$ does NOT imply $u \in W^{1,p}$
- Will associate to M a non-isotropic dimension $Q > \dim_{\mathbb{R}}(M)$ and obtain a corresponding Sobolev inequality
- Recall that in Sobolev inequalities, the bigger the dimension, the less one gains in exponent
- But this is in the nature of subelliptic analysis, and we cannot hope to gain as much as in the elliptic setting

We have the following Sobolev inequality for $\overline{\partial}_b$ on M:

Theorem (Y. 2009)

- *Assume M is of finite commutator type m at every point i.e. Commutators of $Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n$ of length $\leq m$ span the tangent space to M, where $Z_1, \dots, Z_n$ is a basis of holomorphic vector fields tangent to M*
  e.g. strongly pseudoconvex $\Rightarrow$ commutator type 2
- *Also assume M satisfy condition $D(q_0)$ for some $1 \leq q_0 \leq n/2$ i.e. there is a constant $C > 0$ such that for any point $x \in M$, the sum of any q_0 eigenvalues of the Levi form at x is bounded by C times any other such sum.*
  e.g. strongly pseudoconvex $\Rightarrow$ condition $D(1)$.

We have the following Sobolev inequality for $\overline{\partial}_b$ on M:

Theorem (Y. 2009)

- *Assume M is of finite commutator type m at every point*
  i.e. Commutators of $Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n$ of length $\leq m$ span the tangent space to M, where $Z_1, \dots, Z_n$ is a basis of holomorphic vector fields tangent to M
  e.g. strongly pseudoconvex $\Rightarrow$ commutator type 2
- *Also assume M satisfy condition $D(q_0)$ for some $1 \leq q_0 \leq n/2$*
  i.e. there is a constant $C > 0$ such that for any point $x \in M$, the sum of any q_0 eigenvalues of the Levi form at x is bounded by C times any other such sum.
  e.g. strongly pseudoconvex $\Rightarrow$ condition $D(1)$.

We have the following Sobolev inequality for $\overline{\partial}_b$ on M:

Theorem (Y. 2009)

- *Assume M is of finite commutator type m at every point i.e. Commutators of $Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n$ of length $\leq m$ span the tangent space to M, where $Z_1, \dots, Z_n$ is a basis of holomorphic vector fields tangent to M*
  e.g. strongly pseudoconvex $\Rightarrow$ commutator type 2
- *Also assume M satisfy condition $D(q_0)$ for some $1 \leq q_0 \leq n/2$ i.e. there is a constant $C > 0$ such that for any point $x \in M$, the sum of any q_0 eigenvalues of the Levi form at x is bounded by C times any other such sum.*
  e.g. strongly pseudoconvex $\Rightarrow$ condition $D(1)$.

We have the following Sobolev inequality for $\overline{\partial}_b$ on M:

Theorem (Y. 2009)

- *Assume M is of finite commutator type m at every point i.e. Commutators of $Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n$ of length $\leq m$ span the tangent space to M, where $Z_1, \dots, Z_n$ is a basis of holomorphic vector fields tangent to M e.g. strongly pseudoconvex $\Rightarrow$ commutator type 2*
- *Also assume M satisfy condition $D(q_0)$ for some $1 \leq q_0 \leq n/2$ i.e. there is a constant $C > 0$ such that for any point $x \in M$, the sum of any q_0 eigenvalues of the Levi form at x is bounded by C times any other such sum. e.g. strongly pseudoconvex $\Rightarrow$ condition $D(1)$.*

We have the following Sobolev inequality for $\overline{\partial}_b$ on M:

Theorem (Y. 2009)

- *Assume M is of finite commutator type m at every point i.e. Commutators of $Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n$ of length $\leq m$ span the tangent space to M, where $Z_1, \dots, Z_n$ is a basis of holomorphic vector fields tangent to M*
  e.g. strongly pseudoconvex $\Rightarrow$ commutator type 2
- *Also assume M satisfy condition $D(q_0)$ for some $1 \leq q_0 \leq n/2$*
  i.e. there is a constant $C > 0$ such that for any point $x \in M$, the sum of any q_0 eigenvalues of the Levi form at x is bounded by C times any other such sum.
  e.g. strongly pseudoconvex $\Rightarrow$ condition $D(1)$.

We have the following Sobolev inequality for $\overline{\partial}_b$ on M:

Theorem (Y. 2009)

- *Assume M is of finite commutator type m at every point i.e. Commutators of $Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n$ of length $\leq m$ span the tangent space to M, where $Z_1, \dots, Z_n$ is a basis of holomorphic vector fields tangent to M*
  e.g. strongly pseudoconvex $\Rightarrow$ commutator type 2
- *Also assume M satisfy condition $D(q_0)$ for some $1 \leq q_0 \leq n/2$ i.e. there is a constant $C > 0$ such that for any point $x \in M$, the sum of any q_0 eigenvalues of the Levi form at x is bounded by C times any other such sum.*
  e.g. strongly pseudoconvex $\Rightarrow$ condition $D(1)$.

We have the following Sobolev inequality for $\overline{\partial}_b$ on M:

Theorem (Y. 2009)

- *Assume M is of finite commutator type m at every point i.e. Commutators of $Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n$ of length $\leq m$ span the tangent space to M, where $Z_1, \dots, Z_n$ is a basis of holomorphic vector fields tangent to M*
  e.g. strongly pseudoconvex $\Rightarrow$ commutator type 2
- *Also assume M satisfy condition $D(q_0)$ for some $1 \leq q_0 \leq n/2$ i.e. there is a constant $C > 0$ such that for any point $x \in M$, the sum of any q_0 eigenvalues of the Levi form at x is bounded by C times any other such sum.*
  e.g. strongly pseudoconvex $\Rightarrow$ condition $D(1)$.

- Let $Q = 2n + m$.
  (a) Let $u =$ smooth $(0, q)$ form on M orthogonal to Kernel$(\Box_b)$, where $q_0 \leq q \leq n - q_0$ and $q \neq 1$ nor $n - 1$. Then
  $$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}.$$
  (b) Let $v =$ smooth $(0, q_0 - 1)$ form orthogonal to Kernel$(\overline{\partial}_b)$. Then
  $$\|v\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b v\|_{L^1(M)}.$$
  (c) A similar inequality for $(0, n - q_0 + 1)$ forms orthogonal to Kernel$(\overline{\partial}_b^*)$ by duality.

- Let $Q = 2n + m$.
  (a) Let $u =$ smooth $(0, q)$ form on M orthogonal to Kernel$(\Box_b)$, where $q_0 \leq q \leq n - q_0$ and $q \neq 1$ nor $n - 1$. Then

  $$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}.$$

  (b) Let $v =$ smooth $(0, q_0 - 1)$ form orthogonal to Kernel$(\overline{\partial}_b)$. Then

  $$\|v\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b v\|_{L^1(M)}.$$

  (c) A similar inequality for $(0, n - q_0 + 1)$ forms orthogonal to Kernel$(\overline{\partial}_b^*)$ by duality.

- Let $Q = 2n + m$.
  (a) Let $u =$ smooth $(0, q)$ form on M orthogonal to Kernel($\Box_b$), where $q_0 \leq q \leq n - q_0$ and $q \neq 1$ nor $n - 1$. Then

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}.$$

  (b) Let $v =$ smooth $(0, q_0 - 1)$ form orthogonal to Kernel($\overline{\partial}_b$). Then

$$\|v\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b v\|_{L^1(M)}.$$

  (c) A similar inequality for $(0, n - q_0 + 1)$ forms orthogonal to Kernel($\overline{\partial}_b^*$) by duality.

- Let $Q = 2n + m$.
  (a) Let $u =$ smooth $(0, q)$ form on M orthogonal to Kernel$(\Box_b)$, where $q_0 \leq q \leq n - q_0$ and $q \neq 1$ nor $n - 1$. Then
  $$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}.$$
  (b) Let $v =$ smooth $(0, q_0 - 1)$ form orthogonal to Kernel$(\overline{\partial}_b)$. Then
  $$\|v\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b v\|_{L^1(M)}.$$
  (c) A similar inequality for $(0, n - q_0 + 1)$ forms orthogonal to Kernel$(\overline{\partial}_b^*)$ by duality.

- Let $Q = 2n + m$.
  (a) Let $u =$ smooth $(0, q)$ form on M orthogonal to Kernel$(\Box_b)$, where $q_0 \leq q \leq n - q_0$ and $q \neq 1$ nor $n - 1$. Then

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}.$$

  (b) Let $v =$ smooth $(0, q_0 - 1)$ form orthogonal to Kernel$(\overline{\partial}_b)$. Then

$$\|v\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b v\|_{L^1(M)}.$$

  (c) A similar inequality for $(0, n - q_0 + 1)$ forms orthogonal to Kernel$(\overline{\partial}_b^*)$ by duality.

- Let $Q = 2n + m$.
  (a) Let $u =$ smooth $(0, q)$ form on M orthogonal to Kernel$(\Box_b)$, where $q_0 \leq q \leq n - q_0$ and $q \neq 1$ nor $n - 1$. Then
  $$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}.$$
  (b) Let $v =$ smooth $(0, q_0 - 1)$ form orthogonal to Kernel$(\overline{\partial}_b)$. Then
  $$\|v\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b v\|_{L^1(M)}.$$
  (c) A similar inequality for $(0, n - q_0 + 1)$ forms orthogonal to Kernel$(\overline{\partial}_b^*)$ by duality.

- Let $Q = 2n + m$.
  (a) Let $u =$ smooth $(0, q)$ form on M orthogonal to Kernel$(\Box_b)$, where $q_0 \leq q \leq n - q_0$ and $q \neq 1$ nor $n - 1$. Then
  $$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}.$$
  (b) Let $v =$ smooth $(0, q_0 - 1)$ form orthogonal to Kernel$(\overline{\partial}_b)$. Then
  $$\|v\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b v\|_{L^1(M)}.$$
  (c) A similar inequality for $(0, n - q_0 + 1)$ forms orthogonal to Kernel$(\overline{\partial}_b^*)$ by duality.

Corollary

- *M: boundary of a bounded smooth strongly pseudoconvex domain in* $\mathbb{C}^{n+1}$, $n \geq 2$
- $q \neq 1$ *nor* $n-1$
- *Then for any smooth* $(0,q)$ *form u orthogonal to Kernel*$(\Box_b)$,

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}$$

where $Q = 2n+2$.

- *In particular*

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)}$$

for all smooth functions u orthogonal to Kernel$(\overline{\partial}_b)$ *(Gagliardo-Nirenberg for* $\overline{\partial}_b$*).*

Corollary

- *M: boundary of a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$*
- *$q \neq 1$ nor $n-1$*
- *Then for any smooth $(0,q)$ form u orthogonal to Kernel$(\Box_b)$,*

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}$$

where $Q = 2n+2$.

- *In particular*

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)}$$

for all smooth functions u orthogonal to Kernel$(\overline{\partial}_b)$ (Gagliardo-Nirenberg for $\overline{\partial}_b$).

Corollary

- *M: boundary of a bounded smooth strongly pseudoconvex domain in* $\mathbb{C}^{n+1}$, $n \geq 2$
- $q \neq 1$ *nor* $n-1$
- *Then for any smooth* $(0,q)$ *form u orthogonal to Kernel*$(\Box_b)$,

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}$$

where $Q = 2n+2$.

- *In particular*

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)}$$

for all smooth functions u orthogonal to Kernel$(\overline{\partial}_b)$ *(Gagliardo-Nirenberg for* $\overline{\partial}_b$*).*

Corollary

- *M: boundary of a bounded smooth strongly pseudoconvex domain in* $\mathbb{C}^{n+1}$, $n \geq 2$
- $q \neq 1$ *nor* $n-1$
- *Then for any smooth* $(0,q)$ *form u orthogonal to Kernel*$(\Box_b)$,

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}$$

where $Q = 2n+2$.

- *In particular*

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)}$$

for all smooth functions u orthogonal to Kernel$(\overline{\partial}_b)$ *(Gagliardo-Nirenberg for* $\overline{\partial}_b$*).*

Corollary

- *M: boundary of a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$*
- *$q \neq 1$ nor $n-1$*
- *Then for any smooth $(0,q)$ form u orthogonal to Kernel$(\Box_b)$,*

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}$$

where $Q = 2n+2$.

- *In particular*

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)}$$

for all smooth functions u orthogonal to Kernel$(\overline{\partial}_b)$
(Gagliardo-Nirenberg for $\overline{\partial}_b$).

Corollary

- *M: boundary of a bounded smooth strongly pseudoconvex domain in* $\mathbb{C}^{n+1}$, $n \geq 2$
- $q \neq 1$ *nor* $n-1$
- *Then for any smooth* $(0,q)$ *form u orthogonal to Kernel*$(\Box_b)$,

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}$$

  where $Q = 2n+2$.
- *In particular*

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)}$$

  for all smooth functions u orthogonal to Kernel$(\overline{\partial}_b)$ *(Gagliardo-Nirenberg for* $\overline{\partial}_b$*).*

Remarks

- *There is also a version of these Sobolev inequalities for abstract CR manifolds.*
- *The proof of the Sobolev inequality for $\overline{\partial}_b$ relies on a subelliptic version of L^1-duality inequality (to be stated on the next page), and the fact that $\overline{\partial}_b \circ \overline{\partial}_b = 0$.*
- *We assumed $n \geq 2$ because our method does not allow $q = 1$ or $n - 1$.*
- *The conditions of finite commutator type and $D(q_0)$ were made to ensure maximal subellipticity of the solution operator to $\Box_b$ in the L^p sense.*
- *We also need finite commutator type for the following subelliptic L^1-duality inequality that we alluded to.*

Remarks

- *There is also a version of these Sobolev inequalities for abstract CR manifolds.*
- *The proof of the Sobolev inequality for $\overline{\partial}_b$ relies on a subelliptic version of L^1-duality inequality (to be stated on the next page), and the fact that $\overline{\partial}_b \circ \overline{\partial}_b = 0$.*
- *We assumed $n \geq 2$ because our method does not allow $q = 1$ or $n - 1$.*
- *The conditions of finite commutator type and $D(q_0)$ were made to ensure maximal subellipticity of the solution operator to $\Box_b$ in the L^p sense.*
- *We also need finite commutator type for the following subelliptic L^1-duality inequality that we alluded to.*

Remarks

- *There is also a version of these Sobolev inequalities for abstract CR manifolds.*
- *The proof of the Sobolev inequality for $\overline{\partial}_b$ relies on a subelliptic version of L^1-duality inequality (to be stated on the next page), and the fact that $\overline{\partial}_b \circ \overline{\partial}_b = 0$.*
- *We assumed $n \geq 2$ because our method does not allow $q = 1$ or $n - 1$.*
- *The conditions of finite commutator type and $D(q_0)$ were made to ensure maximal subellipticity of the solution operator to $\Box_b$ in the L^p sense.*
- *We also need finite commutator type for the following subelliptic L^1-duality inequality that we alluded to.*

Remarks

- *There is also a version of these Sobolev inequalities for abstract CR manifolds.*
- *The proof of the Sobolev inequality for $\overline{\partial}_b$ relies on a subelliptic version of L^1-duality inequality (to be stated on the next page), and the fact that $\overline{\partial}_b \circ \overline{\partial}_b = 0$.*
- *We assumed $n \geq 2$ because our method does not allow $q = 1$ or $n - 1$.*
- *The conditions of finite commutator type and $D(q_0)$ were made to ensure maximal subellipticity of the solution operator to $\Box_b$ in the L^p sense.*
- *We also need finite commutator type for the following subelliptic L^1-duality inequality that we alluded to.*

Remarks

- *There is also a version of these Sobolev inequalities for abstract CR manifolds.*
- *The proof of the Sobolev inequality for $\overline{\partial}_b$ relies on a subelliptic version of L^1-duality inequality (to be stated on the next page), and the fact that $\overline{\partial}_b \circ \overline{\partial}_b = 0$.*
- *We assumed $n \geq 2$ because our method does not allow $q = 1$ or $n - 1$.*
- *The conditions of finite commutator type and $D(q_0)$ were made to ensure maximal subellipticity of the solution operator to $\Box_b$ in the L^p sense.*
- *We also need finite commutator type for the following subelliptic L^1-duality inequality that we alluded to.*

Theorem (Y. 2009)

- *$X_1, \ldots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume they are linearly independent at 0, and their commutators of length $\leq r$ span at 0.*
- *Let $V_j(0)$ be the span of the restrictions of the commutators of $X_1, \ldots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^r j \cdot (dim V_j(0) - dim V_{j-1}(0))$*
- *Then there is a neighborhood U of 0 and $C > 0$ such that if*

$$X_1 f_1 + \cdots + X_n f_n = 0$$

on U with $f_1, \ldots, f_n \in C^\infty(U)$ and $\Phi \in C_c^\infty(U)$, then

$$\left| \int_U f_1(x)\Phi(x)dx \right| \leq C\|f\|_{L^1(U)} (\sum_{j=1}^n \|X_j \Phi\|_{L^Q(U)} + \|\Phi\|_{L^Q(U)}).$$

Theorem (Y. 2009)

- *$X_1, \ldots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume they are linearly independent at 0, and their commutators of length $\leq r$ span at 0.*
- *Let $V_j(0)$ be the span of the restrictions of the commutators of $X_1, \ldots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^{r} j \cdot (dim V_j(0) - dim V_{j-1}(0))$*
- *Then there is a neighborhood U of 0 and $C > 0$ such that if*

$$X_1 f_1 + \cdots + X_n f_n = 0$$

on U with $f_1, \ldots, f_n \in C^\infty(U)$ and $\Phi \in C_c^\infty(U)$, then

$$\left| \int_U f_1(x)\Phi(x)dx \right| \leq C\|f\|_{L^1(U)}\left(\sum_{j=1}^{n} \|X_j\Phi\|_{L^Q(U)} + \|\Phi\|_{L^Q(U)}\right).$$

Theorem (Y. 2009)

- *$X_1, \dots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume they are linearly independent at 0, and their commutators of length $\leq r$ span at 0.*
- *Let $V_j(0)$ be the span of the restrictions of the commutators of $X_1, \dots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^{r} j \cdot (\dim V_j(0) - \dim V_{j-1}(0))$*
- *Then there is a neighborhood U of 0 and $C > 0$ such that if*

$$X_1 f_1 + \cdots + X_n f_n = 0$$

on U with $f_1, \dots, f_n \in C^\infty(U)$ and $\Phi \in C_c^\infty(U)$, then

$$\left| \int_U f_1(x)\Phi(x)dx \right| \leq C\|f\|_{L^1(U)}\Big(\sum_{j=1}^{n} \|X_j\Phi\|_{L^Q(U)} + \|\Phi\|_{L^Q(U)}\Big).$$

Theorem (Y. 2009)

- *$X_1, \dots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume they are linearly independent at 0, and their commutators of length $\leq r$ span at 0.*
- *Let $V_j(0)$ be the span of the restrictions of the commutators of $X_1, \dots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^r j \cdot (dim V_j(0) - dim V_{j-1}(0))$*
- *Then there is a neighborhood U of 0 and $C > 0$ such that if*

$$X_1 f_1 + \cdots + X_n f_n = 0$$

on U with $f_1, \dots, f_n \in C^\infty(U)$ and $\Phi \in C_c^\infty(U)$, then

$$\left| \int_U f_1(x)\Phi(x)dx \right| \leq C\|f\|_{L^1(U)}\left(\sum_{j=1}^n \|X_j\Phi\|_{L^Q(U)} + \|\Phi\|_{L^Q(U)}\right).$$

Theorem (Y. 2009)

- $X_1, \dots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$
- Assume they are linearly independent at 0, and their commutators of length $\leq r$ span at 0.
- Let $V_j(0)$ be the span of the restrictions of the commutators of $X_1, \dots, X_n$ of length $\leq j$ to 0
- Let $Q = \sum_{j=1}^r j \cdot (dim V_j(0) - dim V_{j-1}(0))$
- Then there is a neighborhood U of 0 and $C > 0$ such that if

$$X_1 f_1 + \cdots + X_n f_n = 0$$

on U with $f_1, \dots, f_n \in C^\infty(U)$ and $\Phi \in C_c^\infty(U)$, then

$$\left| \int_U f_1(x)\Phi(x)dx \right| \leq C\|f\|_{L^1(U)}(\sum_{j=1}^n \|X_j\Phi\|_{L^Q(U)} + \|\Phi\|_{L^Q(U)}).$$

Theorem (Y. 2009)

- *$X_1, \dots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume they are linearly independent at 0, and their commutators of length $\leq r$ span at 0.*
- *Let $V_j(0)$ be the span of the restrictions of the commutators of $X_1, \dots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^{r} j \cdot (dim V_j(0) - dim V_{j-1}(0))$*
- *Then there is a neighborhood U of 0 and $C > 0$ such that if*

$$X_1 f_1 + \cdots + X_n f_n = 0$$

on U with $f_1, \dots, f_n \in C^\infty(U)$ and $\Phi \in C_c^\infty(U)$, then

$$\left| \int_U f_1(x)\Phi(x)dx \right| \leq C\|f\|_{L^1(U)}\left(\sum_{j=1}^{n} \|X_j\Phi\|_{L^Q(U)} + \|\Phi\|_{L^Q(U)}\right).$$

Theorem (Y. 2009)

- *$X_1, \ldots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume they are linearly independent at 0, and their commutators of length $\leq r$ span at 0.*
- *Let $V_j(0)$ be the span of the restrictions of the commutators of $X_1, \ldots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^{r} j \cdot (\dim V_j(0) - \dim V_{j-1}(0))$*
- *Then there is a neighborhood U of 0 and $C > 0$ such that if*

$$X_1 f_1 + \cdots + X_n f_n = 0$$

on U with $f_1, \ldots, f_n \in C^\infty(U)$ and $\Phi \in C_c^\infty(U)$, then

$$\left| \int_U f_1(x)\Phi(x)dx \right| \leq C\|f\|_{L^1(U)}\Big(\sum_{j=1}^{n} \|X_j\Phi\|_{L^Q(U)} + \|\Phi\|_{L^Q(U)}\Big).$$

Remarks

- *This generalizes the L^1-duality inequality we stated at the beginning.*
- *Chanillo-van Schaftingen has proved the theorem above when the underlying space is a homogeneous group and $X_1, \dots, X_n$ is a basis of vector fields of degree 1 on that group.*
- *Difficulty in the current theorem:*
  getting the best (i.e. smallest) possible value of Q.
  The one we had given is the best possible.
  Thus Q should be thought of as the non-isotropic dimension of 0 in such a situation.
- *In fact we have the following subelliptic Sobolev inequality with the best possible exponent:*

Remarks

- *This generalizes the L^1-duality inequality we stated at the beginning.*
- *Chanillo-van Schaftingen has proved the theorem above when the underlying space is a homogeneous group and $X_1, \dots, X_n$ is a basis of vector fields of degree 1 on that group.*
- *Difficulty in the current theorem:*
  getting the best (i.e. smallest) possible value of Q.
  The one we had given is the best possible.
  Thus Q should be thought of as the non-isotropic dimension of 0 in such a situation.
- *In fact we have the following subelliptic Sobolev inequality with the best possible exponent:*

Remarks

- *This generalizes the L^1-duality inequality we stated at the beginning.*
- *Chanillo-van Schaftingen has proved the theorem above when the underlying space is a homogeneous group and $X_1, \dots, X_n$ is a basis of vector fields of degree 1 on that group.*
- *Difficulty in the current theorem:*
  getting the best (i.e. smallest) possible value of Q.
  The one we had given is the best possible.
  Thus Q should be thought of as the non-isotropic dimension of 0 in such a situation.
- *In fact we have the following subelliptic Sobolev inequality with the best possible exponent:*

Remarks

- *This generalizes the L^1-duality inequality we stated at the beginning.*
- *Chanillo-van Schaftingen has proved the theorem above when the underlying space is a homogeneous group and $X_1, \dots, X_n$ is a basis of vector fields of degree 1 on that group.*
- *Difficulty in the current theorem:*
  getting the best (i.e. smallest) possible value of Q.
  The one we had given is the best possible.
  Thus Q should be thought of as the non-isotropic dimension of 0 in such a situation.
- *In fact we have the following subelliptic Sobolev inequality with the best possible exponent:*

Remarks

- *This generalizes the L^1-duality inequality we stated at the beginning.*
- *Chanillo-van Schaftingen has proved the theorem above when the underlying space is a homogeneous group and $X_1, \dots, X_n$ is a basis of vector fields of degree 1 on that group.*
- *Difficulty in the current theorem:*
  getting the best (i.e. smallest) possible value of Q.
  The one we had given is the best possible.
  Thus Q should be thought of as the non-isotropic dimension of 0 in such a situation.
- *In fact we have the following subelliptic Sobolev inequality with the best possible exponent:*

Remarks

- *This generalizes the L^1-duality inequality we stated at the beginning.*
- *Chanillo-van Schaftingen has proved the theorem above when the underlying space is a homogeneous group and $X_1, \dots, X_n$ is a basis of vector fields of degree 1 on that group.*
- *Difficulty in the current theorem:*
  getting the best (i.e. smallest) possible value of Q.
  The one we had given is the best possible.
  Thus Q should be thought of as the non-isotropic dimension of 0 in such a situation.
- *In fact we have the following subelliptic Sobolev inequality with the best possible exponent:*

Proposition (Y. 2009)

- *$X_1, \ldots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume that their commutators of length $\leq r$ span at 0*
- *Let $V_j(0)$ be the span of restrictions of the commutators of $X_1, \ldots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^r j \cdot (dim V_j(0) - dim V_{j-1}(0))$ as before*
- *Then there exists a neighborhood U of 0 and $C > 0$ such that if u is a smooth function on U and $1 \leq p < Q$, then*

$$\|u\|_{L^{p^*}(U)} \leq C(\sum_{j=1}^n \|X_j u\|_{L^p(U)} + \|u\|_{L^p(U)}) \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}.$$

Moreover the inequality cannot hold for any bigger value of p^*.

This generalizes a result of Caponga, Danielli and Garofalo.

Proposition (Y. 2009)

- $X_1, \dots, X_n$ *smooth real vector fields near* 0 *on* $\mathbb{R}^N$
- *Assume that their commutators of length* $\leq r$ *span at* 0
- *Let* $V_j(0)$ *be the span of restrictions of the commutators of* $X_1, \dots, X_n$ *of length* $\leq j$ *to* 0
- *Let* $Q = \sum_{j=1}^r j \cdot (dim V_j(0) - dim V_{j-1}(0))$ *as before*
- *Then there exists a neighborhood* U *of* 0 *and* $C > 0$ *such that if* u *is a smooth function on* U *and* $1 \leq p < Q$, *then*

$$\|u\|_{L^{p^*}(U)} \leq C(\sum_{j=1}^n \|X_j u\|_{L^p(U)} + \|u\|_{L^p(U)}) \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}.$$

Moreover the inequality cannot hold for any bigger value of p^*.

This generalizes a result of Caponga, Danielli and Garofalo.

Proposition (Y. 2009)

- *$X_1, \ldots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume that their commutators of length $\leq r$ span at 0*
- *Let $V_j(0)$ be the span of restrictions of the commutators of $X_1, \ldots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^{r} j \cdot (dimV_j(0) - dimV_{j-1}(0))$ as before*
- *Then there exists a neighborhood U of 0 and $C > 0$ such that if u is a smooth function on U and $1 \leq p < Q$, then*

$$\|u\|_{L^{p^*}(U)} \leq C(\sum_{j=1}^{n} \|X_j u\|_{L^p(U)} + \|u\|_{L^p(U)}) \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}.$$

Moreover the inequality cannot hold for any bigger value of p^*.

This generalizes a result of Caponga, Danielli and Garofalo.

Proposition (Y. 2009)

- *$X_1, \dots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume that their commutators of length $\leq r$ span at 0*
- *Let $V_j(0)$ be the span of restrictions of the commutators of $X_1, \dots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^r j \cdot (dim V_j(0) - dim V_{j-1}(0))$ as before*
- *Then there exists a neighborhood U of 0 and $C > 0$ such that if u is a smooth function on U and $1 \leq p < Q$, then*

$$\|u\|_{L^{p^*}(U)} \leq C(\sum_{j=1}^n \|X_j u\|_{L^p(U)} + \|u\|_{L^p(U)}) \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}.$$

Moreover the inequality cannot hold for any bigger value of p^*.

This generalizes a result of Caponga, Danielli and Garofalo.

Proposition (Y. 2009)

- *$X_1, \dots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume that their commutators of length $\leq r$ span at 0*
- *Let $V_j(0)$ be the span of restrictions of the commutators of $X_1, \dots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^r j \cdot (dim V_j(0) - dim V_{j-1}(0))$ as before*
- *Then there exists a neighborhood U of 0 and $C > 0$ such that if u is a smooth function on U and $1 \leq p < Q$, then*

$$\|u\|_{L^{p^*}(U)} \leq C(\sum_{j=1}^n \|X_j u\|_{L^p(U)} + \|u\|_{L^p(U)}) \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}.$$

Moreover the inequality cannot hold for any bigger value of p^*.

This generalizes a result of Caponga, Danielli and Garofalo.

Proposition (Y. 2009)

- *$X_1, \ldots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume that their commutators of length $\leq r$ span at 0*
- *Let $V_j(0)$ be the span of restrictions of the commutators of $X_1, \ldots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^{r} j \cdot (dim V_j(0) - dim V_{j-1}(0))$ as before*
- *Then there exists a neighborhood U of 0 and $C > 0$ such that if u is a smooth function on U and $1 \leq p < Q$, then*

$$\|u\|_{L^{p^*}(U)} \leq C(\sum_{j=1}^{n} \|X_j u\|_{L^p(U)} + \|u\|_{L^p(U)}) \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}.$$

Moreover the inequality cannot hold for any bigger value of p^*.

This generalizes a result of Caponga, Danielli and Garofalo.

Proposition (Y. 2009)

- *$X_1, \dots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume that their commutators of length $\leq r$ span at 0*
- *Let $V_j(0)$ be the span of restrictions of the commutators of $X_1, \dots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^r j \cdot (dim V_j(0) - dim V_{j-1}(0))$ as before*
- *Then there exists a neighborhood U of 0 and $C > 0$ such that if u is a smooth function on U and $1 \leq p < Q$, then*

$$\|u\|_{L^{p^*}(U)} \leq C\left(\sum_{j=1}^n \|X_j u\|_{L^p(U)} + \|u\|_{L^p(U)}\right) \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}.$$

Moreover the inequality cannot hold for any bigger value of p^*.

This generalizes a result of Caponga, Danielli and Garofalo.

Proposition (Y. 2009)

- *$X_1, \dots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$*
- *Assume that their commutators of length $\leq r$ span at 0*
- *Let $V_j(0)$ be the span of restrictions of the commutators of $X_1, \dots, X_n$ of length $\leq j$ to 0*
- *Let $Q = \sum_{j=1}^{r} j \cdot (dim V_j(0) - dim V_{j-1}(0))$ as before*
- *Then there exists a neighborhood U of 0 and $C > 0$ such that if u is a smooth function on U and $1 \leq p < Q$, then*

$$\|u\|_{L^{p^*}(U)} \leq C(\sum_{j=1}^{n} \|X_j u\|_{L^p(U)} + \|u\|_{L^p(U)}) \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}.$$

Moreover the inequality cannot hold for any bigger value of p^*.

This generalizes a result of Caponga, Danielli and Garofalo.

A Model Example

- *On* $\mathbb{R}^2$*, use coordinates* (x, t)*, and let* $X = \frac{\partial}{\partial x}$*,* $Y = x\frac{\partial}{\partial t}$*.*
- $[X, Y] = \frac{\partial}{\partial t}$*, so finite type 2 at 0;*
  in fact $V_1(0) = span\{\frac{\partial}{\partial x}\big|_0\}$*,* $V_2(0) = span\{\frac{\partial}{\partial x}\big|_0, \frac{\partial}{\partial t}\big|_0\}$*.*
- *Local non-isotropic dimension* Q *at 0 is*
  $1 \cdot dim V_1(0) + 2 \cdot (dim V_2(0) - dim V_1(0)) = 1 \cdot 1 + 2 \cdot 1 = 3.$
- *Previous proposition implies*

$$\|u\|_{L^{p^*}(\mathbb{R}^2)} \leq C\|\nabla_b u\|_{L^p(\mathbb{R}^2)}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$$

where $\nabla_b u = (Xu, Yu)$*, for* $u \in C_c^\infty(\mathbb{R}^2)$*,* $1 \leq p < 3$*.*

A Model Example

- *On $\mathbb{R}^2$, use coordinates (x, t), and let $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$.*
- $[X, Y] = \frac{\partial}{\partial t}$, *so finite type 2 at 0;*
  in fact $V_1(0) = span\{\frac{\partial}{\partial x}\big|_0\}$, $V_2(0) = span\{\frac{\partial}{\partial x}\big|_0, \frac{\partial}{\partial t}\big|_0\}$.
- *Local non-isotropic dimension Q at 0 is*
  $1 \cdot dimV_1(0) + 2 \cdot (dimV_2(0) - dimV_1(0)) = 1 \cdot 1 + 2 \cdot 1 = 3.$
- *Previous proposition implies*

$$\|u\|_{L^{p*}(\mathbb{R}^2)} \leq C\|\nabla_b u\|_{L^p(\mathbb{R}^2)}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$$

where $\nabla_b u = (Xu, Yu)$, *for* $u \in C_c^\infty(\mathbb{R}^2)$, $1 \leq p < 3$.

A Model Example

- *On* $\mathbb{R}^2$*, use coordinates* (x, t)*, and let* $X = \frac{\partial}{\partial x}$*,* $Y = x\frac{\partial}{\partial t}$*.*
- $[X, Y] = \frac{\partial}{\partial t}$*, so finite type 2 at* 0*;*
  in fact $V_1(0) = span\{\frac{\partial}{\partial x}\big|_0\}$*,* $V_2(0) = span\{\frac{\partial}{\partial x}\big|_0, \frac{\partial}{\partial t}\big|_0\}$*.*
- *Local non-isotropic dimension* Q *at* 0 *is* $1 \cdot dim V_1(0) + 2 \cdot (dim V_2(0) - dim V_1(0)) = 1 \cdot 1 + 2 \cdot 1 = 3.$
- *Previous proposition implies*

$$\|u\|_{L^{p^*}(\mathbb{R}^2)} \leq C\|\nabla_b u\|_{L^p(\mathbb{R}^2)}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$$

where $\nabla_b u = (Xu, Yu)$*, for* $u \in C_c^\infty(\mathbb{R}^2)$*,* $1 \leq p < 3$*.*

A Model Example

- *On* $\mathbb{R}^2$*, use coordinates* (x, t)*, and let* $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$.
- $[X, Y] = \frac{\partial}{\partial t}$*, so finite type 2 at* 0*;*
  in fact $V_1(0) = span\{\frac{\partial}{\partial x}\big|_0\}$, $V_2(0) = span\{\frac{\partial}{\partial x}\big|_0, \frac{\partial}{\partial t}\big|_0\}$.
- *Local non-isotropic dimension* Q *at* 0 *is*
  $1 \cdot dim V_1(0) + 2 \cdot (dim V_2(0) - dim V_1(0)) = 1 \cdot 1 + 2 \cdot 1 = 3.$
- *Previous proposition implies*

$$\|u\|_{L^{p^*}(\mathbb{R}^2)} \leq C\|\nabla_b u\|_{L^p(\mathbb{R}^2)}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$$

where $\nabla_b u = (Xu, Yu)$*, for* $u \in C_c^\infty(\mathbb{R}^2)$, $1 \leq p < 3$.

A Model Example

- *On* $\mathbb{R}^2$, *use coordinates* (x, t), *and let* $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$.
- $[X, Y] = \frac{\partial}{\partial t}$, *so finite type 2 at* 0;
  in fact $V_1(0) = span\{\frac{\partial}{\partial x}\big|_0\}$, $V_2(0) = span\{\frac{\partial}{\partial x}\big|_0, \frac{\partial}{\partial t}\big|_0\}$.
- *Local non-isotropic dimension* Q *at* 0 *is*
  $1 \cdot dim V_1(0) + 2 \cdot (dim V_2(0) - dim V_1(0)) = 1 \cdot 1 + 2 \cdot 1 = 3.$
- *Previous proposition implies*

$$\|u\|_{L^{p*}(\mathbb{R}^2)} \leq C\|\nabla_b u\|_{L^p(\mathbb{R}^2)}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$$

where $\nabla_b u = (Xu, Yu)$, *for* $u \in C_c^\infty(\mathbb{R}^2)$, $1 \leq p < 3$.

A Model Example

- *On* $\mathbb{R}^2$, *use coordinates* (x, t), *and let* $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$.
- $[X, Y] = \frac{\partial}{\partial t}$, *so finite type 2 at* 0; *in fact* $V_1(0) = span\{\frac{\partial}{\partial x}\big|_0\}$, $V_2(0) = span\{\frac{\partial}{\partial x}\big|_0, \frac{\partial}{\partial t}\big|_0\}$.
- *Local non-isotropic dimension* Q *at* 0 *is* $1 \cdot dim V_1(0) + 2 \cdot (dim V_2(0) - dim V_1(0)) = 1 \cdot 1 + 2 \cdot 1 = 3$.
- *Previous proposition implies*

$$\|u\|_{L^{p*}(\mathbb{R}^2)} \leq C\|\nabla_b u\|_{L^p(\mathbb{R}^2)}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$$

where $\nabla_b u = (Xu, Yu)$, *for* $u \in C_c^\infty(\mathbb{R}^2)$, $1 \leq p < 3$.

A Model Example

- *On* $\mathbb{R}^2$*, use coordinates* (x, t)*, and let* $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$.
- $[X, Y] = \frac{\partial}{\partial t}$*, so finite type 2 at* 0*;*
  in fact $V_1(0) = span\{\frac{\partial}{\partial x}\big|_0\}$, $V_2(0) = span\{\frac{\partial}{\partial x}\big|_0, \frac{\partial}{\partial t}\big|_0\}$.
- *Local non-isotropic dimension* Q *at* 0 *is*
  $1 \cdot dim V_1(0) + 2 \cdot (dim V_2(0) - dim V_1(0)) = 1 \cdot 1 + 2 \cdot 1 = 3.$
- *Previous proposition implies*

$$\|u\|_{L^{p*}(\mathbb{R}^2)} \leq C\|\nabla_b u\|_{L^p(\mathbb{R}^2)}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$$

where $\nabla_b u = (Xu, Yu)$*, for* $u \in C_c^\infty(\mathbb{R}^2)$, $1 \leq p < 3$.

A Model Example

- *On* $\mathbb{R}^2$*, use coordinates* (x, t)*, and let* $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$.
- $[X, Y] = \frac{\partial}{\partial t}$*, so finite type 2 at* 0*;*
  in fact $V_1(0) = span\{\frac{\partial}{\partial x}\big|_0\}$, $V_2(0) = span\{\frac{\partial}{\partial x}\big|_0, \frac{\partial}{\partial t}\big|_0\}$.
- *Local non-isotropic dimension* Q *at* 0 *is*
  $1 \cdot dim V_1(0) + 2 \cdot (dim V_2(0) - dim V_1(0)) = 1 \cdot 1 + 2 \cdot 1 = 3$.
- *Previous proposition implies*

$$\|u\|_{L^{p*}(\mathbb{R}^2)} \leq C\|\nabla_b u\|_{L^p(\mathbb{R}^2)}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$$

where $\nabla_b u = (Xu, Yu)$*, for* $u \in C_c^\infty(\mathbb{R}^2)$, $1 \leq p < 3$.

We also have

Theorem

If $Xf_1 + Yf_2 = 0$ on $\mathbb{R}^2$, with $f_1, f_2 \in C_c^\infty$, then for all $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^2} f_1 \Phi \right| \leq C \|f\|_{L^1(\mathbb{R}^2)} \|\nabla_b \Phi\|_{L^3(\mathbb{R}^2)}.$$

We also have

Theorem

If $Xf_1 + Yf_2 = 0$ on $\mathbb{R}^2$, with $f_1, f_2 \in C_c^\infty$, then for all $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^2} f_1 \Phi \right| \leq C \|f\|_{L^1(\mathbb{R}^2)} \|\nabla_b \Phi\|_{L^3(\mathbb{R}^2)}.$$

Decomposition Lemma

- Recap: So far we have hinted at that

$$L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } d$$

$$\text{Subelliptic } L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } \overline{\partial}_b$$

because $d \circ d = 0$ and $\overline{\partial}_b \circ \overline{\partial}_b = 0$.

- We have also seen the subelliptic L^1-duality inequality in a model example ($X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$).
- We now turn to the proof of the inequality in this model case.
- Before that it helps to recall how the original L^1-duality inequality was proved.
- The key is a decomposition lemma:

Decomposition Lemma

- Recap: So far we have hinted at that

$$L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } d$$

$$\text{Subelliptic } L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } \overline{\partial}_b$$

because $d \circ d = 0$ and $\overline{\partial}_b \circ \overline{\partial}_b = 0$.

- We have also seen the subelliptic L^1-duality inequality in a model example ($X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$).
- We now turn to the proof of the inequality in this model case.
- Before that it helps to recall how the original L^1-duality inequality was proved.
- The key is a decomposition lemma:

Decomposition Lemma

- Recap: So far we have hinted at that

$$L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } d$$

$$\text{Subelliptic } L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } \overline{\partial}_b$$

because $d \circ d = 0$ and $\overline{\partial}_b \circ \overline{\partial}_b = 0$.

- We have also seen the subelliptic L^1-duality inequality in a model example ($X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$).
- We now turn to the proof of the inequality in this model case.
- Before that it helps to recall how the original L^1-duality inequality was proved.
- The key is a decomposition lemma:

Decomposition Lemma

- Recap: So far we have hinted at that

$$L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } d$$

$$\text{Subelliptic } L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } \overline{\partial}_b$$

because $d \circ d = 0$ and $\overline{\partial}_b \circ \overline{\partial}_b = 0$.

- We have also seen the subelliptic L^1-duality inequality in a model example ($X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$).
- We now turn to the proof of the inequality in this model case.
- Before that it helps to recall how the original L^1-duality inequality was proved.
- The key is a decomposition lemma:

Decomposition Lemma

- Recap: So far we have hinted at that

$$L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } d$$

$$\text{Subelliptic } L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } \overline{\partial}_b$$

because $d \circ d = 0$ and $\overline{\partial}_b \circ \overline{\partial}_b = 0$.

- We have also seen the subelliptic L^1-duality inequality in a model example ($X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$).
- We now turn to the proof of the inequality in this model case.
- Before that it helps to recall how the original L^1-duality inequality was proved.
- The key is a decomposition lemma:

Decomposition Lemma

- Recap: So far we have hinted at that

$$L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } d$$

$$\text{Subelliptic } L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } \overline{\partial}_b$$

because $d \circ d = 0$ and $\overline{\partial}_b \circ \overline{\partial}_b = 0$.

- We have also seen the subelliptic L^1-duality inequality in a model example ($X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$).
- We now turn to the proof of the inequality in this model case.
- Before that it helps to recall how the original L^1-duality inequality was proved.
- The key is a decomposition lemma:

Decomposition Lemma

- Recap: So far we have hinted at that

$$L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } d$$

$$\text{Subelliptic } L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } \overline{\partial}_b$$

  because $d \circ d = 0$ and $\overline{\partial}_b \circ \overline{\partial}_b = 0$.
- We have also seen the subelliptic L^1-duality inequality in a model example ($X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$).
- We now turn to the proof of the inequality in this model case.
- Before that it helps to recall how the original L^1-duality inequality was proved.
- The key is a decomposition lemma:

Lemma (Euclidean Decomposition Lemma)

Given any function $\Phi \in C_c^\infty(\mathbb{R}^{N-1})$ *and any* $\lambda > 0$, *there exists a decomposition* $\Phi = \Phi_1 + \Phi_2$ *such that*

$$\|\Phi_1\|_{L^\infty} \leq C\lambda^{\frac{1}{N}} \|\nabla \Phi\|_{L^N}$$

$$\|\nabla \Phi_2\|_{L^\infty} \leq C\lambda^{\frac{1}{N}-1} \|\nabla \Phi\|_{L^N}.$$

- The original L^1-duality inequality then follows by 'freezing variables'.

Lemma (Euclidean Decomposition Lemma)

Given any function $\Phi \in C_c^\infty(\mathbb{R}^{N-1})$ and any $\lambda > 0$, there exists a decomposition $\Phi = \Phi_1 + \Phi_2$ such that

$$\|\Phi_1\|_{L^\infty} \leq C\lambda^{\frac{1}{N}} \|\nabla\Phi\|_{L^N}$$

$$\|\nabla\Phi_2\|_{L^\infty} \leq C\lambda^{\frac{1}{N}-1} \|\nabla\Phi\|_{L^N}.$$

- The original L^1-duality inequality then follows by 'freezing variables'.

Lemma (Euclidean Decomposition Lemma)

Given any function $\Phi \in C_c^\infty(\mathbb{R}^{N-1})$ *and any* $\lambda > 0$*, there exists a decomposition* $\Phi = \Phi_1 + \Phi_2$ *such that*

$$\|\Phi_1\|_{L^\infty} \leq C\lambda^{\frac{1}{N}} \|\nabla\Phi\|_{L^N}$$

$$\|\nabla\Phi_2\|_{L^\infty} \leq C\lambda^{\frac{1}{N}-1} \|\nabla\Phi\|_{L^N}.$$

- The original L^1-duality inequality then follows by 'freezing variables'.

- Recall that the L^1-duality inequality says that if $f_j \in C_c^\infty$ on $\mathbb{R}^N$ and $\sum_{j=1}^N \frac{\partial f_j}{\partial x_j} = 0$ then for any $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^N} f_1 \Phi dx \right| \leq C \|f\|_{L^1} \|\nabla \Phi\|_{L^N}.$$

- Now

$$\int_{\mathbb{R}^N} f_1 \Phi dx = \int_{-\infty}^{\infty} \int_{\mathbb{R}^{N-1}} f_1 \Phi dx' dx_1.$$

- Freeze $x_1 = a$, restrict Φ to the hyperplane $\{x_1 = a\}$ and for any $\lambda > 0$ decompose $\Phi|_{\{x_1=a\}} = \Phi_1^a + \Phi_2^a$.
-

$$\left| \int_{\{x_1=a\}} f_1 \Phi_1^a \right| \leq \|f_1\|_{L^1(\{x_1=a\})} \|\Phi_1^a\|_{L^\infty(\{x_1=a\})}$$

and $\|\Phi_1^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Recall that the L^1-duality inequality says that if $f_j \in C_c^\infty$ on $\mathbb{R}^N$ and $\sum_{j=1}^N \frac{\partial f_j}{\partial x_j} = 0$ then for any $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^N} f_1 \Phi dx \right| \leq C \|f\|_{L^1} \|\nabla \Phi\|_{L^N}.$$

- Now

$$\int_{\mathbb{R}^N} f_1 \Phi dx = \int_{-\infty}^{\infty} \int_{\mathbb{R}^{N-1}} f_1 \Phi dx' dx_1.$$

- Freeze $x_1 = a$, restrict Φ to the hyperplane $\{x_1 = a\}$ and for any $\lambda > 0$ decompose $\Phi|_{\{x_1=a\}} = \Phi_1^a + \Phi_2^a$.
-

$$\left| \int_{\{x_1=a\}} f_1 \Phi_1^a \right| \leq \|f_1\|_{L^1(\{x_1=a\})} \|\Phi_1^a\|_{L^\infty(\{x_1=a\})}$$

and $\|\Phi_1^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Recall that the L^1-duality inequality says that if $f_j \in C_c^\infty$ on $\mathbb{R}^N$ and $\sum_{j=1}^N \frac{\partial f_j}{\partial x_j} = 0$ then for any $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^N} f_1 \Phi dx \right| \leq C \|f\|_{L^1} \|\nabla \Phi\|_{L^N}.$$

- Now

$$\int_{\mathbb{R}^N} f_1 \Phi dx = \int_{-\infty}^{\infty} \int_{\mathbb{R}^{N-1}} f_1 \Phi dx' dx_1.$$

- Freeze $x_1 = a$, restrict Φ to the hyperplane $\{x_1 = a\}$ and for any $\lambda > 0$ decompose $\Phi|_{\{x_1=a\}} = \Phi_1^a + \Phi_2^a$.

-

$$\left| \int_{\{x_1=a\}} f_1 \Phi_1^a \right| \leq \|f_1\|_{L^1(\{x_1=a\})} \|\Phi_1^a\|_{L^\infty(\{x_1=a\})}$$

and $\|\Phi_1^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Recall that the L^1-duality inequality says that if $f_j \in C_c^\infty$ on $\mathbb{R}^N$ and $\sum_{j=1}^N \frac{\partial f_j}{\partial x_j} = 0$ then for any $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^N} f_1 \Phi dx \right| \leq C \|f\|_{L^1} \|\nabla \Phi\|_{L^N}.$$

- Now

$$\int_{\mathbb{R}^N} f_1 \Phi dx = \int_{-\infty}^{\infty} \int_{\mathbb{R}^{N-1}} f_1 \Phi dx' dx_1.$$

- Freeze $x_1 = a$, restrict Φ to the hyperplane $\{x_1 = a\}$ and for any $\lambda > 0$ decompose $\Phi|_{\{x_1=a\}} = \Phi_1^a + \Phi_2^a$.
- $$\left| \int_{\{x_1=a\}} f_1 \Phi_1^a \right| \leq \|f_1\|_{L^1(\{x_1=a\})} \|\Phi_1^a\|_{L^\infty(\{x_1=a\})}$$

  and $\|\Phi_1^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Next

$$\int_{\mathbb{R}^{N-1}} f_1(a,x')\Phi_2^a(a,x')dx'$$

$$= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} \frac{\partial f_1}{\partial x_1}(x_1,x')\Phi_2^a(a,x')dx'dx_1$$

$$= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} -\sum_{j=2}^{N} \frac{\partial f_j}{\partial x_j}(x_1,x')\Phi_2^a(a,x')dx'dx_1$$

$$= \sum_{j=2}^{N}\int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} f_j(x_1,x')\frac{\partial \Phi_2^a}{\partial x_j}(a,x')dx'dx_1$$

$$\leq \|f\|_{L^1(\mathbb{R}^N)}\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}.$$

and $\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Optimize λ, integrate in a and get the desired estimate.

- Next

$$\int_{\mathbb{R}^{N-1}} f_1(a,x')\Phi_2^a(a,x')dx'$$
$$= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} \frac{\partial f_1}{\partial x_1}(x_1,x')\Phi_2^a(a,x')dx'dx_1$$
$$= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} -\sum_{j=2}^{N}\frac{\partial f_j}{\partial x_j}(x_1,x')\Phi_2^a(a,x')dx'dx_1$$
$$= \sum_{j=2}^{N}\int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} f_j(x_1,x')\frac{\partial \Phi_2^a}{\partial x_j}(a,x')dx'dx_1$$
$$\leq \|f\|_{L^1(\mathbb{R}^N)}\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}.$$

and $\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Optimize λ, integrate in a and get the desired estimate.

- Next

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}} f_1(a,x')\Phi_2^a(a,x')dx'\\&=\int_{-\infty}^a\int_{\mathbb{R}^{N-1}}\frac{\partial f_1}{\partial x_1}(x_1,x')\Phi_2^a(a,x')dx'dx_1\\&=\int_{-\infty}^a\int_{\mathbb{R}^{N-1}}-\sum_{j=2}^N\frac{\partial f_j}{\partial x_j}(x_1,x')\Phi_2^a(a,x')dx'dx_1\\&=\sum_{j=2}^N\int_{-\infty}^a\int_{\mathbb{R}^{N-1}}f_j(x_1,x')\frac{\partial\Phi_2^a}{\partial x_j}(a,x')dx'dx_1\\&\le\|f\|_{L^1(\mathbb{R}^N)}\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}.\end{aligned}$$

and $\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Optimize λ, integrate in a and get the desired estimate.

- Next

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} f_1(a,x')\Phi_2^a(a,x')dx' \\
&= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} \frac{\partial f_1}{\partial x_1}(x_1,x')\Phi_2^a(a,x')dx'dx_1 \\
&= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} -\sum_{j=2}^{N}\frac{\partial f_j}{\partial x_j}(x_1,x')\Phi_2^a(a,x')dx'dx_1 \\
&= \sum_{j=2}^{N}\int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} f_j(x_1,x')\frac{\partial \Phi_2^a}{\partial x_j}(a,x')dx'dx_1 \\
&\leq \|f\|_{L^1(\mathbb{R}^N)}\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}.
\end{aligned}$$

and $\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Optimize λ, integrate in a and get the desired estimate.

- Next

$$
\begin{aligned}
&\int_{\mathbb{R}^{N-1}} f_1(a,x')\Phi_2^a(a,x')dx' \\
&= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} \frac{\partial f_1}{\partial x_1}(x_1,x')\Phi_2^a(a,x')dx'dx_1 \\
&= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} -\sum_{j=2}^{N} \frac{\partial f_j}{\partial x_j}(x_1,x')\Phi_2^a(a,x')dx'dx_1 \\
&= \sum_{j=2}^{N}\int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} f_j(x_1,x')\frac{\partial \Phi_2^a}{\partial x_j}(a,x')dx'dx_1 \\
&\leq \|f\|_{L^1(\mathbb{R}^N)}\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}.
\end{aligned}
$$

and $\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Optimize λ, integrate in a and get the desired estimate.

- Next

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} f_1(a,x')\Phi_2^a(a,x')dx' \\
&= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} \frac{\partial f_1}{\partial x_1}(x_1,x')\Phi_2^a(a,x')dx'dx_1 \\
&= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} -\sum_{j=2}^{N} \frac{\partial f_j}{\partial x_j}(x_1,x')\Phi_2^a(a,x')dx'dx_1 \\
&= \sum_{j=2}^{N}\int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} f_j(x_1,x')\frac{\partial \Phi_2^a}{\partial x_j}(a,x')dx'dx_1 \\
&\leq \|f\|_{L^1(\mathbb{R}^N)}\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}.
\end{aligned}$$

and $\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Optimize λ, integrate in a and get the desired estimate.

- Next

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} f_1(a,x')\Phi_2^a(a,x')dx' \\
&= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} \frac{\partial f_1}{\partial x_1}(x_1,x')\Phi_2^a(a,x')dx'dx_1 \\
&= \int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} -\sum_{j=2}^{N}\frac{\partial f_j}{\partial x_j}(x_1,x')\Phi_2^a(a,x')dx'dx_1 \\
&= \sum_{j=2}^{N}\int_{-\infty}^{a}\int_{\mathbb{R}^{N-1}} f_j(x_1,x')\frac{\partial \Phi_2^a}{\partial x_j}(a,x')dx'dx_1 \\
&\leq \|f\|_{L^1(\mathbb{R}^N)}\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}.
\end{aligned}$$

and $\|\nabla\Phi_2^a\|_{L^\infty(\{x_1=a\})}$ can be estimated by the lemma.

- Optimize λ, integrate in a and get the desired estimate.

To prove Euclidean Decomposition Lemma, it suffices to observe that

- the decomposition is dilation invariant
  $\rightarrow$ reduces to the case $\lambda = 1$
- can do a Littlewood-Paley decomposition, and simply take Φ_2 to be the low-frequency component of Φ
- Equivalently, can take $\Phi_2 = \Phi * \eta$ for a suitable bump function η

To prove Euclidean Decomposition Lemma, it suffices to observe that

- the decomposition is dilation invariant
  $\longrightarrow$ reduces to the case $\lambda = 1$
- can do a Littlewood-Paley decomposition, and simply take Φ_2 to be the low-frequency component of Φ
- Equivalently, can take $\Phi_2 = \Phi * \eta$ for a suitable bump function η

To prove Euclidean Decomposition Lemma, it suffices to observe that

- the decomposition is dilation invariant
  $\rightarrow$ reduces to the case $\lambda = 1$
- can do a Littlewood-Paley decomposition, and simply take Φ_2 to be the low-frequency component of Φ
- Equivalently, can take $\Phi_2 = \Phi * \eta$ for a suitable bump function η

To prove Euclidean Decomposition Lemma, it suffices to observe that

- the decomposition is dilation invariant
  $\rightarrow$ reduces to the case $\lambda = 1$
- can do a Littlewood-Paley decomposition, and simply take Φ_2 to be the low-frequency component of Φ
- Equivalently, can take $\Phi_2 = \Phi * \eta$ for a suitable bump function η

To prove the subelliptic L^1-duality inequality in the model case, we need instead

Lemma (Subelliptic Decomposition Lemma in model example)

Given $\Phi \in C_c^\infty(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi = \Phi_1^a + \Phi_2^a$ on the hyperplane $\{x = a\}$ and an extension of Φ_2^a into the whole $\mathbb{R}^2$ such that

$$\|\Phi_1^a\|_{L^\infty(\{x=a\})} \leq C\lambda^{\frac{1}{3}} M\mathcal{I}(a)$$

$$\|\nabla_b \Phi_2^a\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda^{-\frac{2}{3}} M\mathcal{I}(a)$$

where

$$\mathcal{I}(x) = \|\nabla_b \Phi(x,t)\|_{L^3(dt)}$$

and M is the standard Hardy-Littlewood maximal function on $\mathbb{R}$.

To prove the subelliptic L^1-duality inequality in the model case, we need instead

Lemma (Subelliptic Decomposition Lemma in model example)

Given $\Phi \in C_c^\infty(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi = \Phi_1^a + \Phi_2^a$ on the hyperplane $\{x = a\}$ and an extension of Φ_2^a into the whole $\mathbb{R}^2$ such that

$$\|\Phi_1^a\|_{L^\infty(\{x=a\})} \leq C\lambda^{\frac{1}{3}} M\mathcal{I}(a)$$
$$\|\nabla_b \Phi_2^a\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda^{-\frac{2}{3}} M\mathcal{I}(a)$$

where

$$\mathcal{I}(x) = \|\nabla_b \Phi(x,t)\|_{L^3(dt)}$$

and M is the standard Hardy-Littlewood maximal function on $\mathbb{R}$.

To prove the subelliptic L^1-duality inequality in the model case, we need instead

Lemma (Subelliptic Decomposition Lemma in model example)

Given $\Phi \in C_c^\infty(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi = \Phi_1^a + \Phi_2^a$ on the hyperplane $\{x = a\}$ and an extension of Φ_2^a into the whole $\mathbb{R}^2$ such that

$$\|\Phi_1^a\|_{L^\infty(\{x=a\})} \leq C\lambda^{\frac{1}{3}} M\mathcal{I}(a)$$

$$\|\nabla_b \Phi_2^a\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda^{-\frac{2}{3}} M\mathcal{I}(a)$$

where

$$\mathcal{I}(x) = \|\nabla_b \Phi(x,t)\|_{L^3(dt)}$$

and M is the standard Hardy-Littlewood maximal function on $\mathbb{R}$.

To prove the subelliptic L^1-duality inequality in the model case, we need instead

Lemma (Subelliptic Decomposition Lemma in model example)

Given $\Phi \in C_c^\infty(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi = \Phi_1^a + \Phi_2^a$ on the hyperplane $\{x = a\}$ and an extension of Φ_2^a into the whole $\mathbb{R}^2$ such that

$$\|\Phi_1^a\|_{L^\infty(\{x=a\})} \leq C\lambda^{\frac{1}{3}} M\mathcal{I}(a)$$

$$\|\nabla_b \Phi_2^a\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda^{-\frac{2}{3}} M\mathcal{I}(a)$$

where

$$\mathcal{I}(x) = \|\nabla_b \Phi(x,t)\|_{L^3(dt)}$$

and M is the standard Hardy-Littlewood maximal function on $\mathbb{R}$.

To prove the subelliptic L^1-duality inequality in the model case, we need instead

Lemma (Subelliptic Decomposition Lemma in model example)

Given $\Phi \in C_c^\infty(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi = \Phi_1^a + \Phi_2^a$ on the hyperplane $\{x = a\}$ and an extension of Φ_2^a into the whole $\mathbb{R}^2$ such that

$$\|\Phi_1^a\|_{L^\infty(\{x=a\})} \leq C\lambda^{\frac{1}{3}} M\mathcal{I}(a)$$
$$\|\nabla_b \Phi_2^a\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda^{-\frac{2}{3}} M\mathcal{I}(a)$$

where

$$\mathcal{I}(x) = \|\nabla_b \Phi(x,t)\|_{L^3(dt)}$$

and M is the standard Hardy-Littlewood maximal function on $\mathbb{R}$.

- Key idea in its proof: *lifting* (also important for the general case)
- On $\mathbb{R}^3$ use coordinates (x, y, t). Consider the map

$$\pi\colon \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, t) \mapsto (x, t)$$

- The vector fields $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$ can be *lifted* to vector fields

$$\tilde{X} := \frac{\partial}{\partial x}, \quad \tilde{Y} := \frac{\partial}{\partial y} + x\frac{\partial}{\partial t} \quad \text{on } \mathbb{R}^3$$

such that $d\pi(\tilde{X}) = X, d\pi(\tilde{Y}) = Y$.
- Any function Φ on $\mathbb{R}^2$ can be pulled back to another function $\tilde{\Phi}$ on $\mathbb{R}^3$ by letting

$$\tilde{\Phi} = \Phi \circ \pi.$$

- Key idea in its proof: *lifting*
  (also important for the general case)
- On $\mathbb{R}^3$ use coordinates (x, y, t). Consider the map

$$\pi \colon \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, t) \mapsto (x, t)$$

- The vector fields $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$ can be *lifted* to vector fields

$$\tilde{X} := \frac{\partial}{\partial x}, \quad \tilde{Y} := \frac{\partial}{\partial y} + x\frac{\partial}{\partial t} \quad \text{on } \mathbb{R}^3$$

  such that $d\pi(\tilde{X}) = X, d\pi(\tilde{Y}) = Y$.
- Any function Φ on $\mathbb{R}^2$ can be pulled back to another function $\tilde{\Phi}$ on $\mathbb{R}^3$ by letting

$$\tilde{\Phi} = \Phi \circ \pi.$$

- Key idea in its proof: *lifting*
  (also important for the general case)
- On $\mathbb{R}^3$ use coordinates (x, y, t). Consider the map

$$\pi\colon \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, t) \mapsto (x, t)$$

- The vector fields $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$ can be *lifted* to vector fields

$$\tilde{X} := \frac{\partial}{\partial x}, \quad \tilde{Y} := \frac{\partial}{\partial y} + x\frac{\partial}{\partial t} \quad \text{on } \mathbb{R}^3$$

  such that $d\pi(\tilde{X}) = X, d\pi(\tilde{Y}) = Y$.
- Any function Φ on $\mathbb{R}^2$ can be pulled back to another function $\tilde{\Phi}$ on $\mathbb{R}^3$ by letting

$$\tilde{\Phi} = \Phi \circ \pi.$$

- Key idea in its proof: *lifting*
  (also important for the general case)
- On $\mathbb{R}^3$ use coordinates (x, y, t). Consider the map

$$\pi \colon \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, t) \mapsto (x, t)$$

- The vector fields $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$ on $\mathbb{R}^2$ can be *lifted* to vector fields

$$\tilde{X} := \frac{\partial}{\partial x}, \quad \tilde{Y} := \frac{\partial}{\partial y} + x\frac{\partial}{\partial t} \quad \text{on } \mathbb{R}^3$$

  such that $d\pi(\tilde{X}) = X, d\pi(\tilde{Y}) = Y$.
- Any function Φ on $\mathbb{R}^2$ can be pulled back to another function $\tilde{\Phi}$ on $\mathbb{R}^3$ by letting

$$\tilde{\Phi} = \Phi \circ \pi.$$

- Clearly $\tilde{X}\tilde{\Phi} = \tilde{X}\Phi$ and $\tilde{Y}\tilde{\Phi} = \tilde{Y}\Phi$
- Why is this good? Because $\mathbb{R}^3$ can be endowed with the structure of a *Lie group* such that $\tilde{X}$, $\tilde{Y}$ are left-invariant vector fields: in fact we can define

$$(x, y, t) \cdot (u, v, w) := (x + u, y + v, t + w + xv)$$

(Heisenberg group)
- One advantage of having a group structure is that we can then define convolutions:

$$(F * G)(x, y, t) := \int_{\mathbb{R}^3} F((x, y, t) \cdot (u, v, w)) G(u, v, w) dudvdw$$

- Since $\tilde{X}$, $\tilde{Y}$ are left-invariant, they are very compatible with convolutions: e.g.

$$(\tilde{X}F) * G = -F * (\tilde{X}G), \quad (\tilde{Y}F) * G = -F * (\tilde{Y}G)$$

(Cannot do these on the underlying $\mathbb{R}^2$!)

- Clearly $\tilde{X}\tilde{\Phi} = \tilde{X\Phi}$ and $\tilde{Y}\tilde{\Phi} = \tilde{Y\Phi}$
- Why is this good? Because $\mathbb{R}^3$ can be endowed with the structure of a *Lie group* such that $\tilde{X}$, $\tilde{Y}$ are left-invariant vector fields: in fact we can define

$$(x, y, t) \cdot (u, v, w) := (x + u, y + v, t + w + xv)$$

(Heisenberg group)
- One advantage of having a group structure is that we can then define convolutions:

$$(F * G)(x, y, t) := \int_{\mathbb{R}^3} F((x, y, t) \cdot (u, v, w))G(u, v, w)dudvdw$$

- Since $\tilde{X}$, $\tilde{Y}$ are left-invariant, they are very compatible with convolutions: e.g.

$$(\tilde{X}F) * G = -F * (\tilde{X}G), \quad (\tilde{Y}F) * G = -F * (\tilde{Y}G)$$

(Cannot do these on the underlying $\mathbb{R}^2$!)

- Clearly $\tilde{X}\tilde{\Phi} = \tilde{X\Phi}$ and $\tilde{Y}\tilde{\Phi} = \tilde{Y\Phi}$
- Why is this good? Because $\mathbb{R}^3$ can be endowed with the structure of a *Lie group* such that $\tilde{X}$, $\tilde{Y}$ are left-invariant vector fields: in fact we can define

$$(x, y, t) \cdot (u, v, w) := (x + u, y + v, t + w + xv)$$

(Heisenberg group)

- One advantage of having a group structure is that we can then define convolutions:

$$(F * G)(x, y, t) := \int_{\mathbb{R}^3} F((x, y, t) \cdot (u, v, w)) G(u, v, w) dudvdw$$

- Since $\tilde{X}$, $\tilde{Y}$ are left-invariant, they are very compatible with convolutions: e.g.

$$(\tilde{X}F) * G = -F * (\tilde{X}G), \quad (\tilde{Y}F) * G = -F * (\tilde{Y}G)$$

(Cannot do these on the underlying $\mathbb{R}^2$!)

- Clearly $\tilde{X}\tilde{\Phi} = \tilde{X\Phi}$ and $\tilde{Y}\tilde{\Phi} = \tilde{Y\Phi}$
- Why is this good? Because $\mathbb{R}^3$ can be endowed with the structure of a *Lie group* such that $\tilde{X}$, $\tilde{Y}$ are left-invariant vector fields: in fact we can define
$$(x, y, t) \cdot (u, v, w) := (x + u, y + v, t + w + xv)$$
(Heisenberg group)
- One advantage of having a group structure is that we can then define convolutions:
$$(F * G)(x, y, t) := \int_{\mathbb{R}^3} F((x, y, t) \cdot (u, v, w)) G(u, v, w) dudvdw$$
- Since $\tilde{X}$, $\tilde{Y}$ are left-invariant, they are very compatible with convolutions: e.g.
$$(\tilde{X}F) * G = -F * (\tilde{X}G), \quad (\tilde{Y}F) * G = -F * (\tilde{Y}G)$$
(Cannot do these on the underlying $\mathbb{R}^2$!)

- Clearly $\tilde{X}\tilde{\Phi} = \tilde{X\Phi}$ and $\tilde{Y}\tilde{\Phi} = \tilde{Y\Phi}$
- Why is this good? Because $\mathbb{R}^3$ can be endowed with the structure of a *Lie group* such that $\tilde{X}$, $\tilde{Y}$ are left-invariant vector fields: in fact we can define

$$(x, y, t) \cdot (u, v, w) := (x + u, y + v, t + w + xv)$$

(Heisenberg group)

- One advantage of having a group structure is that we can then define convolutions:

$$(F * G)(x, y, t) := \int_{\mathbb{R}^3} F((x, y, t) \cdot (u, v, w))G(u, v, w)dudvdw$$

- Since $\tilde{X}$, $\tilde{Y}$ are left-invariant, they are very compatible with convolutions: e.g.

$$(\tilde{X}F) * G = -F * (\tilde{X}G), \quad (\tilde{Y}F) * G = -F * (\tilde{Y}G)$$

(Cannot do these on the underlying $\mathbb{R}^2$!)

- Clearly $\tilde{X}\tilde{\Phi} = \tilde{X\Phi}$ and $\tilde{Y}\tilde{\Phi} = \tilde{Y\Phi}$
- Why is this good? Because $\mathbb{R}^3$ can be endowed with the structure of a *Lie group* such that $\tilde{X}$, $\tilde{Y}$ are left-invariant vector fields: in fact we can define

$$(x, y, t) \cdot (u, v, w) := (x + u, y + v, t + w + xv)$$

(Heisenberg group)
- One advantage of having a group structure is that we can then define convolutions:

$$(F * G)(x, y, t) := \int_{\mathbb{R}^3} F((x, y, t) \cdot (u, v, w))G(u, v, w)dudvdw$$

- Since $\tilde{X}$, $\tilde{Y}$ are left-invariant, they are very compatible with convolutions: e.g.

$$(\tilde{X}F) * G = -F * (\tilde{X}G), \quad (\tilde{Y}F) * G = -F * (\tilde{Y}G)$$

(Cannot do these on the underlying $\mathbb{R}^2$!)

- Clearly $\tilde{X}\tilde{\Phi} = \tilde{X\Phi}$ and $\tilde{Y}\tilde{\Phi} = \tilde{Y\Phi}$
- Why is this good? Because $\mathbb{R}^3$ can be endowed with the structure of a *Lie group* such that $\tilde{X}$, $\tilde{Y}$ are left-invariant vector fields: in fact we can define

$$(x, y, t) \cdot (u, v, w) := (x + u, y + v, t + w + xv)$$

(Heisenberg group)

- One advantage of having a group structure is that we can then define convolutions:

$$(F * G)(x, y, t) := \int_{\mathbb{R}^3} F((x, y, t) \cdot (u, v, w)) G(u, v, w) dudvdw$$

- Since $\tilde{X}$, $\tilde{Y}$ are left-invariant, they are very compatible with convolutions: e.g.

$$(\tilde{X}F) * G = -F * (\tilde{X}G), \quad (\tilde{Y}F) * G = -F * (\tilde{Y}G)$$

(Cannot do these on the underlying $\mathbb{R}^2$!)

- Clearly $\tilde{X}\tilde{\Phi} = \tilde{X\Phi}$ and $\tilde{Y}\tilde{\Phi} = \tilde{Y\Phi}$
- Why is this good? Because $\mathbb{R}^3$ can be endowed with the structure of a *Lie group* such that $\tilde{X}$, $\tilde{Y}$ are left-invariant vector fields: in fact we can define

$$(x, y, t) \cdot (u, v, w) := (x + u, y + v, t + w + xv)$$

(Heisenberg group)
- One advantage of having a group structure is that we can then define convolutions:

$$(F * G)(x, y, t) := \int_{\mathbb{R}^3} F((x, y, t) \cdot (u, v, w))G(u, v, w)dudvdw$$

- Since $\tilde{X}$, $\tilde{Y}$ are left-invariant, they are very compatible with convolutions: e.g.

$$(\tilde{X}F) * G = -F * (\tilde{X}G), \quad (\tilde{Y}F) * G = -F * (\tilde{Y}G)$$

(Cannot do these on the underlying $\mathbb{R}^2$!)

- Another observation is that we actually obtained a *homogeneous group*, i.e. a group that carries an automorphic dilation

$$\lambda \cdot (x, y, t) := (\lambda x, \lambda y, \lambda^2 t)$$

- Define a dilation I_λ on functions that preserves L^1 norm:

$$(I_\lambda \eta)(x, y, t) := \lambda^{-4} \eta(\lambda^{-1} x, \lambda^{-1} y, \lambda^{-2} t)$$

- Recall now the decomposition lemma:
  Given $\Phi \in C_c^\infty(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi|_{\{x=a\}} = \Phi_1^a + \Phi_2^a$ on the hyperplane $\{x = a\}$ and an extension of Φ_2^a into the whole $\mathbb{R}^2$ such that

$$\|\Phi_1^a\|_{L^\infty(\{x=a\})} \leq C\lambda^{\frac{1}{3}} M\mathcal{I}(a)$$

$$\|\nabla_b \Phi_2^a\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda^{-\frac{2}{3}} M\mathcal{I}(a)$$

- Another observation is that we actually obtained a *homogeneous group*, i.e. a group that carries an automorphic dilation

$$\lambda \cdot (x, y, t) := (\lambda x, \lambda y, \lambda^2 t)$$

- Define a dilation I_λ on functions that preserves L^1 norm:

$$(I_\lambda \eta)(x, y, t) := \lambda^{-4} \eta(\lambda^{-1} x, \lambda^{-1} y, \lambda^{-2} t)$$

- Recall now the decomposition lemma:
Given $\Phi \in C_c^\infty(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi|_{\{x=a\}} = \Phi_1^a + \Phi_2^a$ on the hyperplane $\{x = a\}$ and an extension of Φ_2^a into the whole $\mathbb{R}^2$ such that

$$\|\Phi_1^a\|_{L^\infty(\{x=a\})} \leq C\lambda^{\frac{1}{3}} M\mathcal{I}(a)$$

$$\|\nabla_b \Phi_2^a\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda^{-\frac{2}{3}} M\mathcal{I}(a)$$

- Another observation is that we actually obtained a *homogeneous group*, i.e. a group that carries an automorphic dilation

$$\lambda \cdot (x, y, t) := (\lambda x, \lambda y, \lambda^2 t)$$

- Define a dilation I_λ on functions that preserves L^1 norm:

$$(I_\lambda \eta)(x, y, t) := \lambda^{-4} \eta(\lambda^{-1} x, \lambda^{-1} y, \lambda^{-2} t)$$

- Recall now the decomposition lemma:
Given $\Phi \in C_c^\infty(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi|_{\{x=a\}} = \Phi_1^a + \Phi_2^a$ on the hyperplane $\{x = a\}$ and an extension of Φ_2^a into the whole $\mathbb{R}^2$ such that

$$\|\Phi_1^a\|_{L^\infty(\{x=a\})} \leq C\lambda^{\frac{1}{3}} M\mathcal{I}(a)$$

$$\|\nabla_b \Phi_2^a\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda^{-\frac{2}{3}} M\mathcal{I}(a)$$

- Another observation is that we actually obtained a *homogeneous group*, i.e. a group that carries an automorphic dilation

$$\lambda \cdot (x, y, t) := (\lambda x, \lambda y, \lambda^2 t)$$

- Define a dilation I_λ on functions that preserves L^1 norm:

$$(I_\lambda \eta)(x, y, t) := \lambda^{-4} \eta(\lambda^{-1} x, \lambda^{-1} y, \lambda^{-2} t)$$

- Recall now the decomposition lemma:
  Given $\Phi \in C_c^\infty(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi|_{\{x=a\}} = \Phi_1^a + \Phi_2^a$ on the hyperplane $\{x = a\}$ and an extension of Φ_2^a into the whole $\mathbb{R}^2$ such that

$$\|\Phi_1^a\|_{L^\infty(\{x=a\})} \leq C\lambda^{\frac{1}{3}} M\mathcal{I}(a)$$

$$\|\nabla_b \Phi_2^a\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda^{-\frac{2}{3}} M\mathcal{I}(a)$$

- To prove lemma, fix $\lambda > 0$, $a \in \mathbb{R}$.
  Let $\eta \in C_c^\infty$ be a bump function on the group $\mathbb{R}^3$, $\int \eta = 1$.
  The desired decomposition of $\Phi|_{\{x=a\}}$ is given by

$$\Phi_2^a(a,t) := \tilde{\Phi} * I_\lambda \eta(a,y,t) \quad \text{for all } t$$

  (the right hand side actually does not depend on y) and

$$\Phi_1^a(a,t) := \Phi(a,t) - \Phi_2^a(a,t)$$

- The desired extension of Φ_2^a is given by

$$\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}} \eta(a,y,t) \quad \text{for all } s, t$$

- Difficulty: Need to integrate away the variable we added during the lifting process

- To prove lemma, fix $\lambda > 0$, $a \in \mathbb{R}$.
  Let $\eta \in C_c^\infty$ be a bump function on the group $\mathbb{R}^3$, $\int \eta = 1$.
  The desired decomposition of $\Phi|_{\{x=a\}}$ is given by

$$\Phi_2^a(a,t) := \tilde{\Phi} * I_\lambda \eta(a,y,t) \quad \text{for all } t$$

  (the right hand side actually does not depend on y) and

$$\Phi_1^a(a,t) := \Phi(a,t) - \Phi_2^a(a,t)$$

- The desired extension of Φ_2^a is given by

$$\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}} \eta(a,y,t) \quad \text{for all } s,t$$

- Difficulty: Need to integrate away the variable we added during the lifting process

- To prove lemma, fix $\lambda > 0$, $a \in \mathbb{R}$.
  Let $\eta \in C_c^\infty$ be a bump function on the group $\mathbb{R}^3$, $\int \eta = 1$.
  The desired decomposition of $\Phi|_{\{x=a\}}$ is given by

$$\Phi_2^a(a,t) := \tilde{\Phi} * I_\lambda \eta(a,y,t) \quad \text{for all } t$$

(the right hand side actually does not depend on y) and

$$\Phi_1^a(a,t) := \Phi(a,t) - \Phi_2^a(a,t)$$

- The desired extension of Φ_2^a is given by

$$\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}} \eta(a,y,t) \quad \text{for all } s,t$$

- Difficulty: Need to integrate away the variable we added during the lifting process

- To prove lemma, fix $\lambda > 0$, $a \in \mathbb{R}$.
  Let $\eta \in C_c^\infty$ be a bump function on the group $\mathbb{R}^3$, $\int \eta = 1$.
  The desired decomposition of $\Phi|_{\{x=a\}}$ is given by

$$\Phi_2^a(a,t) := \tilde{\Phi} * I_\lambda \eta(a,y,t) \quad \text{for all } t$$

  (the right hand side actually does not depend on y) and

$$\Phi_1^a(a,t) := \Phi(a,t) - \Phi_2^a(a,t)$$

- The desired extension of Φ_2^a is given by

$$\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}} \eta(a,y,t) \quad \text{for all } s, t$$

- Difficulty: Need to integrate away the variable we added during the lifting process

- To prove lemma, fix $\lambda > 0$, $a \in \mathbb{R}$.
  Let $\eta \in C_c^\infty$ be a bump function on the group $\mathbb{R}^3$, $\int \eta = 1$.
  The desired decomposition of $\Phi|_{\{x=a\}}$ is given by

$$\Phi_2^a(a,t) := \tilde{\Phi} * I_\lambda \eta(a,y,t) \quad \text{for all } t$$

  (the right hand side actually does not depend on y) and

$$\Phi_1^a(a,t) := \Phi(a,t) - \Phi_2^a(a,t)$$

- The desired extension of Φ_2^a is given by

$$\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}} \eta(a,y,t) \quad \text{for all } s, t$$

- Difficulty: Need to integrate away the variable we added during the lifting process

- To prove lemma, fix $\lambda > 0$, $a \in \mathbb{R}$.
  Let $\eta \in C_c^\infty$ be a bump function on the group $\mathbb{R}^3$, $\int \eta = 1$.
  The desired decomposition of $\Phi|_{\{x=a\}}$ is given by

$$\Phi_2^a(a,t) := \tilde{\Phi} * I_\lambda \eta(a,y,t) \quad \text{for all } t$$

  (the right hand side actually does not depend on y) and

$$\Phi_1^a(a,t) := \Phi(a,t) - \Phi_2^a(a,t)$$

- The desired extension of Φ_2^a is given by

$$\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}} \eta(a,y,t) \quad \text{for all } s, t$$

- Difficulty: Need to integrate away the variable we added during the lifting process

- To prove lemma, fix $\lambda > 0$, $a \in \mathbb{R}$.
  Let $\eta \in C_c^\infty$ be a bump function on the group $\mathbb{R}^3$, $\int \eta = 1$.
  The desired decomposition of $\Phi|_{\{x=a\}}$ is given by

$$\Phi_2^a(a,t) := \tilde{\Phi} * I_\lambda \eta(a,y,t) \quad \text{for all } t$$

  (the right hand side actually does not depend on y) and

$$\Phi_1^a(a,t) := \Phi(a,t) - \Phi_2^a(a,t)$$

- The desired extension of Φ_2^a is given by

$$\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}} \eta(a,y,t) \quad \text{for all } s,t$$

- Difficulty: Need to integrate away the variable we added during the lifting process

- To illustrate the proof of the desired estimates, consider $X\Phi_2^a$.
- Recall $\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$

$$(X\Phi_2^a)(a+s,t) = \frac{d}{ds}\Phi_2^a(a+s,t) = \tilde{\Phi} * \frac{d}{ds} I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$$

$$\frac{d}{ds} I_{\sqrt{\lambda^2+s^2}}\eta = \frac{d}{d\tau} I_\tau \eta \Big|_{\tau=\sqrt{\lambda^2+s^2}} \cdot \frac{s}{\sqrt{\lambda^2+s^2}}$$

$$\frac{d}{d\tau} I_\tau \eta = \tilde{X}(I_\tau \eta_1) + \tilde{Y}(I_\tau \eta_2) \quad \text{for some } \eta_1, \eta_2 \in C_c^\infty$$

$$\begin{aligned} & |(X\Phi_2^a)(a+s,t)| \\ \leq & |\tilde{\Phi} * (\tilde{X} I_\tau \eta_1 + \tilde{Y} I_\tau \eta_2)|(a,y,t) \\ \leq & |\tilde{X}\tilde{\Phi} * I_\tau \eta_1| + |\tilde{Y}\tilde{\Phi} * I_\tau \eta_2|(a,y,t), \quad \tau = \sqrt{\lambda^2+s^2}. \end{aligned}$$

- To illustrate the proof of the desired estimates, consider $X\Phi_2^a$.
- Recall $\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$

$$(X\Phi_2^a)(a+s,t) = \frac{d}{ds}\Phi_2^a(a+s,t) = \tilde{\Phi} * \frac{d}{ds} I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$$

$$\frac{d}{ds} I_{\sqrt{\lambda^2+s^2}}\eta = \left.\frac{d}{d\tau} I_\tau \eta\right|_{\tau=\sqrt{\lambda^2+s^2}} \cdot \frac{s}{\sqrt{\lambda^2+s^2}}$$

$$\frac{d}{d\tau} I_\tau \eta = \tilde{X}(I_\tau \eta_1) + \tilde{Y}(I_\tau \eta_2) \quad \text{for some } \eta_1, \eta_2 \in C_c^\infty$$

$$\begin{aligned} &|(X\Phi_2^a)(a+s,t)| \\ \leq &|\tilde{\Phi} * (\tilde{X} I_\tau \eta_1 + \tilde{Y} I_\tau \eta_2)|(a,y,t) \\ \leq &|\tilde{X}\tilde{\Phi} * I_\tau \eta_1| + |\tilde{Y}\tilde{\Phi} * I_\tau \eta_2|(a,y,t), \quad \tau = \sqrt{\lambda^2+s^2}. \end{aligned}$$

- To illustrate the proof of the desired estimates, consider $X\Phi_2^a$.
- Recall $\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$

$$(X\Phi_2^a)(a+s,t) = \frac{d}{ds}\Phi_2^a(a+s,t) = \tilde{\Phi} * \frac{d}{ds} I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$$

$$\frac{d}{ds} I_{\sqrt{\lambda^2+s^2}}\eta = \left.\frac{d}{d\tau} I_\tau \eta\right|_{\tau=\sqrt{\lambda^2+s^2}} \cdot \frac{s}{\sqrt{\lambda^2+s^2}}$$

$$\frac{d}{d\tau} I_\tau\eta = \tilde{X}(I_\tau\eta_1) + \tilde{Y}(I_\tau\eta_2) \quad \text{for some } \eta_1, \eta_2 \in C_c^\infty$$

$$\begin{aligned} & |(X\Phi_2^a)(a+s,t)| \\ \leq & |\tilde{\Phi} * (\tilde{X} I_\tau \eta_1 + \tilde{Y} I_\tau \eta_2)|(a,y,t) \\ \leq & |\tilde{X}\tilde{\Phi} * I_\tau\eta_1| + |\tilde{Y}\tilde{\Phi} * I_\tau\eta_2|(a,y,t), \quad \tau = \sqrt{\lambda^2+s^2}. \end{aligned}$$

- To illustrate the proof of the desired estimates, consider $X\Phi_2^a$.
- Recall $\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$

$$(X\Phi_2^a)(a+s,t) = \frac{d}{ds}\Phi_2^a(a+s,t) = \tilde{\Phi} * \frac{d}{ds} I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$$

$$\frac{d}{ds} I_{\sqrt{\lambda^2+s^2}}\eta = \left. \frac{d}{d\tau} I_\tau \eta \right|_{\tau=\sqrt{\lambda^2+s^2}} \cdot \frac{s}{\sqrt{\lambda^2+s^2}}$$

$$\frac{d}{d\tau} I_\tau \eta = \tilde{X}(I_\tau \eta_1) + \tilde{Y}(I_\tau \eta_2) \quad \text{for some } \eta_1, \eta_2 \in C_c^\infty$$

$$\begin{aligned} &|(X\Phi_2^a)(a+s,t)| \\ \leq &|\tilde{\Phi} * (\tilde{X} I_\tau \eta_1 + \tilde{Y} I_\tau \eta_2)|(a,y,t) \\ \leq &|\tilde{X}\tilde{\Phi} * I_\tau \eta_1| + |\tilde{Y}\tilde{\Phi} * I_\tau \eta_2|(a,y,t), \quad \tau = \sqrt{\lambda^2+s^2}. \end{aligned}$$

- To illustrate the proof of the desired estimates, consider $X\Phi_2^a$.
- Recall $\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$

$$(X\Phi_2^a)(a+s,t) = \frac{d}{ds}\Phi_2^a(a+s,t) = \tilde{\Phi} * \frac{d}{ds} I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$$

$$\frac{d}{ds} I_{\sqrt{\lambda^2+s^2}}\eta = \frac{d}{d\tau} I_\tau \eta \Big|_{\tau=\sqrt{\lambda^2+s^2}} \cdot \frac{s}{\sqrt{\lambda^2+s^2}}$$

$$\frac{d}{d\tau} I_\tau \eta = \tilde{X}(I_\tau \eta_1) + \tilde{Y}(I_\tau \eta_2) \quad \text{for some } \eta_1, \eta_2 \in C_c^\infty$$

$$\begin{aligned} &|(X\Phi_2^a)(a+s,t)| \\ \leq &|\tilde{\Phi} * (\tilde{X} I_\tau \eta_1 + \tilde{Y} I_\tau \eta_2)|(a,y,t) \\ \leq &|\tilde{X}\tilde{\Phi} * I_\tau \eta_1| + |\tilde{Y}\tilde{\Phi} * I_\tau \eta_2|(a,y,t), \quad \tau = \sqrt{\lambda^2+s^2}. \end{aligned}$$

- To illustrate the proof of the desired estimates, consider $X\Phi_2^a$.
- Recall $\Phi_2^a(a+s,t) := \tilde{\Phi} * I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$

$$(X\Phi_2^a)(a+s,t) = \frac{d}{ds}\Phi_2^a(a+s,t) = \tilde{\Phi} * \frac{d}{ds}I_{\sqrt{\lambda^2+s^2}}\eta(a,y,t)$$

$$\frac{d}{ds}I_{\sqrt{\lambda^2+s^2}}\eta = \left.\frac{d}{d\tau}I_\tau\eta\right|_{\tau=\sqrt{\lambda^2+s^2}} \cdot \frac{s}{\sqrt{\lambda^2+s^2}}$$

$$\frac{d}{d\tau}I_\tau\eta = \tilde{X}(I_\tau\eta_1) + \tilde{Y}(I_\tau\eta_2) \quad \text{for some } \eta_1,\eta_2 \in C_c^\infty$$

$$\begin{aligned} &|(X\Phi_2^a)(a+s,t)| \\ \leq &|\tilde{\Phi} * (\tilde{X}I_\tau\eta_1 + \tilde{Y}I_\tau\eta_2)|(a,y,t) \\ \leq &|\tilde{X}\tilde{\Phi} * I_\tau\eta_1| + |\tilde{Y}\tilde{\Phi} * I_\tau\eta_2|(a,y,t), \quad \tau = \sqrt{\lambda^2+s^2}. \end{aligned}$$

$$|\tilde{X}\tilde{\Phi} * I_\tau \eta_1|(a, y, t)$$

$$|\tilde{X}\Phi * I_\tau \eta_1|(a, y, t)$$

$$
\begin{aligned}
&|\tilde{X}\Phi * I_\tau \eta_1|(a,y,t) \\
= &\int_{\mathbb{R}^3} |X\Phi|(a+u, t+w+av) \left|\eta_1\left(\frac{u}{\tau}, \frac{v}{\tau}, \frac{w}{\tau^2}\right)\right| \frac{1}{\tau^4} dudvdw
\end{aligned}
$$

$$|\tilde{X}\Phi * I_\tau \eta_1|(a, y, t)$$

$$= \int_{\mathbb{R}^3} |X\Phi|(a + u, t + w + av) \left|\eta_1(\frac{u}{\tau}, \frac{v}{\tau}, \frac{w}{\tau^2})\right| \frac{1}{\tau^4} dudvdw$$

Holder in w:

$$\leq \int_{\mathbb{R}^2} \|X\Phi(a + u, w)\|_{L^3(dw)} \|\eta_1(\frac{u}{\tau}, \frac{v}{\tau}, w)\|_{L^{3/2}(dw)} \tau^{-4+\frac{4}{3}} dudv$$

$$|\tilde{X\Phi} * I_\tau \eta_1|(a, y, t)$$

$$= \int_{\mathbb{R}^3} |X\Phi|(a+u, t+w+av) \left|\eta_1(\frac{u}{\tau}, \frac{v}{\tau}, \frac{w}{\tau^2})\right| \frac{1}{\tau^4} dudvdw$$

Holder in w:

$$\leq \int_{\mathbb{R}^2} \mathcal{I}(a+u) \|\eta_1(\frac{u}{\tau}, \frac{v}{\tau}, w)\|_{L^{3/2}(dw)} \tau^{-4+\frac{4}{3}} dudv$$

$$|\tilde{X\Phi} * I_\tau \eta_1|(a, y, t)$$

$$= \int_{\mathbb{R}^3} |X\Phi|(a+u, t+w+av) \left|\eta_1(\frac{u}{\tau}, \frac{v}{\tau}, \frac{w}{\tau^2})\right| \frac{1}{\tau^4} dudvdw$$

Holder in w:

$$\leq \int_{\mathbb{R}^2} \mathcal{I}(a+u) \|\eta_1(\frac{u}{\tau}, \frac{v}{\tau}, w)\|_{L^{3/2}(dw)} \tau^{-4+\frac{4}{3}} dudv$$

Integrate in v: (Important!)

$$\leq \int_{\mathbb{R}} \mathcal{I}(a+u) \|\eta_1(\frac{u}{\tau}, v, w)\|_{L^{3/2}(dw) L^1(dv)} \tau^{-4+\frac{4}{3}+1} du$$

$$|\tilde{X\Phi} * I_\tau \eta_1|(a,y,t)$$

$$= \int_{\mathbb{R}^3} |X\Phi|(a+u, t+w+av) \left|\eta_1(\frac{u}{\tau}, \frac{v}{\tau}, \frac{w}{\tau^2})\right| \frac{1}{\tau^4} dudvdw$$

Holder in w:

$$\leq \int_{\mathbb{R}^2} \mathcal{I}(a+u) \|\eta_1(\frac{u}{\tau}, \frac{v}{\tau}, w)\|_{L^{3/2}(dw)} \tau^{-4+\frac{4}{3}} dudv$$

Integrate in v: (Important!)

$$\leq \int_{\mathbb{R}} \mathcal{I}(a+u) \|\eta_1(\frac{u}{\tau}, v, w)\|_{L^{3/2}(dw)L^1(dv)} \tau^{-4+\frac{4}{3}+1} du$$

Estimate by maximal function:

$$\leq C\frac{1}{\tau} \int_{-C\tau}^{C\tau} \mathcal{I}(a+u) du \cdot \tau^{-4+\frac{4}{3}+1+1}$$

$$|\tilde{X\Phi} * I_\tau \eta_1|(a,y,t)$$
$$= \int_{\mathbb{R}^3} |X\Phi|(a+u, t+w+av) \left|\eta_1(\frac{u}{\tau}, \frac{v}{\tau}, \frac{w}{\tau^2})\right| \frac{1}{\tau^4} dudvdw$$

Holder in w:

$$\leq \int_{\mathbb{R}^2} \mathcal{I}(a+u) \|\eta_1(\frac{u}{\tau}, \frac{v}{\tau}, w)\|_{L^{3/2}(dw)} \tau^{-4+\frac{4}{3}} dudv$$

Integrate in v: (Important!)

$$\leq \int_{\mathbb{R}} \mathcal{I}(a+u) \|\eta_1(\frac{u}{\tau}, v, w)\|_{L^{3/2}(dw)L^1(dv)} \tau^{-4+\frac{4}{3}+1} du$$

Estimate by maximal function:

$$\leq C\frac{1}{\tau} \int_{-C\tau}^{C\tau} \mathcal{I}(a+u) du \cdot \tau^{-4+\frac{4}{3}+1+1} \leq CM\mathcal{I}(a)\lambda^{-\frac{2}{3}} \quad \text{because } \lambda \leq \tau.$$

This basically completes the proof of the model case.

Some difficulties in the general case are:

- In general one cannot expect the lifted vector fields be left-invariant under a group law; rather one can only *approximate* the lifted vector fields by left-invariant vector fields of a homogeneous group. Need to take care of error terms that arise.
- In general it is not possible to put a coordinate system on $\mathbb{R}^N$ so that $X_2, \dots, X_n$ are all tangent to level sets of x_1. When $X_1, \dots, X_n$ are linearly independent, a perturbation argument would work, but it is not clear whether the condition of linear independence is necessary.

This basically completes the proof of the model case.
Some difficulties in the general case are:

- In general one cannot expect the lifted vector fields be left-invariant under a group law; rather one can only *approximate* the lifted vector fields by left-invariant vector fields of a homogeneous group. Need to take care of error terms that arise.
- In general it is not possible to put a coordinate system on $\mathbb{R}^N$ so that $X_2, \dots, X_n$ are all tangent to level sets of x_1. When $X_1, \dots, X_n$ are linearly independent, a perturbation argument would work, but it is not clear whether the condition of linear independence is necessary.

This basically completes the proof of the model case.
Some difficulties in the general case are:

- In general one cannot expect the lifted vector fields be left-invariant under a group law; rather one can only *approximate* the lifted vector fields by left-invariant vector fields of a homogeneous group. Need to take care of error terms that arise.
- In general it is not possible to put a coordinate system on $\mathbb{R}^N$ so that $X_2, \dots, X_n$ are all tangent to level sets of x_1. When $X_1, \dots, X_n$ are linearly independent, a perturbation argument would work, but it is not clear whether the condition of linear independence is necessary.

This basically completes the proof of the model case.
Some difficulties in the general case are:

- In general one cannot expect the lifted vector fields be left-invariant under a group law; rather one can only *approximate* the lifted vector fields by left-invariant vector fields of a homogeneous group. Need to take care of error terms that arise.
- In general it is not possible to put a coordinate system on $\mathbb{R}^N$ so that $X_2, \dots, X_n$ are all tangent to level sets of x_1. When $X_1, \dots, X_n$ are linearly independent, a perturbation argument would work, but it is not clear whether the condition of linear independence is necessary.

This basically completes the proof of the model case.
Some difficulties in the general case are:

- In general one cannot expect the lifted vector fields be left-invariant under a group law; rather one can only *approximate* the lifted vector fields by left-invariant vector fields of a homogeneous group. Need to take care of error terms that arise.
- In general it is not possible to put a coordinate system on $\mathbb{R}^N$ so that $X_2, \dots, X_n$ are all tangent to level sets of x_1. When $X_1, \dots, X_n$ are linearly independent, a perturbation argument would work, but it is not clear whether the condition of linear independence is necessary.

This basically completes the proof of the model case.
Some difficulties in the general case are:

- In general one cannot expect the lifted vector fields be left-invariant under a group law; rather one can only *approximate* the lifted vector fields by left-invariant vector fields of a homogeneous group. Need to take care of error terms that arise.
- In general it is not possible to put a coordinate system on $\mathbb{R}^N$ so that $X_2, \dots, X_n$ are all tangent to level sets of x_1. When $X_1, \dots, X_n$ are linearly independent, a perturbation argument would work, but it is not clear whether the condition of linear independence is necessary.

Further directions of exploration:

- Sobolev inequality for d on bounded smooth domains with boundaries
- Sobolev inequality for $\bar{\partial}$ on bounded pseudoconvex domains of finite type

Further directions of exploration:

- Sobolev inequality for d on bounded smooth domains with boundaries
- Sobolev inequality for $\overline{\partial}$ on bounded pseudoconvex domains of finite type

Thank you!