Subelliptic divergence-curl inequalities

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Introduction

- Part I: Elliptic case
- Some compensation phenomena that has to do with divergence, curl and L^1
- Seems quite different from the classical theory of compensation compactness
- Part II: Subelliptic case

Hodge de-Rham complex on $\mathbb{R}^n$

- To say u is a 0-form means u is a function; then $du = \sum_{i=1}^n \frac{\partial u}{\partial x^i} dx^i$ (gradient of a function)
- To say u is a 1-form means $u = \sum_{i=1}^n u_i dx^i$; then

$$du = \sum_{i<j} \left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} \right) dx^i \wedge dx^j$$

  (curl of a vector field if $n = 3$)
- In general d maps q-forms to $(q+1)$-forms, and

$$du = \sum_{j=1}^n \frac{\partial u_J}{\partial x^j} dx^j \wedge dx^J.$$

- Inner product on q forms:

$$(u, v) = \sum_J \int_{\mathbb{R}^n} u_J \overline{v_J}$$

- We write d^* the formal adjoint of d under this inner product
- e.g. If u is a 1-form, then $d^* u = -\sum_{i=1}^n \frac{\partial u_i}{\partial x^i}$ (divergence of a vector field)

- d forms a complex: $d \circ d = 0$. Same for d^*.
- $dd^* + d^*d = \Delta$ componentwise
- Hodge decomposition: If $u \in C_c^\infty(\Lambda^q)$, then

$$u = d\alpha + d^*\beta$$

where $\alpha = \Delta^{-1}(d^*u)$ and $\beta = \Delta^{-1}(du)$. In particular, u is determined by du and d^*u.

Three pillars of the theory: elementary version

- From now on we work on $\mathbb{R}^n$, $n \geq 2$.
- First pillar is the solution of the following system of equations.

Proposition (Bourgain-Brezis)

For any $f \in L^n$, there exists a vector field $Y \in L^\infty$ such that

$$div\ Y = f$$

with $\|Y\|_{L^\infty} \leq C\|f\|_{L^n}$.

- Can always find $Y \in \dot{W}^{1,n}$ by Hodge decomposition, but $\dot{W}^{1,n}$ fails to embed into L^∞.
- But system is underdetermined: if Y is a solution, so is Y plus any divergence free vector field
- The claim is one can find a solution that is bounded by adding a divergence free vector field

More generally

Theorem (Bourgain-Brezis)

If $q \neq n-1$, then for any $f \in d^*(\dot{W}^{1,n}(\Lambda^{q+1}))$, there exists $Y \in L^\infty(\Lambda^{q+1})$ such that

$$d^* Y = f$$

with $\|Y\|_{L^\infty} \leq C\|f\|_{L^n}$.

- Second pillar is the Gagliardo-Nirenberg inequality and its generalization.
- Recall Gagliardo-Nirenberg: If $u \in C_c^\infty(\Lambda^0)$, then

$$\|u\|_{L^{n/(n-1)}} \leq C\|\nabla u\|_{L^1}.$$

Theorem (Lanzani-Stein)

Suppose $q \neq 1$ nor $n-1$. Then for any $u \in C_c^\infty(\Lambda^q)$,

$$\|u\|_{L^{n/(n-1)}} \leq C(\|du\|_{L^1} + \|d^*u\|_{L^1}).$$

Furthermore, assume $n \geq 3$. Then if $q = 1$, the same inequality holds if $d^*u = 0$; if $q = n-1$, the same inequality holds if $du = 0$.

- Control of u by du and d^*u; since d^* of a function is always zero, when $q = 0$ this is just Gagliardo-Nirenberg
- On the other hand, when $q = 1$, du is curl of u, and d^*u is divergence of u, so this is sometimes called a div-curl inequality.

- Third pillar is the following compensation phenomenon.

Theorem (van Schaftingen)

If $u \in C_c^\infty(\Lambda^1)$ and $d^* u = 0$, then for any function $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^n} u_1 \Phi dx \right| \leq C \|u\|_{L^1} \|\nabla \Phi\|_{L^n}.$$

- Inequality would be trivial if $\dot{W}^{1,n}$ embeds into L^∞. So this is some remedy of failure of this critical Sobolev embedding when one test a $\dot{W}^{1,n}$ function against something divergence free (inequality fails otherwise).

Equivalence of the three pillars

- The three theorems above are all equivalent.
- To illustrate this, assume the following proposition of Bourgain-Brezis (special case of first theorem):

Proposition

For any $f \in L^n$, there exists a vector field $Y \in L^\infty$ such that

$$div\ Y = f$$

with $\|Y\|_{L^\infty} \leq C\|f\|_{L^n}$.

We deduce from this the usual Gagliardo-Nirenberg inequality for functions (special case of second theorem).

Let $u \in C_c^\infty$ function in $\mathbb{R}^n$. We want to prove

$$\|u\|_{L^{n/(n-1)}} \leq C\|\nabla u\|_{L^1}.$$

Use duality: consider $\int_{\mathbb{R}^n} uf$ for $f \in L^n$.
By Proposition, given $f \in L^n$, there is a vector field Y in L^∞ such that div $Y = f$ with $\|Y\|_{L^\infty} \leq C\|f\|_{L^n}$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} uf &= \int_{\mathbb{R}^n} u \operatorname{div} Y \\ &= -\int_{\mathbb{R}^n} \nabla u \cdot Y \\ &\leq \|\nabla u\|_{L^1}\|Y\|_{L^\infty} \\ &\leq C\|\nabla u\|_{L^1}\|f\|_{L^n}.\end{aligned}$$

- Conversely, one can deduce the above Proposition from the Gagliardo-Nirenberg inequality.
  Given function $f \in L^n$, we want to find vector field $Y \in L^\infty$ such that $\operatorname{div} Y = f$. The latter equation can be written

$$-\int_{\mathbb{R}^n} uf = \int_{\mathbb{R}^n} \nabla u \cdot Y$$

  for all functions $u \in C_c^\infty$.
  Let $L^1(\Lambda^1)$ be the space of vector fields in L^1, E be the subspace spanned by ∇u where $u \in C_c^\infty(\Lambda^0)$ (equipped with L^1 norm).
  Define a linear functional T on E by

$$T(\nabla u) = -\int_{\mathbb{R}^n} uf.$$

By Gagliardo-Nirenberg, T is bounded on E with $\|T\| \leq C\|f\|_{L^n}$: this is because

$$|T(\nabla u)| = \left|\int_{\mathbb{R}^n} uf\right| \leq \|u\|_{L^{n/(n-1)}}\|f\|_{L^n} \leq C\|\nabla u\|_{L^1}\|f\|_{L^n}$$

for all $u \in C_c^\infty$.
By Hahn-Banach, we can extend T to $L^1(\Lambda^1)$ without increasing its norm. But all bounded linear functionals on $L^1(\Lambda^1)$ is of the form $v \mapsto \int_{\mathbb{R}^n} v \cdot Y$ for some vector field $Y \in L^\infty$. Thus there is some $Y \in L^\infty$ with

$$T(\nabla u) = \int_{\mathbb{R}^n} \nabla u \cdot Y$$

for all $u \in C_c^\infty$, as desired.

- Similarly one can prove that the first two theorems above are equivalent (although I have not shown you how to prove either of them).

- Next remember there is also a third theorem, which is a compensation phenomenon for divergence-free 1-forms.
- To illustrate why this third theorem is also equivalent to the first two, let's try to deduce from it the following special case of the second theorem:

Proposition

Suppose $n \geq 3$. Then $\|u\|_{L^{n/(n-1)}} \leq C\|du\|_{L^1}$ if u is a 1-form and $d^* u = 0$.

To prove this, use Hodge decomposition: $u = d^*\Delta^{-1}(du)$.

Use duality: Let ϕ be another 1-form, $\phi \in L^n$. Consider

$$(u, \phi) = (d^*\Delta^{-1}(du), \phi) = (du, \Delta^{-1} d\phi)$$

which is equal to

$$\sum_{|J|=2} \int_{\mathbb{R}^n} (du)_J \overline{\Delta^{-1}(d\phi)_J}.$$

Need to estimate this.

Key: One could do so using the third theorem, because for each $|J| = 2$, $(du)_J$ is a component of some divergence free vector field.

Reason: d forms a complex: $d(du) = 0$. So say 1 is not in $J = (j_1, j_2)$ (an index like that exist since $n \geq 3$). Then considering the component $1J$ of $d(du)$, we get

$$\frac{\partial (du)_J}{\partial x^1} \pm \frac{\partial (du)_{1j_1}}{\partial x^{j_2}} \pm \frac{\partial (du)_{1j_2}}{\partial x^{j_1}} = 0.$$

Arguments like this will prove the second theorem from the third.

- To complete this circle of ideas, van Schaftingen provided an elementary (but very beautiful) proof of the third theorem (thus establishes all three theorems).
- Turns out there is a more sophiscated version of the same story, which we describe below.

Three pillars of the theory: sophiscated version

- First pillar is the solution of the following system of equations.

Proposition (Bourgain-Brezis)

For any $f \in L^n$, there exists a vector field $Y \in L^\infty \cap \dot{W}^{1,n}$ such that

$$div\ Y = f$$

with $\|Y\|_{L^\infty} + \|Y\|_{\dot{W}^{1,n}} \leq C\|f\|_{L^n}$.

- Y not only in L^∞, but also in $\dot{W}^{1,n}$!

More generally

Theorem (Bourgain-Brezis)

If $q \neq n-1$, then for any $f \in d^*(\dot{W}^{1,n}(\Lambda^{q+1}))$, there exists $Y \in L^\infty \cap \dot{W}^{1,n}(\Lambda^{q+1})$ such that

$$d^* Y = f$$

with $\|Y\|_{L^\infty} + \|Y\|_{\dot{W}^{1,n}} \leq C\|f\|_{L^n}$.

- Second pillar is the following generalized Gagliardo-Nirenberg inequality.

Theorem (Bourgain-Brezis)

Suppose $q \neq 1$ *nor* $n-1$. *Then for any* $u \in C_c^\infty(\Lambda^q)$,

$$\|u\|_{L^{n/(n-1)}} \leq C(\|du\|_{L^1+(\dot{W}^{1,n})^*} + \|d^*u\|_{L^1+(\dot{W}^{1,n})^*}).$$

Furthermore, assume $n \geq 3$. *Then if* $q = 1$, *the same inequality holds if* $d^*u = 0$; *if* $q = n-1$, *the same inequality holds if* $du = 0$.

- $(\dot{W}^{1,n})^*$ is the dual space of $\dot{W}^{1,n}$. If A and B are Banach spaces, their sum is a Banach space

$$A + B = \{a + b \colon a \in A, b \in B\}$$

with norm

$$\|f\|_{A+B} = \inf\{\|a\|_A + \|b\|_B \colon f = a + b, a \in A, b \in B\}.$$

Note that the dual space of $L^1 + (\dot{W}^{1,n})^*$ is $L^\infty \cap \dot{W}^{1,n}$, which appeared in the previous theorem.

- When $q = 0$, the current theorem says

$$\|u\|_{L^{n/(n-1)}} \leq C\|\nabla u\|_{L^1+(\dot{W}^{1,n})^*}$$

for all functions $u \in C_c^\infty$, which is an improvement of the usual Gagliardo-Nirenberg inequality.

- Third pillar is the following compensation phenomenon.

Theorem (Bourgain-Brezis)

If $u \in C_c^\infty(\Lambda^1)$ and $d^* u = 0$, then for any function $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^n} u_1 \Phi dx \right| \leq C \|u\|_{L^1 + (\dot{W}^{1,n})^*} \|\nabla \Phi\|_{L^n}.$$

- Again these three theorems are equivalent. Bourgain-Brezis gave a constructive proof of the first one directly, thereby proving all three of them.
- The proof of Bourgain-Brezis uses the following approximation lemma, which is of independent interest:

Lemma (Bourgain-Brezis)

Given any $\delta > 0$, there exists a constant C_δ such that for any function $f \in \dot{W}^{1,n}$, there exists a function $F \in L^\infty \cap \dot{W}^{1,n}$ such that

$$\sum_{i=2}^{n} \|\partial_i f - \partial_i F\|_{L^n} \leq \delta \|\nabla f\|_{L^n}$$

and

$$\|\nabla F\|_{L^n} + \|F\|_{L^\infty} \leq C_\delta \|\nabla f\|_{L^n}.$$

- F approximates the derivatives of f in all but one direction!

- Proof of this lemma uses heavily the Littlewood-Paley decomposition of a function, and is highly non-linear. This is part of the nature of the subject matter; in fact, Bourgain-Brezis also proved

Proposition (Bourgain-Brezis)

There is no bounded linear operator $K\colon L^n \to L^\infty(\Lambda^1)$ *such that* $\operatorname{div} Kf = f$ *for all* $f \in L^n$*.*

An approximation lemma for second derivatives

- The original proof of Bourgain-Brezis is restricted to controlling one derivative. In joint work with Yi Wang, we proved:

Theorem (Yi Wang-Y)

Given any $\delta > 0$, there exists a constant C_δ such that for any function $f \in \dot{W}^{2,n/2}$, there exists a function $F \in L^\infty \cap \dot{W}^{2,n/2}$ such that

$$\sum_{i,j=2}^{n} \|\partial_{ij}^2 f - \partial_{ij}^2 F\|_{L^{n/2}} \leq \delta \|\nabla^2 f\|_{L^{n/2}}$$

and

$$\|\nabla^2 F\|_{L^{n/2}} + \|F\|_{L^\infty} \leq C_\delta \|\nabla^2 f\|_{L^{n/2}}.$$

A hyperbolic version: An improved Strichartz estimate

Theorem (Chanillo-Y)

Suppose $u\colon \mathbb{R}^{1+2} \to \mathbb{R}^2$ *is a (weak) solution of the following system of wave equations*

$$\begin{cases} \Box u = f \\ u\big|_{t=0} = u_0 \\ \partial_t u\big|_{t=0} = u_1 \end{cases}$$

where $f = (f_1, f_2)\colon \mathbb{R}^{1+2} \to \mathbb{R}^2$ *is a divergence free vector field at each given time* t*, i.e.*

$$\partial_{x_1} f_1 + \partial_{x_2} f_2 = 0$$

for each t*. Then*

$$\|u\|_{C_t^0 L_x^2} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{-1}} \leq C \left(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \|f\|_{L_t^1 L_x^1} \right).$$

Subelliptic case: Heisenberg group

- Heisenberg group $\mathbb{H}^n$ as the boundary of the upper half space

$$\{\text{Im } z_{n+1} > |z'|^2\}, \quad (z', z_{n+1}) \in \mathbb{C}^{n+1}$$

- $\mathbb{H}^n$ diffeomorphic to $\mathbb{C}^n \times \mathbb{R}$ via

$$\mathbb{C}^n \times \mathbb{R} \to \mathbb{H}^n$$

$$[z, t] \mapsto (z, t + i|z|^2)$$

- $\mathbb{H}^n$ carries the structure of a non-abelian Lie group:

$$[z_1, t_1] \cdot [z_2, t_2] = [z_1 + z_2, t_1 + t_2 + 2\text{Im } (z_1 \bar{z}_2)]$$

- homogeneous in the sense that it carries automorphic dilation

$$\delta_\lambda[z, t] := [\lambda z, \lambda^2 t], \quad \lambda > 0$$

- Haar measure is just Lebesgue measure on $\mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$: it is *dxdydt* if we write a point on $\mathbb{H}^n$ as $[z, t]$, $z = x + iy$
- $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$:

$$\delta_\lambda^*(dxdydt) = \lambda^Q dxdydt$$

- Left-invariant vector fields on $\mathbb{H}^n$:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad 1 \leq j \leq n$$

$$T = \frac{\partial}{\partial t} = -\frac{1}{4}[X_j, Y_j]$$

- Think of X_j, Y_j as of degree 1, T of degree 2. Also write $X_{j+n} = Y_j$ if $1 \leq j \leq n$, and ∇_b for subelliptic gradient:

$$\nabla_b f = (X_1 f, \ldots, X_{2n} f) \quad \text{for functions } f \text{ on } \mathbb{H}^n$$

- Sobolev embedding: $\nabla_b f \in L^p$ implies $f \in L^{p^*}$,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}, \quad 1 \leq p < Q$$

- Fails when $p = Q$; nonetheless we have

Theorem (Chanillo-van Schaftingen)

If $f_1, \dots, f_{2n}$ and Φ are C_c^∞ functions on $\mathbb{H}^n$ and

$$X_1 f_1 + \cdots + X_{2n} f_{2n} = 0,$$

then for any j,

$$\left| \int_{\mathbb{H}^n} f_j \Phi \right| \leq C \|f\|_{L^1} \|\nabla_b \Phi\|_{L^Q}.$$

Here $\|f\|_{L^1} = \sum_{j=1}^{2n} \|f_j\|_{L^1}$.

- Result does not make use of any complex structure; in fact they proved it for more general homogeneous Lie groups
- Proof is more difficult than the abelian case since the vector fields do not commute
- Analog of the first two pillars of the previous theory?

Application to the $\overline{\partial}_b$ complex on $\mathbb{H}^n$

- Notations: $Z_j = X_j + iY_j$, $\overline{Z}_j = X_j - iY_j$, $1 \leq j \leq n$
- Write $[z, t]$ coordinate on $\mathbb{H}^n$. For each multiindex $J = (j_1, \dots, j_q)$, $1 \leq j_k \leq n$ for all k, we write

$$d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

- A $(0, q)$ form on $\mathbb{H}^n$ is an expression of the form

$$u = \sum_{|J|=q} u_J d\bar{z}^J;$$

then $\overline{\partial}_b u$ is a $(0, q+1)$ form given by

$$\overline{\partial}_b u := \sum_{j=1}^{n} \overline{Z}_j(u_J) d\bar{z}^j \wedge d\bar{z}^J$$

- L^2 inner product on space of $(0,q)$ forms:

$$(u,v) = \sum_J \int_{\mathbb{H}^n} u_J \bar{v}_J$$

- $\overline{\partial}_b^*$: formal adjoint of $\overline{\partial}_b$ under this inner product
- e.g. If $u = \sum_{j=1}^n u_j d\bar{z}^j$ is a $(0,1)$ form, then

$$\overline{\partial}_b^* u = -\sum_{j=1}^n Z_j(u_j).$$

We have the following apriori inequalities: (Recall $Q = 2n + 2$.)

Theorem

If u is a $(0, q)$ form on $\mathbb{H}^n$, $2 \leq q \leq n - 2$, then

$$\|u\|_{L^{Q/(Q-1)}} \leq C(\|\overline{\partial}_b u\|_{L^1} + \|\overline{\partial}_b^* u\|_{L^1}).$$

Suppose further that $n \geq 3$. If $q = 1$, the same inequality holds if $\overline{\partial}_b^* u = 0$; if $q = n - 1$, the same result holds if $\overline{\partial}_b u = 0$.

Theorem

Assume $n \geq 2$. If u is a function orthogonal to the kernel of $\overline{\partial}_b$ in L^2, then

$$\|u\|_{L^{Q/(Q-1)}} \leq C\|\overline{\partial}_b u\|_{L^1};$$

an analogous result holds if u is a $(0, n)$ form orthogonal to the kernel of $\overline{\partial}_b^*$ in L^2.

We also have

Theorem

On $\mathbb{H}^n$, if $q \neq n-1$, then for any $f \in \overline{\partial}_b^*(\dot{NL}^{1,Q}(\Lambda^{0,q+1}))$, there exists $Y \in L^\infty(\Lambda^{0,q+1})$ such that

$$\overline{\partial}_b^* Y = f$$

with $\|Y\|_{L^\infty} \leq C\|f\|_{L^Q}$.

Here $\dot{NL}^{1,Q}$ is the space of functions whose ∇_b is in L^Q.

Subelliptic case: Hormander's vector fields

- Question: What happens if there is no group structure? Can one still have a theorem analogous to the one of Chanillo-van Schaftingen?
- Setup: $X_1, \ldots, X_n$ smooth real vector fields near 0 on $\mathbb{R}^N$
- Assume that they are linearly independent at 0, and that their commutators of length $\leq r$ span at 0
- Let $V_j(0)$ be the span of restrictions of the commutators of $X_1, \ldots, X_n$ of length $\leq j$ to 0
- Let $Q = \sum_{j=1}^r j \cdot (\dim V_j(0) - \dim V_{j-1}(0))$

Theorem (Y)

Under the assumptions on the previous slide, there is a neighborhood U of 0 and $C > 0$ such that if

$$X_1 f_1 + \cdots + X_n f_n = 0$$

on U with $f_1, \ldots, f_n \in C_c^\infty(U)$ and $\Phi \in C_c^\infty(U)$, then

$$\left| \int_U f_1(x)\Phi(x)dx \right| \leq C\|f\|_{L^1(U)} \Big(\sum_{j=1}^n \|X_j \Phi\|_{L^Q(U)} + \|\Phi\|_{L^Q(U)} \Big).$$

- Generalizes Chanillo-van Schaftingen
- One difficulty in the current theorem: getting the best (i.e. smallest) possible value of Q. The one we had given is the best possible.

A model example

- On $\mathbb{R}^2$, let $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial t}$, $Q = 3$.

Theorem (Y)

If $Xf_1 + Yf_2 = 0$ on $\mathbb{R}^2$, with $f_1, f_2 \in C_c^\infty$, then for all $\Phi \in C_c^\infty$,

$$\left| \int_{\mathbb{R}^2} f_1 \Phi \right| \leq C \|f\|_{L^1(\mathbb{R}^2)} \|\nabla_b \Phi\|_{L^3(\mathbb{R}^2)}$$

where $\nabla_b \Phi = (X\Phi, Y\Phi)$.

- Strictly speaking this does not fall under the scope of the previous theorem, since Y is zero at 0; but it is where the ideas of the proof is the most transparent.

- The proof of this model theorem is via lifting to the Heisenberg group $\mathbb{H}^1$: there exist a submersion $\pi \colon \mathbb{H}^1 \to \mathbb{R}^2$ such that

$$d\pi(X_1) = X \quad \text{and} \quad d\pi(Y_1) = Y.$$

- One could try to use the result on $\mathbb{H}^1$; but this does not work, since by lifting to the Heisenberg group (which has a higher dimension), one gets less smoothing in any Sobolev-kind inequality.
- The way out: Imitate the argument on $\mathbb{H}^1$; but has to 'integrate away the additional variable' that one adds during the lifting process.
- In general, to prove the general theorem, one can still use the same lifting strategy (Rothschild-Stein), but there will be errors that one has to handle.

Application to $\overline{\partial}_b$ complex on domains of finite type

- M: boundary of a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$
- Assume M is of finite commutator type m and has comparable Levi eigenvalues.

Theorem (Y)

Let $q \neq 1$ nor $n-1$. Then for any smooth $(0,q)$ form u orthogonal to Kernel($\Box_b$),

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)} + \|\overline{\partial}_b^* u\|_{L^1(M)}$$

where $Q = 2n + m$. In particular

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\overline{\partial}_b u\|_{L^1(M)}$$

for all smooth functions u orthogonal to Kernel($\overline{\partial}_b$) (Gagliardo-Nirenberg for $\overline{\partial}_b$).

Under the same assumptions on M, we also have

Theorem (Y)

If $q \neq n-1$, then for any $f \in \overline{\partial}_b^*(\dot{N}L^{1,Q}(\Lambda^{0,q+1}))$ on M, there exists $Y \in L^\infty(\Lambda^{0,q+1})$ such that

$$\overline{\partial}_b^* Y = f$$

with $\|Y\|_{L^\infty} \leq C\|f\|_{L^Q}$.

Here again $Q = 2n + m$.

Sophiscated version of the subelliptic story

- Focus again on the Heisenberg group $\mathbb{H}^n$. We have the following approximation lemma: $(Q = 2n + 2)$

Theorem (Yi Wang-Y)

Given any $\delta > 0$, there exists a constant C_δ such that for any function f with $\nabla_b f \in L^Q$, there exists a function $F \in L^\infty$ with $\nabla_b F \in L^Q$ such that

$$\sum_{j=2}^{2n} \|X_j f - X_j F\|_{L^Q} \leq \delta \|\nabla_b f\|_{L^Q}$$

and

$$\|\nabla_b F\|_{L^Q} + \|F\|_{L^\infty} \leq C_\delta \|\nabla_b f\|_{L^Q}.$$

- F approximates the derivatives of f in all but one good direction!

From this we deduce

Theorem (Yi Wang-Y)

If $q \neq n-1$*, then for any* $f \in \overline{\partial}_b^*(\dot{NL}^{1,Q}(\Lambda^{0,q+1}))$ *on* $\mathbb{H}^n$*, there exists* $Y \in L^\infty \cap \dot{NL}^{1,Q}(\Lambda^{0,q+1})$ *such that*

$$\overline{\partial}_b^* Y = f$$

with $\|Y\|_{L^\infty} + \|\nabla_b Y\|_{L^Q} \leq C\|f\|_{L^Q}$*.*

- This is remarkable since now one has not only $Y \in L^\infty$, but also $\nabla_b Y \in L^Q$.

We further deduced the following apriori inequalities:

Theorem (Yi Wang-Y)

If u is a $(0,q)$ form on $\mathbb{H}^n$, $2 \leq q \leq n-2$, then

$$\|u\|_{L^{Q/(Q-1)}} \leq C(\|\overline{\partial}_b u\|_{L^1+(\dot{NL}^{1,Q})^*} + \|\overline{\partial}_b^* u\|_{L^1+(\dot{NL}^{1,Q})^*}).$$

Suppose further $n \geq 3$. If $q = 1$, the same inequality holds if $\overline{\partial}_b^* u = 0$; if $q = n-1$, the same result holds if $\overline{\partial}_b u = 0$.

Theorem (Yi Wang-Y)

Assume $n \geq 2$. If u is a function orthogonal to the kernel of $\overline{\partial}_b$ in L^2, then

$$\|u\|_{L^{Q/(Q-1)}} \leq C\|\overline{\partial}_b u\|_{L^1+(\dot{NL}^{1,Q})^*};$$

an analogous result holds if u is a $(0,n)$ form orthogonal to the kernel of $\overline{\partial}_b^*$ in L^2.

Bourgain-Brezis's approximation lemma again

Lemma (Bourgain-Brezis)

Given any $\delta > 0$*, there exists a constant* C_δ *such that for any function* $f \in \dot{W}^{1,n}$*, there exists a function* $F \in L^\infty \cap \dot{W}^{1,n}$ *such that*

$$\sum_{i=2}^{n} \|\partial_i f - \partial_i F\|_{L^n} \leq \delta \|\nabla f\|_{L^n}$$

and

$$\|\nabla F\|_{L^n} + \|F\|_{L^\infty} \leq C_\delta \|\nabla f\|_{L^n}.$$

Bourgain-Brezis's proof

- First ingredient: Littlewood-Paley theory
- Every $\dot{W}^{1,n}$ function f can be written

$$f = \sum_{j=-\infty}^{\infty} \Delta_j f$$

$$\widehat{\Delta_j f}(\xi) = \chi_{\{2^j \leq |\xi| \leq 2^{j+1}\}}(\xi)\hat{f}(\xi)$$

- Bernstein inequality:

$$\|\Delta_j f\|_{L^\infty} \leq C\|\nabla f\|_{L^n} \quad \text{for all } j$$

- Thus if $f = \Delta_j f$ for some j, i.e. if f is frequency localized, then the approximation lemma is trivial; one can take $F = f$.
- In general, while each $\Delta_j f$ is in L^∞, one cannot sum all of them in L^∞ since the L^∞ norms do not decay in j.

- Second ingredient: algebraic identity
- Given any N numbers $a_1, \ldots, a_N$, we have

$$1 = \sum_{j=1}^{N} a_j \prod_{j'>j} (1 - a_{j'}) + \prod_{j=1}^{N} (1 - a_j)$$

- This is nothing but

$$\begin{aligned} 1 &= a_N + (1 - a_N) \\ &= a_N + a_{N-1}(1 - a_N) + (1 - a_{N-1})(1 - a_N) \\ &= a_N + a_{N-1}(1 - a_N) + a_{N-2}(1 - a_{N-1})(1 - a_N) \\ &\quad + (1 - a_{N-2})(1 - a_{N-1})(1 - a_N) \ldots \end{aligned}$$

- In particular, if all a_j satisfies $0 \leq a_j \leq 1$, then

$$\sum_{j} a_j \prod_{j'>j} (1 - a_{j'}) \leq 1.$$

- One is now tempted to take

$$F = \sum_j \Delta_j f \prod_{j'>j} (1 - |\Delta_j f|)$$

as an L^∞ approximation to

$$f = \sum_j \Delta_j f \cdot 1$$

- But this is too naive, and in particular one does not gain in any good directions
- Need another controlling function that dominates $|\Delta_j f|$: Bourgain-Brezis introduced

$$\omega_j(x) = \sup_{y \in \mathbb{R}^n} |\Delta_j f(x-y)| e^{-|y_1| - 2^{-\sigma}|y'|}, \quad y = (y_1, y')$$

where σ is a large constant depending on δ. (sup-convolution)
- Bourgain-Brezis also used heavily the Fejer kernels, which are special kernels that one only finds in $\mathbb{R}^n$.

Proof of approximation lemma on the Heisenberg group

- Difficulties:
  - No notion of frequency space; in particular, no special kernels like Fejer kernels
  - Group is non-abelian: in particular, if X is left-invariant vector field, then $X(f * g) = f * (Xg)$ but is not equal to $(Xf) * g$
- Ways to overcome these:
  - Simplify the argument at one crucial point so that we can convolve one fewer times, which allows us to avoid the second problem
  - Price to pay: More errors to control all over the place
  - Introduce two different controlling functions ω_j and $\tilde{\omega}_j$: the first one would be a discrete l^Q convolution, the second one is a continuous ordinary convolution

Epilogue

- We return now to the elliptic setting. A special case of the second theorem of Bourgain-Brezis is the following:

Theorem (Bourgain-Brezis)

On $\mathbb{R}^2$, if u is a function in C_c^∞, then

$$\|u\|_{L^2} \leq C\|\nabla u\|_{L^1+(\dot{W}^{1,2})^*}.$$

- We have discussed how one could prove this by solving d^* (i.e. using the first theorem), but Bourgain-Brezis actually had another direct proof of this inequality, which works only in 2-dimensions.
- To illustrate this, we use their method to give a new proof of the Gagliardo-Nirenberg inequality in $\mathbb{R}^2$: Suppose $\|\nabla u\|_{L^1} = 1$. We want to prove $\|u\|_{L^2} \leq C$.

- Tool: Riesz transforms R_1, R_2 in $\mathbb{R}^2$:

$$\widehat{R_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi), \quad j = 1,2$$

- Fact: $R_1^2 + R_2^2 = -Id$, $[R_1, R_2] = 0$.
- Thus given $u \in C_c^\infty$ with $\|\nabla u\|_{L^1} = 1$, we have

$$\begin{aligned} u &= (R_1^2 - R_2^2)^2 u + 4R_1^2 R_2^2 u \\ &= (R_1 - R_2)^2 (R_1 + R_2)^2 u + 4R_1^2 R_2^2 u \end{aligned}$$

- To show $u \in L^2$, we consider

$$(u,u) = ((R_1 - R_2)^2 (R_1 + R_2)^2 u, u) + 4(R_1^2 R_2^2 u, u).$$

Suffices to bound both terms; by rotating the coordinate axes, need only bound the latter

- Now
$$(R_1^2 R_2^2 u, u) = (\Delta^{-1} R_1 R_2 \partial_1 u, \partial_2 u).$$
If one can show that $\Delta^{-1} R_1 R_2$ maps L^1 boundedly into L^∞, then we are done.
- To do that, let $K(x)$ be the kernel of $\Delta^{-1} R_1 R_2$. One only needs to show that $K \in L^\infty$.
- One uses homogeneity: since $K(x)$ is homogeneous of degree 0, it suffices to show that K is bounded on the unit sphere
- This one can do by using the integral representation
$$K(x) = -\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon < |\xi| < R} \frac{\xi_1 \xi_2}{|\xi|^4} e^{2\pi i x \cdot \xi} d\xi$$
and spliting the integral into integral over small and large ξ's, which works since the multiplier $\frac{\xi_1 \xi_2}{|\xi|^4}$ is odd in both ξ_1 and ξ_2.

Thank you!