DIV-CURL SYSTEMS (PART 1)

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In this sequence of two talks, we describe the work of Bourgain-Brezis, van Schfatingen, and Lanzani-Stein, which represents a new array of compensation estimates related to the divergence and curl operators on $\mathbb{R}^n$. There are three major pillars in the theory. First, we consider the solvability of the equation

$$\operatorname{div} Y = f$$

on $\mathbb{R}^n$, where $f \in L^n$ is given, Y is an unknown vector field on $\mathbb{R}^n$, and $n \geq 2$. Classical theory tells us that for any $f \in L^n$, one can find a solution Y to the equation in $\dot{W}^{1,n}$ (componentwise); in fact, the canonical solution $Y = \nabla u$, where $\Delta u = f$ with $u \in \dot{W}^{2,n}$, is one such solution. However, due to the failure of the Sobolev embedding $\dot{W}^{1,n}$ into L^∞, such solutions may not be bounded. Nevertheless, since the equation $\operatorname{div} Y = f$ is largely underdetermined, the question is whether one can find some solution Y that is in L^∞. If one thinks of the vector field Y as a differential 1-form and the divergence operator as d^* acting on a 1-form, where d^* is the adjoint of the Hodge-de Rham exterior derivative d with respect to the Euclidean metric on $\mathbb{R}^n$, then one can also consider the more general equation

$$d^* Y = f,$$

where now f is a differential l-form on $\mathbb{R}^n$ satisfying the compatibility conditions $d^* f = 0$, and Y is a differential $(l+1)$-form on $\mathbb{R}^n$. One can then ask for the solvability for Y in L^∞ if the given f has components in L^n.

Second, the classical Gagliardo-Nirenberg inequality says that if u is a smooth function with compact support on $\mathbb{R}^n$, then

$$\|u\|_{L^{\frac{n}{n-1}}} \leq C \|\nabla u\|_{L^1}.$$

If we think of u here as a differential 0-form, then the question is whether there is an appropriate generalization of this inequality to differential forms of higher order.

Finally, there is the following inequality, due to van Schaftingen, that remedies the failure of the embedding of $\dot{W}^{1,n}$ into L^∞ on $\mathbb{R}^n$:

Theorem 1 (van Schaftingen)**.** *Suppose f and ϕ are two smooth and compactly supported vector fields on $\mathbb{R}^n$, and suppose that $\operatorname{div} f = 0$. Then*

$$\left| \int_{\mathbb{R}^n} f \cdot \phi \, dx \right| \leq C \|f\|_{L^1} \|\nabla \phi\|_{L^n},$$

where $\|\nabla \phi\|_{L^n}$ is the sum of the $\dot{W}^{1,n}$ norms of the components of ϕ.

If $\dot{W}^{1,n}$ were to embed into L^∞, then this inequality is trivial and holds without requiring the divergence free condition on f. The failure of $\dot{W}^{1,n}$ into L^∞ is what makes this inequality interesting.

It is readily seen from the first and third pillars above that the failure of the Sobolev embedding of $\dot{W}^{1,n}$ into L^∞ plays a key role in our analysis. In fact at the bottom of all these, there is an approximation lemma that describes the extent to which $\dot{W}^{1,n}$ functions can be approximated by L^∞ functions (c.f. Lemma 2 below).

We will begin by describing some results along these lines that can be proved using 'elementary methods'. Then we move on to some far-reaching extensions of such results, whose proofs will have to wait until next time.

1. Elementary results

First, we consider the solvability of the equation

$$(1) \qquad\qquad \operatorname{div} Y = f$$

on $\mathbb{R}^n$ when $f \in L^n$. If u solves $\Delta u = f$, then $Y = \nabla u$ is called the canonical solution to (1). Such solutions are always in $\dot{W}^{1,n}$, but there are indeed situations where it is not bounded, as Bourgain-Brezis demonstrated (c.f. Proposition 3 below; another concrete example is due to L. Nirenberg). Nevertheless, Bourgain-Brezis observed the following:

Proposition 1 (Bourgain-Brezis). *Given any $f \in L^n$, there is always a solution $Y \in L^\infty$ (in the sense of distributions) to the equation $\operatorname{div} Y = f$, with $\|Y\|_{L^\infty} \leq C\|f\|_{L^n}$.*

In fact they proved that the above assertion is equivalent to the usual Gagliardo-Nirenberg inequality by duality:

Proposition 2 (Bourgain-Brezis). *Proposition 1 is equivalent to the fact that*
$$\|u\|_{L^{\frac{n}{n-1}}} \leq C\|\nabla u\|_{L^1}$$
for all smooth functions u with compact support on $\mathbb{R}^n$.

This would give an elementary proof to Proposition 1.

Proof of Proposition 2. First suppose that Proposition 1 is true. Then given $f \in L^n$, there is some $Y \in L^\infty$ such that $\operatorname{div} Y = f$ in distributional sense, and $\|Y\|_{L^\infty} \leq C\|f\|_{L^n}$. Now if u is a smooth function with compact support, then testing u against f, we obtain

$$\int_{\mathbb{R}^n} uf\,dx = \int_{\mathbb{R}^n} u \operatorname{div} Y\,dx = -\int_{\mathbb{R}^n} (\nabla u) \cdot Y\,dx,$$

which in absolute value is bounded by

$$\|\nabla u\|_{L^1}\|Y\|_{L^\infty} \leq C\|\nabla u\|_{L^1}\|f\|_{L^n}.$$

This proves the Gagliardo-Nirenberg inequality on $\mathbb{R}^n$.

Conversely, suppose we begin by assuming the Gagliardo-Nirenberg inequality on $\mathbb{R}^n$. Suppose $f \in L^n$ is given. Let $(L^1)^n$ be the product of n copies of L^1, i.e. the space of vector fields with L^1 coefficients. Let E be the vector subspace of $(L^1)^n$, defined by

$$E = \{\nabla u \colon u \in C_c^\infty(\mathbb{R}^n)\}$$

and equipped with $(L^1)^n$ norm. Then define a linear functional T on E by letting

$$T(\nabla u) = -\int_{\mathbb{R}^n} u f dx.$$

The Gagliardo-Nirenberg inequality on $\mathbb{R}^n$ implies that this is a bounded linear functional on E, with norm $\leq C\|f\|_{L^n}$: in fact

$$|T(\nabla u)| = \left| \int_{\mathbb{R}^n} u f dx \right| \leq \|u\|_{L^{\frac{n}{n-1}}} \|f\|_{L^n} \leq C\|\nabla u\|_{L^1} \|f\|_{L^n}.$$

Now we extend T to a bounded linear functional on $(L^1)^n$ by Hahn-Banach, without increasing the norm of T. Then the new T is represented by some element of $(L^\infty)^n$, i.e. there is some vector field Y with coefficients in L^∞ such that

$$T(v) = \int_{\mathbb{R}^n} Y \cdot v dx$$

for all $v \in (L^1)^n$. In particular, restricting back to E, we have

$$-\int_{\mathbb{R}^n} u f dx = \int_{\mathbb{R}^n} Y \cdot \nabla u dx$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. This says Y solves $\operatorname{div} Y = f$ in the distribution sense. Furthermore, we have

$$\|Y\|_{L^\infty} \leq \|T\| \leq C\|f\|_{L^n}.$$

This finishes the proof of Proposition 1. $\qquad\square$

By refining the above duality arguments, Bourgain-Brezis has also proved that the solution Y in Proposition 1 can be taken to be continuous.

Next, it is possible to generalize Proposition 1 to differential forms of higher order. Consider now the equation $d^*Y = f$, where f is a differential l form on $\mathbb{R}^n$ with L^n coefficients, and Y is an unknown differential $(l+1)$-form. For the equation to be solvable, we must require the compatibility conditions $d^* f = 0$, since d^* forms a complex. Now by standard Hodge theory on $\mathbb{R}^n$, this means $f = d^* X$ for some differential $(l+1)$-form X with coefficients in $\dot{W}^{1,n}$. The question is then whether it is possible to find $Y \in L^\infty$ solving $d^* Y = d^* X$ if X is a given $(l+1)$-form with $\dot{W}^{1,n}$ coefficients. The theorem is the following.

Theorem 2. *Suppose $l \neq n-1$. Then for any $(l+1)$-form X on $\mathbb{R}^n$ with $\dot{W}^{1,n}$ coefficients, there is some $(l+1)$-form Y with coefficients in L^∞ such that*

$$d^* Y = d^* X$$

in the sense of distributions, with $\|Y\|_{L^\infty} \le C\|d^*X\|_{L^n}$.

By the same duality argument as above (which we shall not repeat), Theorem 2 is equivalent to the following Gagliardo-Nirenberg inequality for differential l-forms:

Theorem 3. *Suppose $l \ne n-1$. Then for all smooth l-forms u with compact support on $\mathbb{R}^n$, one has*

$$\|u\|_{L^{\frac{n}{n-1}}} \le C\|du\|_{L^1}$$

if $d^*u = 0$. (Here the norms on the differential forms are taken component-wise.)

Note that the condition on d^*u here is necessary, since du alone does not determine u.

It thus remains to prove Theorem 3. We can prove something slightly more general:

Theorem 4 (Lanzani-Stein). *Suppose u is a smooth l-forms with compact support on $\mathbb{R}^n$.*

(a) If $l \ne 1$ nor $n-1$, then one has

$$\text{(2)} \qquad\qquad \|u\|_{L^{\frac{n}{n-1}}} \le C\left(\|du\|_{L^1} + \|d^*u\|_{L^1}\right).$$

(b) If $l = 1$ or $n-1$ (or both), then the same inequality holds, except that one has to replace $\|du\|_{L^1}$ by $\|du\|_{\mathcal{H}^1}$ if du is a top form, and $\|d^*u\|_{L^1}$ by $\|d^*u\|_{\mathcal{H}^1}$ if d^*u is a function. Here $\mathcal{H}^1$ is the Hardy $\mathcal{H}^1$ norm.

This turns out to be a consequence of Theorem 1 we stated at the beginning. We shall first show how this implication works, and then prove Theorem 1.

First, it is clear that Theorem 1 has the following equivalent formulation:

Theorem 5 (van Schaftingen). *Let $f = (f_1, \dots, f_n)$ be a divergence-free vector field on $\mathbb{R}^n$, and Φ be a function on $\mathbb{R}^n$. Suppose they are all smooth and compactly supported. Then*

$$\left|\int_{\mathbb{R}^n} f_1 \Phi dx\right| \le C\|f\|_{L^1}\|\nabla\Phi\|_{L^n}.$$

We use this to prove Theorem 4.

Proof of Theorem 4. The proof proceeds by duality. The case $l = 0$ or n is just the ordinary Gagliardo-Nirenberg inequality. So from now on we assume $1 \le l \le n-1$. Write

$$(u, \phi) := \sum_I \int_{\mathbb{R}^n} u_I \phi_I dx$$

for any differential forms u and ϕ of the same order. If u is a differential l-form on $\mathbb{R}^n$, then for any C_c^∞ test l-form ϕ,

$$(u, \phi) = (du, d\Delta^{-1}\phi) + (d^*u, d^*\Delta^{-1}\phi)$$

$$= \sum_I \int_{\mathbb{R}^n} (du)_I (d\Delta^{-1}\phi)_I + \sum_J \int_{\mathbb{R}^n} (d^*u)_J (d^*\Delta^{-1}\phi)_J.$$

For each multiindex I, if $l \leq n-2$ then $(du)_I$ is a component of a divergence-free vector field; this is because if i is a label from 1 to n that is not in I (which exists because $|I| \leq n-1$), we have, from $d(du) = 0$, that

$$\partial_i(du)_I = \sum_{j \in I} \partial_j(du)_{I_j}$$

where I_j is the multiindex obtained from I by replacing j by i. Hence by Theorem 5, we have

$$\left|(du, d\Delta^{-1}\phi)\right| \leq C\|du\|_{L^1}\|\nabla d\Delta^{-1}\phi\|_{L^n} \leq C\|du\|_{L^1}\|\phi\|_{L^n}$$

by the boundedness of the Riesz transforms on L^n. If $l \geq 2$ as well, then using $d^*d^* = 0$ we get similarly the estimate

$$\left|(d^*u, d^*\Delta^{-1}\phi)\right| \leq C\|d^*u\|_{L^1}\|\phi\|_{L^n}$$

from the lemma. Hence when $2 \leq l \leq n-2$,

$$|(u, \phi)| \leq C\left(\|du\|_{L^1} + \|d^*u\|_{L^1}\right)\|\phi\|_{L^n}.$$

This proves the inequality (2).

If d^*u is a 0-form, then

$$\left|(d^*u, d^*\Delta^{-1}\phi)\right| \leq \|d^*u\|_{\mathcal{H}^1}\|d^*\Delta^{-1}\phi\|_{BMO}$$

$$\leq C\|d^*u\|_{\mathcal{H}^1}\|d^*\Delta^{-1}\phi\|_{\dot{W}^{1,n}}$$

$$\leq C\|d^*u\|_{\mathcal{H}^1}\|\phi\|_{L^n}$$

(Alternatively,

$$\left|(d^*u, d^*\Delta^{-1}\phi)\right| = \left|(\Delta^{-\frac{1}{2}}d^*u, d^*\Delta^{-\frac{1}{2}}\phi)\right|$$

$$\leq C\|\Delta^{-\frac{1}{2}}d^*u\|_{L^{\frac{n}{n-1}}}\|d^*\Delta^{-\frac{1}{2}}\phi\|_{L^n}$$

$$\leq C\|d^*u\|_{\mathcal{H}^1}\|\phi\|_{L^n}$$

by the boundedness of the Riesz potential from $\mathcal{H}^1$ to $L^{\frac{n}{n-1}}$ and the boundedness of Riesz transform on L^n.) Similarly if du is an n-form. This completes the proof of Theorem 4. $\qquad\square$

Next we turn to the proof of Theorem 5. The key is the following decomposition lemma:

Lemma 1 (van Schaftingen). *For any Schwartz function Φ on $\mathbb{R}^N$ and $p > N$, if $\delta > 0$ is given, then there exists a decomposition*

$$\Phi = \Phi_1 + \Phi_2$$

such that

$$\begin{cases} \|\Phi_1\|_{L^\infty} \le C\delta^{1-\frac{N}{p}}\|\nabla\Phi\|_{L^p} \\ \|\nabla\Phi_2\|_{L^\infty} \le C\delta^{-\frac{N}{p}}\|\nabla\Phi\|_{L^p}. \end{cases}$$

Proof. It suffices to take Φ_1 and Φ_2 to be the high and low-frequency components of Φ respectively, say

$$\Phi_1 = \sum_{j>M} \Delta_j\Phi, \quad \text{and} \quad \Phi_2 = \sum_{j\le M} \Delta_j\Phi$$

where Δ_j is the Littlewood-Paley projection onto an annulus of size 2^j, and M is such that $2^M \simeq \delta^{-1}$. Then $\Phi_1 \in L^\infty$ by Bernstein inequality, because

$$\|\Delta_j\Phi\|_{L^\infty} \le C2^{-j(1-\frac{N}{p})}\|\nabla\Phi\|_{L^p};$$

also $\nabla\Phi_2 = \sum_{j\le M}\Delta_j(\nabla\Phi)$, and

$$\|\Delta_j(\nabla\Phi)\|_{L^\infty} \le C2^{\frac{jN}{p}}\|\nabla\Phi\|_{L^p}.$$

Summing these over the corresponding ranges of j gives the desired estimates. $\square$

Remark. The decomposition of Φ used in the above proof is very flexible. One could have used the heat kernel decomposition

$$\Phi = (1 - e^{t\Delta})\Phi + e^{t\Delta}\Phi = \Phi_1 + \Phi_2, \quad t = \delta^2$$

or more generally one could have decomposed Φ as

$$\Phi = (\Phi - \Phi * \rho_\delta) + \Phi * \rho_\delta = \Phi_1 + \Phi_2$$

where ρ is a bump function and $\rho_\delta(x) = \delta^{-N}\rho(\delta^{-1}x)$. The latter was the original one given by van Schaftingen.

It is also instructive to see that the decomposition we have stated here is dilation invariant; thus the lemma would follow once we could prove it for $\delta = 1$.

Proof of Theorem 5. Let us write $x \in \mathbb{R}^n$ as $x = (x_1, x')$ and $\Phi^{x_1}(x') = \Phi(x_1, x')$ etc. Fix x_1 and consider

$$\int_{\mathbb{R}^{n-1}} f_1(x_1, x')\Phi(x_1, x')dx' = \int_{\mathbb{R}^{n-1}} f_1^{x_1}(x')\Phi^{x_1}(x')dx'.$$

Decompose the function Φ^{x_1} as a function on $\mathbb{R}^{n-1}$ using Lemma 1 with $N = n-1$ and $p = n$. (The decomposition depends on x_1.) Then we obtain

$$\Phi^{x_1} = \Phi_1^{x_1} + \Phi_2^{x_1}$$

where

$$\begin{cases} \|\Phi_1^{x_1}\|_{L^\infty(\mathbb{R}^{n-1})} \le C\delta^{\frac{1}{n}}\|\nabla'\Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})} \\ \|\nabla'\Phi_2^{x_1}\|_{L^\infty(\mathbb{R}^{n-1})} \le C\delta^{\frac{1}{n}-1}\|\nabla'\Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})}. \end{cases}$$

and ∇' denotes derivatives in the x' directions. Now

$$\int_{\mathbb{R}^{n-1}} f_1^{x_1}\Phi^{x_1}dx' = \int_{\mathbb{R}^{n-1}} f_1^{x_1}\Phi_1^{x_1}dx' + \int_{\mathbb{R}^{n-1}} f_1^{x_1}\Phi_2^{x_1}dx' = I + II.$$

I is estimated by

$$|I| \leq C\delta^{\frac{1}{n}}\|f^{x_1}\|_{L^1(\mathbb{R}^{n-1})}\|\nabla'\Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})};$$

while

$$II = \int_{\mathbb{R}^{n-1}}\int_{-\infty}^{x_1}\partial_t f_1(t,x')\Phi_2^{x_1}(x')dtdx'$$

$$= \int_{\mathbb{R}^{n-1}}\int_{-\infty}^{x_1}-\sum_{j=2}^{n}\partial_{x_j}f_j(t,x')\Phi_2^{x_1}(x')dtdx'$$

$$= \int_{\mathbb{R}^{n-1}}\int_{-\infty}^{x_1}\sum_{j=2}^{n}f_j(t,x')\partial_{x_j}\Phi_2^{x_1}(x')dtdx'$$

(the second equality following from $d^*f = 0$) so

$$|II| \leq \|f\|_{L^1(\mathbb{R}^n)}\|\nabla'\Phi_2^{x_1}\|_{L^\infty(\mathbb{R}^{n-1})} \leq C\delta^{\frac{1}{n}-1}\|f\|_{L^1(\mathbb{R}^n)}\|\nabla'\Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})}.$$

Optimize by setting $\delta = \|f\|_{L^1(\mathbb{R}^n)}/\|f^{x_1}\|_{L^1(\mathbb{R}^{n-1})}$, we get

$$\left|\int_{\mathbb{R}^{n-1}}f_1^{x_1}\Phi^{x_1}dx'\right| \leq C\|f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}}\|f^{x_1}\|_{L^1(\mathbb{R}^{n-1})}^{\frac{n-1}{n}}\|\nabla'\Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})}.$$

Unfreeze x_1 now and integrate with respect to x_1, we get

$$\left|\int_{\mathbb{R}^n}f_1\Phi dx\right| \leq C\|f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}}\int_{\mathbb{R}}\|f^{x_1}\|_{L^1(\mathbb{R}^{n-1})}^{\frac{n-1}{n}}\|\nabla'\Phi^{x_1}\|_{L^n(\mathbb{R}^{n-1})}dx_1$$

$$\leq C\|f\|_{L^1(\mathbb{R}^n)}\|\nabla'\Phi\|_{L^n(\mathbb{R}^n)}$$

upon invoking Holder's inequality. $\square$

Finally, to complete this circle of ideas, we show that Theorem 1 can also be deduced from Theorem 2. Thus Theorems 1, 2, 3, 4, and 5 are all equivalent.

Proof that Theorem 2 $\Rightarrow$ Theorem 1. First, we are going to interprete our vector fields f and ϕ as $(n-1)$-forms on $\mathbb{R}^n$. The divergence of f is then df, and by assumption this is zero.

Now given an $(n-1)$-form $\phi \in L^n$, one can write, via Hodge decomposition,

$$\phi = d\alpha + d^*\beta$$

and from

$$(f, d^*\beta) = (df, \beta) = 0$$

we have

$$(f, \phi) = (f, d\alpha).$$

However, by Theorem 2 in the case $l = n - 2$, there exists an $(n - 1)$-form $\psi \in L^\infty$ such that

$$\begin{cases} d^*\psi = d^*(d\alpha) \\ \|\psi\|_{L^\infty} \le C\|d^*(d\alpha)\|_{L^n} = C\|d^*\phi\|_{L^n}. \end{cases}$$

The first equation says that $d\alpha = \psi + d^*\gamma$ for some γ, so from $df = 0$ we have

$$(f, d\alpha) = (f, \psi) + (f, d^*\gamma) = (f, \psi).$$

Thus the second equation implies

$$|(f, \phi)| = |(f, \psi)| \le \|f\|_{L^1}\|\psi\|_{L^\infty} \le C\|f\|_{L^1}\|d^*\phi\|_{L^n}$$

which implies the desired estimate. $\qquad\square$

It looks like we have proved something stronger, namely

$$(3) \qquad\qquad |(f, \phi)| \le C\|f\|_{L^1}\|\operatorname{curl}\phi\|_{L^n}$$

under the assumptions of Theorem 1. But this is not really stronger, because in any case (f, ϕ) depends only on $\operatorname{curl}\phi$ only if f is divergence free; in fact $(f, \phi) = (f, d(-\Delta)^{-1}(d^*\phi))$ if $df = 0$ (again we identify the vector fields f and ϕ with $(n - 1)$-forms). Now if one can prove $|(f, \phi)| \le C\|f\|_{L^1}\|\nabla\phi\|_{L^n}$ under the assumption $\operatorname{div} f = 0$, then by applying this estimate to $d(-\Delta)^{-1}(d^*\phi)$ instead of ϕ, we obtain

$$|(f, \phi)| \le C\|f\|_{L^1}\|\nabla d(-\Delta)^{-1}(d^*\phi)\|_{L^n}.$$

But since the operator $\nabla d(-\Delta)^{-1}$ is bounded on L^n, one obtains again (3). (The author thanks van Schaftingen for pointing out this argument.)

Finally, we remark that in our current proof of Proposition 1 and Theorem 2, the Hahn-Banach theorem was used, which relies on the axiom of choice and is not constructive. One interesting feature here is, for instance, that the solution Y in Proposition 1 cannot arise as the image of f under any *linear* operator. We have the following Proposition:

Proposition 3 (Bourgain-Brezis)**.** *There does not exist any bounded linear operator $K\colon L^n \to L^\infty$ such that $\operatorname{div} Kf = f$ for all $f \in L^n$.*

This says that any *constructive* proof of Proposition 1 has to be *non-linear* in nature.

The proof of this Proposition makes use of an averaging argument, which relies on the fact that div commutes with translation. In fact if an operator K as in the Proposition exists, then it can be taken to be a convolution operator. Further analysis reveals that this is not possible. We omit the details.

What is truly remarkable here is that nonlinear *constructive* proofs of Proposition 1 and Theorem 2 are possible. Bourgain-Brezis gave one such proof in their original papers, and indeed a proof of a far reaching extension of all the theorems in this section. We present their results in the next section, and defer the essence of the proofs to the next talk.

2. Extensions of elementary results

First, recall that if B_1 and B_2 are two Banach spaces, then their intersection $B_1 \cap B_2$ can be equipped with the norm $\| \cdot \|_{B_1} + \| \cdot \|_{B_2}$, and their sum $B_1 + B_2$ can be equipped with the norm

$$\|f\|_{B_1+B_2} := \inf\{\|g\|_{B_1} + \|h\|_{B_2} : g \in B_1, h \in B_2, f = g + h\}.$$

The dual of $B_1 \cap B_2$ is $B_1^* + B_2^*$. The first result below says that when one solves the equation $d^*Y = d^*X$ with $X \in \dot{W}^{1,n}$, Y can not only be taken in L^∞, but it can be taken to be in $L^\infty \cap \dot{W}^{1,n}$. The predual of this space, namely $L^1 + \dot{W}^{-1,\frac{n}{n-1}}$, will also be of interest to us.

The results we have are the following; in essence we replace, in the theorems of the previous section, all the L^∞ norms by $L^\infty \cap \dot{W}^{1,n}$, and all the L^1 norms by $L^1 + \dot{W}^{-1,\frac{n}{n-1}}$.

Theorem 6 (Bourgain-Brezis). *Given any q-form $X \in \dot{W}^{1,n}$, $q \neq n$, there is always a distributional solution Y to the equation $d^*Y = d^*X$ that is both in L^∞ and $\dot{W}^{1,n}$, with $\|Y\|_{L^\infty} + \|\nabla Y\|_{L^n} \leq C\|d^*X\|_{L^n}$.*

Theorem 7 (Bourgain-Brezis). *Suppose u is a smooth l-forms with compact support on $\mathbb{R}^n$.*

(a) If $l \neq 1$ nor $n - 1$, then one has

$$\|u\|_{L^{\frac{n}{n-1}}} \leq C \left(\|du\|_{L^1+\dot{W}^{-1,\frac{n}{n-1}}} + \|d^*u\|_{L^1+\dot{W}^{-1,\frac{n}{n-1}}} \right).$$

(b) If $l = 1$ or $n - 1$ (or both), then the same inequality holds, except that one has to replace $\|du\|_{L^1+\dot{W}^{-1,\frac{n}{n-1}}}$ by $\|du\|_{\mathcal{H}^1}$ if du is a top form, and $\|d^*u\|_{L^1+\dot{W}^{-1,\frac{n}{n-1}}}$ by $\|d^*u\|_{\mathcal{H}^1}$ if d^*u is a function. Here $\mathcal{H}^1$ is the Hardy $\mathcal{H}^1$ norm.

Theorem 8 (Bourgain-Brezis). *Suppose f and ϕ are two smooth and compactly supported vector fields on $\mathbb{R}^n$, and suppose that $\mathrm{div}\, f = 0$. Then*

$$\left| \int_{\mathbb{R}^n} f \cdot \phi\, dx \right| \leq C\|f\|_{L^1+\dot{W}^{-1,\frac{n}{n-1}}} \|\mathrm{curl}\, \phi\|_{L^n},$$

where $\|\mathrm{curl}\, \phi\|_{L^n}$ is the sum of the L^n norms of the components of $\mathrm{curl}\, \phi$, and $\mathrm{curl}\, \phi$ is the $(n-2)$ form given by $d^*\phi$ if we identify the vector field ϕ with an $(n-1)$-form on $\mathbb{R}^n$.

These three theorems are all equivalent, by essentially the same proofs that we have given in the last section. Thus it suffices to prove one of them. We will prove Theorem 6. The proof is constructive, and it relies on the following approximation lemma for functions in $\dot{W}^{1,n}$ on $\mathbb{R}^n$:

Lemma 2 (Bourgain-Brezis). *For any $\delta > 0$, there exists $A_\delta > 0$, such that for any $f \in \dot{W}^{1,n}$, there exists $F \in \dot{W}^{1,n} \cap L^\infty$ satisfying*

$$\begin{cases} \sum_{i=2}^n \|\partial_i(f - F)\|_{L^n} \leq \delta\|\nabla f\|_{L^n} \\ \|\nabla F\|_{L^n} + \|F\|_{L^\infty} \leq A_\delta\|\nabla f\|_{L^n}. \end{cases}$$

Roughly speaking, this says one can approximate a function in $\dot{W}^{1,n}$ by a function in L^∞ if one is willing to give up the derivative in one direction.

Proof of Theorem 6. Assume Lemma 2 for the moment. The key idea is that when one computes d^* of a q form on $\mathbb{R}^n$, only q of the n directional derivatives of each component is involved. So if $q < n$, then for each component of the q form, there will be some directional derivatives that is irrelevant in computing d^*, and we can give up estimates in those directions when we apply Lemma 2.

We shall use the bounded inverse theorem and an argument closely related to the usual proof of the open mapping theorem.

Consider the map $d^*\colon \dot{W}^{1,n}(\Lambda^q\mathbb{R}^n) \to L^n(\Lambda^{q-1}\mathbb{R}^n)$. It is bounded and has closed range. Hence it induces a bounded linear bijection between the Banach spaces $\dot{W}^{1,n}(\Lambda^q\mathbb{R}^n)/\ker(d^*)$ and $\text{Image}(d^*) \subseteq L^n(\Lambda^{q-1}\mathbb{R}^n)$. By the bounded inverse theorem, this map has a bounded inverse; hence for any $X \in \dot{W}^{1,n}(\Lambda^q\mathbb{R}^n)$, there exists $\alpha^{(0)} \in \dot{W}^{1,n}(\Lambda^q\mathbb{R}^n)$ such that

$$\begin{cases} d^*\alpha^{(0)} = d^*X \\ \|\nabla\alpha^{(0)}\|_{L^n} \leq C\|d^*X\|_{L^n}. \end{cases}$$

Now for $q < n$, if I is a multiindex of length q, then one can pick $i \notin I$ and approximate $\alpha_I^{(0)}$ by Lemma 2 in all but the i-th direction; more precisely, for any $\delta > 0$, there exists $\beta_I^{(0)} \in \dot{W}^{1,n} \cap L^\infty$ such that

$$\sum_{j \neq i} \left\| \partial_j \left(\alpha_I^{(0)} - \beta_I^{(0)} \right) \right\|_{L^n} \leq \delta \left\| \nabla\alpha_I^{(0)} \right\|_{L^n} \leq C\delta \left\| d^*X \right\|_{L^n}$$

and

$$\left\| \beta_I^{(0)} \right\|_{L^\infty} + \left\| \nabla\beta_I^{(0)} \right\|_{L^n} \leq A_\delta \left\| \nabla\alpha_I^{(0)} \right\|_{L^n} \leq CA_\delta \left\| d^*X \right\|_{L^n}.$$

Then if δ is picked so that $C\delta \leq \frac{1}{2}$, we have $\beta^{(0)} := \sum_I \beta_I^{(0)} dx^I \in \dot{W}^{1,n} \cap L^\infty(\Lambda^q\mathbb{R}^n)$ satisfying

$$\begin{cases} \|d^*(X - \beta^{(0)})\|_{L^n} \leq \frac{1}{2}\|d^*X\|_{L^n} \\ \|\beta^{(0)}\|_{L^\infty} + \|\nabla\beta^{(0)}\|_{L^n} \leq A\|d^*X\|_{L^n} \end{cases}$$

(the first equation holds because $\|d^*(X - \beta^{(0)})\|_{L^n} = \|d^*(\alpha^{(0)} - \beta^{(0)})\|_{L^n}$, and A here is a fixed constant). In other words, we have sacrificed the property $d^*X = d^*\alpha^{(0)}$ by replacing $\alpha^{(0)} \in \dot{W}^{1,n}$ with $\beta^{(0)}$, which in addition to being in $\dot{W}^{1,n}$ is in L^∞. Now we repeat the process, with $X - \beta^{(0)}$ in place of X, so that we obtain $\beta^{(1)} \in \dot{W}^{1,n} \cap L^\infty(\Lambda^q\mathbb{R}^n)$ with

$$\begin{cases} \|d^*(X - \beta^{(0)} - \beta^{(1)})\|_{L^n} \leq \frac{1}{2}\|d^*(X - \beta^{(0)})\|_{L^n} \leq \frac{1}{2^2}\|d^*X\|_{L^n} \\ \|\beta^{(1)}\|_{L^\infty} + \|\nabla\beta^{(1)}\|_{L^n} \leq A\|d^*(X - \beta^{(0)})\|_{L^n} \leq \frac{A}{2}\|d^*X\|_{L^n}. \end{cases}$$

Iterating, we get $\beta^{(k)} \in \dot{W}^{1,n} \cap L^\infty(\Lambda^q \mathbb{R}^n)$ such that

$$\begin{cases} \|d^*(X - \beta^{(0)} - \cdots - \beta^{(k)})\|_{L^n} \le \frac{1}{2^k} \|d^*X\|_{L^n} \\ \|\beta^{(k)}\|_{L^\infty} + \|\nabla\beta^{(k)}\|_{L^n} \le \frac{A}{2^k} \|d^*X\|_{L^n}. \end{cases}$$

Hence

$$Y = \sum_{k=0}^{\infty} \beta^{(k)}$$

satisfies $Y \in \dot{W}^{1,n} \cap L^\infty(\Lambda^q \mathbb{R}^n)$ with

$$\begin{cases} d^*X = d^*Y \\ \|Y\|_{L^\infty} + \|\nabla Y\|_{L^n} \le 2A\|d^*X\|_{L^n} \end{cases}$$

as desired. $\qquad\square$

While the statement of the above approximation lemma looks linear, the proof proceeds via the following *non-linear*[1] statement:

Lemma 3 (Bourgain-Brezis)**.** *There exists $c_n < 1$ such that for $\delta > 0$, there exists $C_\delta > 0$, such that for any $f \in \dot{W}^{1,n}$ with $\|\nabla f\|_{L^n} \le c_n$, there exists $F \in \dot{W}^{1,n} \cap L^\infty$ such that*

$$\begin{cases} \|F\|_{L^\infty} \le C_\delta \\ \|\nabla F\|_{L^n} \le C_\delta \|\nabla f\|_{L^n} \\ \sum_{i=2}^{n} \|\partial_i(f - F)\|_{L^n} \le \delta \|\nabla f\|_{L^n} + C_\delta \|\nabla f\|_{L^n}^2. \end{cases}$$

We end today by showing how Lemma 2 and Lemma 3 are equivalent.

Proof that Lemma 3 $\Rightarrow$ Lemma 2. Since the statement of Lemma 2 is invariant under dilation, and only small δ's needs to be considered, one can without loss of generality assume that the given $f \in \dot{W}^{1,n}$ and δ satisfies $\|\nabla f\|_{L^n} = \delta C_\delta^{-1} \le c_n$, where C_δ is the constant arising in Lemma 3. The one can apply Lemma 3, and obtain some $F \in \dot{W}^{1,n} \cap L^\infty$ such that

$$\begin{cases} \|F\|_{L^\infty} \le C_\delta = \delta^{-1}C_\delta^2 \|\nabla f\|_{L^n} \\ \|\nabla F\|_{L^n} \le C_\delta \|\nabla f\|_{L^n} \\ \sum_{i=2}^{n} \|\partial_i(f - F)\|_{L^n} \le \delta \|\nabla f\|_{L^n} + C_\delta \|\nabla f\|_{L^n}^2 = 2\delta \|\nabla f\|_{L^n}. \end{cases}$$

Replacing δ by $\delta/2$ and choosing the appropriate A_δ, this finishes the proof of Lemma 2. $\qquad\square$

The proof of the converse is obvious, which we omit.

Next time the effort will go into proving Lemma 3. Once that is done, we will have proved all the theorems we stated in this section.

We remark that when $n = 2$, there is a direct proof of Theorem 7 using Phancherel's theorem (which is possible because then $\frac{n}{n-1}$ is equal to 2), but in higher dimensions there has been no direct proofs of that so far.

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[1]Note the square of $\|\nabla f\|_{L^n}$ in the last inequality of Lemma 3.

References

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