SOME SPECIAL ISOMORPHISMS OF LIE ALGEBRAS IN LOW DIMENSIONS

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In this note, we present a more geometric construction of some special isomorphisms between Lie algebras in low dimensions. For simplicity our Lie algebras will be defined over $\mathbb{C}$; the statements and the proofs will all go through if $\mathbb{C}$ is replaced by an algebraically closed field k with $\operatorname{char} k \neq 2$.

First we recall the definitions of some standard matrix Lie algebras:

$$\mathfrak{sl}_n = \{x \in \mathfrak{gl}_n \colon \operatorname{tr} x = 0\}$$
$$\mathfrak{so}_n = \{x \in \mathfrak{gl}_n \colon x + x^t = 0\}$$
$$\mathfrak{sp}_{2n} = \{x \in \mathfrak{gl}_{2n} \colon xJ_{2n} + J_{2n}x^t = 0\}$$

where

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and I_n is the $n \times n$ identity matrix. It follows that

$$\mathfrak{sp}_{2n} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathfrak{gl}_n, a = -d^t, b = b^t, c = c^t \right\}.$$

Next, let V be a finite dimensional vector space over $\mathbb{C}$. A symmetric bilinear form $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$ is said to be *non-degenerate*, if for every non-zero $v \in V$, there exists some $w \in V$ such that $\langle v, w \rangle \neq 0$. It is known that all non-degenerate symmetric bilinear forms on V are equivalent: if $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two non-degenerate symmetric bilinear forms on V, then there exists a linear isomorphism $T \colon V \to V$ such that $\langle v, w \rangle_1 = \langle Tv, Tw \rangle_2$ for all $v, w \in V$. In particular, if $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$ is a non-degenerate symmetric bilinear form on V, then there exists a basis $\{e_1, \dots, e_n\}$ of V such that $\langle \cdot, \cdot \rangle$ becomes diagonal in this basis, i.e. $\langle e_j, e_k \rangle = \delta_{jk}$ (for example, if $\{v_1, \dots, v_n\}$ is any basis of V, and A is the invertible symmetric matrix given by $A_{jk} = \langle v_j, v_k \rangle$, then $e_j := \sum_{k=1}^n (A^{-1/2})_{jk} v_k$ for $j = 1, \dots, n$ gives the desired basis. The square root of A exists since one can write A in Jordan normal form, noting $\mathbb{C}$ is algebraically closed; each Jordan block has a square root since it is a non-zero multiple of the identity plus a nilpotent matrix).

Now suppose $\mathfrak{g}$ is a complex Lie algebra, and V is a complex vector space with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$. Suppose we also have a representation $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ preserving $\langle \cdot, \cdot \rangle$, i.e.

$$\langle \rho(x)v, w \rangle + \langle v, \rho(x)w \rangle = 0 \quad \text{for all } x \in \mathfrak{g} \text{ and all } v, w \in V.$$

Then picking a special basis $\{e_1, \dots, e_n\}$ as above, so that $\langle \cdot, \cdot \rangle$ becomes diagonal in this basis, one can identify V with $\mathbb{C}^n$, and identify ρ as a representation $\rho \colon \mathfrak{g} \to \mathfrak{so}_n$. More precisely, we define a linear map $T \colon V \to \mathbb{C}^n$ by $T(\sum_{j=1}^n a_j e_j) = (a_1, \dots, a_n)$. Then $\langle v, w \rangle = (Tv)^t (Tw)$. The linear map $\tilde{\rho} \colon \mathfrak{g} \to \mathfrak{so}_n$, defined by

$$\tilde{\rho}(x) = T \circ \rho(x) \circ T^{-1}$$

is then a representation of $\mathfrak{g}$ induced from ρ: we have

$$\tilde{\rho}([x,y]) = T \circ \rho([x,y]) \circ T^{-1} = [\tilde{\rho}(x), \tilde{\rho}(y)]$$

for all $x, y \in \mathfrak{g}$ (since conjugation commutes with Lie brackets), and since ρ preserves $\langle \cdot, \cdot \rangle$, one has

$$(T\rho(x)T^{-1}z)^t w + z^t(T\rho(x)T^{-1}w) = 0 \quad \text{for all } x \in \mathfrak{g} \text{ and } z, w \in \mathbb{C}^n,$$

which implies $\tilde{\rho}(x) \in \mathfrak{so}_n$. We will make repeated use of this fact below.

Theorem 1. $\mathfrak{sl}_2 = \mathfrak{sp}_2 \simeq \mathfrak{so}_3$.

Proof. From definition of $\mathfrak{sl}_2$ and $\mathfrak{sp}_2$, it is clear that the two are identical.

Now let $V = \mathfrak{sl}_2$. On V there is a non-degenerate symmetric bilinear form

$$\langle y, z \rangle = \mathrm{tr}\,(yz).$$

(This is a multiple of the Killing form of $\mathfrak{sl}_2$.) The adjoint action $\mathrm{ad}\colon \mathfrak{sl}_2 \to \mathfrak{gl}(\mathfrak{sl}_2)$ preserves this non-degenerate symmetric bilinear form:

$$\langle \mathrm{ad}(x)y, z \rangle + \langle y, \mathrm{ad}(x)z \rangle = 0 \quad \text{for all } x, y, z \in \mathfrak{sl}_2.$$

In fact,

$$\langle \mathrm{ad}(x)y, z \rangle + \langle y, \mathrm{ad}(x)z \rangle = \mathrm{tr}\,((xy - yx)z + y(xz - zx)) = \mathrm{tr}\,(x(yz) - (yz)x) = 0$$

for all $x, y, z \in \mathfrak{sl}_2$. It follows that the adjoint action ad induces a Lie homomorphism of $\mathfrak{sl}_2$ into $\mathfrak{so}_3$. This is an injective homomorphism, since its kernel is a proper ideal of $\mathfrak{sl}_2$, and $\mathfrak{sl}_2$ is simple; since both $\mathfrak{sl}_2$ and $\mathfrak{so}_3$ are 3-dimensional, it follows that this is an isomorphism of Lie algebras. □

Theorem 2. $\mathfrak{sl}_4 \simeq \mathfrak{so}_6$.

Proof. Let $V = \Lambda^2\mathbb{C}^4$ be the vector space of skew-symmetric 2-tensors on $\mathbb{C}^4$. In other words, V is the span of $z \wedge w$ over all $z, w \in \mathbb{C}^4$, where $z \wedge w := z \otimes w - w \otimes z$. Then V is 6-dimensional. Furthermore, there is a natural non-degenerate symmetric bilinear form on V: if $\iota\colon \Lambda^4\mathbb{C}^4 \to \mathbb{C}$ is an isomorphism of the vector space of alternating 4-tensors on $\mathbb{C}^4$ with $\mathbb{C}$, then one can define a non-degenerate symmetric bilinear form on V by

$$\langle u, v \rangle = \iota(u \wedge v) \quad \text{for } u, v \in V.$$

(Both symmetry and non-degeneracy of the bilinear form can be checked by hand easily.) Now the vector representation of $\mathfrak{sl}_4$ on $\mathbb{C}^4$ naturally induces a representation $\rho\colon \mathfrak{sl}_4 \to \mathfrak{gl}(V)$. Moreover, this representation preserves the non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$:

$$\langle \rho(x)u, v \rangle + \langle u, \rho(x)v \rangle = 0 \quad \text{for all } x \in \mathfrak{sl}_4 \text{ and all } u, v \in V.$$

In fact, by linearity, it suffices to check this when $u = z_1 \wedge z_2$ and $v = z_3 \wedge z_4$, where each z_i is one of the standard basis vectors e_1, e_2, e_3, e_4 of $\mathbb{C}^4$. Then

$$\begin{aligned} &\langle \rho(x)u, v \rangle + \langle u, \rho(x)v \rangle \\ =&\iota\left[(xz_1) \wedge z_2 \wedge z_3 \wedge z_4 + z_1 \wedge (xz_2) \wedge z_3 \wedge z_4 + z_1 \wedge z_2 \wedge (xz_3) \wedge z_4 + z_1 \wedge z_2 \wedge z_3 \wedge (xz_4)\right]; \end{aligned}$$

Here xz_i is the natural action of $x \in \mathfrak{sl}_4$ on $z_i \in \mathbb{C}^4$. Hence by skew-symmetry, this is zero unless $\{z_1, z_2, z_3, z_4\}$ is a re-ordering of $\{e_1, e_2, e_3, e_4\}$. By relabelling the basis $\{e_1, e_2, e_3, e_4\}$, we may assume that $z_i = e_i$ for $i = 1, \ldots, 4$. In that case,

$$\langle \rho(x)u, v \rangle + \langle u, \rho(x)v \rangle = \iota\left[(\mathrm{tr}\, x)e_1 \wedge e_2 \wedge e_3 \wedge e_4\right] = 0$$

as desired as well. Hence ρ induces a representation of $\mathfrak{sl}_4$ into $\mathfrak{so}_6$. By simplicity of $\mathfrak{sl}_4$, the latter is an injective Lie homomorphism; since both $\mathfrak{sl}_4$ and $\mathfrak{so}_6$ are 15 dimensional, it follows that they are isomorphic. ☐

Theorem 3. $\mathfrak{sp}_4 \simeq \mathfrak{so}_5$.

Proof. The isomorphism between $\mathfrak{sp}_4$ and $\mathfrak{so}_5$ is obtained by restricting the isomorphism in the previous theorem. In fact, $\mathfrak{sp}_4 \subset \mathfrak{sl}_4$, so if $V = \Lambda^2\mathbb{C}^4$, $\langle\cdot,\cdot\rangle$ and $\rho\colon \mathfrak{sl}_4 \to \mathfrak{gl}(V)$ is as in the previous theorem, then it induces a representation $\rho_0\colon \mathfrak{sp}_4 \to \mathfrak{gl}(V)$ preserving $\langle\cdot,\cdot\rangle$. Now let

$$v_0 = e_1 \wedge e_3 + e_2 \wedge e_4.$$

If $x \in \mathfrak{sp}_4$, then $\rho_0(x)v_0$ is a multiple of v_0. Hence if W is the orthogonal complement of v_0 in V, i.e.

$$W = \{w \in V\colon \langle w, v_0\rangle = 0\},$$

then $\rho_0(x)$ restricts to a map from W into W for all $x \in \mathfrak{sp}_4$. It follows that ρ_0 induces a representation $\rho_1\colon \mathfrak{sp}_4 \to \mathfrak{gl}(W)$. Furthermore, one can restrict $\langle\cdot,\cdot\rangle$ to W, and the restriction gives a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_1$ on the 5-dimensional vector space W. Since ρ_1 preserves $\langle\cdot,\cdot\rangle_1$, it induces a Lie homomorphism of $\mathfrak{sp}_4$ into $\mathfrak{so}_5$. Since $\mathfrak{sp}_4$ is simple, the kernel of this map is trivial; since $\mathfrak{sp}_4$ and $\mathfrak{so}_5$ are both 10-dimensional, it follows that they are isomorphic. ☐

Theorem 4. $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Proof. Let $V = \mathfrak{gl}_2$ be our 4-dimensional vector space. First, $\mathfrak{sl}_2$ acts on V on the left. In other words, there is a representation $\rho_1\colon \mathfrak{sl}_2 \to \mathfrak{gl}(V)$, given by

$$\rho_1(x)v = xv \quad \text{for all } x \in \mathfrak{sl}_2 \text{ and all } v \in V.$$

Similarly, $\mathfrak{sl}_2$ acts on V on the right. In other words, there is a representation $\rho_2\colon \mathfrak{sl}_2 \to \mathfrak{gl}(V)$, defined by

$$\rho_2(y)v = -vy \quad \text{for all } y \in \mathfrak{sl}_2 \text{ and all } v \in V.$$

Note $[\rho_1(x), \rho_2(y)] = 0$ for all $x, y \in \mathfrak{sl}_2$. Thus one can define a representation $\rho\colon \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{gl}(V)$, namely

$$\rho(x, y) = \rho_1(x) + \rho_2(y) \quad \text{for all } (x, y) \in \mathfrak{sl}_2 \oplus \mathfrak{sl}_2.$$

More explicitly,

$$\rho(x, y)v = xv - vy \quad \text{for all } (x, y) \in \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \text{ and all } v \in V. \tag{1}$$

Now let

$$J = J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and define a bilinear form on V by[1]

$$\langle v, w\rangle = \operatorname{tr}(vJw^tJ). \tag{2}$$

[1]More explicitly, if

$$v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad w = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then this bilinear form is given by

$$\langle v, w\rangle = -aD + bC + cB - dA.$$

This bilinear form is symmetric, since

$$\langle v, w\rangle = \mathrm{tr}\,(vJw^tJ) = \mathrm{tr}\,(vJw^tJ)^t = \mathrm{tr}\,(JwJv^t) = \mathrm{tr}\,(wJv^tJ) = \langle w, v\rangle.$$

Furthermore, this bilinear form is non-degenerate on V, because the bilinear form $(v, w) := \mathrm{tr}\,(vw)$ is non-degenerate, and the map $w \mapsto Jw^tJ$ is a linear isomorphism of V onto itself. We claim that ρ preserves this non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. In fact, by definition of ρ, it suffices to show that both ρ_1 and ρ_2 preserves $\langle\cdot,\cdot\rangle$. To see the latter, note that for any $x \in \mathfrak{sl}_2$ and any $v, w \in V$, we have

$$\langle \rho_1(x)v, w\rangle = \mathrm{tr}\,(xvJw^tJ) = \mathrm{tr}\,(vJw^tJx),$$

and

$$\langle v, \rho_1(x)w\rangle = \mathrm{tr}\,(vJ(xw)^tJ) = \mathrm{tr}\,(vJw^tx^tJ).$$

But from $x \in \mathfrak{sl}_2 = \mathfrak{sp}_2$, we have $xJ + Jx^t = 0$. Hence

$$\langle \rho_1(x)v, w\rangle + \langle v, \rho_1(x)w\rangle = \mathrm{tr}\,(vJw^t(x^tJ + Jx)) = 0$$

as desired. Similarly, for any $y \in \mathfrak{sl}_2$ and any $v, w \in V$, we have

$$\begin{aligned}\langle \rho_2(y)v, w\rangle + \langle v, \rho_2(y)w\rangle &= -\mathrm{tr}\,(vyJw^tJ) - \mathrm{tr}\,(vJ(wy)^tJ) \\ &= -\mathrm{tr}\,(v(yJ + Jy^t)w^tJ) = 0.\end{aligned}$$

Thus ρ preserves $\langle\cdot,\cdot\rangle$, and induces a map $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{so}_4$. The kernel of this map is an ideal of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and by simplicity of $\mathfrak{sl}_2$ can only be $\{0\}$, $\mathfrak{sl}_2 \oplus \{0\}$, $\{0\} \oplus \mathfrak{sl}_2$, or $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. It is then clear that the kernel of this map is trivial, and since both $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{so}_4$ are 6-dimensional, it follows that they are isomorphic.

We remark that one could rephrase the above proof by identifying $V = \mathfrak{gl}_2$ naturally as $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$. In fact, from the vector representation of $\mathfrak{sl}_2$ on $\mathbb{C}^2$, one can induce naturally an action of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ on $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$, and that induced representation agrees with the representation ρ we defined in (**??**). Furthermore, the bilinear form on V defined by (**??**) is just the one defined by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle = -\omega(v_1, v_2)\omega(w_1, w_2)$$

for all $v_1, v_2 \in \mathbb{C}^2$ and all $w_1, w_2 \in (\mathbb{C}^2)^*$, where ω is the symplectic form on $\mathbb{C}^2$ (and on $(\mathbb{C}^2)^*$ by abuse of notation). Now $\langle\cdot,\cdot\rangle$ is symmetric on V since ω is anti-symmetric on $\mathbb{C}^2$, and $\langle\cdot,\cdot\rangle$ is non-degenerate on V since ω is non-degenerate on $\mathbb{C}^2$. Furthermore, $\langle\cdot,\cdot\rangle$ is preserved by the action of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, since ω is preserved by $\mathfrak{sl}_2 = \mathfrak{sp}_2$. This gives us a more conceptual way of presenting the above argument. □