THE JOINTS AND THE MULTIJOINTS THEOREM

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Abstract. The following are notes (taken by Po-Lam Yung) of a mini-course by Ruixiang Zhang on the joints theorem and the multijoints theorem. The note taker has since supplemented these notes with more references to the existing literature.

Let L be a set of N straight lines in $\mathbb{R}^d$. A joint formed by these lines is a point in $\mathbb{R}^d$ that belongs to at least d of these lines, whose directions form a linearly independent set in $\mathbb{R}^d$. The main theorem about joints is the following.

Theorem 1. *The number of joints formed by N lines in $\mathbb{R}^d$ is at most $C_d N^{d/(d-1)}$, for some dimensional constant C_d.*

This was first proved by Katz and Guth [4] in dimension $d = 3$, and by Quilodrán [14] and Kaplan, Sharir and Shustin [13] in higher dimensions. We refer to the introduction in the notes of Carbery and Iliopoulou [2], and also the introduction of Iliopoulou [11], for some prior partial results such as [5], [15], [16], [7] and [1] (see also [6] for an alternative proof in 3 dimensions).

Proof. We use the polynomial method. Let L be a set of N lines in $\mathbb{R}^d$, and J be the set of joints formed by L. We assume that J is non-empty, for otherwise there is nothing to prove.

We claim that there exists a line $\ell \in L$ that contains at most $A_d|J|^{1/d}$ joints, where A_d is some dimensional constant. Indeed, let $Q(x)$ be a non-zero polynomial in $\mathbb{R}[x_1, \dots, x_d]$, of minimal total degree, that vanishes at all points in J. Then the degree of Q is at most

$$\deg(Q) \leq A_d|J|^{1/d}$$

for some dimensional constant A_d. If all lines in L contains more than $A_d|J|^{1/d}$ points in J, then Q restricted to any line in L would have more than $\deg(Q)$ roots, so Q would vanish identically on all lines in L. This shows that the gradient ∇Q vanishes at all joints formed by L, i.e. ∇Q vanishes at all points in J. But one of $\partial_{x_1} Q, \dots, \partial_{x_d} Q$ is non-zero (for otherwise Q is zero, contradicting our choice of Q). Hence one of them a non-zero polynomial of degree $< \deg(Q)$, that vanishes at all points of J. This contradicts the minimality of the degree of Q, and we obtain our claim.

Now we may finish in two ways. One is to use induction on the number of lines in L. Let $\ell \in L$ be a line that contains at most $A_d|J|^{1/d}$ points in J. Then the number of elements of J is bounded by $A_d|J|^{1/d}$ plus the number of elements of J that are not on ℓ. Now any element

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of J that is not on ℓ is a joint of $L \setminus \{\ell\}$, which contains only $N-1$ lines. So induction hypothesis applies, and we get

$$|J| \leq A_d |J|^{1/d} + C_d (N-1)^{d/(d-1)}. \tag{1}$$

We claim that if C_d were chosen large enough initially, then this implies $|J| \leq C_d N^{d/(d-1)}$ as well. Indeed, if not, then $|J| > C_d N^{d/(d-1)}$, so using the fact that $t \mapsto t - A_d t^{1/d}$ is a strictly increasing function for $t \geq a_d$ for some dimensional constant a_d, we have

$$|J| - A_d |J|^{1/d} > C_d N^{d/(d-1)} - A_d (C_d N^{d/(d-1)})^{1/d}.$$

(we just choose C_d large enough so that $C_d \geq a_d$, so that $C_d N^{d/(d-1)} \geq a_d$ for all positive integers N). It follows that

$$\begin{aligned} |J| - A_d |J|^{1/d} - C_d (N-1)^{d/(d-1)} &> C_d (N^{d/(d-1)} - (N-1)^{d/(d-1)}) - A_d C_d^{1/d} N^{1/(d-1)} \\ &> \frac{dC_d}{d-1} (N-1)^{1/(d-1)} - 2A_d C_d^{1/d} (N-1)^{1/(d-1)} \\ &> 0 \end{aligned}$$

if we had chosen the dimensional constant C_d sufficiently large, so that

$$\frac{dC_d}{d-1} - 2A_d C_d^{1/d} > 0.$$

This contradicts (1), so $|J| \leq C_d N^{d/(d-1)}$ as well, as desired.

Alternatively, let $j(N)$ be the maximum number of joints formed by N lines in $\mathbb{R}^d$. Then the previous argument shows that

$$j(N) \leq A_d [j(N)]^{1/d} + j(N-1).$$

Iterating, we get

$$j(N) \leq A_d [j(N)]^{1/d} + A_d [j(N-1)]^{1/d} + \cdots + A_d [j(d)]^{1/d}$$

(note $j(d-1) = 0$). Since $j(N-i) \leq j(N)$ for all $1 \leq i \leq N$, we have

$$j(N) \leq A_d N [j(N)]^{1/d},$$

i.e.

$$j(N) \leq (A_d N)^{d/(d-1)}$$

for all $N \in \mathbb{N}$. □

The above proof uses a few properties of polynomials over the reals: for instance, we used that the gradient of a polynomial is a limit of difference quotients, and we used that the gradient of a non-zero polynomial is non-zero unless it is constant. This is no longer true over a general field (e.g. over a field of characteristic p, the polynomial x^p differentiates to zero while x^p is not identically zero). Nevertheless, Theorem 1 holds if one replaces $\mathbb{R}$ by any general field $\mathbb{F}$: indeed one can fix the above proof so that it works without too many changes. This result was a folklore for a while, and is recorded in the notes of Carbery and Iliopoulou [2].

Following ideas from [19], we provide another proof of the joints theorem, that works equally well for all fields, and that generalizes easily to the study of *multijoints* later on.

Alternative proof of Theorem 1. Let L be a set of N lines in $\mathbb{R}^d$, and J be the set of joints formed by L. Let $Q \in \mathbb{R}[x_1, \dots, x_d]$ be a non-zero polynomial of degree $\leq A_d |J|^{1/d}$, that vanishes at all points of J; we denote its zero set by $Z(Q)$. Given a line ℓ and a point $p \in \ell$, we will define what we call the *residual multiplicity* of ℓ at p with respect to Q as follows.

Let's begin with some intuition. If $Z(Q)$ is just a hyperplane and $\ell \subset Z(Q)$, then all points on ℓ have the same "multiplicity" with respect to Q, so the "residual multiplicity" of ℓ at any point p on ℓ should be set to zero. If on the other hand, $Z(Q)$ is the union of two hyperplanes and ℓ is contained in one of them, then ℓ may intersect the other hyperplane transversely at a point p, so p have one higher "multiplicity" with respect to Q than nearby points on ℓ, and in this case we would declare the "residual multiplicity" of ℓ at p to be 1.

More precisely, suppose for the moment that ℓ is the x_d axis, and $p = 0 \in \ell$. We Taylor expand $Q(x)$ as

$$Q(x) = \sum_{\alpha'} x^{\alpha'} q_{\alpha'}(x_d)$$

where the sum is over all multiindices $\alpha' = (\alpha_1, \dots, \alpha_{d-1})$, $x^{\alpha'} := x_1^{\alpha_1} \dots x_{d-1}^{\alpha_{d-1}}$, and $q_{\alpha'}$ is a polynomial of x_d only. We let $a_0 = \min\{|\alpha'| \colon q_{\alpha'}(x_d) \text{ is not identically zero}\}$. Then the *residual multiplicity* of ℓ at p with respect to Q is defined to be the minimum order of vanishing of $q_{\alpha'}(x_d)$ at $x_d = 0$ among all multiindices α' with $|\alpha'| = a_0$. More generally, given any line ℓ and any point $p \in \ell$, let T be an invertible affine map on $\mathbb{R}^d$ that maps ℓ to the x_d axis, and the point p to the point 0. Taylor expand

$$Q(T^{-1}x) = \sum_{\alpha'} x^{\alpha'} q_{\alpha'}(x_d),$$

and let $a_0 = \min\{|\alpha'| \colon q_{\alpha'}(x_d) \text{ is not identically zero}\}$. Then the *residual multiplicity* of ℓ at p with respect to Q is defined to be the minimum order of vanishing of $q_{\alpha'}(x_d)$ at $x_d = 0$ among all multiindices α' with $|\alpha'| = a_0$.

Since the polynomial Q is fixed in this argument, and since we will only discuss residual multiplicities, instead of the residual multiplicity of ℓ at p with respect to Q, and we will simply say the *multiplicity* of ℓ at p.

One can show that this notion of multiplicity is well-defined, independent of the choice of the affine map T on $\mathbb{R}^d$. Indeed, if T and S are both invertible affine maps on $\mathbb{R}^d$ such that $T(p) = S(p) = 0$ and $T(\ell) = S(\ell) =$ the x_d axis, then $S \circ T^{-1}$ is an invertible linear map on $\mathbb{R}^d$ that fixes the x_d axis. If $y = (S \circ T^{-1})x$, then there exists an invertible linear map U on $\mathbb{R}^{d-1}$, and a vector $v = (v', v_d)$ on $\mathbb{R}^d$ whose last component v_d is non-zero, such that

$$y' = Ux', \quad \text{and} \quad y_d = v \cdot x = v_d x_d + v' \cdot x'.$$

Hence if $Q(S^{-1}y) = \sum_{\beta'} y^{\beta'} \tilde{q}_{\beta'}(y_d)$, then

$$Q(T^{-1}x) = \sum_{\beta'} (Ux')^{\beta'} \tilde{q}_{\beta'}(v_d x_d + v' \cdot x').$$

We can collect terms and write this as $Q(T^{-1}x) = \sum_{\alpha'} x^{\alpha'} q_{\alpha'}(x_d)$. Then

$$\min\{|\alpha'| \colon q_{\alpha'}(x_d) \text{ is not identically zero}\} = \min\{|\beta'| \colon \tilde{q}_{\beta'}(y_d) \text{ is not identically zero}\}$$

Let a_0 be this common minimum, and λ be the number of multiindices of $(d-1)$ variables of length a_0. Then U induces an invertible linear map $\bar{U} = (\bar{U}_{\alpha'}^{\beta'})$ on $\mathbb{R}^\lambda$, such that for every α' with $|\alpha'| = a_0$, we have

$$q_{\alpha'}(x_d) = \sum_{|\beta'|=a_0} U_{\alpha'}^{\beta'} \tilde{q}_{\beta'}(v_d x_d).$$

It follows that

$$\begin{aligned}&\min\{\text{ order of vanishing of } q_{\alpha'} \text{ at } 0 : |\alpha'| = a_0\}\\ =&\min\{\text{ order of vanishing of } \tilde{q}_{\beta'} \text{ at } 0 : |\beta'| = a_0\}.\end{aligned}$$

Thus the multiplicity of ℓ at p is well-defined, independent of the choice of the affine map T on $\mathbb{R}^d$.

In the proof of the current theorem, we will only concern ourselves whether the multiplicity of ℓ at p is positive or zero. If the multiplicity of ℓ at p is positive, we call p a *special* point of ℓ; if the multiplicity is zero, we call p an *ordinary* point of ℓ.

The key is to show that for every joint $p \in J$, there is some line $\ell \in L$ passing through p, so that p is a special point of ℓ. Indeed, since p is a joint formed by L, one can find d lines in L that passes through p, whose directions are linearly independent. Let T be an invertible affine map on $\mathbb{R}^d$ that maps p to 0, and maps these d lines from L to the d coordinate axes. We expand

$$Q(T^{-1}x) = \sum_\alpha c_\alpha x^\alpha, \tag{2}$$

and let $\bar{\alpha}$ be a multiindex of minimal length, such that $c_{\bar{\alpha}} \neq 0$. By renaming the coordinate axes, we may assume that $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_d)$, with $\bar{\alpha}_1 \leq \bar{\alpha}_2 \leq \cdots \leq \bar{\alpha}_d$. Let $\ell_0 \in L$ be the image of the x_d axis under T^{-1}. We claim that p is a special point on ℓ_0.

Indeed, we may recollect the terms in the expression for $Q(T^{-1}x)$, and write it as

$$Q(T^{-1}x) = \sum_{\alpha'} x^{\alpha'} q_{\alpha'}(x_d)$$

for some polynomials $q_{\alpha'}(x_d)$ for each multiindex α' of the first $(d-1)$ variables. Let $a_0 = \min\{|\alpha'| \colon q_{\alpha'}(x_d) \text{ is not identically zero}\}$. Then $a_0 \leq |\bar{\alpha}'|$. Suppose now α' is a multiindex of $(d-1)$ variables with $|\alpha'| = a_0$. If $q_{\alpha'}(x_d)$ has a zero of order α_d at 0, then $\alpha := (\alpha', \alpha_d)$ is a multiindex, for which the coefficient of x^α in (2) is non-zero. So by the choice of $\bar{\alpha}$, we have $|\alpha| \geq |\bar{\alpha}|$. Since we also have $|\alpha'| = a_0 \leq |\bar{\alpha}'|$, this shows $\alpha_d \geq \bar{\alpha}_d$, which is positive: indeed, since $\bar{\alpha}_1 \leq \bar{\alpha}_2 \leq \cdots \leq \bar{\alpha}_d$, if $\bar{\alpha}_d$ were 0, then $|\bar{\alpha}| = 0$, which says $Q(T^{-1}0) \neq 0$, contradicting our choice of Q. This shows that $q_{\alpha'}(x_d)$ vanishes at 0 to a positive order. Since this is true for all multiindices α' with $|\alpha'| = a_0$, we conclude that the multiplicity of ℓ at p is positive, and hence p is a special point on ℓ_0.

Finally, note that for every $\ell \in L$, there are at most $\deg(Q)$ special points on ℓ. This is because each $q_{\alpha'}(x_d)$ has at most $\deg(Q)$ zeroes (being polynomials of one variable of degree $\leq \deg(Q)$). Since there are only N lines in L, it follows that

$$|J| \leq N \deg(Q) \leq A_d N |J|^{1/d},$$

which shows

$$|J| \leq (A_d N)^{d/(d-1)}.$$

$\square$

Let now $L_1, \dots, L_d$ be d family of straight lines in $\mathbb{R}^d$. If $p \in \mathbb{R}^d$, and there exists a line from each of these d families, all of which containing p, such that the directions of these d lines are linearly independent, then p is called a multijoint of $L_1, \dots, L_d$. We then have the following multijoints theorem:

Theorem 2. *The number of multijoints formed by $L_1, \dots, L_d$ is at most $C_d(|L_1| \dots |L_d|)^{\frac{1}{d-1}}$, for some dimensional constant C_d.*

This was first proved by Iliopoulou in [12], who also established the same result when $\mathbb{R}^d$ is replaced by $\mathbb{F}^3$ for an arbitrary field $\mathbb{F}$ (see also [11] for an earlier result in $\mathbb{R}^3$). Carbery and Valdimarsson [3] considered colorings of multijoints. Finally, Zhang [19] gave a proof of this theorem that works for arbitrary fields $\mathbb{F}$ and arbitrary dimensions d. We present his proof below, which is a refinement of the second proof we gave above of Theorem 1.

Proof. (Taken from [19]) Let J be the set of all multijoints formed by $L_1, \dots, L_d$. For each $p \in J$, choose lines $\ell_{1,p} \in L_1, \dots, \ell_{d,p} \in L_d$ such that p lies on all of $\ell_{1,p}, \dots, \ell_{d,p}$, and such that the directions of $\ell_{1,p}, \dots, \ell_{d,p}$ are all linearly independent. Choose also an invertible affine map T_p on $\mathbb{R}^d$ that maps p to 0, and that maps $\ell_{i,p}$ to the x_i axis for all $i = 1, \dots, d$. We choose a polynomial Q on $\mathbb{R}^d$, such that for all $p \in J$, when we expand $Q(T_p^{-1}x)$ in monomials in x, the coefficients of x^α is zero whenever $\alpha_i < |L_i|$ for all $i = 1, \dots, d$. This is putting $|L_1| \dots |L_d|$ conditions at each of the $|J|$ points in J, and one can find a polynomial Q of degree

$$\deg(Q) \lesssim A_d(|J||L_1| \dots |L_d|)^{1/d}$$

that achieves this, where A_d is some dimensional constant. We define the residual multiplicity of a line ℓ at a point p with respect to this Q as in the previous proof; we will denote this as $m(\ell, p)$, suppressing in the notation the dependence on Q since Q will be fixed throughout our proof.

Now for any line ℓ, we clearly have

$$\sum_{p \in \ell} m(\ell, p) \leq \deg(Q). \tag{3}$$

Furthermore, for each $p \in J$, there exists $i = i(p) \in \{1, \dots, d\}$ such that

$$m(\ell_{i,p}, p) \geq |L_i|. \tag{4}$$

Indeed, given $p \in J$, we Taylor expand

$$Q(T_p^{-1}x) = \sum_\alpha c_\alpha x^\alpha,$$

and let $\bar{\alpha}$ be a multiindex of minimal length, such that $c_{\bar{\alpha}} \neq 0$. By following the argument in the previous proof, one sees that $m(\ell_{i,p}, p) \geq \bar{\alpha}_i$ for all $i = 1, \dots, d$. But by our choice of

Q, there exists $i = i(p) \in \{1, \ldots, d\}$ such that $\bar{\alpha}_i \geq |L_i|$. It follows that for this i, we have $m(\ell_{i,p}, p) \geq |L_i|$ as desired.

We can now finish the proof in a few strokes. First, by pigeonhole principle, there exists $i_0 \in \{1, \ldots, d\}$ for which the number of points p in J with $i(p) = i_0$ is at least $|J|/d$; in other words, there are at least $|J|/d$ points in J, for which $m(\ell_{i_0,p}, p) \geq |L_{i_0}|$. Let J_{i_0} be the set of all such points $p \in J$. Then

$$\sum_{p \in J_{i_0}} m(\ell_{i_0,p}, p) \geq |J_{i_0}||L_{i_0}| \geq |J||L_{i_0}|/d.$$

But if we sum (3) over all lines in L_{i_0}, we get

$$\sum_{p \in J_{i_0}} m(\ell_{i_0,p}, p) \leq \sum_{\ell \in L_{i_0}} \sum_{p \in \ell} m(\ell_{i_0,p}, p) \leq |L_{i_0}| \deg(Q).$$

Combining the two inequalities, we see that

$$|J| \leq d \deg(Q).$$

Since $\deg(Q) \leq A_d(|J||L_1| \ldots |L_d|)^{1/d}$, we see that

$$|J| \leq (dA_d)^{\frac{d}{d-1}} (|L_1| \ldots |L_d|)^{\frac{1}{d-1}},$$

as desired. □

We close by mentioning that Carbery considered the problem of counting joints and multijoints with multiplicities. See Iliopoulou [9] and [10] for positive results in $\mathbb{R}^3$, Hablicsek [8] for results about generic joints, Yang [18] for some almost sharp results, and finally Zhang [19] for the resolution of Carbery's conjecture. The problem of counting multijoints with multiplicities can be seen as a discrete analog of the multilinear Kakeya problem (see Iliopoulou [11], [12]). The problem of counting joints is also connected to the (linear) Kakeya problem, albeit less directly; see some heuristic observations by Schlag and Wolff in the Further Remark 3.4 of [17].

References

[1] Jonathan Bennett, Anthony Carbery, and Terence Tao, *On the multilinear restriction and Kakeya conjectures*, Acta Math. **196** (2006), no. 2, 261–302.

[2] Anthony Carbery and Marina Iliopoulou, *Counting joints in vector spaces over arbitrary fields*, arXiv **1403:6438v2** (2014).

[3] Anthony Carbery and Stefán Ingi Valdimarsson, *Colouring multijoints*, Discrete Comput. Geom. **52** (2014), no. 4, 730–742.

[4] Larry Guth and Nets Hawk Katz, *Algebraic methods in discrete analogs of the Kakeya problem*, Adv. Math. **225** (2010), no. 5, 2828–2839.

[5] Bernard Chazelle, Herbert Edelsbrunner, Leonidas J. Guibas, R. Pollack, R. Seidel, M. Sharir, and J. Snoeyink, *Counting and cutting cycles of lines and rods in space*, Comput. Geom. **1** (1992), no. 6, 305–323.

[6] György Elekes, Haim Kaplan, and Micha Sharir, *On lines, joints, and incidences in three dimensions*, J. Combin. Theory Ser. A **118** (2011), no. 3, 962–977.

[7] Sharona Feldman and Micha Sharir, *An improved bound for joints in arrangements of lines in space*, Discrete Comput. Geom. **33** (2005), no. 2, 307–320.

[8] Márton Hablicsek, *On the joints problem with multiplicities*, arXiv:1408.5791.
[9] Marina Iliopoulou, *Counting joints with multiplicities*, Proc. Lond. Math. Soc. (3) **106** (2013), no. 3, 675–702.
[10] ______, *Discrete analogues of Kakeya problems*, ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–The University of Edinburgh (United Kingdom).
[11] ______, *Counting multijoints*, J. Combin. Theory Ser. A **136** (2015), 143–163.
[12] ______, *Incidence bounds on multijoints and generic joints*, Discrete Comput. Geom. **54** (2015), no. 2, 481–512.
[13] Haim Kaplan, Micha Sharir, and Eugenii Shustin, *On lines and joints*, Discrete Comput. Geom. **44** (2010), no. 4, 838–843.
[14] René Quilodrán, *The joints problem in* $\mathbb{R}^n$, SIAM J. Discrete Math. **23** (2009/10), no. 4, 2211–2213.
[15] Micha Sharir, *On joints in arrangements of lines in space and related problems*, J. Combin. Theory Ser. A **67** (1994), no. 1, 89–99.
[16] Micha Sharir and Emo Welzl, *Point-line incidences in space*, Combin. Probab. Comput. **13** (2004), no. 2, 203–220.
[17] Thomas Wolff, *Recent work connected with the Kakeya problem*, Prospects in mathematics (Princeton, NJ, 1996), Amer. Math. Soc., Providence, RI, 1999, pp. 129–162.
[18] Ben Yang, *Generalizations of Joints Problem*, arXiv **1606.08525** (2016).
[19] Ruixiang Zhang, *A proof of the multijoints conjecture and Carbery's generalization*, arXiv **1612:05717** (2016).