OPTIMALITY OF ℓ^p DECOUPLING FOR THE LIGHT CONE IN $\mathbb{R}^{n+1}$

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Let S_R be the truncated light cone in $\mathbb{R}^{n+1}$ given by

$$S_R := \{(\xi, |\xi|) \in \mathbb{R}^{n+1} \colon |\xi| \simeq R\}.$$

Let $\mathcal{B}_R$ be a covering of S_R by finitely overlapping rectangular boxes in $\mathbb{R}^{n+1}$, of dimensions comparable to $1 \times R^{1/2} \times \cdots \times R^{1/2} \times R$, such that for every $\Theta \in \mathcal{B}_R$, the center c_Θ of Θ lies on S_R, and the longest side of Θ is parallel to the line connecting the origin to c_Θ, whereas the shortest side of Θ is parallel to the normal to S_R at c_Θ. Let $D_p(R)$ be the best constant so that the inequality

$$\Big\| \sum_{\Theta \in \mathcal{B}_R} F_\Theta \Big\|_{L^p(\mathbb{R}^{n+1})} \leq D_p(R) \left(\sum_{\Theta \in \mathcal{B}_R} \|F_\Theta\|_{L^p(\mathbb{R}^{n+1})}^p \right)^{1/p} \tag{1}$$

holds for all family of functions $\{F_\Theta\}_{\Theta \in \mathcal{B}_R}$ on $\mathbb{R}^{n+1}$ so that the Fourier support of F_Θ is contained in Θ for every $\Theta \in \mathcal{B}_R$. Below are examples showing that

$$D_p(R) \gtrsim \max\{R^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{p}}, R^{\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\}. \tag{2}$$

Hence the ℓ^p decoupling inequality of Bourgain and Demeter is optimal up to R^ε loss, for $2 \leq p \leq \infty$.

We begin with some preparation.

Let $\mathcal{P}_R$ be a covering of the annulus $\{|\xi| \simeq R\}$ by finitely overlapping rectangles in $\mathbb{R}^n$, of dimensions $R^{1/2} \times \cdots \times R^{1/2} \times R$, such that for every $\theta \in \mathcal{P}_R$, the center c_θ of θ lies in the annulus $\{|\xi| \simeq R\}$, and the longest side of θ is parallel to the line connecting the origin to c_θ. We write $e_\theta := \frac{c_\theta}{|c_\theta|}$.

Let $\phi(x)$ be a non-negative Schwartz function on $\mathbb{R}^n$, with $\widehat{\phi}(\xi)$ compactly supported on $[-1/2, 1/2]^n$, so that $\phi(x) \geq 1$ for $|x| \leq 1$. Let $\eta(t)$ be a Schwartz function on $\mathbb{R}$ whose Fourier transform is supported on $[-c, c]$ for some small absolute constant $c > 0$, and for which $|\eta(t)| \geq 1$ for all $t \in [-1, 1]$.

Step 1. We prove

$$D_p(R) \gtrsim R^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{p}}. \tag{3}$$

Let δ_R be the dilation $\delta_R(x) = (Rx_1, R^{1/2}x')$. For $\theta \in \mathcal{P}_R$, let L_θ be the rotation that rotates the long side of θ to $(1, 0, \ldots, 0)$, and rotates the short sides of θ to the remaining coordinate directions in $\mathbb{R}^n$. Define

$$f_\theta(x) := e^{2\pi i c_\theta \cdot x}\phi(\delta_R L_\theta x).$$

The Fourier transform of f_θ is

$$\det(\delta_R)^{-1}\widehat{\phi}(\delta_R^{-1}L_\theta(\xi - c_\theta))$$

which is supported on $c_\theta + L_\theta^{-1}\delta_R[-1/2, 1/2]^n \simeq \theta$. The (space-time) Fourier transform of

$$\eta(t)e^{it\sqrt{-\Delta}}f_\theta(x)$$

is supported in a rectangular box Θ of dimensions $1 \times R^{1/2} \times \cdots \times R^{1/2} \times R$ like one of those in $\mathcal{B}_R$, so if (1) were to hold at an exponent p, then

$$\left\|\eta(t)\sum_{\theta\in\mathcal{P}_R} e^{it\sqrt{-\Delta}}f_\theta(x)\right\|_{L^p(\mathbb{R}^{n+1})} \leq D_p(R)\left(\sum_{\theta\in\mathcal{P}_R}\|\eta(t)e^{it\sqrt{-\Delta}}f_\theta(x)\|^p_{L^p(\mathbb{R}^{n+1})}\right)^{1/p}. \tag{4}$$

But

$$\begin{aligned} e^{it\sqrt{-\Delta}}f_\theta(x) &= \int_{\mathbb{R}^n}\det(\delta_R)^{-1}\widehat{\phi}(\delta_R^{-1}L_\theta(\xi - c_\theta))e^{2\pi i(t|\xi|+x\cdot\xi)}d\xi \\ &= e^{2\pi i(x\cdot c_\theta + t|c_\theta|)}\int_{\mathbb{R}^n}\widehat{\phi}(\xi)e^{2\pi i\delta_R L_\theta(x - te_\theta)\cdot\xi}e^{2\pi it(|c_\theta + L_\theta^{-1}\delta_R\xi| - |c_\theta| - \delta_R L_\theta e_\theta\cdot\xi)}d\xi \end{aligned} \tag{5}$$

Writing

$$e^{2\pi it(|c_\theta + L_\theta^{-1}\delta_R\xi| - |c_\theta| - \delta_R L_\theta e_\theta\cdot\xi)} = 1 + O(tR^{-1})$$

on the support of ϕ, we see that

$$e^{it\sqrt{-\Delta}}f_\theta(x) = e^{2\pi i(x\cdot c_\theta + t|c_\theta|)}\phi(\delta_R L_\theta(x - te_\theta)) + O(tR^{-1}).$$

Hence there exists some $c \in (0, 1)$ so that

$$\inf_{|x|\leq cR^{-1}, |t|\leq cR^{-1}} \operatorname{Re} e^{it\sqrt{-\Delta}}f_\theta(x) \geq \frac{1}{2}.$$

It follows that

$$\operatorname{Re}\sum_{\theta\in\mathcal{P}_R} e^{it\sqrt{-\Delta}}f_\theta(x) \gtrsim |\mathcal{P}_R| \simeq R^{\frac{n-1}{2}} \quad \text{whenever } |x| \leq cR^{-1}, |t| \leq cR^{-1}.$$

Hence the left hand side of (4) is at least

$$\gtrsim R^{\frac{n-1}{2}}R^{-\frac{n+1}{p}}. \tag{6}$$

On the other hand, (5) shows that

$$|e^{it\sqrt{-\Delta}}f_\theta(x)| \lesssim_N |t|^N(1 + R|x\cdot e_\theta - t| + R^{1/2}|x\cdot e_\theta^\perp|)^{-N}$$

for any unit vector $e_\theta^\perp$ orthogonal to e_θ and any positive integer N, because one can write

$$e^{2\pi i\delta_R L_\theta(x - te_\theta)\cdot\xi} = \frac{\Delta_\xi e^{2\pi i\delta_R L_\theta(x - te_\theta)\cdot\xi}}{|\delta_R L_\theta(x - te_\theta)|^2}$$

and integrate by parts; note that

$$\begin{aligned}|\delta_R L_\theta(x-te_\theta)| &\simeq R|L_\theta(x-te_\theta)\cdot(1,0,\dots,0)| + R^{1/2}|L_\theta(x-te_\theta)\cdot(0,1,0,\dots,0)| + \dots\\ &= R|(x-te_\theta)\cdot e_\theta| + R^{1/2}|(x-te_\theta)\cdot L_\theta^{-1}(0,1,0,\dots)| + \dots\\ &\geq R|x\cdot e_\theta - t| + R^{1/2}|x\cdot e_\theta^\perp|\end{aligned}$$

for any choice of $e_\theta^\perp$. Thus

$$\|\eta(t)e^{it\sqrt{-\Delta}}f_\theta(x)\|_{L^p(\mathbb{R}^{n+1})} \lesssim R^{-\frac{n+1}{2p}},$$

and the right hand side of (4) is

$$\lesssim D_p(R)\left(R^{\frac{n-1}{2}}\right)^{\frac{1}{p}} R^{-\frac{n+1}{2p}}. \tag{7}$$

Combining the lower bound (6) and the upper bound (7), we see that

$$D_p(R) \gtrsim \frac{R^{\frac{n-1}{2}}R^{-\frac{n+1}{p}}}{\left(R^{\frac{n-1}{2}}\right)^{\frac{1}{p}} R^{-\frac{n+1}{2p}}} = R^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{p}},$$

as was to be proved.

Remark. *Sometimes such a $\eta(t)e^{it\sqrt{-\Delta}}f_\theta(x)$ (or translations thereof in physical space-time) is called a wave packet. It has a designated compact support in the frequency space, and is concentrated in the physical space-time to the extent possible by the uncertainty principle (in this case, to a rectangular box in space-time of dimensions $R^{-1}\times R^{-1/2}\times\dots\times R^{-1/2}\times 1$. In addition, in this example it is like a plane wave propagating with velocity c_θ inside the rectangular box.*

Step 2. We prove

$$D_p(R) \gtrsim R^{\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}. \tag{8}$$

For $\theta\in\mathcal{P}_R$, let

$$f_\theta(x) := \varepsilon_\theta e^{2\pi i c_\theta\cdot x}\phi(x)$$

where ε_θ is a random choice of ±1 and c_θ is the center of θ so that $\operatorname{supp}\widehat{f_\theta}\subset\theta$. The (space-time) Fourier transform of

$$\varepsilon_\theta\eta(t)e^{it\sqrt{-\Delta}}f_\theta(x)$$

is supported in a rectangular box Θ of dimensions $1\times R^{1/2}\times\dots\times R^{1/2}\times R$ like one of those in $\mathcal{B}_R$, so if (1) were to hold at an exponent p, then

$$\left\|\eta(t)\sum_{\theta\in\mathcal{P}_R}\varepsilon_\theta e^{it\sqrt{-\Delta}}f_\theta(x)\right\|_{L^p(\mathbb{R}^{n+1})} \leq D_p(R)\left(\sum_{\theta\in\mathcal{P}_R}\|\eta(t)e^{it\sqrt{-\Delta}}f_\theta(x)\|^p_{L^p(\mathbb{R}^{n+1})}\right)^{1/p}. \tag{9}$$

We evaluate the expectation of the p-th power of both sides as the sign ε_θ varies. Then Klintchine's inequality gives

$$\Big\|\eta(t)\Big(\sum_{\theta\in\mathcal{P}_R}|e^{it\sqrt{-\Delta}}f_\theta(x)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^{n+1})} \le D_p(R)\left(\sum_{\theta\in\mathcal{P}_R}\|\eta(t)e^{it\sqrt{-\Delta}}f_\theta(x)\|^p_{L^p(\mathbb{R}^{n+1})}\right)^{1/p}. \tag{10}$$

But

$$\begin{aligned} e^{it\sqrt{-\Delta}}f_\theta(x) &= \varepsilon_\theta\int_{\mathbb{R}^n}\widehat{\phi}(\xi)e^{2\pi i[(\xi+c_\theta)\cdot x+t|\xi+c_\theta|]}d\xi \\ &= \varepsilon_\theta e^{2\pi i(x\cdot c_\theta+t|c_\theta|)}\int_{\mathbb{R}^n}\widehat{\phi}(\xi)e^{2\pi i[\xi\cdot(x+te_\theta)]}e^{2\pi it(|\xi+c_\theta|-|c_\theta|-\xi\cdot e_\theta)}d\xi. \end{aligned} \tag{11}$$

By expanding the last exponential in the integral as 1+ error, we have

$$e^{it\sqrt{-\Delta}}f_\theta(x) = \varepsilon_\theta e^{2\pi i(x\cdot c_\theta+t|c_\theta|)}\phi(x+te_\theta)+O(|t|).$$

Since $\phi(x)\ge 1$ for $|x|\le 1$, we see that

$$|e^{it\sqrt{-\Delta}}f_\theta(x)|\ge\frac{1}{2}$$

whenever $|(x,t)|\le c$ for some small positive constant c (which depends on ϕ, but which crucially is independent of R). The left hand side of (10) is thus

$$\gtrsim|\mathcal{P}_R|^{1/2}\simeq R^{\frac{n-1}{2}\cdot\frac{1}{2}}. \tag{12}$$

To estimate the right hand side of (10), note that from (11), we have

$$|e^{it\sqrt{-\Delta}}f_\theta(x)|\lesssim_N |t|^N(1+|x+te_\theta|)^{-N}$$

for every non-negative integer N. As a result,

$$\|\eta(t)e^{it\sqrt{-\Delta}}f_\theta(x)\|_{L^p(\mathbb{R}^{n+1})}\lesssim 1,$$

which shows that the right hand side of (10) is

$$\lesssim D_p(R)|\mathcal{P}_R|^{1/p}\simeq D_p(R)R^{\frac{n-1}{2}\cdot\frac{1}{p}}. \tag{13}$$

The lower bound (12) and the upper bound (13) together implies (8), as desired.

Remark. *Another way of seeing that* (8) *holds is to take* F_Θ *to be a tiling of* $B(0,1)\times[0,1]$ *with wave packets of sizes* $R^{-1}\times R^{-1/2}\times\cdots\times R^{-1/2}\times 1$ *whose Fourier support fill up* Θ*. By randomizing the signs of* F_Θ *as above, we yield* (8)*.*