MAT 102 SPRING 2008
REVIEW FOR THE FINAL

This is a review for the whole course, and contains a collection of some random questions that can be tackled using what has been covered. Good luck on your final exam!

1. Highlights of the course

- Increasing/decreasing functions and simple graphing
- Optimization
- L'Hospital's rule
- Definite/indefinite integrals
- Areas under curves
- (Very) simple initial value problems / differential equations
- Techniques in integration: substitution (including trigonometric ones), integration by parts, partial fractions, trigonometric integrals, and other assorted techniques (like completing the square to facilitate a change of variable when one integrates a rational function; splitting the integrals)

2. Key concepts

Can you define the following terms rigorously?

- critical point of a function
- increasing function
- local maximum of a function
- global maximum of a function (and an illustrative example that demonstrates the difference between global and local maximums?)
- a function that is concave up
- tangent line for a graph
- antiderivative of a function
- indefinite integral of a function
- definite integral of a function
- area under a curve

Can you write down (in your own words) what the following theorems tell you, and how they can be applied? Illustrate with a simple example in each case.

- the first derivative test
- the second derivative test
- the procedure for finding global maxima
- the L'Hospital's rule
- the fundamental theorem of calculus (both first and second!)
- the chain rule
- integration by parts

Can you explain, in your own words, when the following techniques of integration can be used, and how? Give a simple illustrative example of your own in each case.

- substitution
- trigonometric substitution

- partial fractions
- trigonometric identities
- integration by parts
- other techniques (like what?)

Do you remember how many techniques you have just listed? These are the tools that help you, so you may want to know them well enough that you can come up with them on your own!

Can you list all the basic integral formula that we have come across? (Including the ones with the hyperbolic sines, cosines, etc)

Can you list all the trigonometric identities that we have come across? (Again, don't forget those for the hyperbolic ones)

Can you explain how in general you manipulate rational functions (like how you add them, multiply them, resolve them into partial fractions, etc)?

Can you list all the basic trigonometric substitutions?

3. Practice exercises

1. You are standing at the edge of a slow-moving river which is 5 miles wide and wish to return to your campground on the opposite side of the river. You can swim at 2 mph and walk at 3 mph. You must first swim across the river to any point on the opposite bank. From there walk to the campground, which is 10 miles from the point directly across the river from where you start your swim. What route will take the least amount of time?
2. Let $f(t)$ be the fortune of your best friend at time t. Suppose
$$f(t) = \left(1 + \frac{2}{t}\right)^t$$
for $0 < t < \infty$. Show that the fortune is increasing, and compute the limit of the fortune as t goes to positive infinity. Does it become infinite too?
3. Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum.
4. Graph $f(x) = x \ln |x|$. (What do you need to do? What concepts are relevant here?)
5. Find all local and global minima and maxima of $f(x) = x^2 \ln x$ on the interval $e^{-3} \leq x \leq 1$. (Where are they achieved and what are the function values there?)
6. Let C be the curve $y = -x^2$ and L be the straight line $y = x - 2$. Make a rough sketch of the two figures on the same set of axes. Then find the area of the region lying below C and above L.
7. Compute the following limits:
   (a) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^3}$
   (b) $\displaystyle\lim_{x\to 0} x^{-2}(\ln(1-x) + x)$
   (c) $\displaystyle\lim_{x\to 0} x^{\sin x}$
   (d) $\displaystyle\lim_{x\to 0} \frac{1}{x^3} \int_0^x \frac{\sin^2 t}{1+t^4} dt$
8. Compute the total area of the propeller-shaped region enclosed by the curves $y = x^3$ and $y = x^5$. (Hint: Remember, you will always have to figure out where the curves intersect and draw a quick sketch of the regions involved...)
9. Find the area under the curve $y = \cos^2 2x \sin^4 2x$, from $x = 0$ to $x = \pi$.
10. Solve the following initial value problems:

(a) $\dfrac{dy}{dx} = \sinh x, \quad y(0) = -1$
(b) $\dfrac{dy}{dx} = x\cosh x, \quad y(0) = 0$
(c) $\dfrac{d^2y}{dt^2} = t\cos t, \quad y(0) = 1, \quad y'(0) = 0$
(d) $\dfrac{dy}{dx} = \cos^3 2x \sin^2 2x, \quad y(0) = 1$
(e) $\dfrac{dy}{dx} = \dfrac{2x^2+3x}{x^2+4x+5}, \quad y(0) = 0$
(f) $\dfrac{dy}{dx} = \sqrt{1-4x^2}, \quad y(0) = 1$
(g) $\dfrac{dy}{dx} = \dfrac{1}{\cos^4 x}, \quad y(0) = 1$

11. Compute the following integrals:
(a) $\displaystyle\int_0^{\pi/4} \sqrt{2+2\cos 4x}\,dx$
(b) $\displaystyle\int_1^4 \frac{x^2-3x+2}{\sqrt{x}}dx$
(c) $\displaystyle\int_{\pi/2}^{\pi} \cot\frac{x}{2}dx$
(d) $\displaystyle\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}}$
(e) $\displaystyle\int_{-2}^{-1} \frac{x^2+3}{x^3-2x^2+x}dx$
(f) $\displaystyle\int_0^{\pi/2} e^{\sin t}\cos(2\sin t)\cos t\,dt$
(g) $\displaystyle\int_1^2 \frac{2x^4-3x+2}{x(x^2+3x+2)}dx$
(h) $\displaystyle\int_0^{\pi} \frac{4\cos t\sin t + 12\sin t}{5\cos^2 t - 4\cos t + 4\sin^2 t}dt$
(i) $\displaystyle\int_{\ln 3}^{\ln 5} \frac{e^{3t}dt}{(e^t+1)(e^{2t}-3e^t+2)}$
(j) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{(\cos t+1)\sin t\cos t}{(\cos t-1)(3+2\cos t-\sin^2 t)}dt$

12. Let $f(x)$ be a smooth function, and $y = f(x)$ be its graph. Suppose at each point x the tangent line to the graph has slope $\sin x\cos 2x$. What can you say about $f(x)$?

13. If
$$\int_0^x f(t)dt = xe^x$$
for all real numbers x, find $f(1)$.

14. Compute
$$\frac{d}{dx}\int_0^{\sin^2 x} \sin(e^t(1-t))dt.$$

15. Let
$$A(t) = \int_0^t f(x)dx$$
for some smooth function $f(x)$. Show that if $A(t)$ is an increasing function of t, then $f(x) \geq 0$ for all x, and conversely.