MAT 102 SPRING 2008
REVIEW FOR THE FINAL - SOLUTIONS

1. Errata in the review exercises

- In Q6, the line $y = x - 5$ should be $y = x - 2$ instead.
- In Q11(j), the integral should be from $\frac{\pi}{2}$ to π (instead of from 0 to π).

Sorry for the typos!

2. Outline of solutions

Please kindly let me know of any typos. Thanks!

1. Let x miles be the distance you walk on the other side of the river. The total time you need to reach the campsite is
$$f(x) = \frac{1}{2}\sqrt{5^2 + (10 - x)^2} + \frac{x}{3}.$$
Minimize this, get $x = 10 - 2\sqrt{5}$.
2. Note that to show that $f(t)$ is increasing it suffices to show that $\ln f(t)$ is increasing. So one just needs to show $\frac{d}{dt} \ln f(t) > 0$. Next to compute $\lim_{t\to\infty} f(t)$, first compute
$$\lim_{t\to\infty} \ln f(t) = 2$$
using L'Hospitals, then exponentiate to get
$$\lim_{t\to\infty} f(t) = e^2.$$
3. The product is maximized when you multiply the square of 6 with 3.
4. Observe that the function is odd. Find the critical points, see where the function is increasing, etc.
5. The maximum occurs at $x = e^{-1/2}$, with value $f(e^{-1/2}) = -\frac{1}{2e}$.
6. First solve for the intersection of the two graphs: they intersect when
$$-x^2 = x - 2,$$
i.e. $x = -2$ or $x = 1$. Next, observe that the graph $y = -x^2$ lies above that of $y = x - 2$ when $-2 \le x \le 1$. Hence the area enclosed between them is
$$\int_{-2}^{1} ((-x^2) - (x - 2))dx = \frac{9}{2}.$$
7. (a) $-1/6$ (b) $-1/2$ (c) 1 (d) $1/3$
8. The area is
$$2\int_0^1 (x^3 - x^5)dx = \frac{1}{6}.$$
9. The area is
$$\int_0^{\pi} \cos^2 2x \sin^4 2x dx = \frac{\pi}{16}.$$
10. (a) $y = \cosh x - 2$
(b) $y = x \sinh x - \cosh x + 1$
(c) $y = -t \cos t + 2 \sin t - t + 1$
(d) $y = \frac{1}{6}\sin^3 2x - \frac{1}{10}\sin^5 2x + 1$

(e) $y = 2x - \frac{5}{2}\ln(x^2+4x+5) + \frac{5}{2}\ln 5$

(f) $y = \frac{1}{4}\sin(2\sin^{-1}(2x)) + \frac{1}{4}\sin^{-1}(2x) + 1$

(or $y = \frac{1}{2}x\sqrt{1-4x^2} + \frac{1}{4}\sin^{-1}(2x) + 1$)

(g) $y = \tan x \sec^2 x - \frac{2}{3}\tan^3 x + 1$

11. (a) 1 (b) $\frac{12}{5}$ (c) $\ln 2$ (d) $\ln(1+\sqrt{2})$ (e) $\frac{2}{3} + 2\ln 3 - 5\ln 2$
(f) $\frac{e\cos 2}{5} + \frac{2e\sin 2}{5} - \frac{1}{5}$ (g) $-3 + 24\ln 2 - 12\ln 3$ (h) $\frac{40}{3} - 4\ln 3$ (i) $\frac{3}{2}\ln 3 - \frac{2}{3}\ln 2$
(j) $\frac{\pi}{20} - \frac{\ln 2}{10}$

12. $f'(x) = \sin x \cos 2x$, so

$$f(x) = \frac{1}{2}\cos x - \frac{1}{6}\cos 3x + C$$

for some constant C, and any constant C will do because no initial condition is specified here.

13. Differentiate the identity with respect to x, get $f(x) = \frac{d}{dx}(xe^x) = (x+1)e^x$. Hence $f(1) = 2e$.

14.

$$\begin{aligned}\frac{d}{dx}\int_0^{\sin^2 x} \sin(e^t(1-t))dt &= \sin(e^{\sin^2 x}(1-\sin^2 x))\frac{d}{dx}\sin^2 x \\ &= 2\sin(e^{\sin^2 x}\cos^2 x)\sin x\cos x.\end{aligned}$$

15. $A(t)$ is increasing precisely when $A'(t) \geq 0$ for all t. However, by the fundamental theorem of calculus, $A'(t) = f(t)$. Hence $A(t)$ is increasing precisely when $f(t) \geq 0$ for all t (which is the same as saying $f(x) \geq 0$ for all x).